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IMPROVING STRONG NEGATION

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Abstract. Strong negation is a well-known alternative to the standard negation in intuitionistic logic. It is defined virtually by giving falsity conditions to each of the connectives. Among these, the falsity condition for implication appears to unnecessarily deviate from the standard negation. In this paper, we introduce a slight modification to strong negation, and observe its comparative advantages over the original notion. In addition, we consider the paraconsistent variants of our modification, and study their relationship with non-constructive principles and connexivity.

§1. Introduction. The notion of *strong negation*, originally known as *constructible falsity*, was introduced by David Nelson [20] and Andrei Andreevich Markov [15]. The basic idea of strong negation is to constructivize the notion of falsity or refutation. To illustrate, in intuitionistic logic the proof of a disjunction warrants the proof of either of the disjuncts, i.e.,

$$\vdash A \vee B \Longrightarrow \vdash A \text{ or } \vdash B$$

holds (*disjunction property*). On the other hand, the proof of the falsity of a conjunction, that is,

$$\vdash \neg(A \wedge B)$$

does not in general allow us to derive that at least one of the conjuncts is false: i.e., it does not always warrant

$$\vdash \neg A \text{ nor } \vdash \neg B.$$

The insight of Nelson and Markov was to define another kind of negation ($\sim\!A$), which satisfies

$$\vdash \sim\!(A \wedge B) \leftrightarrow (\sim\!A \vee \sim\!B),$$

so that the above property for falsity is satisfied with respect to this strong negation $\sim\!A$. The name 'strong negation' comes from the fact that in the system **N3** of Nelson–Markov, it is assumed that $\sim\!A$ implies the intuitionistic negation $\neg A$. However this assumption can be dropped, in which case we obtain the system **N4** of Almukdad & Nelson [1]. For more information on strong negation, cf. for instance [10, 13, 21, 22, 29, 31].

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In **N3** and **N4**, the condition for (constructive) falsity is in effect defined for each connective, rather than a general condition for falsity being given. Therefore in addition to the condition above for conjunction, we have the equivalences

$$\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B),$$
$$\sim(A \to B) \leftrightarrow (A \wedge \sim B).$$

Now we may ask how these conditions compare to the theorems for intuitionistic negation. Because strong negation is motivated in constructive reasoning, it does not seem too controversial to claim that it ought to resemble intuitionistic negation as much as possible. This would mean that except for conjunction, the intuitionistic formulas corresponding to the falsity conditions should be theorems of intuitionistic logic.[1] Indeed for disjunction, it holds that

$$\vdash \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$$

in intuitionistic logic. On the contrary, for implication, in general

$$\nvdash \neg(A \to B) \leftrightarrow (A \wedge \neg B)$$

for we cannot derive A from $\neg(A \to B)$. This appears to suggest the falsity condition for implication is perhaps not ideal. To remedy this situation, we note the following equivalence that holds in intuitionistic logic:

$$\vdash \neg(A \to B) \leftrightarrow (\neg\neg A \wedge \neg B).$$

We would like to interpret this equivalence in terms of $\sim$. Here, a double negation for $\sim$ cannot meaningfully capture a double negation for $\neg$, because it holds in **N3** and **N4** that $\sim\sim A \leftrightarrow A$. Consequently we must leave $\neg\neg A$ as it is. Then we arrive at the next falsity condition for implication.

$$\vdash \sim(A \to B) \leftrightarrow (\neg\neg A \wedge \sim B).$$

Informally, this condition says that an implication is falsified, if its premise is eventually true and the conclusion is falsified. (This interpretation will be made rigorous using Kripke semantics.) Hence the new condition does not require a conflict at present between the premise and the conclusion in order to refute an implication. This should not be too controversial from the intuitionistic point of view, because it in general allows taking future situations into account.

The idea of altering the falsity condition for implication has been pursued by a number of authors. One such direction is taken by H. Wansing [32] toward connexive logic, which supports contra-classical theorems (cf. [16]). Adopting a modal language is also considered by H. Omori [23] in this relation. Another direction is to use co-implication of C. Rauszer [26] in the falsity condition, as can be seen in Wansing [33]. N. Kamide [12] formulates a variant of **N4** perhaps most similar to ours, in which $\neg\sim A$ instead of $\neg\neg A$ is used, following the ideas of De & Omori [7].

In this paper, we shall look at the effects of this falsity condition in **N3** and **N4**. In particular we observe that the new condition makes explicit the relationship between the two negations for the variant of **N3**, while it depends on the precise formulation in

[1] We refer to [25, 30] for the details of intuitionistic logic.

case of the variants for **N4**. In relation to this, we shall show the completeness of the variants with respect to Kripke semantics.

We then look at some of the properties of the variants. We begin with looking at the conditions under which the two negations become equivalent. Then we shall observe how our variants refute more intuitionistic propositions than **N3** and **N4**. Furthermore, we show that taking the variants does not affect the set of provable formulas of the form $\neg A$. Finally, we see how the new conditions allow us to obtain the contraposability of strong negation with less restrictions.

This is followed by the study of the interaction between the new condition and the law of excluded middle for strong negation. We shall find that different non-constructive principles are justified, depending on which of the variants of **N4** is used as the basis.

Lastly, we shall attempt to give an analysis of the variant of our conditions, obtained by following the paradigm of Wansing [32]. We shall observe that the resulting system fails to satisfy the formulas characterizing connexive logics. We then discuss an alternative approach which enables to partially regain connexivity.

§2. Variants for N3 and N4.

In this section, we introduce the formalizations for the variants of **N3** and **N4** with an alternative falsity condition for implication, which we shall call **DN3** and **DN4**. Here we shall argue in the following language $\mathcal{L}$:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid {\sim}A.$$

We shall use $\neg A$ and $A \leftrightarrow B$ as abbreviations for $A \to \bot$ and $(A \to B) \wedge (B \to A)$, respectively. The set of formulas in $\mathcal{L}$ will be denoted by **Fm**, and $\equiv$ will be used for graphical equality.

2.1. Proof theory.

We start with introducing the axiomatization of **N3** in $\mathcal{L}$.

DEFINITION 2.1 (**N3**).

$\bot \to A$	(EFQ)	${\sim}\bot$	(BF)
$A \to (B \to A)$	(K)	${\sim}(A \wedge B) \leftrightarrow ({\sim}A \vee {\sim}B)$	(CF)
$(A\to(B\to C))\to((A\to B)\to(A\to C))$	(S)	${\sim}(A \vee B) \leftrightarrow ({\sim}A \wedge {\sim}B)$	(DF)
$A \to (B \to (A \wedge B))$	(CI)	${\sim}(A \to B) \leftrightarrow (A \wedge {\sim}B)$	(IF)
$(A_1 \wedge A_2) \to A_i$	(CE)	${\sim}{\sim}A \leftrightarrow A$	(FF)
$A_i \to (A_1 \vee A_2)$	(DI)	${\sim}A \to \neg A$	(SN)
$(A\to C)\to((B\to C)\to(A\vee B\to C))$	(DE)	$\dfrac{A \quad A \to B}{B}$	(MP)

where $i \in \{1, 2\}$.

We shall write $\Gamma \vdash A$ if there is a finite sequence of formulas $B_1, \dots, B_n \equiv A$ such that each B_i is either an element of Γ, an instance of one of the axioms, or obtained by (MP) from B_j and B_k where $j, k < i$. We shall sometimes denote the proof system explicitly, e.g., $\mathbf{N3} \vdash A$. Similar remarks will apply to the other systems we shall introduce.

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The system **N4**[2] is now defined by removing (SN) from the above axiomatization. If we further restrict the formulas of the form $\sim A$ to $\sim\bot$ [3], and remove the axioms (BF), (CF), (DF), (IF) and (FF), then we obtain the intuitionistic propositional calculus **IPC**.

Now, as explained in the introduction, our intent is to replace (IF) in **N3** and **N4** with the following equivalence:

$$\sim(A \to B) \leftrightarrow (\neg\neg A \wedge \sim B). \tag{IF2}$$

Henceforth we shall designate the resulting systems by **DN3** and **DN4**.

Before moving on, we have something to observe in **DN3**. In the system, the relationship between $\neg(A \to B)$ and $\sim(A \to B)$ can be made explicit by the following equivalence.

Proposition 2.2. ***DN3*** $\vdash \sim(A \to B) \leftrightarrow (\neg(A \to B) \wedge \sim B)$.

Proof. In **IPC**, we can show

$$\textbf{IPC} \vdash (\neg\neg A \wedge \neg B) \leftrightarrow (\neg(A \to B) \wedge \neg B),$$

which consequently holds in **DN3** as well. Thus by (SN),

$$\textbf{DN3} \vdash (\neg\neg A \wedge \sim B) \to \neg(A \to B),\ (\neg(A \to B) \wedge \sim B) \to \neg\neg A.$$

Hence it follows that **DN3** $\vdash (\neg\neg A \wedge \sim B) \leftrightarrow (\neg(A \to B) \wedge \sim B)$. Now use (IF2) to obtain the desired equivalence. □

We may indeed take

$$\sim(A \to B) \leftrightarrow (\neg(A \to B) \wedge \sim B) \tag{IF3}$$

to be the falsity condition for implication, as the resulting system is identical to **DN3**. However, if (SN) is absent, then the resulting system (let us name it **DN4'**) will turn out to be non-identical to **DN4**, as we shall see later.

2.2. Semantics. We next look at the semantical side of the systems. The systems **DN3**, **DN4** and **DN4'** can be characterized with Kripke semantics which are slight variants of the semantics for **N3** and **N4** in [10, 13, 29]. The following completeness proof is also based on their methodology, and is similar to those of **N3** and **N4**.

Definition 2.3 (Kripke semantics for **DN3**). *A* Kripke frame $\mathcal{F}$ *of **DN3** is an inhabited pre-ordered set* $(W, \leq)$*. A* Kripke model $\mathcal{M}$ *of **DN3** is a pair* $(\mathcal{F}, \mathcal{V})$ *where* $\mathcal{V} = \{\mathcal{V}^+, \mathcal{V}^-\}$*. Each* $\mathcal{V}^*$ $(* \in \{+,-\})$ *is a mapping assigning a subset* $\mathcal{V}^*(p)$ *to each propositional variable* p*. Here each* $\mathcal{V}^*(p)$ *is required to be upward closed, i.e.,*

$$w \in \mathcal{V}^*(p) \text{ and } w' \geq w \text{ implies } w' \in \mathcal{V}^*(p).$$

We further assume $\mathcal{V}^+(p) \cap \mathcal{V}^-(p) = \emptyset$*. Then the forcings* $\Vdash^+$ *and* $\Vdash^-$ *of formulas are uniquely extended from* $\mathcal{V}^+$ *and* $\mathcal{V}^-$ *by the following clauses.*

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[2] This formulation with $\bot$ in the language is also commonly called $\mathbf{N4}^\bot$ [22].

[3] That is to say, $\sim\bot$ is still a well-formed formula in **IPC**, but other negated formulas like $\sim p$, $\sim(A \to B)$ are not. This in effect means we treat $\sim\bot$ as a true proposition $\top$, when it comes to **IPC**. We choose this formulation in order to make some arguments simpler.

$$w \Vdash^* p \Leftrightarrow w \in \mathcal{V}^*(p).$$
$$w \Vdash^+ \bot \textit{ for no } w \in W.$$
$$w \Vdash^- \bot \textit{ for all } w \in W.$$
$$w \Vdash^+ A \wedge B \Leftrightarrow w \Vdash^+ A \textit{ and } w \Vdash^+ B.$$
$$w \Vdash^- A \wedge B \Leftrightarrow w \Vdash^- A \textit{ or } w \Vdash^- B.$$
$$w \Vdash^+ A \vee B \Leftrightarrow w \Vdash^+ A \textit{ or } w \Vdash^+ B.$$
$$w \Vdash^- A \vee B \Leftrightarrow w \Vdash^- A \textit{ and } w \Vdash^- B.$$
$$w \Vdash^+ A \to B \Leftrightarrow \forall w' \geq w(w' \Vdash^+ A \textit{ implies } w' \Vdash^+ B).$$
$$w \Vdash^- A \to B \Leftrightarrow \forall w' \geq w \exists w'' \geq w'(w'' \Vdash^+ A) \textit{ and } w \Vdash^- B.$$
$$w \Vdash^+ {\sim}A \Leftrightarrow w \Vdash^- A.$$
$$w \Vdash^- {\sim}A \Leftrightarrow w \Vdash^+ A.$$

Occasionally we denote the model explicitly, as $\mathcal{M}, w \Vdash^* A$. *We shall write* $\mathcal{M} \vDash A$ *if* $\mathcal{M}, w \Vdash^+ A$ *for all* w. *We shall write* $\Gamma \vDash A$ *if for all* $\mathcal{M}$:

$$\forall B \in \Gamma(\mathcal{M} \vDash B) \textit{ implies } \mathcal{M} \vDash A.$$

In particular, when $\Gamma = \emptyset$ *we write* $\vDash A$. *When necessary, we shall denote the system explicitly, e.g.,* ***DN3*** $\vDash A$.

We note that $w \Vdash^- A \to B$ iff $w \Vdash^+ \neg\neg A$ and $w \Vdash^- B$ with the above condition. This is because, as in intuitionistic logic, we have:

$$w \Vdash^+ \neg\neg A \Leftrightarrow \forall w' \geq w \exists w'' \geq w'(w'' \Vdash^+ A).$$

In particular, if $w \Vdash^+ \neg\neg A$ then $w' \nVdash^+ \neg A$ for any $w' \geq w$, which forces the existence of $w'' \geq w$ such that $w'' \Vdash^+ A$. This forcing condition formalizes the informal interpretation of the falsity condition for implication discussed in the introduction.

The semantics for **N3** can be obtained by making the next change to the clauses.

$$w \Vdash^- A \to B \Leftrightarrow (w \Vdash^+ A \text{ and } w \Vdash^- B).$$

The semantics for **N4** and **DN4** are then obtained by removing the condition $\mathcal{V}^+(p) \cap \mathcal{V}^-(p) = \emptyset$ from the respective semantics. In addition, the semantics of **DN4'** is obtained by altering the falsity condition of implication to:

$$w \Vdash^- A \to B \Leftrightarrow \forall w' \geq w \exists w'' \geq w'(w'' \Vdash^+ A \text{ and } w'' \nVdash^+ B) \text{ and } w \Vdash^- B.$$

That is to say, $w \Vdash^+ \neg(A \to B)$ and $w \Vdash^- B$. Finally, the semantics for **IPC** can be obtained if we restrict the language and our attention to $\Vdash^+$ alone, with the additional condition that $w \Vdash^+ {\sim}\bot$ always holds. In what follows, we shall omit the superscript $+$ from $\mathcal{V}^+$ and $\Vdash^+$ when we talk about **IPC**.

THEOREM 2.4 (Completeness of **IPC**). $\Gamma \vdash A$ *in* ***IPC*** *if and only if* $\Gamma \vDash A$ *in the Kripke semantics for* ***IPC***.

Proof. See [30]. □

It is routine to establish for the semantics of **DN3**, **DN4** and **DN4'** that the forcing of a formula is closed upward, i.e.:

PROPOSITION 2.5. *For* $* \in \{+, -\}$, *if* $w \Vdash^* A$ *and* $w' \geq w$ *then* $w' \Vdash^* A$.

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Proof. By induction on the complexity of A. □

Moreover, we readily observe that the condition $\mathcal{V}^+(p) \cap \mathcal{V}^-(p) = \emptyset$ in the semantics of **DN3** is extended to all formulas.

PROPOSITION 2.6. *In the semantics for **DN3**, for no w it holds that $w \Vdash^+ A$ and $w \Vdash^- A$.*

Proof. Again by induction on the complexity of A. □

We shall establish the completeness of **DN3** with respect to the Kripke semantics via the completeness of **IPC**. For this purpose, we need some preparations. To begin with, we assign to each formula A its *negation normal form* $r(A)$.

DEFINITION 2.7 (Negation normal form). *We define a mapping $r : \mathbf{Fm} \to \mathbf{Fm}$ inductively by the following clauses.*

$$\begin{aligned} r(p) &= p, \\ r(\bot) &= \bot, \\ r(A \circ B) &= r(A) \circ r(B), \\ r(\sim p) &= \sim p, \\ r(\sim \bot) &= \sim \bot, \end{aligned} \qquad \begin{aligned} r(\sim(A \wedge B)) &= r(\sim A) \vee r(\sim B), \\ r(\sim(A \vee B)) &= r(\sim A) \wedge r(\sim B), \\ r(\sim(A \to B)) &= \neg\neg r(A) \wedge r(\sim B), \\ r(\sim\sim A) &= r(A), \end{aligned}$$

where $\circ \in \{\wedge, \vee, \to\}$. We shall call a formula of the form $r(A)$ a reduced formula.

In the above, if we use the clause

$$r(\sim(A \to B)) = r(A) \wedge r(\sim B),$$

then we obtain the negation normal form for **N3** and **N4**. Let us call a propositional variable or $\bot$ a *prime* formula. We note that $r(A)$ has all occurrences of $\sim$ in front of prime formulas. Moreover, we observe the next lemmas.

LEMMA 2.8. ***DN3*** $\vdash A \leftrightarrow r(A)$.

Proof. By induction on the complexity of A. □

LEMMA 2.9. *If **DN3** $\vdash A$, then **DN3** $\vdash r(A)$; moreover, one may assume all the formulas in the proof are reduced. (That is, it has a reduced proof.)*

Proof. By induction on the depth of derivation. That is to say, we have to show that for each axiom A, its negation normal form $r(A)$ is derivable with all the formulas in the proof reduced. Then

$$\frac{r(A) \qquad r(A) \to r(B)}{r(B)}$$

is an instance of (MP) and so the statement follows.

For the case for (SN), we have to argue by induction on the complexity of the formula A. When $A \equiv B \to C$, we have to show $r(\sim(B \to C) \to \neg(B \to C))$ is derivable. This is syntactically equivalent to

$$(\neg\neg r(B) \wedge r(\sim C)) \to \neg(r(B) \to r(C)).$$

By I.H., there is a reduced proof of $r(\sim C) \to \neg r(C)$. Also

$$\neg r(C) \to (\neg\neg r(B) \to \neg(r(B) \to r(C)))$$

is an instance of an intuitionistically derivable formula; so the proof can be assumed to be reduced. Then it is easy to construct a reduced proof of $(\neg\neg r(B) \wedge r({\sim}C)) \rightarrow \neg(r(B) \rightarrow r(C))$. □

Let $A[{\sim}p_1, \ldots, {\sim}p_n]$ be a reduced formula, with the occurrences of subformulas of the form ${\sim}p$ marked by ${\sim}p_1, \ldots, {\sim}p_n$. We shall assume that for each p_i there is a new propositional variable q_i in the language of **IPC**. Then we define a formula

$$E := \bigwedge_{1 \leq i \leq n} (q_i \rightarrow \neg p_i).$$

This allows us to assert the next proposition, which is analogous to the case for **N3** originally treated by Vorob'ev [31].[4]

Proposition 2.10. *Let $A[{\sim}p_1, \ldots, {\sim}p_n]$ be a reduced formula. Then the following are equivalent.*

(i) ***DN3*** $\vdash A[{\sim}p_1, \ldots, {\sim}p_n]$.
(ii) ***IPC*** $\vdash E \rightarrow A[{\sim}p_1, \ldots, {\sim}p_n / q_1, \ldots, q_n]$.

Proof. From (i) to (ii), first note that we can w.l.o.g. assume all the formulas of the form ${\sim}p$ occurring in the derivation occurs in $A[{\sim}p_1, \ldots, {\sim}p_n]$; for otherwise we can consider the derivation of an equivalent formula

$$A[{\sim}p_1, \ldots, {\sim}p_n] \wedge ({\sim}p \rightarrow {\sim}p),$$

whose reduced proof requires no more occurrences of formulas of the mentioned form. With this proviso it is sufficient to observe that a reduced proof in **DN3** can be replicated in **IPC** with the aid of E. In particular, for the axiom ${\sim}B \rightarrow \neg B$, because the proof is reduced, B has to be prime. If $B \equiv \bot$, then ${\sim}\bot \rightarrow \neg\bot$ is a theorem of **IPC** (recall ${\sim}\bot$ is in the language of **IPC**); if $B \equiv p_i$ for some $1 \leq i \leq n$ (as we may assume by the comment above), then

$$\mathbf{IPC} \vdash (q_i \rightarrow \neg p_i) \rightarrow (({\sim}p_i \rightarrow \neg p_i)[{\sim}p_i / q_i]),$$

and so E is sufficient for the replication.

From (ii) to (i), if

$$\mathbf{IPC} \vdash E \rightarrow A[q_1, \ldots, q_n],$$

then **DN3** can replicate the proof except for the unavailable propositional variables $q_1, \ldots, q_n$. Therefore we replace the occurrences of $q_1, \ldots, q_n$ by ${\sim}p_1, \ldots, {\sim}p_n$ from the proof in **IPC**, which allows us to obtain the proof witnessing

$$\mathbf{DN3} \vdash (E \rightarrow A)[q_1, \ldots, q_n / {\sim}p_1, \ldots, {\sim}p_n].$$

That is, **DN3** $\vdash A[{\sim}p_1, \ldots, {\sim}p_n]$. □

Now we are ready to establish the soundness and completeness of **DN3** with the Kripke semantics. Firstly, it is straightforward to check the soundness direction.

Theorem 2.11 (Soundness of **DN3**). *If **DN3** $\vdash A$ then **DN3** $\vDash A$.*

Proof. By induction on the depth of derivation. In particular, the case for (SN) follows from Proposition 2.6. □

[4] We thank one of the anonymous referees for this information.

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For the completeness, we shall use the next lemma.

LEMMA 2.12.

(i) $w \Vdash^+ A$ *if and only if* $w \Vdash^+ r(A)$.
(ii) $w \Vdash^- A$ *if and only if* $w \Vdash^+ r(\sim A)$.

Proof. By simultaneous induction on the complexity of A. For instance, in (ii), if $A \equiv B \to C$, $w \Vdash^- B \to C$ iff $w \Vdash^+ \neg\neg B$ and $w \Vdash^- C$. By I.H. this is equivalent to $w \Vdash^+ \neg\neg r(B)$ and $w \Vdash^+ r(\sim C)$, which is by definition equivalent to $w \Vdash^+ r(\sim(B \to C))$. □

THEOREM 2.13 (Completeness of **DN3**). *If* **DN3** $\vDash A$ *then* **DN3** $\vdash A$.

Proof. We prove the contrapositive. Suppose **DN3** $\nvdash A$. Then by Lemma 2.8,

$$\mathbf{DN3} \nvdash r(A)[\sim p_1, \dots, \sim p_n].$$

Then by Proposition 2.10 and the deduction theorem for **IPC**,

$$E \nvdash r(A)[\sim p_1, \dots, \sim p_n / q_1, \dots, q_n]$$

in **IPC**. Hence by the completeness of **IPC**,

$$E \nvDash r(A)[\sim p_1, \dots, \sim p_n / q_1, \dots, q_n]$$

in the Kripke semantics for **IPC**. That is, there is a model $(\mathcal{F}, \mathcal{V})$ of **IPC** and a world w such that

$$(\mathcal{F}, \mathcal{V}) \vDash E,$$

but

$$(\mathcal{F}, \mathcal{V}), w \nVdash r(A)[\sim p_1, \dots, \sim p_n / q_1, \dots, q_n]. \circledast .$$

Then we construct a corresponding model of DN3 $(\mathcal{F}_2, \mathcal{V}_2)$ such that

- $\mathcal{F}_2 = \mathcal{F}$;
- $u \in \mathcal{V}_2^+(p_i)$ iff $u \in \mathcal{V}(p_i)$;
- $u \in \mathcal{V}_2^-(p_i)$ iff $u \in \mathcal{V}(q_i)$.

Note here if $u \in \mathcal{V}_2^+(p_i) \cap \mathcal{V}_2^-(p_i)$ then $(\mathcal{F}, \mathcal{V}), u \Vdash p_i \wedge q_i$, so $(\mathcal{F}, \mathcal{V}), u \Vdash \bot$, a contradiction. Hence $\mathcal{V}_2^+(p_i) \cap \mathcal{V}_2^-(p_i) = \emptyset$, and consequently $(\mathcal{F}_2, \mathcal{V}_2)$ is well-defined.

Now it suffices to observe (by an easy induction on the complexity of the formula) that for any subformula B of $r(A)$,

$$(\mathcal{F}, \mathcal{V}), w \Vdash B[\sim p_1, \dots, \sim p_n / q_1, \dots, q_n] \text{ iff } (\mathcal{F}_2, \mathcal{V}_2), w \Vdash^+ B,$$

because then $\circledast$ implies $(\mathcal{F}_2, \mathcal{V}_2), w \nVdash^+ r(A)$ and so $(\mathcal{F}_2, \mathcal{V}_2), w \nVdash^+ A$ by Lemma 2.12. □

The soundness and completeness for **DN4** and **DN4'** are provable in a similar manner. For **DN4'**, the notion of negation normal form is altered to

$$r(\sim(A \to B)) = \neg r(A \to B) \wedge r(\sim B).$$

In addition, for both **DN4** and **DN4'**, E is not assumed in establishing the equivalence in the analogue of Proposition 2.10. Thence we establish the relationship between the two systems.

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PROPOSITION 2.14. ***DN4** and **DN4’** are incomparable.*

Proof. We claim

$$\textbf{DN4} \nvdash \sim(p \to q) \to (\neg(p \to q) \wedge \sim q)$$

and

$$\textbf{DN4'} \nvdash (\neg\neg p \wedge \sim q) \to \sim(p \to q).$$

For this purpose, take the next model $(\mathcal{F}, \mathcal{V})$ of **IPC** refuting $\neg\neg p \to \neg(p \to q)$.

- $W = \{w\}$;
- $\mathcal{V}(p) = \mathcal{V}(q) = W$.

Then construct a model $(\mathcal{F}_2, \mathcal{V}_2)$ of **DN4’** by letting

- $\mathcal{F}_2 = \mathcal{F}$;
- $\mathcal{V}_2^+ = \mathcal{V}$ and $\mathcal{V}_2^-(q) = W$.

Then $(\mathcal{F}_2, \mathcal{V}_2), w \Vdash^+ \neg\neg p \wedge \sim q$, but $(\mathcal{F}_2, \mathcal{V}_2), w \nVdash^+ \neg(p \to q) \wedge \sim q$. Hence

$$(\mathcal{F}_2, \mathcal{V}_2) \nvDash \sim(p \to q) \to (\neg(p \to q) \wedge \sim q).$$

So by soundness

$$\textbf{DN4} \nvdash \sim(p \to q) \to (\neg(p \to q) \wedge \sim q).$$

Similarly we can show

$$\textbf{DN4'} \nvdash (\neg\neg p \wedge \sim q) \to \sim(p \to q).$$

□

The above in particular implies that while

$$\textbf{DN4'} \vdash \sim(A \to B) \to \neg(A \to B),$$

(SN) in full generality is not derivable in **DN4’**, because then it would be identical to **DN3**, which contains **DN4**.

From this point on, we shall occasionally write **DN4(’)** to mean both **DN4** and **DN4’**, to state propositions that hold for both of the systems. Before moving on, we observe the *constructible falsity* property for the systems as a special case for disjunction property, which can be proved by a semantical argument as in **IPC**.

THEOREM 2.15 (Disjunction property).

(i) ***DN3** $\vdash A \vee B$ then **DN3** $\vdash A$ or **DN3** $\vdash B$.*
(ii) ***DN4(’)** $\vdash A \vee B$ then **DN4(’)** $\vdash A$ or **DN4(’)** $\vdash B$.*

Proof. We prove by contradiction. Since **IPC** is complete with respect to *rooted* frames (i.e., frames with the least element), **DN3** and **DN4(’)** are complete with respect to rooted frames as well. Now suppose **DN3** $\vdash A_1 \vee A_2$, but **DN3** $\nvdash A_1$ and **DN3** $\nvdash A_2$. Then by completeness, there are models $(\mathcal{F}_i, \mathcal{V}_i)$ and the worlds r_i, where $i \in \{1, 2\}$ and $(\mathcal{F}_i, \mathcal{V}_i), r_i \nVdash^+ A_i$. We may assume r_i to be the least element of the set W_i of the respective model. Then define a new model $(\mathcal{F}_3, \mathcal{V}_3)$ by taking $W_3 = W_1 \cup W_2 \cup \{r_3\}$, where r_3 is a new least element of W_3. Set

$$r_3 \in \mathcal{V}_3^*(p) \text{ iff } W_1 = \mathcal{V}_1^*(p) \text{ and } W_2 = \mathcal{V}_2^*(p).$$

If $w \neq r_3$ and $w \in W_i$, set

$$w \in \mathcal{V}_3^*(p) \text{ iff } w \in \mathcal{V}_i^*(p).$$

It is easy to check $\mathcal{V}_3^*(p)$ is upward closed, and $\mathcal{V}_3^+(p) \cap \mathcal{V}_3^-(p) = \emptyset$. Now since **DN3** $\vdash A_1 \vee A_2$, by soundness

$$(\mathcal{F}_3, \mathcal{V}_3), r_3 \Vdash^+ A_1 \text{ or } (\mathcal{F}_3, \mathcal{V}_3), r_3 \Vdash^+ A_2.$$

On the other hand, $(\mathcal{F}_3, \mathcal{V}_3), r_i \nVdash^+ A_i$ from our choice of $\mathcal{V}_3^+$, a contradiction. The case for **DN4(')** is similar. □

Corollary 2.16 (Constructible falsity).

(i) ***DN3*** $\vdash \sim(A \wedge B)$ *then* ***DN3*** $\vdash \sim A$ *or* ***DN3*** $\vdash \sim B$.
(ii) ***DN4(')*** $\vdash \sim(A \wedge B)$ *then* ***DN4(')*** $\vdash \sim A$ *or* ***DN4(')*** $\vdash \sim B$.

Proof. Immediate from the previous theorem and (CF). □

§3. Properties of DN3 and DN4(').

In this section, we look at the properties of **DN3** and **DN4(')** in order to demonstrate the advantages of the logics compared with **N3** and **N4**.

We begin with observing how the alternation in the falsity condition for implication allows the strong negation to retain more similarity with intuitionistic negation. More precisely, we shall see that the equivalence of strong and intuitionistic negation can be shown for a wider class of formulas, once we assume the equivalence for the atomic formulas.[5] For this purpose, we consider a limited language $\mathcal{L}^*$, which does not contain $\wedge$ and $\sim$ that allow inferences not allowed for intuitionistic negation.

$$A ::= p \mid \bot \mid A \vee A \mid A \to A.$$

Then we shall see $\neg A$ and $\sim A$ become equivalent if we assume the equivalence between $\neg p$ and $\sim p$ for each p from a certain subset of the set of all propositional variables occurring in A.

Definition 3.17. *Let A be a formula in $\mathcal{L}^*$. We inductively define a class Γ_A by the following clauses*:

$$\Gamma_p \equiv \{p\}, \quad \Gamma_{A\vee B} \equiv \Gamma_A \cup \Gamma_B,$$
$$\Gamma_\bot \equiv \emptyset, \quad \Gamma_{A\to B} \equiv \Gamma_B.$$

Now let

$$\mathcal{I}_A := \{\neg p \to \sim p : p \in \Gamma_A\} \text{ and } \mathcal{E}_A := \{\neg p \leftrightarrow \sim p : p \in \Gamma_A\}.$$

Then we observe that $\neg A$ and $\sim A$ become equivalent under a suitable class of assumptions and a restriction on language.

Proposition 3.18. *Let A be a formula in $\mathcal{L}^*$. Then*

- $\mathcal{I}_A \vdash \neg A \to \sim A$ *in* ***DN3***.
- $\mathcal{E}_A \vdash \neg A \leftrightarrow \sim A$ *in* ***DN4(')***.

5 This approach bears a resemblance with the topic of Ishihara [11].

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Proof. Here we prove for **DN3**; the case for **DN4(')** is analogous. We show by induction on the complexity of A.

When $A \equiv p$, then $\mathcal{I}_p = \{\neg p \to \sim p\}$. Thus the statements hold, as $\neg p \to \sim p \vdash \neg p \to \sim p$.

When $A \equiv \bot$, then $\vdash \neg\bot \to \sim\bot$, hence $\mathcal{I}_\bot = \emptyset$ suffices.

When $A \equiv B \vee C$, then $\mathcal{I}_A = \mathcal{I}_B \cup \mathcal{I}_C$. By I.H.

$$\mathcal{I}_B \vdash \neg B \to \sim B \text{ and } \mathcal{I}_C \vdash \neg C \to \sim C.$$

Since $\neg(B \vee C) \leftrightarrow (\neg B \wedge \neg C)$ holds in **IPC**, it follows that

$$\mathcal{I}_A \vdash \neg(B \vee C) \to \sim(B \vee C).$$

When $A \equiv B \to C$, then $\mathcal{I}_A = \mathcal{I}_C$. Then by I.H.

$$\mathcal{I}_C \vdash (\neg\neg B \wedge \neg C) \to (\neg\neg B \wedge \sim C).$$

Hence in view of the fact that **IPC** $\vdash \neg(B \to C) \leftrightarrow (\neg\neg B \wedge \neg C)$, it follows that

$$\mathcal{I}_A \vdash \neg(B \to C) \to \sim(B \to C). \qquad \square$$

On the other hand, clearly $\neg p \leftrightarrow \sim p \nvdash \neg\neg p \to \sim\neg p$ in **N3** and **N4**, since $\sim\neg p$ is equivalent to p in **N3** and **N4**. Hence the difference between the cases for **N3** and **N4** is the inclusion of implication, which is significant as a fragment.

We next give an example of formulas provable in **DN3** and **DN4(')** but not in **N3** nor **N4**.

PROPOSITION 3.19.

(i) ***DN4(')*** $\vdash \neg\neg A \leftrightarrow \sim\neg A$.
(ii) ***DN4(')*** $\vdash \sim\neg(A \vee \neg A)$.

Proof. (i) is immediate from **DN4** $\vdash \sim\neg A \leftrightarrow (\neg\neg A \wedge \sim\bot)$. (ii) then follows from **IPC** $\vdash \neg\neg(A \vee \neg A)$. $\square$

It is apparent that $\sim\neg(A \vee \neg A)$ is not provable in **N3** nor **N4**, because it is equivalent to $A \vee \neg A$. On the other hand, it is straightforward to check that $\sim\neg p \to p$ is provable in **N3** and **N4** but not in **DN3** nor **DN4(')**. Hence **N3** and **DN3** are incomparable, and similarly for **N4** and **DN4(')**.

Nonetheless, we can still establish a certain relationship between the systems, with the exception of **DN4'**. For this we first require establishing a Glivenko-like lemma.

LEMMA 3.20.

(i) *If* ***N3*** $\vdash A$ *then* ***DN3*** $\vdash \neg\neg A$.
(ii) *If* ***N4*** $\vdash A$ *then* ***DN4*** $\vdash \neg\neg A$.

Proof. We argue by induction on the depth of deduction. In each case, it suffices to show the derivability of

$$\neg\neg(\sim(A \to B) \leftrightarrow (A \wedge \sim B))$$

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in the target system. Since **IPC** $\vdash \neg\neg(A \leftrightarrow B) \leftrightarrow (\neg\neg A \leftrightarrow \neg\neg B)$, we aim to show

$$\neg\neg{\sim}(A \to B) \to \neg\neg(A \wedge {\sim}B)$$

and

$$\neg\neg(A \wedge {\sim}B) \to \neg\neg{\sim}(A \to B).$$

Indeed, the latter is an immediate consequence of

$$(A \wedge {\sim}B) \to {\sim}(A \to B).$$

As for the former, note $\neg\neg{\sim}(A \to B) \leftrightarrow \neg\neg(\neg\neg A \wedge {\sim}B)$ holds in both **DN3** and **DN4**. Then as **IPC** $\vdash \neg\neg(A \wedge B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$,

$$\neg\neg{\sim}(A \to B) \to (\neg\neg A \wedge \neg\neg{\sim}B),$$

and using the above equivalence again, we obtain $\neg\neg{\sim}(A \to B) \to \neg\neg(A \wedge {\sim}B)$ as desired. □

Now we can infer the next theorem, showing that any formulas in the language of **IPC** (i.e., not containing $\sim$ except for ${\sim}\bot$) refutable in **N3** and **N4** are also refutable in **DN3** and **DN4**. Given the preceding observation that ${\sim}\neg(A \vee \neg A)$ is refutable only in the latter pair, we can conclude that they refute strictly more propositions that are in the language of **IPC** than **N3** and **N4**.

THEOREM 3.21. *Let A be a formula not containing a subformula of the form ${\sim}B$ except for ${\sim}\bot$. Then*:

(i) *If* ***N3*** $\vdash {\sim}A$ *then* ***DN3*** $\vdash {\sim}A$.
(ii) *If* ***N4*** $\vdash {\sim}A$ *then* ***DN4*** $\vdash {\sim}A$.

Proof. Here we look at the case for **N3**. The case for **N4** is analogous. We prove by induction on the complexity of A.

When $A \equiv p$, then ${\sim}p$ is not a theorem of **N3**, so the statement vacuously holds. Similarly when $A \equiv {\sim}\bot$.

When $A \equiv \bot$, then **DN3** $\vdash {\sim}\bot$.

When $A \equiv B \wedge C$, then if **N3** $\vdash {\sim}(B \wedge C)$, either

$$\textbf{N3} \vdash {\sim}B \text{ or } \textbf{N3} \vdash {\sim}C,$$

by the constructive falsity property of **N3**. Thus by I.H.

$$\textbf{DN3} \vdash {\sim}B \text{ or } \textbf{DN3} \vdash {\sim}C,$$

and so **DN3** $\vdash {\sim}(B \wedge C)$ in each case.

When $A \equiv B \vee C$, then if **N3** $\vdash {\sim}(B \vee C)$,

$$\textbf{N3} \vdash {\sim}B \text{ and } \textbf{N3} \vdash {\sim}C,$$

and so by I.H.

$$\textbf{DN3} \vdash {\sim}B \text{ and } \textbf{DN3} \vdash {\sim}C.$$

Hence **DN3** $\vdash {\sim}(B \vee C)$.

When $A \equiv B \to C$, then if **N3** $\vdash {\sim}(B \to C)$,

$$\textbf{N3} \vdash B \text{ and } \textbf{N3} \vdash {\sim}C.$$

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Then by I.H. and Lemma 3.20

$$\mathbf{DN3} \vdash \neg\neg B \text{ and } \mathbf{DN3} \vdash {\sim} C,$$

and so $\mathbf{DN3} \vdash {\sim}(B \to C)$. □

Note here that the above theorem cannot be extended to the full language, because when $A \equiv {\sim}B$, we would need $\mathbf{N3} \vdash B$ implying $\mathbf{DN3} \vdash B$.

Before moving on to another topic, let us look back at Lemma 3.20. Unlike Glivenko's theorem, in each case we cannot show the converse direction, for $\neg\neg(A \vee \neg A)$ is intuitionistically derivable but not $A \vee \neg A$. However, we may still observe a correspondence for propositions of the form $\neg A$. In order to state the next result, we shall use $\Vdash_{\mathbf{N4}}$ and $\Vdash_{\mathbf{DN4}}$ for the forcings of the respective semantics, and r_1 and r_2 for the negation normal forms in **N4** and **DN4**, respectively. Also note that a model of **N3** or **N4** can be seen as a model of **DN3** and **DN4** but with a different forcing condition.

Lemma 3.22. *Let $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ be a model of **N4**. Then for any A and any world w in the model,*

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1(A) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2(A).$$

Proof. We prove by induction on the complexity of A.

If $A \equiv p$, then $r_1(A) = r_2(A) = p$. We have $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg p$ if and only if $\forall w' \geq w (w' \notin \mathcal{V}^+(p))$ if and only if $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg p$.

If $A \equiv \bot$, then $r_1(A) = r_2(A) = \bot$, and $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg\bot$ and $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg\bot$ hold for all w.

If $A \equiv B \wedge C$, then $r_i(A) = r_i(B) \wedge r_i(C)$ for $i \in \{1, 2\}$. By I.H.,

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1(B) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2(B),$$
$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1(C) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2(C).$$

We need to show

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1(B) \wedge r_1(C)) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2(B) \wedge r_2(C)).$$

For the left-to-right direction, assume $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1(B) \wedge r_1(C))$. Suppose that $\mathcal{M}, w' \Vdash^+_{\mathbf{DN4}} r_2(B) \wedge r_2(C)$ for $w' \geq w$. Then if $\mathcal{M}, w'' \Vdash^+_{\mathbf{N4}} \neg r_1(B)$ for $w'' \geq w'$, by I.H. $\mathcal{M}, w'' \Vdash^+_{\mathbf{DN4}} \neg r_2(B)$, a contradiction. Hence $\mathcal{M}, w' \Vdash^+_{\mathbf{N4}} \neg\neg r_1(B)$. Similarly $\mathcal{M}, w' \Vdash^+_{\mathbf{N4}} \neg\neg r_1(C)$. Consequently $\mathcal{M}, w' \Vdash^+_{\mathbf{N4}} \neg\neg(r_1(B) \wedge r_1(C))$, another contradiction. Therefore $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2(B) \wedge r_2(C))$. The converse direction is analogous.

If $A \equiv B \vee C$, then $r_i(A) = r_i(B) \vee r_i(C)$ for $i \in \{1, 2\}$. We have the same I.H. as the previous case. We need to show

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1(B) \vee r_1(C)) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2(B) \vee r_2(C)).$$

For the left-to-right direction, assume $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1(B) \vee r_1(C))$. Then $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1(B)$ and $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1(C)$. Hence by I.H. $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2(B)$ and $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2(C)$. Therefore $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2(B) \vee r_2(C))$. The converse direction is analogous.

If $A \equiv B \to C$, then $r_i(A) = r_i(B) \to r_i(C)$ for $i \in \{1, 2\}$. We have the same I.H. as the previous case. We need to show

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1(B) \to r_1(C)) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2(B) \to r_2(C)).$$

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For the left-to-right direction, assume $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1(B) \to r_1(C))$. Suppose $\mathcal{M}, w' \Vdash^+_{\mathbf{DN4}} r_2(B) \to r_2(C)$ for $w' \geq w$. Then if $\mathcal{M}, w'' \Vdash^+_{\mathbf{N4}} \neg r_1(C)$ for $w'' \geq w'$, by I.H. $\mathcal{M}, w'' \Vdash^+_{\mathbf{DN4}} \neg r_2(C)$. Hence by our supposition $\mathcal{M}, w'' \Vdash^+_{\mathbf{DN4}} \neg r_2(B)$. By I.H. again, $\mathcal{M}, w'' \Vdash^+_{\mathbf{N4}} \neg r_1(B)$. Thus $\mathcal{M}, w' \Vdash^+_{\mathbf{N4}} \neg r_1(C) \to \neg r_1(B)$. So $\mathcal{M}, w' \Vdash^+_{\mathbf{N4}} \neg\neg(r_1(B) \to r_1(C))$, which contradicts with our assumption. Therefore $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2(B) \to r_2(C))$. The converse direction is analogous.

If $A \equiv {\sim}B$, we argue by induction on the complexity of B.

If $B \equiv p$, then $r_i(A) = {\sim}p$ for $i \in \{1, 2\}$. We have $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg{\sim}p$ if and only if $\forall w' \geq w(w' \notin \mathcal{V}^-(p))$ if and only if $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg{\sim}p$.

If $B \equiv \bot$, then $r_1(A) = r_2(A) = {\sim}\bot$, and $\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg{\sim}\bot$ and $\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg{\sim}\bot$ never hold for any w.

If $B \equiv C \wedge D$, then $r_i(A) = r_i({\sim}C) \vee r_i({\sim}D)$ for $i \in \{1, 2\}$. By I.H.

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1({\sim}C) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2({\sim}C),$$
$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1({\sim}D) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2({\sim}D).$$

We need to show

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1({\sim}C) \vee r_1({\sim}D)) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2({\sim}C) \vee r_2({\sim}D)).$$

The case is thus similar to the case $A \equiv B \vee C$.

If $B \equiv C \vee D$, then $r_i(A) = r_i({\sim}C) \wedge r_i({\sim}D)$ for $i \in \{1, 2\}$. We have the same I.H. as the previous case. We need to show

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1({\sim}C) \wedge r_1({\sim}D)) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2({\sim}C) \wedge r_2({\sim}D)).$$

The case is thus similar to the case $A \equiv B \wedge C$.

If $B \equiv C \to D$, then $r_1(A) = r_1(C) \wedge r_1({\sim}D)$ and $r_2(A) = \neg\neg r_2(C) \wedge r_2({\sim}D)$. By I.H.,

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1(C) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2(C),$$
$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1({\sim}D) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2({\sim}D).$$

We need to show

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1(C) \wedge r_1({\sim}D)) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(\neg\neg r_2(C) \wedge r_2({\sim}D)).$$

We first note that (because both formulas equal $r_2({\sim}D) \to \neg r_2({\sim}C)$)

$$\mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(\neg\neg r_2(C) \wedge r_2({\sim}D)) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2(C) \wedge r_2({\sim}D)).$$

Hence it suffices to show

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg(r_1(C) \wedge r_1({\sim}D)) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg(r_2(C) \wedge r_2({\sim}D)).$$

Thus the case is similar to the case $A \equiv B \wedge C$.

If $B \equiv {\sim}C$, then $r_i(A) = r_i(C)$ for $i \in \{1, 2\}$. We need to show

$$\mathcal{M}, w \Vdash^+_{\mathbf{N4}} \neg r_1(C) \text{ if and only if } \mathcal{M}, w \Vdash^+_{\mathbf{DN4}} \neg r_2(C),$$

which already holds by I.H. □

In particular, the statement holds between **N3** and **DN3**, since a model of **N3** is a model of **N4**. Then we can demonstrate the next theorem.

THEOREM 3.23.

(i) ***N3*** $\vdash \neg A$ *if and only if* ***DN3*** $\vdash \neg A$.
(ii) ***N4*** $\vdash \neg A$ *if and only if* ***DN4*** $\vdash \neg A$.

Proof. We look at the case for (ii). For the right-to-left direction, if **DN4** $\vdash \neg A$ then **DN4** $\vdash \neg r_1(A)$ by the analogue of Lemma 2.8 for **DN4**. Then **DN4** $\vDash \neg r_1(A)$ by the soundness of **DN4**. Hence by the previous lemma, **N4** $\vDash \neg r_2(A)$. So by the completeness of **N4**, **N4** $\vdash \neg r_2(A)$. Again by the analogue of Lemma 2.8, this time for **N4**, we conclude **N4** $\vdash \neg A$. The converse direction holds analogously. The case for (i) is similarly demonstrated. □

Moving on to the next topic, a notable feature of **N3** and **N4** is that they do not allow the contraposition of an implication.[6] For **DN3** and **DN4(')**, they are still not provable. However it is provable in partial forms.

PROPOSITION 3.24.

(i) ***DN3*** $\vdash (\neg A \to B) \to (\sim B \to \sim\neg A)$.
(ii) ***DN4(')*** $\vdash (\neg A \to \neg B) \to (\sim\neg B \to \sim\neg A)$.

Proof. For (i), by a theorem of **IPC**,

$$\mathbf{DN3} \vdash (\neg A \to B) \to (\neg B \to \neg\neg A).$$

Then by (SN) and Proposition 3.19 we obtain the stated formula. For (ii), similarly we have

$$\mathbf{DN4(')} \vdash (\neg A \to \neg B) \to (\neg\neg B \to \neg\neg A).$$

Then use Proposition 3.19 for both $\neg\neg B$ and $\neg\neg A$. □

In comparison, **N3** and **N4** do not prove $(\neg A \to \neg B) \to (B \to A)$, so we can see that the same restrictions do not work. On the other hand, we can derive $(A \to B) \to (\sim B \to \sim\neg\neg A)$ and $(A \to B) \to (\sim\neg\neg B \to \sim\neg\neg A)$ in **N3** and **N4**, respectively. However it is easy to check that each of the formulas is also derivable respectively in **DN3** and **DN4**. Hence **DN3** and **DN4(')** seem to fare better in this regard.

§4. Law of excluded middle for strong negation in DN4('). As is well-known, the addition of the law of excluded middle over intuitionistic logic defines classical logic. Similarly, the addition of the law of excluded middle for strong negation, i.e.,

$$A \vee \sim A$$

to **N3** will result in classical logic, as the strong negation becomes identical to the classical negation. The situation does not change when we move from **N3** to **DN3**; from $A \vee \sim A$ and $\sim A \to \neg A$, all the same we can derive both $A \vee \neg A$ and $\sim A \leftrightarrow \neg A$.

More interesting are the cases for the systems **DN4** and **DN4'**. What kind of systems are obtained when we add $A \vee \sim A$ to these logics? For **N4**, the resulting system will be the system **CLuNs** of Batens & De Clercq [2], as mentioned in [24]. **CLuNs** may be seen as a three-valued logic, with the truth-tables (Table 1) given in terms of values

[6] For some approaches to regain contraposition, cf. [14, 21].

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Table 1. *Truth tables for **CLuNs***

A	$\sim A$
T	**F**
I	**I**
F	**T**

$A \vee B$	**T**	**I**	**F**
T	**T**	**T**	**T**
I	**T**	**I**	**I**
F	**T**	**I**	**F**

$A \wedge B$	**T**	**I**	**F**
T	**T**	**I**	**F**
I	**I**	**I**	**F**
F	**F**	**F**	**F**

$A \to B$	**T**	**I**	**F**
T	**T**	**I**	**F**
I	**T**	**I**	**F**
F	**T**	**T**	**T**

$\{\mathbf{T}, \mathbf{I}, \mathbf{F}\}$. The value of $\bot$ is set to be **F** and the designated values for this semantics are taken to be **T** and **I**.

That the above truth table works as a semantics for **CLuNs** means that $A \vee \neg A$ is derivable in it, i.e., the fragment without $\sim$ is classical. Indeed, it is easily seen that $(A \to \neg A) \vee \sim(A \to \neg A)$ derives the excluded middle with respect to the intuitionistic negation. Here it is important that $\sim(A \to \neg A)$ is equivalent to A.

Going back to **DN4**, the same formula $\sim(A \to \neg A)$ is equivalent only to $\neg\neg A$, and as a result $(A \to \neg A) \vee \sim(A \to \neg A)$ derives only the *weak law of excluded middle* $\neg A \vee \neg\neg A$. This suggests the following axiomatization of the system $\mathbf{DN4} + (A \vee \sim A)$, which we shall call $\mathbf{DN4}_+$.

Definition 4.25 ($\mathbf{DN4}_+$). *The system* $\mathbf{DN4}_+$ *is defined by the following addition of axioms to* ***DN4***.

$$\neg A \vee \neg\neg A \tag{WLEM}$$

$$A \vee \sim A \tag{SLEM}$$

We first note that as a proof system, $\mathbf{DN4}_+$ has a redundancy: it follows from our discussion above that (WLEM) is provable from (SLEM). Our decision to include (WLEM) in the list of axioms is motivated solely by the convenience it offers in proving the completeness.

The semantics for $\mathbf{DN4}_+$ is obtained from that of **DN4** by slight modifications.

Definition 4.26 (Kripke semantics for $\mathbf{DN4}_+$). *The Kripke semantics for* $\mathbf{DN4}_+$ *is defined from the one for* ***DN4*** *by*:

- *the frames satisfy the condition*

$$\forall u, v, w(u \geq w \text{ and } v \geq w \text{ implies } \exists w'(w' \geq u \text{ and } w' \geq v)),$$

- *the models satisfy the condition* $\mathcal{V}^+(p) \cup \mathcal{V}^-(p) = W$.

It is well-known (cf. [4, 9]) that the system defined by the addition of (WLEM) to intuitionistic logic (to be denoted **IPC** + (WLEM)[7]) is sound and complete with the

[7] In the literature, this famous intermediate logic is also known by the name **KC**.

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class of Kripke frames satisfying the above frame conditions. We shall utilize this fact in order to prove the completeness for $\mathbf{DN4}_+$.

We begin with observing the soundness of $\mathbf{DN4}_+$.

PROPOSITION 4.27. *If* $\mathbf{DN4}_+ \vdash A$ *then* $\mathbf{DN4}_+ \vDash A$.

Proof. We argue by induction on the depth of deduction. It suffices to show that $w \Vdash^+ A \vee {\sim}A$ for all w and A. We show this by induction on the complexity of A.

When $A \equiv p$, then since $\mathcal{V}^+(p) \cup \mathcal{V}^-(p) = W$, either

$$w \Vdash^+ p \text{ or } w \Vdash^- p$$

for each w. Hence $w \Vdash^+ p \vee {\sim}p$ for all w.

When $A \equiv \bot$, since $w \Vdash^- \bot$, it follows that $w \Vdash^+ \bot \vee {\sim}\bot$ for all w.

When $A \equiv A_1 \wedge A_2$, by I.H.

$$w \Vdash^+ A_1 \vee {\sim}A_1 \text{ and } w \Vdash^+ A_2 \vee {\sim}A_2.$$

If $w \Vdash^+ A_1$ and $w \Vdash^+ A_2$, then $w \Vdash^+ A_1 \wedge A_2$. Otherwise, $w \Vdash^- A_1$ or $w \Vdash^- A_2$, so $w \Vdash^- A_1 \wedge A_2$. Hence $w \Vdash^+ {\sim}(A_1 \wedge A_2)$. Therefore either way, $w \Vdash^+ (A_1 \wedge A_2) \vee {\sim}(A_1 \wedge A_2)$.

When $A \equiv A_1 \vee A_2$, we have the same I.H. as above. If $w \Vdash^+ A_1$ or $w \Vdash^+ A_2$, then $w \Vdash^+ A_1 \vee A_2$. Otherwise, $w \Vdash^- A_1$ and $w \Vdash^- A_2$. So $w \Vdash^- A_1 \vee A_2$. Thus $w \Vdash^+ {\sim}(A_1 \vee A_2)$. Therefore $w \Vdash^+ (A_1 \vee A_2) \vee {\sim}(A_1 \vee A_2)$.

When $A \equiv A_1 \to A_2$, we wish to show

$$w \Vdash^+ (A_1 \to A_2) \vee {\sim}(A_1 \to A_2).$$

Assume otherwise. Then

$$w \nVdash^+ (A_1 \to A_2) \text{ and } w \nVdash^+ {\sim}(A_1 \to A_2).$$

From the former, there is $w' \geq w$ such that

$$w' \Vdash^+ A_1 \text{ and } w' \nVdash^+ A_2.$$

It follows then that $w \nVdash^+ A_2$. From the latter, either

$$w \nVdash^+ \neg\neg A_1 \text{ or } w \nVdash^- A_2.$$

Now, by I.H. $w \Vdash^+ A_2 \vee {\sim}A_2$; but as $w \nVdash^+ A_2$, we need to infer that $w \Vdash^+ {\sim}A_2$. Thus $w \Vdash^- A_2$, and so it cannot be the case that $w \nVdash^- A_2$. Therefore $w \nVdash^+ \neg\neg A_1$. Hence there is $w'' \geq w$ such that for all $w''' \geq w''$, it holds that $w''' \nVdash^+ A_1$. Then by the frame property, there is $u \geq w', w''$ such that $u \Vdash^+ A_1$ and $u \nVdash^+ A_1$, a contradiction. Therefore $w \Vdash^+ (A_1 \to A_2) \vee {\sim}(A_1 \to A_2)$.

When $A \equiv {\sim}A_1$, by I.H. we have $w \Vdash^+ A_1 \vee {\sim}A_1$. Hence $w \Vdash^+ {\sim}A_1 \vee {\sim}{\sim}A_1$. □

COROLLARY 4.28 (Conservative extension). *Let* A *be a formula in the language of* ***IPC****. Then* $\mathbf{DN4}_+ \vdash A$ *implies* $\mathbf{IPC} + (\mathit{WLEM}) \vdash A$.

Proof. By soundness, if $\mathbf{DN4}_+ \vdash A$ then $\mathbf{DN4}_+ \vDash A$. Let $\mathcal{M}$ be a model of **IPC** + (WLEM). Then we can define a model $\mathcal{M}'$ of $\mathbf{DN4}_+$ from $\mathcal{M}$ by stipulating $\mathcal{V}^-(p) = W$. Then since the forcing of A does not rely on $\mathcal{V}^-$, $\mathcal{M}' \vDash A$ implies $\mathcal{M} \vDash A$. Thus $\mathbf{IPC} + (\text{WLEM}) \vDash A$; so $\mathbf{IPC} + (\text{WLEM}) \vdash A$ by the completeness of **IPC** + (WLEM). □

Next we move on to prove the completeness. The argument is similar to that of **DN3**. First, we look at the analogue of Lemma 2.9.

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LEMMA 4.29. *If* $\boldsymbol{DN4}_+ \vdash A$, *then* $\boldsymbol{DN4}_+ \vdash r(A)$; *moreover, one may assume all the formulas in the proof are reduced. (That is, it has a reduced proof.)*

Proof. We argue by induction on the depth of derivation. It suffices to check the case for the axiom (SLEM). This we show by induction on the complexity of A.

When $A \equiv p$, then $r(p \vee {\sim}p)$ is $p \vee {\sim}p$, which is an instance of (SLEM). Similarly when $A \equiv \bot$.

When $A \equiv A_1 \wedge A_2$, then $r(A \vee {\sim}A)$ is

$$(r(A_1) \wedge r(A_2)) \vee (r({\sim}A_1) \vee r({\sim}A_2)).$$

By I.H., there are reduced proofs of

$$r(A_1) \vee r({\sim}A_1) \text{ and } r(A_2) \vee r({\sim}A_2),$$

and so we can also obtain a reduced proof of $r(A \vee {\sim}A)$.

When $A \equiv A_1 \vee A_2$, the argument is similar to the previous case.

When $A \equiv A_1 \to A_2$, $r(A \vee {\sim}A)$ is

$$(r(A_1) \to r(A_2)) \vee (\neg\neg r(A_1) \wedge r({\sim}A_2)).$$

By I.H. there is a reduced proof of $r(A_2) \vee r({\sim}A_2)$. In addition, $\neg r(A_1) \vee \neg\neg r(A_1)$ is an instance of (WLEM). Thus there is a reduced proof of $r(A \vee {\sim}A)$.

When $A \equiv {\sim}A_1$, then $r(A \vee {\sim}A)$ is $r({\sim}A_1) \vee r(A_1)$, which has a reduced proof by I.H. □

We shall next show the analogue of Proposition 2.10. For this purpose, given a reduced formula $A[{\sim}p_1, \dots, {\sim}p_n]$, we define a formula

$$F := \bigwedge_{1 \leq i \leq n} (p_i \vee q_i).$$

PROPOSITION 4.30. *Let* $A[{\sim}p_1, \dots, {\sim}p_n]$ *be a reduced formula. Then the following are equivalent.*

(i) $\boldsymbol{DN4}_+ \vdash A[{\sim}p_1, \dots, {\sim}p_n]$.
(ii) $\boldsymbol{IPC} + (\mathit{WLEM}) \vdash F \to A[{\sim}p_1, \dots, {\sim}p_n/q_1, \dots, q_n]$.

Proof. From (i) to (ii), we argue by induction on the depth of derivation in $\mathbf{DN4}_+$. It suffices to show the case for (SLEM). Then A has to be prime.

When $A \equiv p_i$, we have $\mathbf{DN4}_+ \vdash p_i \vee {\sim}p_i$. Correspondingly, $\mathbf{IPC} + (\mathrm{WLEM}) \vdash (p_i \vee q_i) \to (p \vee ({\sim}p_i[{\sim}p_i/q_i]))$.

When $A \equiv \bot$ We have $\mathbf{DN4}_+ \vdash \bot \vee {\sim}\bot$. Correspondingly, $\mathbf{IPC} + (\mathrm{WLEM}) \vdash \bot \vee {\sim}\bot$.

From (ii) to (i), the argument is as in Proposition 2.10. □

Now we are ready to conclude the completeness proof.

THEOREM 4.31. *If* $\boldsymbol{DN4}_+ \vDash A$ *then* $\boldsymbol{DN4}_+ \vdash A$.

Proof. Our argument is mostly identical to Theorem 2.13. Instead of Lemma 2.9 and Proposition 2.10, we use Lemma 4.29 and Proposition 4.27. In addition, we appeal to the completeness of $\mathbf{IPC}$ + (WLEM). Finally, to see that the constructed model of $\mathbf{DN4}_+$ satisfies the condition that $\mathcal{V}^+(p) \cup \mathcal{V}^-(p) = W$. it suffices to observe that the original model of $\mathbf{IPC}$ + (WLEM) validates $p_i \vee q_i$. □

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Comparing $\mathbf{DN4}_+$ to **CLuNs**, it may be taken as advantageous that (SLEM) does not necessitate the full law of excluded middle in the former. For instance, one may wish to introduce a decidable notion of negation while keeping the constructive setting as much as possible.[8]

We next look at the case for **DN4'**. For this purpose we define a new system $\mathbf{DN4'}_+$.

DEFINITION 4.32 ($\mathbf{DN4'}_+$). *The system* $\mathbf{DN4'}_+$ *is defined by the addition of* (*SLEM*) *and the next axiom to* ***DN4'***.

$$A \vee \neg A \tag{LEM}$$

This time, as was the case for **N4**, we require (LEM) to be derivable in the system, thus committing to a less constructive principle than $\mathbf{DN4}_+$. It does not mean, however, that $\mathbf{DN4'}_+$ becomes identical to **CLuNs**, as we shall see below.

The Kripke semantics for $\mathbf{DN4'}_+$ is quite similar to the semantics for classical logic.

DEFINITION 4.33 (Kripke semantics for $\mathbf{DN4'}_+$). *The Kripke semantics for* $\mathbf{DN4'}_+$ *is defined from the one for* ***DN4'*** *by*:

- *the frames satisfy the condition*

$$W = \{w\},$$

- *the models satisfy the condition* $\mathcal{V}^+(p) \cup \mathcal{V}^-(p) = W$.

Then we start with the soundness of the semantics with respect to $\mathbf{DN4'}_+$.

PROPOSITION 4.34. *If* $\mathbf{DN4'}_+ \vdash A$ *then* $\mathbf{DN4'}_+ \vDash A$.

Proof. We show by induction on the depth of derivation. We concentrate on the case for (SLEM), with $A \equiv A_1 \to A_2$. In this case, by I.H.

$$w \Vdash^+ A_1 \vee {\sim} A_1 \text{ and } w \Vdash^+ A_2 \vee {\sim} A_2.$$

Now if $w \nVdash^+ A_1 \to A_2$ and $w \nVdash^+ {\sim}(A_1 \to A_2)$, then $w \Vdash^+ A_1$ and $w \nVdash^+ A_2$ from the former. Also, $w \nVdash^- A_1 \to A_2$ from the latter; so $w \nVdash^+ \neg(A_1 \to A_2)$ or $w \nVdash^- A_2$. But the former means $w \Vdash^+ A_1 \to A_2$, a contradiction. On the other hand, the latter means $w \nVdash^+ A_2 \vee {\sim} A_2$, another contradiction. Therefore $w \Vdash^+ (A_1 \to A_2) \vee {\sim}(A_1 \to A_2)$. □

Next we shall treat the completeness direction. The outline is identical to that of $\mathbf{DN4}_+$.

LEMMA 4.35. *If* $\mathbf{DN4'}_+ \vdash A$, *then* $\mathbf{DN4'}_+ \vdash r(A)$; *moreover, one may assume all the formulas in the proof are reduced.*

Proof. Again, it suffices to check the case for (SLEM), and we concentrate on the case $A \equiv A_1 \to A_2$. Then

$$r((A_1 \to A_2) \vee {\sim}(A_1 \to A_2)) = (r(A_1) \to r(A_2)) \vee (\neg(r(A_1) \to r(A_2)) \wedge r({\sim} A_2)).$$

By I.H., there is a reduced proof of $r(A_2) \vee r({\sim} A_2)$, and thus of $(r(A_1) \to r(A_2)) \vee r({\sim} A_2)$. Moreover, $(r(A_1) \to r(A_2)) \vee \neg(r(A_1) \to r(A_2))$ is an instance of (LEM). Hence there is a reduced proof of

$$(r(A_1) \to r(A_2)) \vee (\neg(r(A_1) \to r(A_2)) \wedge r({\sim} A_2)).$$ □

[8] An example of this type of attitude can be found in [6].

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Table 2. *Truth table of implication for* ***DN4'***$_+$

$A \to B$	**T**	**I**	**F**
T	**T**	**T**	**F**
I	**T**	**T**	**F**
F	**T**	**T**	**T**

Let **CPC** to be the formalization of classical logic, defined as **IPC** + (LEM).

PROPOSITION 4.36. *Let* $A[\sim p_1, \dots, \sim p_n]$ *be a reduced formula. Then the following are equivalent.*

(i) $\boldsymbol{DN4'}_+ \vdash A[\sim p_1, \dots, \sim p_n]$.
(ii) $\boldsymbol{CPC} \vdash F \to A[\sim p_1, \dots, \sim p_n / q_1, \dots, q_n]$.

Proof. Analogous to Proposition 4.30. □

THEOREM 4.37. *If* $\boldsymbol{DN4'}_+ \vDash A$ *then* $\boldsymbol{DN4'}_+ \vdash A$.

Proof. Analogous to Theorem 4.31. □

Having established the completeness, we shall see how the semantics for **DN4'**$_+$ compares to that of **CLuNs** in terms of truth tables.

We shall define an *assignment* $\mathcal{A}$ to be a mapping from the set of prime formulas to the set of *truth values* $\{\mathbf{T}, \mathbf{I}, \mathbf{F}\}$. We in particular set $\mathcal{A}(\bot) := \mathbf{F}$.

The truth value for a general formula is determined by the truth tables for each of the connectives. The truth tables for conjunction, disjunction and strong negation are identical to those of **CLuNs**. Table 2 gives the table of implication for **DN4'**$_+$.

The only changes from the truth table of implication for **CLuNs** are when the antecedent of an implication has the value **T** or **I**, and the succedent the value **I**. As a consequence, an implication always has the value **T** or **F**[9].

We define the validity with respect to the truth tables, to be denoted with $\vDash_3$, by the next clause.

$$\vDash_3 A \text{ iff for all assignment } \mathcal{A}, \mathcal{A}(A) \in \{T, I\}.$$

We shall now establish the correspondence of the truth tables with the Kripke semantics.

LEMMA 4.38. *Let* $\mathcal{M}$ *be a model of* ***DN4'***$_+$. *Define an assignment* $\mathcal{A}_{\mathcal{M}}$ *by:*

$$\mathcal{A}_{\mathcal{M}} := \begin{cases} \boldsymbol{T} & \text{if } \mathcal{M}, w \Vdash^+ p \text{ and } \mathcal{M}, w \nVdash^- p, \\ \boldsymbol{I} & \text{if } \mathcal{M}, w \Vdash^+ p \text{ and } \mathcal{M}, w \Vdash^- p, \\ \boldsymbol{F} & \text{if } \mathcal{M}, w \nVdash^+ p \text{ and } \mathcal{M}, w \Vdash^- p. \end{cases}$$

Then the following statements hold.

(i) $\mathcal{A}_{\mathcal{M}}(A) = \boldsymbol{T}$ *iff* $\mathcal{M}, w \Vdash^+ A$ *and* $\mathcal{M}, w \nVdash^- A$.
(ii) $\mathcal{A}_{\mathcal{M}}(A) = \boldsymbol{I}$ *iff* $\mathcal{M}, w \Vdash^+ A$ *and* $\mathcal{M}, w \Vdash^- A$.
(iii) $\mathcal{A}_{\mathcal{M}}(A) = \boldsymbol{F}$ *iff* $\mathcal{M}, w \nVdash^+ A$ *and* $\mathcal{M}, w \Vdash^- A$.

9 We note that an identical table is used in [27].

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Proof. By induction on the complexity of A. □

For the converse direction, we have the following lemma.

LEMMA 4.39. *Let $\mathcal{A}$ be an assignment. Define a **DN4'**$_+$ model $\mathcal{M}_{\mathcal{A}}$ by:*

$$w \in \mathcal{V}^+(p) \textit{ if } \mathcal{A}(p) = \boldsymbol{T} \textit{ or } \boldsymbol{I},$$
$$w \in \mathcal{V}^-(p) \textit{ if } \mathcal{A}(p) = \boldsymbol{F} \textit{ or } \boldsymbol{I}.$$

Then the following statements hold.

(i) $\mathcal{M}_{\mathcal{A}}, w \Vdash^+ A$ *if and only if* $\mathcal{A}(A) = \boldsymbol{T}$ *or* $\boldsymbol{I}$.
(ii) $\mathcal{M}_{\mathcal{A}}, w \Vdash^- A$ *if and only if* $\mathcal{A}(A) = \boldsymbol{F}$ *or* $\boldsymbol{I}$.

Proof. By induction on the complexity of A. We look at the case when $A \equiv A_1 \to A_2$. For (i) and (ii), we observe the following equivalences.

$$\begin{aligned}\text{(i) } \mathcal{M}_{\mathcal{A}}, w \Vdash^+ A_1 \to A_2 &\Leftrightarrow \mathcal{M}_{\mathcal{A}}, w \nVdash^+ A_1 \text{ or } \mathcal{M}_{\mathcal{A}}, w \Vdash^+ A_2.\\ &\Leftrightarrow \mathcal{A}(A_1) = \mathbf{F} \text{ or } \mathcal{A}(A_2) \neq \mathbf{F}.\\ &\Leftrightarrow \mathcal{A}(A_1 \to A_2) = \mathbf{T} \text{ (or } \mathbf{I}).\end{aligned}$$

In the last equivalence, note it is impossible that $\mathcal{A}(A_1 \to A_2) = \mathbf{I}$.

$$\begin{aligned}\text{(ii) } \mathcal{M}_{\mathcal{A}}, w \Vdash^- A_1 \to A_2 &\Leftrightarrow \mathcal{M}_{\mathcal{A}}, w \Vdash^+ \neg(A_1 \to A_2) \text{ and } \mathcal{M}_{\mathcal{A}}, w \Vdash^- A_2.\\ &\Leftrightarrow (\mathcal{A}(A_1) = \mathbf{T} \text{ or } \mathbf{I}) \text{ and } \mathcal{A}(A_2) = \mathbf{F}.\\ &\Leftrightarrow \mathcal{A}(A_1 \to A_2) = \mathbf{F} \text{ (or } \mathbf{I}).\end{aligned}$$

□

Therefore we can establish the desired correspondence.

THEOREM 4.40. ***DN4'***$_+ \vDash A \Leftrightarrow \vDash_3 A$.

Proof. For the left-to-right direction, if $\mathcal{A}$ is an assignment, then by assumption, $\mathcal{M}_{\mathcal{A}}, w \Vdash^+ A$. Thus by Lemma 4.39, $\mathcal{A}(A) = \mathbf{T}$ or $\mathbf{I}$. Therefore $\vDash_3 A$.

For the right-to-left direction, if $\mathcal{M}$ is a model of **DN4'**$_+$, then by assumption $\mathcal{A}_{\mathcal{M}}(A) = \mathbf{T}$ or $\mathbf{I}$. Thus by Lemma 4.38, $\mathcal{M} \vDash A$. Therefore **DN4'**$_+ \vDash A$. □

Having obtained the completeness of **DN4'**$_+$ with the three-valued truth-tables, a natural question now would be to ask what may be the sense of the tables. In this respect, it is helpful to note that the same tables are used in [5] as a semantics for the logic **LPT**, which is defined by (K), (S), (CI), (CE), (DI), (DE), (FF), (SLEM) and the following axioms, with the rule (MP).

- $A \vee (A \to B)$,
- $\circ A \to (A \to ({\sim}A \to B))$,
- ${\sim} \circ A \to (A \wedge {\sim}A)$,
- $\circ(A \to B)$,
- $\circ A \wedge \circ B \to \circ(A \wedge B)$,
- $(A \wedge {\sim}A \wedge B) \to {\sim}(A \wedge B) \wedge {\sim}(B \wedge A)$,

where $\circ := \neg(A \wedge {\sim}A)$ is what is known as the *consistency operator.*

LPT is designed to capture the notion of quasi-truth, which was originally introduced in [17]. Its setting differs from ours only in that they take $\bot$ and $\vee$ as defined by the clauses $\bot := {\sim}(A \to A)$ and $A \vee B := {\sim}({\sim}A \wedge {\sim}B)$. We shall see the precise correspondence of the two systems.

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LEMMA 4.41.

(i) ***DN4'***$_+ \vdash \bot \leftrightarrow \sim(A \to A)$ *and* ***DN4'***$_+ \vdash \sim\bot \leftrightarrow \sim\sim(A \to A)$.
(ii) ***DN4'***$_+ \vdash (A \vee B) \leftrightarrow \sim(\sim A \wedge \sim B)$ *and* ***DN4'***$_+ \vdash \sim(A \vee B) \leftrightarrow \sim\sim(\sim A \wedge \sim B)$.

Proof. Straightforward. □

Then we appeal to the well-known replacement rule for strong negation (cf. for instance [22, proposition 8.1.3]). Let $*$ be a fixed propositional variable taking a position in a formula C. (That is, $*$ occurs once in C.)

LEMMA 4.42. *If* ***DN4'***$_+ \vdash A \leftrightarrow B$ *and* ***DN4'***$_+ \vdash \sim A \leftrightarrow \sim B$, *then* ***DN4'***$_+ \vdash C[*/A] \leftrightarrow C[*/B]$.

Proof. By induction on the complexity of C. □

The above lemmas imply that occurrences of $\bot$ and disjunctions in a formula can be replaced with equivalent formulas without the connectives. Then it is straightforward to observe the following.

PROPOSITION 4.43. ***DN4'***$_+ \vdash A$ *if and only if* ***LPT*** $\vdash A'$, *where* A' *is a formula in the language of* ***LPT*** (*i.e., without* $\bot, \vee$) *satisfying* ***DN4'***$_+ \vdash A \leftrightarrow A'$.

Proof. If **LPT** $\vdash A'$, then **DN4'**$_+ \vdash A'$ by the completeness of **LPT** with respect to the truth-tables [5, theorem 5.12] and Theorem 4.40. On the other hand, if **DN4'**$_+ \vdash A$, then by the above lemmas, there is a formula A' in the language of **LPT** such that **DN4'**$_+ \vdash A \leftrightarrow A'$. Then again by completeness, **LPT** $\vdash A'$. □

§5. Connexivizing the variants.. Having looked at the behavior of strong negation in **DN3** and **DN4(')**, one natural development would be to consider how a similar change would affect the sibling notion of connexive negation in the system **C** of Wansing [32]. **C** may be obtained[10] from **N4** by changing the falsity condition of implication to

$$\sim(A \to B) \leftrightarrow (A \to \sim B).$$

That is to say, by replacing $\wedge$ with $\to$ in the condition. Then the contra-classical principles called *Aristotle's theses* and *Boethius' theses* below become valid.

$$\sim(\sim A \to A) \qquad \text{(AT)}$$

$$\sim(A \to \sim A) \qquad \text{(AT')}$$

$$(A \to B) \to \sim(A \to \sim B) \qquad \text{(BT)}$$

$$(A \to \sim B) \to \sim(A \to B) \qquad \text{(BT')}$$

When we consider similar alternations in the falsity condition for implication, we have a choice between starting from **DN4**-type or **DN4'**-type falsity condition. Selecting the latter would mean employing the axiom

$$\sim(A \to B) \leftrightarrow (\neg(A \to B) \to \sim B).$$

[10] Note however that $\bot$ is absent from the language in the original formulation.

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This however does not seem satisfactory, because this would imply the derivability of

$$(A \to B) \to \sim(A \to B),$$

that is to say all implication implies its falsity, which is odd and rather strong. **DN4**-type condition, on the other hand, gives rise to the next axiom

$$\sim(A \to B) \leftrightarrow (\neg\neg A \to \sim B).$$

Then one may ask whether Aristotle's and Boethius' theses hold as in **C**. Now in our setting, these are equivalent to:

- $\neg\neg\sim A \to \sim A$,
- $\neg\neg A \to A$,
- $(A \to B) \to (\neg\neg A \to B)$,
- $(A \to \sim B) \to (\neg\neg A \to \sim B)$.

Therefore we observe that the validity of the connexive theses in this formulation is tied with the validity of non-constructive principles such as double negation elimination. This is a rather interesting phenomenon in that it ties *contra-classical* characteristics for negation with *classical* characteristics, thus offering a different view of connexivity from that of **C**.

If we restrict A to be of the form $\neg A$, then (AT)–(BT') indeed hold, but this rests on the fact that $\sim(A \to \bot)$ is a theorem for any A. This is an odd feature, and one may wish not to have $\bot$ in the language for this reason, as the original formulation did not. A possible remedy then is to change the language to a modal language $\mathcal{L}_\Box$

$$A ::= p \mid \Box A \mid A \wedge A \mid A \vee A \mid A \to A \mid \sim A,$$

with the modal axioms for double negation in [3, 28]:

$$\Box(A \to B) \to (\Box A \to \Box B) \tag{M1}$$

$$A \to \Box A \tag{M2}$$

$$\Box(((A \to B) \to A) \to A) \tag{M3}$$

$$\Box(\Box A \to A) \tag{M4}$$

We can set the falsity conditions for implication and modal formulas to be

- $\sim(A \to B) \leftrightarrow (\Box A \to \sim B)$,
- $\sim\Box A \leftrightarrow \Box\sim A$.

Then for formulas of the form $\Box A$, (AT)–(BT') become equivalent to the following:

- $\Box\sim\Box A \to \sim\Box A$,
- $\Box\Box A \to \Box A$,
- $(\Box A \to \Box B) \to (\Box\Box A \to \Box B)$,
- $(\Box A \to \sim\Box B) \to (\Box\Box A \to \sim\Box B)$,

which all hold. (Note $\Box\Box A \leftrightarrow \Box A$ because $\neg\neg\neg\neg A \leftrightarrow \neg\neg A$.) Therefore even when we adopt the **DN4**-type condition, one can retain connexivity with respect to the formulas of form $\Box A$, i.e.,

- $\sim(\sim\Box A \to \Box A)$,
- $\sim(\Box A \to \sim\Box A)$,
- $(\Box A \to \Box B) \to \sim(\Box A \to \sim\Box B)$,
- $(\Box A \to \sim\Box B) \to \sim(\Box A \to \Box B)$.

In the above, one may also take $\neg$ to be primitive and alter the falsity condition for $\neg A$ to

$$\sim\neg A \leftrightarrow \neg\sim A$$

in order to obtain a similar result.[11] Here one might argue that such a move is unwarranted, because the falsity condition for $\neg A$ should not be different from that of $A \to \bot$, for semantically their truth conditions are equivalent. We may, however, associate a different truth condition for intuitionistic negation. For example, Došen [8] observes that the intuitionistic negation may be seen as a modal operator with a modal accessibility relation R (with certain conditions imposed), s.t.

$$w \Vdash \neg A \Longleftrightarrow \forall w'(wRw' \Rightarrow w' \nVdash A).$$

From such a point of view, it does not appear as justified that the above falsity condition for primitive negation should be identical to that of $A \to \bot$. Similarly, for $\Box$ the truth condition in [3] is given by $\forall w'(wRw' \Rightarrow w' \Vdash A)$, which when looked independently does not seem to support that the corresponding falsity condition should be that of $(A \to \bot) \to \bot$.

§6. Concluding remarks. In this paper, we looked at the strong negation of Nelson–Markov, and identified the problem that the falsity condition for implication does not reflect the intuitionistic equivalence between $\neg(A \to B)$ and $\neg\neg A \wedge \neg B$. For this reason we set up the systems **DN3** and **DN4(')** by replacing A in the falsity condition with $\neg\neg A$. We established the soundness and completeness of the systems with respect to Kripke semantics, and then made comparisons with **N3** and **N4**.

From the results we obtained, it is possible to summarize the advantages of **DN3** and **DN4(')** over **N3** and **N4** as follows.

- **DN3** and **DN4** refute strictly more propositions in the language of intuitionistic logic.
- More general forms of contraposition are available in **DN3** and **DN4(')**.
- Unlike **N4**, adding $A \vee \sim A$ to **DN4** does not force $A \vee \neg A$ to be valid.

For these reasons, we wish to claim that our systems are improvements over **N3** and **N4**, at least if one believes that strong negation should resemble intuitionistic negation as much as possible (while retaining the constructive falsity property).

In addition to the above, our observations included that the provability of propositions of the form $\neg A$ in **N3** (**N4**) corresponds with that of **DN3** (**DN4**); how **DN4'** may be seen as a generalization of a logic of quasi-truth; and how the resulting systems fare with connexivity when a change is made corresponding to that of **DN4** to **C**.

For the future works, an obvious direction would be to consider the predicate extension of **DN3** and **DN4(')**. It is expected that not all of the observations we made in this paper are replicable in the predicate logic, at least because we appealed to a

[11] We note that the negation of the system **BDi** in [12] satisfies this condition.

Glivenko-like proposition. Therefore it is interesting to study how much of our results can be elevated to the predicate level. It is also worthwhile to explore whether the change in the falsity condition for implication creates a difference that is unique to the predicate language.

Another fruitful direction would be to relate **DN4'** with the notion of quasi-truth. We started from the notion of strong negation and reached **DN4'**$_+$. It is then a natural question to ask whether one can start from the theory of quasi-truth and shed light on **DN4'** from that perspective. In particular, it should be of considerable interest if one could give an interpretation of the difference between **DN4** and **DN4'** through such a route.

Finally, it is intriguing to ask how the same change in the falsity condition would affect the logic of contraposable strong negation explored in [21], which is recently studied in depth in [18, 19].

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