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UPPER AND LOWER LIMIT OSCILLATION CONDITIONS FOR FIRST-ORDER DIFFERENCE EQUATIONS

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Abstract In this work, we consider the first-order difference equation with general argument

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \geq 0,$$

where $(p(n))$ is a sequence of non-negative real numbers, $(\tau(n))$ is a sequence of integers such that $\tau(n) < n$ for $n \in \mathbf{N}$, and $\lim_{n\to\infty} \tau(n) = \infty$. Under the assumption that the deviating argument is not necessarily monotone, we obtain some new oscillation conditions and improve the all known results for the above equation in the literature, involving only upper and only lower limit conditions. Two examples illustrating the results are also given.

Keywords: delay difference equation; general argument; oscillation

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1. Introduction

For a long time, the oscillation theory of differential and difference equations has attracted many researchers. In recent years, there has been much research activity concerning the oscillation and non-oscillation of solutions of delay differential and difference equations. For these oscillatory and non-oscillatory results, we refer, for instance [1–19].

Consider the retarded difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $(p(n))$ is a sequence of non-negative real numbers, $(\tau(n))$ is a sequence of integers such that

$$\tau(n) < n \text{ for } n \geq 0, \text{ and } \lim_{n\to\infty} \tau(n) = \infty. \tag{1.2}$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$. Define

$$k = -\min_{n\geq 0} \tau(n) \quad \text{(clearly, } k \text{ is a positive integer).}$$

By a solution of the difference equation (1.1), we mean a sequence of real numbers $(x(n))$ which satisfies (1.1) for all $n \geq 0$. It is clear that, for each choice of real numbers $c_{-k}, c_{-k+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))$ of (1.1) which satisfies the initial conditions $x(-k) = c_{-k},\ x(-k+1) = c_{-k+1},\ \ldots,\ x(-1) = c_{-1},\ x(0) = c_0$.

A solution $(x(n))$ of the difference equation (1.1) is called oscillatory, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be non-oscillatory.

When $\tau(n) = n - l$ where $l \geq 0$ an integer number, then Equation (1.1) reduces to

$$\Delta x(n) + p(n)x(n-l) = 0. \tag{1.3}$$

Strong interest in the delay difference equation (1.1) is motivated by the fact that it represents a discrete analogue of the delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.4}$$

where $p(t) \in C([t_0, \infty), [0, \infty)),\ \tau(t) \in C([t_0, \infty), \mathbb{R}),\ \tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

In particular, the delay difference equation (1.3) represents a discrete analogue of the (first-order) delay differential equation

$$x'(t) + p(t)x(t-T) = 0, \quad t \geq t_0, \tag{1.5}$$

where T is a positive real constant. For Equations (1.4) and (1.5), see [8, 10, 13].

The problem of establishing sufficient conditions for the oscillation of all solutions of the difference equations (1.1) and (1.3) has been the subject of many investigations, for instance, in 1989, Erbe and Zhang [7] proved that each one of the conditions

$$\liminf_{n\to\infty} p(n) > \frac{l^l}{(l+1)^{l+1}} \tag{1.6}$$

or

$$\limsup_{n\to\infty} \sum_{j=n-l}^{n} p(j) > 1 \tag{1.7}$$

is sufficient for all solutions of (1.3) to be oscillatory. In the same year, 1989, Ladas, Philos and Sficas [12] established that all solutions of (1.3) are oscillatory if

$$\liminf_{n\to\infty} \sum_{j=n-l}^{n-1} p(j) > \left(\frac{l}{l+1}\right)^{l+1}. \tag{1.8}$$

Clearly, the condition (1.8) improves the condition (1.6).

We now turn to the general case of the delay difference equation (1.1). The condition (1.7) can be extended to Equation (1.1). More precisely, if the sequence $(\tau(n))$ is assumed to be increasing, then from Chatzarakis et al. [3], it follows that all solutions of (1.1) are oscillatory if

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1. \tag{1.9}$$

In 1991, Philos [14] extended the oscillation criterion (1.8) to the general case of Equation (1.1), by establishing that, if the sequence $(\tau(n))$ is non-decreasing, then the condition

$$\liminf_{n\to\infty} [\frac{1}{n-\tau(n)} \sum_{j=\tau(n)}^{n-1} p(j)] > \limsup_{n\to\infty} \frac{(n-\tau(n))^{n-\tau(n)}}{(n-\tau(n)+1)^{n-\tau(n)+1}} \tag{1.10}$$

suffices for the oscillation of all solutions of Equation (1.1).

In 1998, Zhang and Tian [19] obtained that if $(\tau(n))$ is non-decreasing,

$$\lim_{n\to\infty} (n-\tau(n)) = \infty, \tag{1.11}$$

and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.12}$$

then all solutions of (1.1) are oscillatory.

In 1998, Zhang and Tian [18] obtained that if $(\tau(n))$ is not necessarily monotone and

$$\limsup_{n\to\infty} p(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.13}$$

then all solutions of (1.1) oscillate.

In 2008, Chatzarakis et al. [2, 3], when $(\tau(n))$ is not necessarily monotone, studied Equation (1.1) and proved that, if one of the following conditions

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) > 1, \quad \text{where } h(n) = \max_{0\le s\le n} \tau(s), n \ge 0, \tag{1.14}$$

or

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) < \infty \text{ and } \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.15}$$

is satisfied, then all solutions of (1.1) oscillate.

Set,

$$k(n) := \left(\frac{n-\tau(n)+1}{n-\tau(n)}\right)^{n-\tau(n)+1}, \quad n \geq 0. \tag{1.16}$$

Clearly,

$$e < k(n) \leq 4, \quad n \geq 0.$$

In 2006, W. Yan, Q. Meng and J. Yan [17] obtained that if $(\tau(n))$ is non-decreasing and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) \left(\frac{j-\tau(j)+1}{j-\tau(j)}\right)^{j-\tau(j)+1} > 1, \tag{1.17}$$

then all solutions of (1.1) are oscillatory.

Observe that, it is easy to see that

$$\sum_{j=\tau(n)}^{n-1} p(j)k(j) > e \sum_{j=\tau(n)}^{n-1} p(j)$$

and therefore the condition (1.17) is better than the condition (1.12).

In 2016, Öcalan [15], when $(\tau(n))$ is not necessarily monotone, established the following result; if

$$\begin{aligned} &\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j) \left(\frac{j-\tau(j)+1}{j-\tau(j)}\right)^{j-\tau(j)+1} \\ &= \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) \left(\frac{j-\tau(j)+1}{j-\tau(j)}\right)^{j-\tau(j)+1} > 1, \end{aligned} \tag{1.18}$$

where $h(n) = \max_{0\leq s\leq n} \tau(s)$, $n \geq 0$, then all solutions of (1.1) are oscillatory.

In 2011, Braverman and Karpuz [1] proved that if $(\tau(n))$ is not necessarily monotone and

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > 1, \tag{1.19}$$

then all solutions of (1.1) oscillate. Evidently, condition (1.19) has improved condition (1.14).

In [16], Öcalan proved that if $(\tau(n))$ is not necessarily monotone and

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j) \prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)} > \frac{1}{e}, \tag{1.20}$$

then all solutions of (1.1) oscillate. It can be seen immediately that if $(\tau(n))$ is non-decreasing, then condition (1.20) returns to condition (1.15). However, if $(\tau(n))$ is strictly non-monotone, then condition (1.20) has improved condition (1.15).

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The main aim of this paper is to improve, involving only upper and only lower limit conditions, the all known results for Equation (1.1) in the literature.

Throughout this paper, we are going to use the following notation:

$$\prod_{i=k}^{k-1} A(i) = 1 \quad \text{and} \quad \sum_{i=k}^{k-1} A(i) = 0.$$

2. Main results

We present some new sufficient conditions for the oscillation of all solutions of Equation (1.1), under the assumption that the arguments $(\tau(n))$ is not necessarily monotone.

Let,

$$h(n) = \max_{s \le n} \tau(s), \quad n \ge 0. \tag{2.1}$$

Clearly, $h(n)$ is non-decreasing and $\tau(n) \le h(n)$ for all $n \ge 0$.

The following Lemma was given in [3], which is needed to prove our next theorem.

Lemma 2.1. *Assume that (1.2) holds and $p(n) \ge 0$. Thus, we have*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j),$$

where $(h(n))$ is defined by (2.1).

Theorem 2.2. *Assume that (1.2) holds and $p(n) \ge 0$. Furthermore, assume that*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) \left(\frac{j-h(j)+1}{j-h(j)} \right)^{j-h(j)+1} \prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)} > 1, \tag{2.2}$$

where $(h(n))$ is defined by (2.1). If $\lim_{n\to\infty}(n-h(n)) = \infty$ or $h(n) = n-m,\ m \ge 1 \in \mathbb{N}$, then all solutions of Equation (1.1) oscillate.

Proof. Assume, for the sake of contradiction, that $(x(n))$ be an eventually positive solution of Equation (1.1). Let $n_1 \ge -k$ be an integer such that $x(n)$, $x(\tau(n)) > 0$ for all $n \ge n_1$. Thus, from Equation (1.1), we have

$$\Delta x(n) = -p(n)x(\tau(n)) \le 0, \quad n \ge n_1,$$

which means that the sequence $(x(n))$ is eventually non-increasing. In view of this and taking into account that $\tau(n) < n$, Equation (1.1) gives

$$\Delta x(n) + p(n)x(n) \le 0, \quad n \ge n_1.$$

If we apply the discrete Grönwall inequality to this inequality, we obtain

$$x(m) \ge x(n) \prod_{i=m}^{n-1} \frac{1}{1-p(i)}, \quad n_1 \le m \le n.$$

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On the other hand, we know from Lemma 2.1 that

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1} p(j)\left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1}\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)}$$
$$=\liminf_{n\to\infty}\sum_{j=h(n)}^{n-1} p(j)\left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1}\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)}.$$

Now, we define

$$s(n) := \left(\frac{n-h(n)+1}{n-h(n)}\right)^{n-h(n)+1}, \quad n \geq 0. \tag{2.3}$$

Thus, by (2.2), it follows that there exists a constant d such that

$$\begin{aligned}\sum_{j=h(n)}^{n} p(j)\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)} &\geq \sum_{j=h(n)}^{n} p(j)s(j)\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)}\\ &\geq \sum_{j=h(n)}^{n-1} p(j)s(j)\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)} \geq d > 1.\end{aligned} \tag{2.4}$$

Now, in view of (2.4), and for all large n, there exists $n^* \in [h(n), n)$ such that

$$4\sum_{j=h(n)}^{n^*} p(j)\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)} \geq \frac{d}{2} \text{ and } 4\sum_{j=n^*}^{n} p(j)\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)} \geq \frac{d}{2}. \tag{2.5}$$

From the fact that $(h(n))$ is non-decreasing and $(x(n))$ is non-increasing, summing up (1.1) from $h(n)$ to $n^*$and applying the discrete Grönwall inequality, we obtain

$$x(n^*+1) - x(h(n)) + \sum_{j=h(n)}^{n^*} p(j)x(h(j))\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)} \leq 0$$

and

$$x(n^*+1) - x(h(n)) + x(h(n^*))\sum_{j=h(n)}^{n^*} p(j)\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)} \leq 0 \tag{2.6}$$

Also, summing up Equation (1.1) from n^* to n, and using the discrete Grönwall inequality, will yield

$$x(n+1) - x(n^*) + x(h(n))\sum_{j=n^*}^{n} p(j)\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)} \leq 0. \tag{2.7}$$

By omitting the first terms in (2.6) and (2.7) and by using (2.5), we obtain

$$-x(h(n)) + x(h(n^*))\frac{d}{8} \leq 0$$

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and

$$-x(n^*) + x(h(n))\frac{d}{8} \leq 0.$$

Thus, we have

$$x(n^*) \geq x(h(n))\frac{d}{8} \geq x(h(n^*))\left(\frac{d}{8}\right)^2,$$

and so

$$\frac{x(h(n^*))}{x(n^*)} \leq \left(\frac{8}{d}\right)^2. \tag{2.8}$$

On the other hand, by (1.1), we obtain

$$\frac{x(n+1)}{x(n)} - 1 + p(n)\frac{x(\tau(n))}{x(n)} = 0. \tag{2.9}$$

Applying the discrete Grönwall inequality to (2.9), we obtain

$$\frac{x(n+1)}{x(n)} - 1 + p(n)\frac{x(h(n))}{x(n)} \prod_{i=\tau(n)}^{h(n)-1} \frac{1}{1-p(i)} \leq 0. \tag{2.10}$$

Set

$$y(n) = 1 - \frac{x(n+1)}{x(n)}, \tag{2.11}$$

then (2.11) yields,

$$y(n) \geq p(n) \prod_{i=\tau(n)}^{h(n)-1} \frac{1}{1-p(i)} \prod_{j=h(n)}^{n-1} [1 - y(j)]^{-1}. \tag{2.12}$$

Now, using the well-known inequality between the arithmetic and geometric means, we find that

$$y(n) \geq p(n) \prod_{i=\tau(n)}^{h(n)-1} \frac{1}{1-p(i)} \left[1 - \frac{1}{n-h(n)} \sum_{j=h(n)}^{n-1} y(j)\right]^{-(n-h(n))}. \tag{2.13}$$

So, using the inequality

$$\alpha(1-\alpha)^k \leq \frac{k^k}{(k+1)^{k+1}}, \quad \alpha \in (0,1),\ k \in \mathbf{N}$$

inequality (2.13) gives

$$y(n) \geq p(n) \prod_{i=\tau(n)}^{h(n)-1} \frac{1}{1-p(i)} \left(\frac{n-h(n)+1}{n-h(n)}\right)^{n-h(n)+1} \sum_{j=h(n)}^{n-1} y(j). \tag{2.14}$$

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Let $\liminf_{n\to\infty} \sum\limits_{j=h(n)}^{n-1} y(j) = c$. We know that

$$\frac{x(h(n))}{x(n)} = \prod_{j=h(n)}^{n-1} [1 - y(j)]^{-1} \geq \left[1 - \frac{1}{n-h(n)} \sum_{j=h(n)}^{n-1} y(j)\right]^{-(n-h(n))}$$

and

$$\frac{x(h(n))}{x(n)} \geq \left(\frac{n-h(n)+1}{n-h(n)}\right)^{n-h(n)+1} \sum_{j=h(n)}^{n-1} y(j) \geq \sum_{j=h(n)}^{n-1} y(j). \tag{2.15}$$

From (2.10) and (2.15), we have

$$\infty > \liminf_{n\to\infty} \frac{x(h(n))}{x(n)} \geq \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} y(j) = c. \tag{2.16}$$

Now, from (2.16) we get

$$\sum_{j=h(n)}^{n-1} y(j) \geq c - \varepsilon, \quad \text{for } n \geq n_2, \tag{2.17}$$

where ε is an arbitrary real number with $0 < \varepsilon < c$. So, from (2.14) and (2.17), we have

$$y(n) \geq p(n) \left(\frac{n-h(n)+1}{n-h(n)}\right)^{n-h(n)+1} (c-\varepsilon) \prod_{i=\tau(n)}^{h(n)-1} \frac{1}{1-p(i)}. \tag{2.18}$$

Summing up (2.18) from $h(n)$ to $n-1$, we have

$$\sum_{j=h(n)}^{n-1} y(j) \geq (c-\varepsilon) \sum_{j=h(n)}^{n-1} p(j) \left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1} \prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)}. \tag{2.19}$$

Thus, by (2.19), we obtain

$$c = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} y(j) \geq (c-\varepsilon) \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j) \left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1} \prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)}$$

and as $\varepsilon \to 0$, the above inequality yields

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j) \left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1} \prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)} \leq 1,$$

which contradicts to (2.2). The proof of the theorem is complete. □

Remark 2.1. It can be seen immediately that if $\tau(n) < n$, $\lim_{n\to\infty}(n - h(n)) = \infty$ or $h(n) = n - m$, $m \geq 1 \in \mathbb{N}$, then

$$\sum_{j=h(n)}^{n-1} p(j)s(j) \prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)} > e \sum_{j=h(n)}^{n-1} p(j) \prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)},$$

and therefore condition (2.2) is better than condition (1.20).

Moreover, when $(\tau(n))$ is strictly non-monotone and $\prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)} = 1$, since

$$\left(\frac{n-h(n)+1}{n-h(n)}\right)^{n-h(n)+1} \geq \left(\frac{n-\tau(n)+1}{n-\tau(n)}\right)^{n-\tau(n)+1},$$

condition (2.2) is better than condition (1.18).

Theorem 2.3. *Assume that* (1.2) *holds. Moreover, we suppose that*

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1} \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > e, \tag{2.20}$$

where $(h(n))$ is defined by (2.1). *If $\lim_{n\to\infty}(n - h(n)) = \infty$ or $h(n) = n - m$, $m \geq 1 \in \mathbb{N}$, then all solutions of Equation* (1.1) *oscillate.*

Proof. Assume, for the sake of contradiction, that $(x(n))$ be an eventually positive solution of Equation (1.1). Let $n_1 \geq -k$ be an integer such that $x(n)$, $x(\tau(n)) > 0$ for all $n \geq n_1$. Thus, by equation (1.1), we have

$$\Delta x(n) = -p(n)x(\tau(n)) \leq 0, \quad n \geq n_1,$$

which means that the sequence $(x(n))$ is non-increasing. It is clear that if $\tau(n) < n$ and $\lim_{n\to\infty}(n - h(n)) = \infty$, or $h(n) = n - m$ with $m \geq 1 \in \mathbb{N}$, then

$$e < s(n) \leq 4, \quad n \geq 0.$$

From Equation (1.1), we have

$$s(n)\Delta x(n) + p(n)s(n)x(\tau(n)) = 0, \quad n \geq n_1. \tag{2.21}$$

Now, we assume that $\tau(n) < n$ and $\lim_{n\to\infty}(n - h(n)) = \infty$. Summing up (2.21) from $h(n)$ to n, we obtain

$$\sum_{j=h(n)}^{n} s(j)\Delta x(j) + \sum_{j=h(n)}^{n} p(j)s(j)x(\tau(j)) = 0. \tag{2.22}$$

Applying the discrete Grönwall inequality to (2.22), we obtain

$$\sum_{j=h(n)}^{n} s(j)\Delta x(j) + \sum_{j=h(n)}^{n} p(j)s(j)x(h(n)) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} \leq 0,$$

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and from the fact that $(h(n))$ is non-decreasing and $(x(n))$ is non-increasing, we have

$$s(h(n))x(n+1) - s(h(n))x(h(n)) + x(h(n)) \sum_{j=h(n)}^{n} p(j)s(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} \leq 0, \quad (2.23)$$

or

$$- s(h(n))x(h(n)) + x(h(n)) \sum_{j=h(n)}^{n} p(j)s(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} \leq 0. \quad (2.24)$$

So, from (2.24), we obtain

$$x(h(n)) \left[\sum_{j=h(n)}^{n} p(j)s(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} - s(h(n)) \right] \leq 0, \quad (2.25)$$

and

$$\sum_{j=h(n)}^{n} p(j)s(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} \leq s(h(n)). \quad (2.26)$$

Since $\lim_{n\to\infty}(n - h(n)) = \infty$, we have $\lim_{n\to\infty} s(h(n)) = e$. So, by (2.26), we obtain

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \left(\frac{j-h(j)+1}{j-h(j)} \right)^{j-h(j)+1} \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} \leq e,$$

which contradicts to (2.20).

Now, we assume that $h(n) = n - m,\ \ m \geq 1 \in \mathbb{N}$. So, condition (2.20) is equivalent to

$$\limsup_{n\to\infty} \sum_{j=n-m}^{n} p(j) \prod_{i=j-m}^{n-m-1} \frac{1}{1-p(i)} > e \left(\frac{m}{m+1} \right)^{m+1}. \quad (2.27)$$

In view of (2.23), we have

$$s(h(n))x(n+1) - s(h(n))x(h(n)) + x(h(n)) \sum_{j=n-m}^{n} p(j)s(j) \prod_{i=j-m}^{n-m-1} \frac{1}{1-p(i)} \leq 0. \quad (2.28)$$

On the other hand, since $\lim_{n\to\infty} x(n) = l \geq 0$, we can find a constant $c > 0$ such that

$$c > [s(h(n)) - e] \text{ and } s(h(n))x(n+1) > cx(h(n)). \quad (2.29)$$

Thus, from (2.28) and (2.29), we obtain

$$cx(h(n)) - s(h(n))x(h(n)) + x(h(n)) \left(\frac{m+1}{m} \right)^{m+1} \sum_{j=n-m}^{n} p(j) \prod_{i=j-m}^{n-m-1} \frac{1}{1-p(i)} \leq 0,$$

or

$$x(h(n))[c - s(h(n))] + x(h(n))\left(\frac{m+1}{m}\right)^{m+1} \sum_{j=n-m}^{n} p(j) \prod_{i=j-m}^{n-m-1} \frac{1}{1-p(i)} \leq 0,$$

or

$$-ex(h(n)) + x(h(n))\left(\frac{m+1}{m}\right)^{m+1} \sum_{j=n-m}^{n} p(j) \prod_{i=j-m}^{n-m-1} \frac{1}{1-p(i)} \leq 0,$$

and

$$x(h(n))\left[\left(\frac{m+1}{m}\right)^{m+1} \sum_{j=n-m}^{n} p(j) \prod_{i=j-m}^{n-m-1} \frac{1}{1-p(i)} - e\right] \leq 0. \tag{2.30}$$

So, by (2.30), we obtain

$$\limsup_{n\to\infty} \sum_{j=n-m}^{n} p(j) \prod_{i=j-m}^{n-m-1} \frac{1}{1-p(i)} \leq e\left(\frac{m}{m+1}\right)^{m+1},$$

which contradicts to (2.20). The proof of theorem is complete. □

Remark 2.2. Observe that, it is easy to see that if $\tau(n) < n$, $\lim_{n\to\infty}(n - h(n)) = \infty$ or $h(n) = n - m$, $m \geq 1 \in \mathbb{N}$, then

$$\sum_{j=h(n)}^{n} p(j)s(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > e \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)},$$

and therefore condition (2.20) is better than condition (1.19).

Now, we present two examples to show the significance of our results. The first example is for comparing lim sup conditions.

Example 2.1. Consider the following difference equation

$$\Delta x(n) + (0.22)x(n-2) = 0, \quad n = 0, 1, \ldots \tag{2.31}$$

Let us first show that the lim sup tests mentioned in the introduction fail for this equation. Clearly,

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) = \limsup_{n\to\infty} \sum_{j=n-2}^{n} p(j) = 0.66 \not> 1,$$

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which means that the condition (1.14) is not applicable for this equation. Moreover,

$$\limsup_{n\to\infty}\sum_{j=h(n)}^{n}p(j)\prod_{i=\tau(j)}^{h(n)-1}\frac{1}{1-p(i)}=\limsup_{n\to\infty}\sum_{j=n-2}^{n}p(j)\prod_{i=j-2}^{n-3}\frac{1}{1-p(i)}$$
$$=\limsup_{n\to\infty}\left[p(n-2)\prod_{i=n-4}^{n-3}\frac{1}{1-p(i)}+p(n-1)\prod_{i=n-3}^{n-3}\frac{1}{1-p(i)}+p(n)\prod_{i=n-2}^{n-3}\frac{1}{1-p(i)}\right]$$
$$=\limsup_{n\to\infty}\left[p(n-2)\left(\frac{1}{1-p(n-4)}\right)\left(\frac{1}{1-p(n-3)}\right)+p(n-1)\left(\frac{1}{1-p(n-3)}\right)+p(n)\right]$$
$$=0.863\,66\not>1,$$

which means that the condition (1.19) is not applicable for this equation. However,

$$\limsup_{n\to\infty}\sum_{j=h(n)}^{n}p(j)\left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1}\prod_{i=\tau(j)}^{h(n)-1}\frac{1}{1-p(i)}$$
$$=0.863\,66\left(\frac{3}{2}\right)^{3}$$
$$=2.914\,9>e\cong 2.71.$$

That is, condition (2.20) of Theorem 2.3 is satisfied. Therefore, all solutions of (2.31) oscillate.

The second example is for comparing lim inf conditions.

Example 2.2. Consider the following difference equation

$$\Delta x(n)+(0.3)x(\tau(n))=0,\quad n\geq 0,\tag{2.32}$$

with

$$\tau(n)=\begin{cases}n-3, & \text{if } n \text{ is even,}\\ n-1 & \text{if } n \text{ is odd.}\end{cases}$$

Here, it is clear that (1.2) is satisfied and $(\tau(n))$ is strictly non-monotone. Also, by (2.1), we have

$$h(n)=\max_{s\leq n}\tau(s)=\begin{cases}n-2, & \text{if } n \text{ is even,}\\ n-1 & \text{if } n \text{ is odd.}\end{cases}$$

Let us first show that the lim inf tests mentioned in the introduction fail for this equation. Clearly,

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}p(j)=\liminf_{n\to\infty}\sum_{j=n-1}^{n-1}(0.3)=0.3\not>\frac{1}{e},$$

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which means that condition (1.15) is not applicable for this equation. Moreover,

$$\liminf_{n\to\infty}\sum_{j=h(n)}^{n-1}p(j)\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)}=\liminf_{n\to\infty}\sum_{j=n-1}^{n-1}p(j)\prod_{i=j-1}^{j-2}\frac{1}{1-p(i)}$$
$$=0.3\not>\frac{1}{e},$$

which means that condition (1.20) is not applicable for this equation. Also,

$$\liminf_{n\to\infty}\sum_{j=h(n)}^{n-1}p(j)\left(\frac{j-\tau(j)+1}{j-\tau(j)}\right)^{j-\tau(j)+1}=\liminf_{n\to\infty}\sum_{j=n-1}^{n-1}p(j)\left(\frac{j-\tau(j)+1}{j-\tau(j)}\right)^{j-\tau(j)+1}$$
$$=(0.3)\times\left(\frac{4}{3}\right)^{4}\cong 0.94\not>1,$$

which means that condition (1.18) is not applicable for this equation. However,

$$\liminf_{n\to\infty}\sum_{j=h(n)}^{n-1}p(j)\left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1}\prod_{i=\tau(j)}^{h(j)-1}\frac{1}{1-p(i)}$$
$$=\liminf_{n\to\infty}\sum_{j=n-1}^{n-1}p(j)\left(\frac{j-h(j)+1}{j-h(j)}\right)^{j-h(j)+1}$$
$$=(0.3)\times\left(\frac{3}{2}\right)^{3}\cong 1.01>1.$$

That is, condition (2.2) of Theorem 2.2 is satisfied. Therefore, all solutions of (2.32) oscillate.

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