Bull. Aust. Math. Soc. (First published online 2025), page 1 of 11*
doi:10.1017/S0004972725000036
*Provisional—final page numbers to be inserted when paper edition is published

IRRATIONALITY OF ZEROS OF POLYGAMMA FUNCTIONS

PREETI GOYAT

(Received 15 November 2024; accepted 2 January 2025)

Abstract

Our work owes its origin to a recent note of Ram Murty ['Irrationality of zeros of the digamma function', *Number Theory in Memory of Eduard Wirsing* (eds. H. Maier, R. Steuding and J. Steuding) (Springer, Cham, 2023), 237–243], in which he proves that all the zeros of the digamma function are irrational with at most one possible exception. We extend this investigation to higher-order polygamma functions.

2020 *Mathematics subject classification*: primary 11J72; secondary 11J81, 11J86.

Keywords and phrases: polygamma function, zeros, irrationality.

1. Introduction

A meromorphic function on $\mathbb{C}$ is said to be transcendental if it is transcendental over the field of rational functions $\mathbb{C}(z)$. It is a guiding principle in number theory that naturally occurring transcendental functions take transcendental values at algebraic points, with obvious exceptions. The obvious exceptions typically emerge from purely analytic reasons, for example to cancel poles of some other transcendental function.

We investigate the zeros of derivatives of the classical Gamma function, $\Gamma(z)$. It is a classical result of Hölder [7] that the Gamma function does not satisfy any algebraic differential equation whose coefficients are rational functions over the complex numbers. This lack of a differential structure is one of the reasons that makes investigating the nature of values of the Gamma function rather difficult.

Our work is motivated by a recent elegant note of Murty [10] in which he proves the following result for the digamma function $\psi(z) = \psi_0(z) = \Gamma'(z)/\Gamma(z)$.

THEOREM 1.1. *All zeros of the digamma function are irrational with at most one possible exception.*

The proof involves tools from analytic number theory as well as the theory of linear forms in logarithms of algebraic numbers. In fact, the fugitive exceptional rational zero is not expected to exist.

We extend this investigation to the higher-order polygamma functions. For a nonnegative integer k, the polygamma function $\psi_k(z)$ is the kth derivative of $\psi(z)$. Thus,

$$\psi_k(z) = (-1)^{k+1} k! \sum_{n=0}^{\infty} \frac{1}{(n+z)^{k+1}}.$$

We prove the following results.

THEOREM 1.2. *Let k be a positive integer.*

(a) *The function ψ_{2k+1} has no real zeros. In particular, there are no rational zeros.*
(b) *The real zeros of ψ_{2k} lie in the intervals $(-m + 1/2, -m + 114/227)$ for all $m > 0$.*

THEOREM 1.3. *Concerning the complex zeros of the function ψ_1:*

(a) *ψ_1 has no zeros in $\mathbb{Z}[i]$; and*
(b) *ψ_1 has no zeros in $\{z : \mathrm{Re}(z) \geq 0\}$.*

It would be interesting to study the complex zeros of ψ_{2k+1} and, in particular, to see whether ψ_1 has any zeros in $\mathbb{Q}(i)$.

THEOREM 1.4. *For an integer $q > 1$, consider the vector spaces over $\mathbb{Q}$,*

$$V_o(q) := \mathrm{span}\{\psi_{2k+1}(a/q) : k \geq 0, 1 \leq a \leq q, (a, q) = 1\}$$

and

$$V_e(q) := \mathrm{span}\{\psi_{2k}(a/q) : k \geq 1, 1 \leq a \leq q, (a, q) = 1\}.$$

Let V_o and V_e be the $\mathbb{Q}$ vector spaces generated by $V_o(q)$ and $V_e(q)$, respectively, over all q. Then both V_o and V_e have infinite dimension.

We prove a stronger theorem (Theorem 5.1) of which Theorem 1.4 is an immediate consequence. As we see, the parity of k plays an important role in studying the zeros of ψ_k. We formulate an analogue of Theorem 1.1 and an irrationality result for $\psi_k(x)$ in Section 6. In place of Baker's theorem on linear forms in logarithms, we appeal to a strong version of a conjecture by Chowla and Milnor on the linear independence of values of the Hurwitz zeta function.

The numbers $\psi_k(a/q)$ for $k > 0$ occur naturally in the context of special values of periodic L-functions, via the Hurwitz zeta function. The digamma function also occurs in the recent work of Radchenko and Zagier [11] on the series expansion of the mysterious Herglotz function. Further related results are given by David *et al.* [4], who study the linear independence of values of the Lerch functions $\Phi_s(x, z) = \sum_{k=0}^{\infty} z^{k+1}/(k + x + 1)^s$ at algebraic points and give a criterion for the linear independence of the numbers $\Phi_i(x, \alpha_j)$ at algebraic points α_j.

https://doi.org/10.1017/S0004972725000036 Published online by Cambridge University Press

2. Prerequisites

The polygamma function, $\psi_k(z)$, is a meromorphic function on $\mathbb{C}$ defined by

$$\psi_k(z) = \frac{d^{k+1}}{dz^{k+1}} \log\Gamma(z) = (-1)^{k+1}k! \sum_{n=0}^{\infty} \frac{1}{(n+z)^{k+1}}, \quad z \neq 0, -1, -2, \ldots.$$

If m is a natural number, then

$$\psi_k(m) = \psi_k(1) + (-1)^{k+2}k! \sum_{n=1}^{m-1} \frac{1}{(n+1)^{k+1}}.$$

The digamma function, $\psi(z) = \psi_0(z) = \Gamma'(z)/\Gamma(z)$, satisfies the functional equations

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \quad \psi(1-z) = \psi(z) + \pi\cot\pi z.$$

Consequently, the polygamma function satisfies the functional equations

$$\psi_k(z+1) = \psi_k(z) + \frac{(-1)^k k!}{z^{k+1}}, \quad (-1)^k\psi_k(1-z) = \psi_k(z) + \frac{d^k}{dz^k}(\pi\cot\pi z).$$

Recall that, for z not an integer,

$$\pi\cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\Big(\frac{1}{z-n} + \frac{1}{z+n}\Big).$$

We end this section by listing some irrationality results relevant to our work. In 1978, Apéry astonished the mathematics community by proving the irrationality of the function $\zeta(3)$. Following that, Ball and Rivoal [2, 12] made the next remarkable breakthrough.

THEOREM 2.1 (Ball and Rivoal, [2]). *Let $\epsilon > 0$. For any $s \geq 3$ odd and sufficiently large with respect to ϵ,*

$$\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{1, \zeta(3), \zeta(5), \ldots, \zeta(s)\} \geq \frac{1-\epsilon}{1+\log 2}\log s.$$

Fischler *et al.* [5] recently proved the following result.

THEOREM 2.2. *Let $\epsilon > 0$ and $s \geq 3$ be an odd integer sufficiently large with respect to ϵ. Then, among the numbers $\zeta(3), \zeta(5), \ldots, \zeta(s)$, at least $2^{(1-\epsilon)\log s/\log\log s}$ are irrational.*

3. Proof of Theorem 1.2

We begin with the odd case. For $z = x + iy$ with $x, y \in \mathbb{R}$,

$$\begin{aligned}\psi_{2k+1}(x+iy) &= (2k+1)! \sum_{n=0}^{\infty} \frac{(n+x-iy)^{2k+2}}{((n+x)^2+y^2)^{2k+2}} \\ &= (2k+1)! \sum_{n=0}^{\infty} \frac{\sum_{r=0}^{2k+2}\binom{2k+2}{r}(n+x)^{2k+2-r}(-iy)^r}{((n+x)^2+y^2)^{2k+2}}.\end{aligned}$$

https://doi.org/10.1017/S0004972725000036 Published online by Cambridge University Press

The real part of $\psi_{2k+1}(x+iy)$ is given by

$$(2k+1)!\Big[\sum_{n=0}^{\infty}\frac{\sum_{r=0,4|r}^{2k+2}\binom{2k+2}{r}(n+x)^{2k+2-r}(y)^r}{((n+x)^2+y^2)^{2k+2}}\Big]$$
$$-(2k+1)!\Big[\sum_{n=0}^{\infty}\frac{\sum_{r=2,4|(r-2)}^{2k+2}\binom{2k+2}{r}(n+x)^{2k+2-r}(y)^r}{((n+x)^2+y^2)^{2k+2}}\Big].$$

When z is a real number (which is not a pole), that is, when $y=0$,

$$\mathrm{Re}\,\psi_{2k+1}(x)=(2k+1)!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{2k+2}}>0.$$

Consequently, $\psi_{2k+1}(x)\neq 0$ and so ψ_{2k+1} has no real zeros.

We now turn to the more involved case when k is even. We see that $\psi_{2k}(x)$ has no real zeros with $x\geq 0$. However, unlike the odd case, ψ_{2k} does have real zeros. We divide our investigation into two cases.

Case 1: ψ_{2k} has no real zeros in the intervals $(-m,-m+1/2]$ for all integers $m\geq 1$. The derivative of ψ_{2k} is given by

$$\psi_{2k+1}(x)=(2k+1)!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{2k+2}},$$

which is positive. Thus, the function ψ_{2k} is a strictly increasing function of x in each of the intervals $I_n=(-n,-n+1)$ for $n=1,2,3,\ldots$. Appealing to the functional equation, we show that $\psi_{2k}(-m+1/2)<0$. This, along with the strict monotonicity noted above, shows that ψ_{2k} has no zero in $(-m,-m+1/2]$ for all integers $m\geq 1$.

To this end, we observe that

$$\psi_{2k}\Big(-m+\frac{1}{2}\Big)=-2k!\sum_{n=0}^{\infty}\frac{1}{(n-m+1/2)^{2k+1}}=-2k!\;2^{2k+1}\sum_{n=0}^{\infty}\frac{1}{(2(n-m)+1)^{2k+1}}$$
$$=-2k!\;2^{2k+1}\Big[\Big(\sum_{n=0}^{m-1}+\sum_{n=m}^{2m-1}+\sum_{n=2m}^{\infty}\Big)\frac{1}{(2(n-m)+1)^{2k+1}}\Big].$$

The terms in the first two sums cancel in pairs and we are left with

$$\psi_{2k}\Big(-m+\frac{1}{2}\Big)=2k!\;2^{2k+1}\Big[-\frac{1}{(2m+1)^{2k+1}}-\frac{1}{(2m+3)^{2k+1}}-\cdots\Big]<0,$$

which proves the claim.

Case 2: Real zeros in intervals of the form $(-m+1/2,-m+1)$. Consider

$$\frac{-p}{q}=-m+\frac{a}{q},\quad \frac{q}{2}<a\leq q-1,$$

https://doi.org/10.1017/S0004972725000036 Published online by Cambridge University Press

where the denominator q is an odd positive integer. We show that

$$\psi_{2k}\Big(-m+\frac{q+1}{2q}\Big)>0$$

for $q \le 228$. This, along with the fact that ψ_{2k} is strictly increasing in each interval $(-m, 1-m)$, ensures that

$$\psi_{2k}\Big(-m+\frac{a}{q}\Big)\ge\psi_{2k}\Big(-m+\frac{114}{227}\Big)>0.$$

Since $2qm-q-1<2mq-q+1$,

$$\begin{aligned}
&\frac{\psi(-m+q+1/2q)}{2k!\,(2q)^{2k+1}}\\
&\quad\ge\Big[\frac{1}{(q-1)^{2k+1}}-\frac{1}{(q+1)^{2k+1}}-\frac{1}{(2qm+q+1)^{2k+1}}-\frac{1}{(2mq+3q+1)^{2k+1}}-\cdots\Big]\\
&\quad\ge\Big[\frac{1}{(q-1)^{2k+1}}-\frac{1}{(q+1)^{2k+1}}-\frac{1}{(2qm)^{2k+1}}-\frac{1}{(2q(m+1))^{2k+1}}-\cdots\Big]\\
&\quad>\Big[\frac{1}{(q-1)^{2k+1}}-\frac{1}{(q+1)^{2k+1}}-\frac{1}{(2q)^{2k+1}}\Big(\zeta(2k+1)-\sum_{n=1}^{m-1}\frac{1}{n^{2k+1}}\Big)\Big]\\
&\quad\ge\Big[\frac{1}{(q-1)^{2k+1}}-\frac{1}{(q+1)^{2k+1}}-\frac{1}{(2q)^{2k+1}}(\zeta(2k+1)-1)\Big]\\
&\quad\ge\Big[\frac{1}{(q-1)^{2k+1}}-\frac{1}{(q+1)^{2k+1}}-\frac{0.21}{(2q)^{2k+1}}\Big]\quad(\text{since } \zeta(2k+1)\le 1.21 \text{ for all } k\in\mathbb{N})\\
&\quad\ge\Big[\frac{1}{(q-1)^{2k+1}}-\frac{1}{(q+1)^{2k+1}}-\frac{0.02625}{q^{2k+1}}\Big].
\end{aligned}$$

We claim that the last quantity is positive for $q \le 228$ and all k. Consider

$$f(x)=\frac{1}{(x-1)^m}-\frac{1}{(x+1)^m}-\frac{0.02625}{x^m}.$$

Now, $f(x) > 0$ is equivalent to

$$1>\Big(1-\frac{2}{x+1}\Big)^m+0.02625\Big(1-\frac{1}{x}\Big)^m.$$

The last inequality holds for $x = 228$ and $m = 3$ and the function on the right-hand side is an increasing function of x, so it holds for all $x \le 228$. For this range of x, the function on the right-hand side is a decreasing function of m for $m \ge 3$. This implies that $f(x) > 0$ for $x \le 228$ and $m \ge 3$. Since q is an odd integer,

https://doi.org/10.1017/S0004972725000036 Published online by Cambridge University Press

$$\psi_{2k}\big(-m+\tfrac{114}{227}\big)>0.$$

So the real zeros of ψ_{2k} will lie in the intervals $(-m+1/2,-m+114/227)$ for all $m>0$.

Next, we show that ψ_{2k} will have a zero in the interval $(-n,-n+1)$ for all $n=1,2,3,\dots$. In the preceding calculations, we saw that

$$\psi_{2k}\Big(\frac{-p}{q}\Big)<0,\quad \text{where } \frac{-p}{q}=-m+\frac{a}{q}, a\le\frac{q}{2}$$

and

$$\psi_{2k}\Big(-m+\frac{a}{q}\Big)\ge\psi_{2k}\Big(-m+\frac{114}{227}\Big)>0.$$

Since $\psi_{2k}(x)$ is continuous and strictly increasing in each interval $I_n=(-n,-n+1)$ for $n=1,2,3,\dots$, it follows that $\psi_{2k}(x)$ has exactly one zero in each of these intervals.

REMARK 3.1. We have proved that ψ_{2k} has one real zero, say, x_m, in the interval $(-m+1/2,-m+114/227)$. Write $x_m=-m+y_m$, where $y_m\in(0,1)$.

We claim that the sequence $\{y_m\}$ is strictly decreasing and that $\psi_{2k}(-m+1/2)\to 0$ as $m\to\infty$.

PROOF. Suppose that $x_m=-m+y_m$ and $x_{m'}=-m'+y_{m'}$ are both zeros of ψ_{2k}, where $m>m'$ and $y_m\ge y_{m'}$. From the functional equation,

$$\begin{aligned}0&=\psi_{2k}(-m+y_m)-\psi_{2k}(-m'+y_{m'})\\&=\psi_{2k}(-m+y_m)-\psi_{2k}(-m+y_{m'})-\sum_{n=m'+1}^{m}\frac{2k!}{(-n+y_{m'})^{2k+1}}\\&=\psi_{2k}(-m+y_m)-\psi_{2k}(-m+y_{m'})+\sum_{n=m'+1}^{m}\frac{2k!}{(n-y_{m'})^{2k+1}}>0\end{aligned}$$

because ψ_{2k} is strictly increasing in $(-m,-m+1)$. This is a contradiction, so $y_m<y_{m'}$.

Now, we come to the second assertion,

$$\begin{aligned}\psi_{2k}\Big(-m+\frac{1}{2}\Big)&=-2k!\sum_{n=0}^{\infty}\frac{1}{(n-m+1/2)^{2k+1}}=-2k!\;2^{2k+1}\sum_{n=0}^{\infty}\frac{1}{(2(n-m)+1)^{2k+1}}\\&=2k!\;2^{2k+1}\Big[\sum_{n=0}^{m}\frac{1}{(2n+1)^{2k+1}}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k+1}}\Big]\to 0\end{aligned}$$

as $m\to\infty$ □

REMARK 3.2. It would be interesting to see whether the real zeros of ψ_{2k} and ψ_{2l} are distinct for $k\ne l$. If x_m, x'_m are the zeros of ψ_{2k} and ψ_{2l}, respectively, in $(-m,-m+1)$ and $l>k$, is $x_m>x'_m$? We have verified this for the zeros of ψ_2 and ψ_4 in $(-1,0)$.

https://doi.org/10.1017/S0004972725000036 Published online by Cambridge University Press

4. Proof of Theorem 1.3

We begin with a proof of Theorem 3(b). For $x \geq 0$,

$$\psi_1(x+iy) = \sum_{n=0}^{\infty} \frac{1}{(n+x+iy)^2} = \sum_{n=0}^{\infty} \frac{(n+x-iy)^2}{((n+x)^2+y^2)^2}$$
$$= \sum_{n=0}^{\infty} \frac{(n+x)^2-y^2}{((n+x)^2+y^2)^2} - 2iy \sum_{n=0}^{\infty} \frac{n+x}{((n+x)^2+y^2)^2}.$$

Now, $\operatorname{Im} \psi_1(x+iy)$ will be zero if and only if either

$$y = 0 \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{n+x}{((n+x)^2+y^2)^2} = 0.$$

If $y = 0$, then by Theorem 1.2, $\operatorname{Re} \psi_1(x+iy) \neq 0$, so $\psi_1(x+iy) \neq 0$. If $y \neq 0$, then

$$\sum_{n=0}^{\infty} \frac{n+x}{((n+x)^2+y^2)^2} > 0$$

for $x \geq 0$ and, again, $\operatorname{Im} \psi_1(x+iy) \neq 0$.

Now, we turn to part (a) of the theorem. It is enough to show that $\psi_1(-m+iy) \neq 0$, where $m \in \mathbb{N}, y \in \mathbb{R}$. As before,

$$\operatorname{Im} \psi_1(-m+iy) = -2y \sum_{n=0}^{\infty} \frac{n-m}{((n-m)^2+y^2)^2}.$$

Hence, $\operatorname{Im} \psi_1(-m+iy)$ will be zero if and only if either

$$y = 0 \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{n-m}{((n-m)^2+y^2)^2} = 0.$$

If $y = 0$, then $\psi_1(z)$ has a pole at $z = -m$, so this cannot be a zero of $\psi_1(z)$. If $y \neq 0$, then

$$\sum_{n=0}^{\infty} \frac{n-m}{((n-m)^2+y^2)^2} = \frac{m+1}{((m+1)^2+y^2)^2} + \frac{m+2}{((m+2)^2+y^2)^2} \cdots > 0,$$

because the first $2m$ terms of the series cancel in pairs (for $n = k$ and $n = 2m-k$), so $\operatorname{Im} \psi_1(-m+iy) \neq 0$. Hence, $\psi_1(z)$ has no zero in $\mathbb{Z} + i\mathbb{R}$. This completes the proof.

5. Proof of Theorem 1.4

THEOREM 5.1. *For positive integers N and $q > 2$, let $V_{o,N}(q)$ and $V_{e,N}(q)$ denote the $\mathbb{Q}$-vector spaces*

$$V_{o,N}(q) := \operatorname{span}_{\mathbb{Q}}\{\psi_{2k+1}(a/q) : 0 \leq k \leq N, 1 \leq a \leq q, (a,q) = 1\}$$

and

$$V_{e,N}(q) := \operatorname{span}_{\mathbb{Q}}\{1, \psi_{2k}(a/q) : 1 \leq k \leq N, 1 \leq a \leq q, (a,q) = 1\}.$$

Then $\dim_{\mathbb{Q}} V_{o,N}(q) \gg N$ *and* $\dim_{\mathbb{Q}} V_{e,N}(q) \gg \log N$.

https://doi.org/10.1017/S0004972725000036 Published online by Cambridge University Press

PROOF. We show that

$$\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{\psi_{2k+1}(a/q) : 0 \le k \le N, 1 \le a < q, (a,q) = 1\} \ge N+1.$$

By the definition of $\psi_k(z)$,

$$\psi_{2k+1}(a/q) = (2k+1)! \sum_{n=0}^{\infty} \frac{1}{(n+a/q)^{2k+2}} = (2k+1)!\, \zeta(2k+2, a/q).$$

Since

$$q^k \zeta(k) \prod_{p|q} (1 - p^{-k}) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \zeta(k, a/q),$$

it follows that $\operatorname{span}_{\mathbb{Q}}\{\zeta(2k+2, a/q) : 0 \le k \le N, 1 \le a < q, (a,q) = 1\}$ contains $\operatorname{span}_{\mathbb{Q}}\{\zeta(2k+2) : 0 \le k \le N\}$. Now, $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{\zeta(2k+2) : 0 \le k \le N\} = N+1$, so we get the required result.

Similarly,

$$\begin{aligned} &\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{1, \psi_{2k}(a/q) : 1 \le k \le N, 1 \le a < q, (a,q) = 1\} \\ &\quad = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{1, \zeta(2k+1, a/q) : 1 \le k \le N, 1 \le a < q, (a,q) = 1\} \\ &\quad \ge \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}}\{1, \zeta(2k+1) : 1 \le k \le N\}, \end{aligned}$$

and the theorem of Ball and Rivoal (Theorem 2.1) gives the desired result. □

Theorem 1.4 and the following corollary are immediate consequences of Theorem 5.1.

COROLLARY 5.2. *Let $k, q > 1$ be integers. Then the $\mathbb{Q}$-linear space generated by*

$$\{\psi_k(a/q) : k \ge 1, 1 \le a < q, (a,q) = 1\}$$

has infinite dimension over $\mathbb{Q}$.

6. The conjecture of Chowla–Milnor and its ramifications

Gun *et al.* [6] discuss several conjectures on the linear independence of values of the Hurwitz zeta function. The starting point is a question of P. and S. Chowla.

CONJECTURE 6.1 (Chowla and Chowla, [3]). Let p be any prime and let f be any rational-valued periodic function with period p. Then

$$L(2, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^2} \ne 0,$$

except for the case when

$$f(1) = f(2) = \cdots = f(p-1) = \frac{f(p)}{1-p^2}.$$

https://doi.org/10.1017/S0004972725000036 Published online by Cambridge University Press

Milnor interpreted this conjecture in terms of the linear independence of the Hurwitz zeta function and generalised it for all $k > 1$.

CONJECTURE 6.2 (Milnor, [9]). For any integer $k > 1$, the real numbers

$$\zeta(k, 1/p), \zeta(k, 2/p), \ldots, \zeta(k, (p-1)/p)$$

are linearly independent over $\mathbb{Q}$.

Further, for q not necessarily prime, Milnor suggested the following generalisation of the Chowla conjecture.

CONJECTURE 6.3 (Chowla–Milnor, [9]). Let $k > 1, q > 2$ be integers. Then the $\phi(q)$ real numbers $\zeta(k, a/q)$, for $(a, q) = 1, 1 \le a \le q$, are linearly independent over $\mathbb{Q}$.

Following Gun *et al.* [6], for any integer $k > 1$, define the $\mathbb{Q}$-linear space $V_k(q)$ by

$$V_k(q) = \mathrm{span}_{\mathbb{Q}}\{\zeta(k, a/q) : 1 \le a \le q, (a, q) = 1\}.$$

The Chowla–Milnor conjecture asserts that the dimension of $V_k(q)$ for $k > 1$ is $\phi(q)$. Gun *et al.* [6] gave the following nontrivial lower bound for this dimension.

THEOREM 6.4 [6]. *Let $k > 1, q > 2$. Then $\dim_{\mathbb{Q}} V_k(q) \ge \phi(q)/2$.*

They also proposed a stronger version of the Chowla–Milnor conjecture.

CONJECTURE 6.5 (Strong Chowla–Milnor; see [6]). For any $k, q > 1$, the $\phi(q) + 1$ real numbers 1 and $\zeta(k, a/q)$ with $1 \le a \le q, (a, q) = 1$ are linearly independent over the rational numbers.

We now indicate the relationship between the Chowla–Milnor conjecture and our investigation, which is facilitated by the relationship

$$\psi_k\left(\frac{a}{q}\right) = (-1)^{k+1} k! \sum_{n=0}^{\infty} \frac{1}{(n + a/q)^{k+1}} = (-1)^{k+1} k!\, \zeta(k+1, a/q).$$

Consequently, the Chowla–Milnor conjecture is equivalent to the assertion that, for any $k > 1$, the set

$$\{\psi_k(a/q) : 1 \le a \le q, (a, q) = 1\}$$

is linearly independent over $\mathbb{Q}$.

THEOREM 6.6. *Under the Chowla–Milnor conjecture, the set of rationals $I(q) = \{p/q, p \in \mathbb{Z}, (p, q) = 1\}$ contains at most one zero of $\psi_{2k}(x)$.*

PROOF. Since $\psi_{2k}(p/q) \ne 0$ if $p/q > 0$, it enough to consider negative values of p. Suppose that there exist two zeros in this set, say,

$$p_1/q = -m + a/q \quad \text{and} \quad p_2/q = -m' + b/q, \quad \text{with } (a, q) = (b, q) = 1.$$

Note that the functional equation for ψ_{2k} ensures that both $\psi_{2k}(a/q)$ and $\psi_{2k}(b/q)$ are rational numbers. This shows that $\zeta(2k+1, a/q)$ and $\zeta(2k+1, b/q)$ are linearly dependent over $\mathbb{Q}$, which is in contradiction to the Chowla–Milnor conjecture. □

REMARK 6.7. The strong Chowla–Milnor conjecture implies that, for any $x \in \mathbb{Q} \setminus \mathbb{Z}$, $\psi_k(x)$ is irrational.

PROOF. By the Strong Chowla–Milnor Conjecture, we see that $\psi_k(a/q)$ is irrational for all $(a, q) = 1, 1 \leq a < q$. By the first functional equation for ψ_k, for any nonintegral rational number x, $\psi_k(x)$ is irrational. □

DEFINITION 6.8. For an integer $k \geq 2$ and complex number $z \in \mathbb{C}$ with $|z| \leq 1$, the polylogarithm function of order k is $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} z^n/n^k$.

For $k = 1$, the series is $\mathrm{Li}_1(z) = -\log(1 - z)$, provided that $|z| < 1$.

Analogous to Baker's theorem on linear forms in logarithms [1], Gun *et al.* [6] proposed the following conjecture for polylogarithms.

CONJECTURE 6.9 (Polylogarithm conjecture). Any linear combination of polylogarithms of algebraic numbers of modulus less than or equal to one with algebraic coefficients is either zero or transcendental.

We end our paper with the following theorem.

THEOREM 6.10. *Under the polylogarithm conjecture, for any rational $x \in \mathbb{Q} \setminus \mathbb{Z}$, $\psi_k(x)$ is transcendental.*

PROOF. We derive an extension of Gauss's formula by an argument of Jensen using roots of unity. This identity is known (see, for example, [8]).

We begin with Simpson's formula. For a power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$,

$$\sum_{n=0}^{\infty} a_{qn+m} t^{qn+m} = \frac{1}{q} \sum_{j=0}^{q-1} \omega^{-jm} f(\omega^j t),$$

where ω is a primitive qth root of unity. This follows easily by orthogonality. Now,

$$\psi_k\Big(\frac{a}{q}\Big) = (-1)^{k+1} k!\, q^{k+1} \sum_{n=0}^{\infty} \frac{1}{(qn + a)^{k+1}}.$$

From the series $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} z^n/n^k$ and Simpson's formula with $\omega = e^{2\pi i/q}$ and $t = 1$,

$$\psi_k\Big(\frac{a}{q}\Big) = (-1)^{k+1} k!\, q^{k} \sum_{j=0}^{q-1} \omega^{-ja} \mathrm{Li}_{k+1}(\omega^j).$$

By using this identity and the polylogarithm conjecture, along with

$$\psi_k\Big(\frac{a}{q}\Big) = (-1)^{k+1} k! \sum_{n=0}^{\infty} \frac{1}{(n + a/q)^{k+1}} = (-1)^{k+1} k!\, \zeta(k + 1, a/q) \neq 0,$$

we deduce that $\psi_k(a/q)$ is transcendental for $1 \leq a \leq q$ and $(a, q) = 1$. The first of the two functional equation for $\psi_k(z)$ immediately implies that $\psi_k(x)$ is transcendental for any nonintegral rational number x. □

https://doi.org/10.1017/S0004972725000036 Published online by Cambridge University Press

Acknowledgements

The author is grateful to Prof. Sanoli Gun for showing her the paper of Prof. Ram Murty and suggesting that it could be extended for higher-order gamma functions. The author is also grateful to Prof. Purusottam Rath for helpful discussions and suggesting some other problems. Finally, the author thanks the referee for carefully reading the article and for pointing out an inaccuracy in an earlier version.

References

[1] A. Baker, *Transcendental Number Theory* (Cambridge University Press, Cambridge, 1975).
[2] K. Ball and T. Rivoal, 'Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs', *Invent. Math.* **146** (2001), 193–207.
[3] P. Chowla and S. Chowla, 'On irrational numbers', *Norske Vid. Selsk. Skr. (Trondheim)* **3** (1982), 1–5.
[4] S. David, N. Hirata-Kohno and M. Kawashima, 'Linear forms in polylogarithms', *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **23**(3) (2022), 1447–1490.
[5] S. Fischler, J. Sprang and W. Zudilin, 'Many odd zeta values are irrational', *Compos. Math.* **155** (2019), 938–952.
[6] S. Gun, M. R. Murty and P. Rath, 'On a conjecture of Chowla and Milnor', *Canad. J. Math.* **63**(6) (2011), 1328–1344.
[7] O. Hölder, 'Über die Eigenschaft der γ-Funktion, keiner algebraischen Differentialgleichung zu genügen', *Math. Ann.* **28** (1887), 1–13.
[8] K. S. Kölbig, 'The polygamma function and derivatives of the cotangent function for rational arguments', CERN-IT-Reports, CERN-CN-96-005.
[9] J. Milnor, 'On polylogarithms, Hurwitz zeta functions and their Kubert identities', *Enseign. Math. (2)* **29** (1983), 281–322.
[10] M. R. Murty, 'Irrationality of zeros of the digamma function', in: *Number Theory in Memory of Eduard Wirsing* (eds. H. Maier, R. Steuding and J. Steuding) (Springer, Cham, 2023), 237–243.
[11] D. Radchenko and D. Zagier, 'Arithmetic properties of the Herglotz function', *J. reine angew. Math.* **797** (2023), 229–253.
[12] T. Rivoal, 'La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs', *C. R. Acad. Sci. Paris, Ser. I* **331**(4) (2000), 267–270.

PREETI GOYAT, Chennai Mathematical Institute,
Sipcot IT Park, Siruseri, Kelambakkam 603103, India
e-mail: preeti@cmi.ac.in