A SIMPLE AND NEARLY OPTIMAL INVESTMENT STRATEGY TO MINIMIZE THE PROBABILITY OF LIFETIME RUIN

BY

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ABSTRACT

We study the optimal investment strategy to minimize the probability of lifetime ruin under a general mortality hazard rate. We explore the error between the minimum probability of lifetime ruin and the achieved probability of lifetime ruin if one follows a simple investment strategy inspired by earlier work in this area. We also include numerical examples to illustrate the estimation. We show that the nearly optimal probability of lifetime ruin under the simplified investment strategy is quite close to the original minimum probability of lifetime ruin under reasonable parameter values.

KEYWORDS

Probability of lifetime ruin, optimal investment, stochastic control, comparison results

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JEL codes: C61, D81, G22

1. INTRODUCTION

Milevsky and Robinson (2000) proposed the probability of lifetime ruin as an important way to measure the risk that individuals become bankrupt before they die. Young (2004) extended the problem and considered how individuals should optimally invest their wealth in a financial market with one

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risky asset and one risk-free asset to minimize this probability. By applying optimal stochastic control, Young derived the associated Hamilton–Jacobi–Bellman(HJB) equation for the problem. Under the assumption of constant mortality rate, the HJB equation reduces to a fully nonlinear, ordinary differential equation, from which one can obtain a closed-form expression for the minimal ruin probability and the optimal investment strategy, which has an easy-to-follow form. Since then, this problem was extended to more complicated settings, such as adding borrowing constraints (Bayraktar and Young 2007), assuming drift uncertainty (Bayraktar and Zhang 2015a) and ambiguous mortality hazard rate (Young and Zhang 2016), allowing stochastic consumption (Bayraktar and Young 2011) and stochastic volatility (Bayraktar *et al*. 2011), and injecting transaction costs (Bayraktar and Zhang 2015b; Liang and Young 2019). However, in order to deduce the explicit form of solutions, most of the above papers worked by assuming that the individual's future lifetime random variable is exponentially distributed, that is, the mortality hazard rate is constant, an assumption, that is simple but unrealistic.

Motivated by Moore and Young (2006), in this article, we consider the minimal probability of lifetime ruin under more general mortality assumptions. We first prove a comparison principle for the general problem, then apply this comparison result to quantify the error between the minimum probability of lifetime ruin and the achieved probability of lifetime ruin if one follows an investment strategy inspired by Young (2004). We also analyze the detailed characteristics of the error bounds when the mortality rate follows DeMoivre's law and Makeham's law. Finally, we present numerical examples to illustrate the estimation and analyze the errors.

One can think of this article as a natural continuation of Moore and Young (2006). Moore and Young (2006) used the Legendre transform to convert the boundary value problem for the minimum probability of ruin to a free-boundary problem in the so-called *dual world*. In that dual world, they used the projected successive over-relaxation method to numerically compute the solution of the free-boundary problem; then, they inverted the Legendre transform to obtain the minimum probability of ruin ψ and the corresponding optimal investment strategy π^*. In Section 6 of their paper, under the Gompertz model for mortality, Moore and Young (2006) numerically analyzed how ψ and π^* change with attained age and with the risky asset's volatility.

In Section 7 of their paper, Moore and Young (2006) showed that by updating the mortality hazard rate each year and treating it as a constant, the agent can quite closely obtain the minimal probability of ruin when the true hazard rate is Gompertz. Specifically, (1) at the beginning of each year, they approximated the hazard rate using four different methods (for example, by setting it equal to the probability of dying in that year); then, (2) they computed the corresponding optimal investment strategy as given in Young (2004), and applied that strategy for the year. According to the numerical work of Moore and Young (2006), their scheme resulted in a probability of ruin close to the minimum probability of ruin.

Our article picks up where Moore and Young (2006) left off. We prove *analytic* bounds between the minimum probability of ruin ψ and the probability of ruin $\tilde{\psi}$ when we use a simple investment strategy, one motivated by the optimal investment strategy when the mortality hazard rate is constant. By contrast, Moore and Young (2006) only obtained *numerical* bounds between ψ and the nearly optimal $\tilde{\psi}$. Also, our investment strategy is slightly different than the ones proposed by Moore and Young (2006), but one could adapt our proofs to analyze their investment strategies.

The remainder of this article is organized as follows. In Section 2, we describe the financial market in which the individual invests, and we define the problem of minimizing the probability of lifetime ruin with a general mortality rate. We then provide a comparison principle and use it to prove some properties for the minimum probability of lifetime ruin. In Section 3, we apply the comparison result to analyze the error between the minimum probability of lifetime ruin and the probability of lifetime ruin if one follows a simple investment strategy motivated by the one in Young (2004), that is, by replacing the constant mortality hazard rate by the general mortality hazard rate function. In order to find more concise expression of the error bounds, in Section 4, we consider two special cases for the future lifetime random variable, specifically, DeMoivre's and Makeham's laws, and we present numerical examples to illustrate how well our proposed simple investment strategy works compared with the optimal one. Section 5 concludes the article.

2. Financial market and probability of lifetime ruin

In this section, we first present the financial ingredients that affect the individual's wealth, namely, consumption, a riskless asset, and a risky asset. Then, we define the minimum probability of lifetime ruin and end the section with a comparison result. We assume that the individual invests in a riskless asset that earns interest at a constant rate $r > 0$. Also, the individual invests in a risky asset whose price at time t, S_t, follows geometric Brownian motion with dynamics

$$dS_t = S_t(\mu dt + \sigma dB_t),$$

in which $\mu > r$, $\sigma > 0$, and $B = \{B_t\}_{t \geq 0}$ is a standard Brownian motion with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let X_t denote the wealth of the individual's investment account at time $t \geq 0$. Let π_t denote the dollar amount invested in the risky asset at time $t \geq 0$. An investment policy $\Pi = \{\pi_t\}_{t \geq 0}$ is *admissible* if it is an $\mathbb{F}$-progressively measurable process satisfying $\int_0^t \pi_s^2 \, ds < \infty$ almost surely, for all $t \in [0, T(y) \wedge \tau_0)$, in which $T(y)$ is the future lifetime random variable of the individual aged y and τ_0 is the time of ruin. (We define both random variables in the following paragraphs.) Denote the set of admissible strategies by $\mathcal{A}$. We assume that the individual consumes at a (net) constant rate $c > 0$. Therefore, the wealth process follows the dynamics

$$\mathrm{d}X_t = (rX_t + (\mu - r)\pi_t - c)\,\mathrm{d}t + \sigma\pi_t \mathrm{d}B_t.$$

Let $\lambda(t)$ denote the individual's (deterministic) hazard rate of mortality at time $t \ge 0$, and we assume there exists $\lambda_0 > 0$ such that $\lambda(t) \ge \lambda_0$ for all $t \ge 0$. Furthermore, we assume $\lambda(t)$ is a continuous function of t. Let $T(y)$ denote the random future lifetime of the individual, who is age y at time $t = 0$. Then,

$${}_tp_y = \mathbb{P}(T(y) > t) = \mathrm{e}^{-\int_0^t \lambda(s)\mathrm{d}s}.$$

We assume that the Brownian motion B and the future lifetime random variable $T(y)$ are independent.

By *lifetime ruin*, we mean that the individual's wealth reaches 0 before she dies. Define the corresponding hitting time by $\tau_0 := \inf\{t \ge 0 : X_t \le 0\}$. Denote the minimum probability of lifetime ruin by ψ; thus, ψ equals

$$\psi(x, t) = \inf_{\Pi \in \mathcal{A}} \mathbb{P}\left[\tau_0 < T(y) \,\middle|\, X_t = x,\ \tau_0 \ge t,\ T(y) > t\right], \tag{2.1}$$

for all $(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+$. It is straightforward to show that one can express ψ as follows:

$$\psi(x, t) = \inf_{\Pi \in \mathcal{A}} \mathbb{E}\left[\mathrm{e}^{-\int_t^{\tau_0} \lambda(u)\mathrm{d}u} \,\middle|\, X_t = x,\ \tau_0 \ge t\right]. \tag{2.2}$$

By following the convex–duality relationship, as discussed in Bayraktar and Young (2011), by using standard techniques in the theory of viscosity solutions for optimal stopping problems, and by further upgrading ψ's regularity, one can show that ψ is the unique, classical, decreasing-in-x, convex-in-x solution of the following boundary-value problem (BVP) on $[0, c/r] \times \mathbb{R}^+$; see Appendix A for an outline of the proof: For $(x, t) \in [0, c/r] \times \mathbb{R}^+$,

$$\begin{cases} \lambda(t)u = u_t + (rx - c)u_x + \inf\limits_{\pi}\left[(\mu - r)\pi u_x + \dfrac{1}{2}\sigma^2\pi^2 u_{xx}\right], \\ u(0, t) = 1, \qquad u(c/r, t) = 0. \end{cases} \tag{2.3}$$

Note that $\psi(x, t) \equiv 0$ for $x \ge c/r$ because the individual can invest all her wealth in the riskless asset, and the interest earnings cover her rate of consumption c.

Our focus in this article is not to find an expression for ψ. Instead, we wish to evaluate the error of using a simple investment strategy in place of the optimal one. Our main tool is comparison of PDEs, so we prove the following comparison theorem. But, first, we define the operator F via its action on test functions $u \in \mathcal{C}^{2,1}([0, c/r] \times \mathbb{R}^+)$ as follows:

$$F(x, t, u(x, t), u_t(x, t), u_x(x, t), u_{xx}(x, t)) = \lambda(t)u(x, t) - u_t(x, t) - (rx - c)u_x(x, t)$$
$$- \inf_{\pi}\left[(\mu - r)\pi u_x(x, t) + \frac{1}{2}\sigma^2\pi^2 u_{xx}(x, t)\right]. \tag{2.4}$$

For simplicity, at times, we write $F|_{(x,t,u)}$ instead of $F\big(x, t, u(x, t), u_t(x, t), u_x(x, t), u_{xx}(x, t)\big)$.

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Theorem 2.1. *Suppose* $u, v \in \mathcal{C}^{2,1}([0, c/r] \times \mathbb{R}^+)$ *satisfy the following conditions:*

(a) *u and v are uniformly bounded on* $[0, c/r] \times \mathbb{R}^+$*, that is, there exists* $M > 0$ *such that*

$$\max\{||u||_\infty,\ ||v||_\infty\} < M.$$

(b) $u(0, t) \leq v(0, t)$ *for all* $t \in \mathbb{R}^+$.

(c) $u(c/r, t) \leq v(c/r, t)$ *for all* $t \in \mathbb{R}^+$.

(d) *For all* $(x, t) \in [0, c/r] \times \mathbb{R}^+$,

$$\begin{aligned} &F(x, t, u(x, t), u_t(x, t), u_x(x, t), u_{xx}(x, t)) \\ &\quad \leq F(x, t, v(x, t), v_t(x, t), v_x(x, t), v_{xx}(x, t))\,, \end{aligned} \tag{2.5}$$

with the left side finite on $[0, c/r] \times \mathbb{R}^+$.

Then, $u \leq v$ *on* $[0, c/r] \times \mathbb{R}^+$.

Proof. Define S by

$$S = \sup_{(x,t)\in[0,c/r]\times\mathbb{R}^+} (u(x, t) - v(x, t)). \tag{2.6}$$

We wish to show that $S \leq 0$; suppose, on the contrary, that $S > 0$. Note that S is finite because u and v are bounded. We, next, approximate S. To that end, define the function $q \in \mathcal{C}^1(\mathbb{R}^+)$ by

$$q(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ M(1 + \cos(\pi t)), & 1 < t \leq 2, \\ 2M, & t > 2, \end{cases} \tag{2.7}$$

in which $M > 0$ is the bound in condition (a). Then, for $m \in \mathbb{N}$, define the function q_m on $\mathbb{R}^+$ by

$$q_m(t) = q(t/m). \tag{2.8}$$

Define S_m, which we will use to approximate S, as follows:

$$S_m = \sup_{(x,t)\in[0,c/r]\times\mathbb{R}^+} (u(x, t) - v(x, t) - q_m(t))\,. \tag{2.9}$$

Because $S > 0$, there exists $(x', t') \in [0, c/r] \times \mathbb{R}^+$ such that

$$u(x', t') - v(x', t') \geq \frac{S}{2}.$$

Let $m > t'$ for the remainder of this proof; then,

$$\begin{aligned} S_m &= \sup_{(x,t)\in[0,c/r]\times\mathbb{R}^+} (u(x,t) - v(x,t) - q_m(t)) \\ &\geq \sup_{(x,t)\in[0,c/r]\times[0,m]} (u(x,t) - v(x,t) - q_m(t)) \\ &= \sup_{(x,t)\in[0,c/r]\times[0,m]} (u(x,t) - v(x,t)) \\ &\geq u\left(x', t'\right) - v\left(x', t'\right) \geq \frac{S}{2}. \end{aligned} \tag{2.10}$$

Among other things, we have $S_m > 0$, which implies that the supremum defining S_m is achieved in $[0, c/r] \times [0, 2m]$ because $q_m(t) > ||u||_\infty + ||v||_\infty$ for $t > 2m$. Let $(x_m, t_m) \in [0, c/r] \times [0, 2m]$ be a point at which the supremum S_m is realized.

We can obtain even more from the inequalities in (2.10). First, note that

$$\lim_{m\to\infty} \sup_{(x,t)\in[0,c/r]\times[0,m]} (u(x,t) - v(x,t)) = \sup_{(x,t)\in[0,c/r]\times\mathbb{R}^+} (u(x,t) - v(x,t)), \tag{2.11}$$

in which the right side equals S. Indeed, because $u - v$ is continuous, it follows that, on the set $[0, c/r] \times [0, m]$, $u - v$ achieves its supremum at, say, $(\hat{x}_m, \hat{t}_m)$. Then, because the interval [0, m] increases with m, the sequence $\{u(\hat{x}_m, \hat{t}_m) - v(\hat{x}_m, \hat{t}_m)\}$ is nondecreasing. Also, this sequence is bounded above by S; therefore, it has a limit S'. Clearly, $S' \leq S$, and we wish to show that $S' = S$. Suppose, on the contrary, that $S' < S$, and define $\delta' = (S - S')/2$. By the definition of S, there exists $(\tilde{x}, \tilde{t}) \in [0, c/r] \times \mathbb{R}^+$, such that

$$u\left(\tilde{x}, \tilde{t}\right) - v\left(\tilde{x}, \tilde{t}\right) > S - \delta' = \frac{S + S'}{2} > S'.$$

Moreover, if we set $\hat{m} = \left\lceil \max\left(t', \tilde{t}\right) \right\rceil$, then

$$S' \geq u\left(\hat{x}_{\hat{m}}, \hat{t}_{\hat{m}}\right) - v\left(\hat{x}_{\hat{m}}, \hat{t}_{\hat{m}}\right) \geq u\left(\tilde{x}, \tilde{t}\right) - v\left(\tilde{x}, \tilde{t}\right),$$

which gives us $S' > S'$, a contradiction. Thus, $S' = S$.

Now, because $q_m \geq 0$, from the inequalities in (2.10), we deduce

$$\begin{aligned} \sup_{(x,t)\in[0,c/r]\times\mathbb{R}^+} (u(x,t) - v(x,t)) &\geq \sup_{(x,t)\in[0,c/r]\times\mathbb{R}^+} (u(x,t) - v(x,t) - q_m(t)) \\ &\geq \sup_{(x,t)\in[0,c/r]\times[0,m]} (u(x,t) - v(x,t)), \end{aligned}$$

or equivalently,

$$S \geq S_m \geq \sup_{(x,t)\in[0,c/r]\times[0,m]} (u(x,t) - v(x,t)). \tag{2.12}$$

Then, by taking the limit as m goes to ∞ in (2.12) and by using (2.11), we obtain

$$S \geq \limsup_{m\to\infty} S_m \geq \liminf_{m\to\infty} S_m \geq S,$$

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which implies

$$
\begin{aligned}
S &\geq \lim_{m\to\infty} \left(u(x_m, t_m) - v(x_m, t_m) \right) \\
&\geq \lim_{m\to\infty} \left(u(x, t) - v(x, t) - q_m(t_m) \right) = \lim_{m\to\infty} S_m = S; \qquad (2.13)
\end{aligned}
$$

thus, we also have $\lim_{m\to\infty} q_m(t_m) = 0$. From the definition of q_m, we further deduce that $\lim_{m\to\infty} q'_m(t_m) = 0$.

Recall that, for $m > t'$, $(x_m, t_m) \in [0, c/r] \times [0, 2m]$ is a point at which the supremum S_m is realized on $[0, c/r] \times \mathbb{R}^+$. From conditions (b) and (c), we know that $x_m \in (0, c/r)$, and from $q_m(2m) > ||u||_\infty + ||v||_\infty$, we know that $t_m \in [0, 2m)$, from which we deduce

$$
u_x(x_m, t_m) - v_x(x_m, t_m) = 0, \qquad (2.14)
$$

$$
u_{xx}(x_m, t_m) - v_{xx}(x_m, t_m) \leq 0, \qquad (2.15)
$$

and

$$
u_t(x_m, t_m) - v_t(x_m, t_m) - q'_m(t_m) \leq 0, \qquad (2.16)
$$

in which we have inequality in the last expression because t_m might equal 0.

Recall that we write $F|_{(x,t,u)}$ in place of $F\big(x, t, u(x, t), u_t(x, t), u_x(x, t), u_{xx}(x, t)\big)$; similarly, for v. Because we assume $F|_{(x_m,t_m,u)} < \infty$, either (1) $u_{xx}(x_m, t_m) > 0$ or (2) $u_x(x_m, t_m) = u_{xx}(x_m, t_m) = 0$. In the case (1), condition (d) implies

$$
\begin{aligned}
0 &\leq F|_{(x_m,t_m,v)} - F|_{(x_m,t_m,u)} \\
&= \lambda(t_m)(v(x_m, t_m) - u(x_m, t_m)) - (v_t(x_m, t_m) - u_t(x_m, t_m)) \\
&\quad - (rx_m - c)(v_x(x_m, t_m) - u_x(x_m, t_m)) + \delta \, \frac{v_x^2(x_m, t_m)}{v_{xx}(x_m, t_m)} - \delta \, \frac{u_x^2(x_m, t_m)}{u_{xx}(x_m, t_m)},
\end{aligned}
$$

in which δ equals

$$
\delta = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2. \qquad (2.17)
$$

The expressions in (2.14) and (2.15) and the relationship $S_m = u(x_m, t_m) - v(x_m, t_m) - q_m(t_m)$ imply

$$
\begin{aligned}
0 &\leq -\lambda(t_m)(S_m + q_m(t_m)) + (u_t(x_m, t_m) - v_t(x_m, t_m)) \\
&\quad + \delta \, \frac{v_x^2(x_m, t_m)(u_{xx}(x_m, t_m) - v_{xx}(x_m, t_m))}{u_{xx}(x_m, t_m) \, v_{xx}(x_m, t_m)} \\
&\leq -\lambda(t_m)(S_m + q_m(t_m)) + (u_t(x_m, t_m) - v_t(x_m, t_m)) \, .
\end{aligned}
$$

In case (2), the expressions in (2.14) and (2.15) imply $v_x(x_m, t_m) = 0$ and $v_{xx}(x_m, t_m) \geq 0$, from which we deduce

$$
\begin{aligned}
0 &\leq F|_{(x_m,t_m,v)} - F|_{(x_m,t_m,u)} \\
&= -\lambda(t_m)(S_m + q_m(t_m)) + (u_t(x_m, t_m) - v_t(x_m, t_m)) .
\end{aligned}
$$

In either case, we have, for all $m > t'$,

$$\lambda(t_m)(S_m + q_m(t_m)) \leq u_t(x_m, t_m) - v_t(x_m, t_m) ,$$

and inequality (2.16) implies

$$\lambda(t_m)(S_m + q_m(t_m)) \leq q_m'(t_m) .$$

If we take a limit as m goes to ∞ in this inequality, then we deduce (because $0 < \lambda_0 \leq \lambda(t)$ for all $t \geq 0$)

$$0 < \lambda_0 S \leq \liminf_{m\to\infty} \lambda(t_m) \cdot S \leq 0,$$

a contradiction. Thus, we must have $S \leq 0$, which implies $u \leq v$ on $[0, c/r] \times \mathbb{R}^+$. □

In the next proposition, we use Theorem 2.1 to reprove Theorem 5.1 of Moore and Young (2006), who proved the result via a probabilistic argument.

Proposition 2.1. *Suppose $\lambda^{(1)} \geq \lambda^{(2)}$ on $\mathbb{R}^+$, and let $\psi^{(i)}$ denote the minimum probability of lifetime ruin associated with the mortality law $\lambda^{(i)}$ for $i = 1, 2$. Then,*

$$\psi^{(1)} \leq \psi^{(2)}, \tag{2.18}$$

on $[0, c/r] \times \mathbb{R}$.

Proof. Both $\psi^{(1)}$ and $\psi^{(2)}$ are bounded by $M = 1$ and satisfy the boundary conditions in (2.3) at $x = 0$ and $x = c/r$ with equality, so conditions (a), (b), and (c) of Theorem 2.1 hold. Let $F^{(i)}$ denote the operator F in (2.4) with λ replaced by $\lambda^{(i)}$ for $i = 1, 2$; thus, $F^{(i)}|_{(x,t,\psi^{(i)})} = 0$ on $[0, c/r] \times \mathbb{R}$ for $i = 1, 2$. Next, we compute

$$
\begin{aligned}
F^{(1)}|_{(x,t,\psi^{(2)})} &= \lambda^{(1)}(t)\psi^{(2)}(x, t) - \psi_t^{(2)}(x, t) - (rx - c)\psi_x^{(2)}(x, t) \\
&\quad - \inf_{\pi}\left[(\mu - r)\pi\,\psi_x^{(2)}(x, t) + \frac{1}{2}\,\sigma^2\pi^2\psi_{xx}^{(2)}(x, t)\right] \\
&= \lambda^{(1)}(t)\psi^{(2)}(x, t) - \psi_t^{(2)}(x, t) - (rx - c)\psi_x^{(2)}(x, t) \\
&\quad - \left[\lambda^{(2)}(t)\psi^{(2)}(x, t) - \psi_t^{(2)}(x, t) - (rx - c)\psi_x^{(2)}(x, t)\right] \\
&= \left(\lambda^{(1)}(t) - \lambda^{(2)}(t)\right)\psi^{(2)}(x, t) \geq 0,
\end{aligned}
$$

in which the second equality follows from $F^{(2)}|_{(x,t,\psi^{(2)})} = 0$. Thus, we have $F^{(1)}|_{(x,t,\psi^{(1)})} = 0 \leq F^{(1)}|_{(x,t,\psi^{(2)})}$, so condition (d) in Theorem 2.1 holds with $u = \psi^{(1)}$ and $v = \psi^{(2)}$. Inequality (2.18), then, follows from Theorem 2.1. □

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We, next, prove an intuitively pleasing corollary of Proposition 2.1.

Corollary 2.1. *If $\lambda(t)$ is nondecreasing with respect to $t \in \mathbb{R}^+$, then $\psi(x, t)$ is nonincreasing with respect to $t \in \mathbb{R}^+$.*

Proof. Suppose $t_1 \leq t_2$, and define $\lambda^{(1)}$ on $\mathbb{R}^+$ by $\lambda^{(1)}(t) = \lambda(t + (t_2 - t_1))$, that is, $\lambda^{(1)}$ is a shifted version of λ, which implies that the corresponding minimum probability of lifetime ruin $\psi^{(1)}$ is a shifted version of ψ, that is, $\psi^{(1)}(x, t) = \psi(x, t + (t_2 - t_1))$. Because λ is nondecreasing on $\mathbb{R}^+$, we have $\lambda^{(1)} \geq \lambda$ on $\mathbb{R}^+$, and Proposition 2.1 implies

$$\psi^{(1)}(x, t) \leq \psi(x, t),$$

for all $(x, t) \in [0, c/r] \times \mathbb{R}^+$, or equivalently,

$$\psi(x, t + (t_2 - t_1)) \leq \psi(x, t),$$

which implies (with $t = t_1$)

$$\psi(x, t_2) \leq \psi(x, t_1)\,.$$

Thus, we have proved $\psi(x, t)$ is nonincreasing with respect to $t \in \mathbb{R}^+$. □

We use Theorem 2.1 in the following section to analyze the efficacy of a proposed simple, but suboptimal, investment strategy.

3. Analyzing a simple investment strategy

For the remainder of the article, we impose the following assumption:

Assumption 3.1. *The hazard rate of mortality $\lambda(t)$ is a nondecreasing function on $\mathbb{R}^+$.*

Assumption 3.1 is reasonable because $\lambda(t)$ is usually increasing for people over age 30 or so, so we are effectively assuming that the age of the individual at time $t = 0$, namely, y, is at least 30.

For a general mortality law, there is no closed-form expression for the optimal investment strategy to minimize the probability of lifetime ruin; see Moore and Young (2006) for numerical work in this regard. However, when the hazard rate of mortality is constant, the optimal investment strategy is particularly simple. Indeed, Young (2004) shows that if $\lambda(t) \equiv \lambda$, then the optimal amount to invest in the risky asset, when wealth equals x, is

$$\frac{\mu - r}{\sigma^2} \cdot \frac{c/r - x}{p - 1}\,, \tag{3.1}$$

in which p equals

$$p = \frac{1}{2r}\left[(r + \lambda + \delta) + \sqrt{(r + \lambda + \delta)^2 - 4r\lambda}\,\right] > 1. \tag{3.2}$$

and δ is defined in (2.17). In this case, the corresponding probability of lifetime ruin equals

$$\left(1 - \frac{rx}{c}\right)^p, \tag{3.3}$$

for all $x \in [0, c/r]$, independent of time (because $\lambda(t) \equiv \lambda$ is independent of time).

For a general mortality law embodied by $\lambda(t)$, we modify the expressions in (3.1) and (3.2) by proposing the following feedback investment strategy:

$$\tilde{\pi}(x, t) = \frac{\mu - r}{\sigma^2} \cdot \frac{c/r - x}{p(t) - 1}, \tag{3.4}$$

when wealth equals x at time t. In (3.4), the function $p(t)$ equals

$$p(t) = \frac{1}{2r}\left[(r + \lambda(t) + \delta) + \sqrt{(r + \lambda(t) + \delta)^2 - 4r\lambda(t)}\right] > 1, \tag{3.5}$$

for all $t \geq 0$. We know that this strategy is suboptimal, and the purpose of this section is to determine the degree to which it is suboptimal.

Define the operator G via its action on test functions $u \in \mathcal{C}^{2,1}([0, c/r] \times \mathbb{R}^+)$ as follows:

$$G(x, t, u(x, t), u_t(x, t), u_x(x, t), u_{xx}(x, t)) = \lambda(t)u(x, t) - u_t(x, t) - (rx - c)u_x(x, t)$$
$$- \left[(\mu - r)\tilde{\pi}(x, t)u_x(x, t) + \frac{1}{2}\sigma^2\tilde{\pi}^2(x, t)u_{xx}(x, t)\right]. \tag{3.6}$$

Let $\tilde{\psi}$ denote the corresponding probability of lifetime ruin under the strategy given by $\tilde{\pi}$. One can show that $\tilde{\psi}$ is the unique, classical, decreasing-in-x solution of the following second-order parabolic BVP on $[0, c/r] \times \mathbb{R}^+$:

$$\begin{cases} G\Big(x, t, \tilde{\psi}(x, t), \tilde{\psi}_t(x, t), \tilde{\psi}_x(x, t), \tilde{\psi}_{xx}(x, t)\Big) = 0, \\ \tilde{\psi}(0, t) = 1, \qquad \tilde{\psi}(c/r, t) = 0. \end{cases} \tag{3.7}$$

One can show that Theorem 2.1 holds with F replaced by G. Indeed, the proof of the G-comparison result is similar to that of Theorem 2.1 and somewhat easier because G has no infimum. As a result, Proposition 2.1 and Corollary 2.1 hold with ψ replaced by $\tilde{\psi}$.

We wish to compare ψ, the *minimum* probability of lifetime ruin, with $\tilde{\psi}$, the probability of lifetime ruin under the strategy given by $\tilde{\pi}$ in (3.4). First, because ψ is the minimum probability of lifetime ruin, we know that

$$\psi(x, t) \leq \tilde{\psi}(x, t), \tag{3.8}$$

for all $(x, t) \in [0, c/r] \times \mathbb{R}^+$. In the following proposition, we provide an upper bound for $\tilde{\psi}$, which is inspired by the expression in (3.3).

Proposition 3.1. *For all* $(x, t) \in [0, c/r] \times \mathbb{R}^+$,

$$\tilde{\psi}(x, t) \le \left(1 - \frac{rx}{c}\right)^{p(t)}, \tag{3.9}$$

in which $p(t)$ *is given in (3.5).*

Proof. A direct calculation gives us

$$G|_{(x,t,(1-rx/c)^{p(t)})} = -\left(1 - \frac{rx}{c}\right)^{p(t)} p'(t) \ln\left(1 - \frac{rx}{c}\right) \ge 0,$$

for all $(x, t) \in [0, c/r] \times [t_0, T]$, in which the inequality follows from Assumption 3.1 and from the fact that $p(t)$ increases with $\lambda(t)$. Also, $(1 - rx/c)^{p(t)}$ satisfies the boundary conditions in (3.7) at $x = 0$ and $x = c/r$. Thus, the analog of Theorem 2.1 implies

$$\tilde{\psi}(x, t) \le \left(1 - \frac{rx}{c}\right)^{p(t)},$$

on $[0, c/r] \times \mathbb{R}^+$, which proves inequality (3.9). □

It remains for us to find a lower bound of ψ, one that will help us relate ψ and $\tilde{\psi}$, and we do that in the following proposition.

Proposition 3.2. *For all* $(x, t) \in [0, c/r] \times \mathbb{R}^+$,

$$\left(\left(1 - \frac{rx}{c}\right)^{p(t)} - g(t)\right)_+ \le \psi(x, t), \tag{3.10}$$

in which the function p is given in (3.5) and the function g is given by

$$g(t) = \frac{1}{\mathrm{e}} \int_t^\infty \frac{p'(s)}{p(s)} \, \mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v} \mathrm{d}s. \tag{3.11}$$

Proof. A direct calculation gives us

$$F|_{(x,t,(1-rx/c)^{p(t)}-g(t))} = -\lambda(t)g(t) - \left(1 - \frac{rx}{c}\right)^{p(t)} p'(t) \ln\left(1 - \frac{rx}{c}\right) + g'(t). \tag{3.12}$$

The function of x

$$\left(1 - \frac{rx}{c}\right)^{p(t)} \ln\left(1 - \frac{rx}{c}\right)$$

achieves its minimum on $[0, c/r]$ at x_{min} that solves

$$\ln\left(1 - \frac{rx_{min}}{c}\right) = -\frac{1}{p(t)},$$

and the minimum equals

$$-\frac{1}{\mathrm{e}} \cdot \frac{1}{p(t)}.$$

Thus,

$$F|_{(x,t,(1-rx/c)^{p(t)}-g(t))} \leq -\lambda(t)g(t) + \frac{1}{\mathrm{e}} \cdot \frac{p'(t)}{p(t)} + g'(t) = 0,$$

in which the equality follows from the expression for g in (3.11).

Because $g(t) \geq 0$, we have

$$\left(1 - \frac{rx}{c}\right)^{p(t)} \Bigg|_{x=0} - g(t) = 1 - g(t) \leq 1 = \psi(0, t),$$

and

$$\left(1 - \frac{rx}{c}\right)^{p(t)} \Bigg|_{x=c/r} - g(t) = -g(t) \leq 0 = \psi(c/r, t).$$

Theorem 2.1 implies

$$\left(1 - \frac{rx}{c}\right)^{p(t)} - g(t) \leq \psi(x, t),$$

on $[0, c/r] \times \mathbb{R}^+$. Moreover, $\psi \geq 0$ because it is a probability. We have, thereby, proved inequality (3.10). □

Inequalities (3.9) and (3.10) in Propositions (3.1) and (3.2) immediately give us the following theorem, which we state without proof.

Theorem 3.1. *For all* $(x, t) \in [0, c/r] \times \mathbb{R}^+$,

$$0 \leq \tilde{\psi}(x, t) - \psi(x, t) \leq \left(\frac{1}{\mathrm{e}} \int_t^{\infty} \frac{p'(s)}{p(s)} \, \mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v} \mathrm{d}s\right) \wedge 1. \tag{3.13}$$

Moreover, if $\lambda(t) \equiv \lambda$, *then* $g(t) \equiv 0$, *and we have*

$$\psi(x, t) = \tilde{\psi}(x, t) = \left(1 - \frac{rx}{c}\right)^p, \tag{3.14}$$

for all $(x, t) \in [0, c/r] \times \mathbb{R}^+$, *in which the constant* p *is given in (3.2).*

Remark 3.1. *Theorem 3.1 shows us that the bounds in (3.9) and (3.10) are tight in the sense that if* $\lambda(t) \equiv \lambda$, *then the upper bound in (3.9) equals the lower bound in (3.10), and we obtain (3.14).*

In the next lemma, we provide an upper bound of g, which is also tight in the sense of Remark 3.1.

Lemma 3.1. *The function* g *defined in (3.11) satisfies the following inequality:*

$$g(t) \leq \frac{1}{\mathrm{e}} \mathbb{E}\left(\ln \frac{\lambda(T(y))}{\lambda(t)} \,\Bigg|\, T(y) > t\right) = \frac{1}{\mathrm{e}} \mathbb{E}\left(\ln \frac{\lambda(t + T(y+t))}{\lambda(t)}\right), \tag{3.15}$$

in which $T(y+t)$ *denotes the future lifetime random variable of a person aged* $y+t$.

Proof. The function $p(t) > 1$ satisfies the following quadratic equation:

$$rp^2(t) = (r + \lambda(t) + \delta)p(t) - \lambda(t). \tag{3.16}$$

By differentiating this quadratic expression, we obtain

$$2rp(t)p'(t) = (r + \lambda(t) + \delta)p'(t) + (p(t) - 1)\lambda'(t),$$

or equivalently,

$$\frac{p'(t)}{p(t)} = \lambda'(t)\,\frac{p(t)-1}{2rp^2(t) - (r+\lambda(t)+\delta)p(t)} = \frac{\lambda'(t)}{\lambda(t)}\,\frac{(p(t)-1)\lambda(t)}{(r+\lambda(t)+\delta)p(t) - 2\lambda(t)}, \tag{3.17}$$

in which the last equality follows from (3.16). It is straightforward to show

$$0 \le \frac{(p(t)-1)\lambda(t)}{(r+\lambda(t)+\delta)p(t) - 2\lambda(t)} \le 1,$$

which implies

$$\frac{p'(t)}{p(t)} \le \frac{\lambda'(t)}{\lambda(t)}.$$

Thus,

$$\begin{aligned} g(t) &= \frac{1}{\mathrm{e}}\int_t^\infty \frac{p'(s)}{p(s)}\,\mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v}\mathrm{d}s \le \frac{1}{\mathrm{e}}\int_t^\infty \frac{\lambda'(s)}{\lambda(s)}\,\mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v}\mathrm{d}s \\ &= \frac{1}{\mathrm{e}}\left[\ln(\lambda(s))\mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v}\Big|_{s=t}^{\infty} + \int_t^\infty \ln(\lambda(s))\cdot\lambda(s)\mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v}\mathrm{d}s\right] \\ &= \frac{1}{\mathrm{e}}\left[-\ln(\lambda(t)) + \mathbb{E}(\ln(\lambda(T(y))) \mid T(y) > t)\right] \\ &= \frac{1}{\mathrm{e}}\,\mathbb{E}\left(\ln\frac{\lambda(t+T(y+t))}{\lambda(t)}\right), \end{aligned}$$

in which the second line follows from integration by parts, the third line follows from the fact that the probability density function of $T(y)|(T(y) > t)$ is $\lambda(s)\mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v}$ for $s \ge t$, and the last line follows from $T(y)|(T(y) > t) \sim t + T(y+t)$. □

As we noted before the statement of Lemma 3.1, if $\lambda(t) \equiv \lambda$, then the upper bound of g in (3.15) is identically zero. As a corollary to Theorem 3.1 and Lemma 3.1, we present the following without proof.

Corollary 3.1. *For all* $(x, t) \in [0, c/r] \times \mathbb{R}^+$,

$$0 \le \tilde{\psi}(x,t) - \psi(x,t) \le \left(\frac{1}{\mathrm{e}}\,\mathbb{E}\left(\ln\frac{\lambda(t+T(y+t))}{\lambda(t)}\right)\right) \wedge 1. \tag{3.18}$$

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As a second corollary to Theorem 3.1, we prove the following.

Corollary 3.2. *Suppose $\lambda(t) = \lambda + \varepsilon f(t)$ for some $\lambda > 0$, for some non-decreasing function f, and for $\varepsilon > 0$ small enough so that $\lambda + \varepsilon f(0) > 0$. Then, we have*

$$\psi(x, t) = \tilde{\psi}(x, t) + \mathcal{O}(\varepsilon), \tag{3.19}$$

as $\varepsilon \to 0^+$, uniformly for all $x \in [0, c/r]$.

Proof. By Corollary 3.1, it is enough to show that

$$\mathbb{E}\left(\ln \frac{\lambda(t + T(y+t))}{\lambda(t)} \right) = \mathcal{O}(\varepsilon) \tag{3.20}$$

as $\varepsilon \to 0^+$, and the expression on the left is independent of x, so if we show (3.20), then uniformity with respect to $x \in [0, c/r]$ is automatic. Now, because $\lambda(t)$ is nondecreasing,

$$\begin{aligned} 0 \le \mathbb{E}\left(\ln \frac{\lambda(t + T(y+t))}{\lambda(t)} \right) &= \mathbb{E}\left(\ln \frac{\lambda + \varepsilon f(t + T(y+t))}{\lambda + \varepsilon f(t)} \right) \\ &= \mathbb{E}\left(\ln \left(1 + \varepsilon \frac{f(t + T(y+t)) - f(t)}{\lambda + \varepsilon f(t)} \right) \right) \\ &\le \frac{\varepsilon}{\lambda + \varepsilon f(t)} \left(\mathbb{E}(f(t + T(y+t))) - f(t) \right) \\ &\le \frac{\varepsilon}{\lambda + \varepsilon f(t)} \left(\mathbb{E}(f(t + T(y+t)_\lambda)) - f(t) \right), \end{aligned}$$

which is of order $\mathcal{O}(\varepsilon)$. In the above, $T(y+t)_\lambda$ denotes the future lifetime random variable of an individual subject to the constant mortality law λ. $T(y+t)$ is stochastically dominated by $T(y+t)_\lambda$, so $\mathbb{E}\big(f(t + T(y+t))\big) \le \mathbb{E}\big(f(t + T(y+t)_\lambda\big)$. □

Remark 3.2 *We do not obtain uniformity with respect to $t \ge 0$ in Corollary 3.2 because we do not impose a bound on the growth of f.*

In the next theorem, we add the hypothesis that $\lambda(t) \equiv \lambda$ for all $t \ge T$, for some $T > 0$. This assumption is borne out by some mortality studies; see, for example, Barbi *et al*. (2018) and Milevsky (2020).

Theorem 3.2. *Suppose $\lambda(t) \equiv \lambda$ for all $t \ge T$, for some $T > 0$. Then, for all $(x, t) \in [0, c/r] \times \mathbb{R}^+$,*

$$0 \le \tilde{\psi}(x, t) - \psi(x, t) \le \left(1 - \frac{rx}{c} \right)^{p(t)} - \left(1 - \frac{rx}{c} \right)^{p(T)} \le \frac{p(T) - p(t)}{p(T)}. \tag{3.21}$$

In particular, for $t \ge T$, we have

$$\psi(x, t) = \tilde{\psi}(x, t) = \left(1 - \frac{rx}{c} \right)^{p(T)}, \tag{3.22}$$

for all $x \in [0, c/r]$.

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Proof. From the definition of ψ and from Proposition 3.1, we know

$$\psi(x, t) \le \tilde{\psi}(x, t) \le \left(1 - \frac{rx}{c}\right)^{p(t)}.$$

To show that

$$\left(1 - \frac{rx}{c}\right)^{p(T)} \le \psi(x, t),$$

compute

$$F|_{(x,t,(1-rx/c)^{p(T)})} = \left(1 - \frac{rx}{c}\right)^{p(T)} (\lambda(t) - \lambda) \le 0,$$

for all $(x, t) \in [0, c/r] \times \mathbb{R}^+$, in which the inequality follows from the assumption that $\lambda(t)$ is nondecreasing with $\lambda(t) \equiv \lambda$ for all $t \ge T$. Thus, we have proved the first and second inequalities in (3.21).

To show the third inequality in (3.21), for a fixed value of $t \ge 0$, define h by

$$h(x) = \left(1 - \frac{rx}{c}\right)^{p(t)} - \left(1 - \frac{rx}{c}\right)^{p(T)},$$

and we wish to show that the maximum of h on $[0, c/r]$ is bounded above by $1 - p(t)/p(T)$. First, differentiate h to get

$$h'(x) = -\frac{rp(t)}{c}\left(1 - \frac{rx}{c}\right)^{p(t)-1} + \frac{rp(T)}{c}\left(1 - \frac{rx}{c}\right)^{p(T)-1},$$

and

$$h''(x) = \frac{r^2 p(t)(p(t)-1)}{c^2}\left(1 - \frac{rx}{c}\right)^{p(t)-2} - \frac{r^2 p(T)(p(T)-1)}{c^2}\left(1 - \frac{rx}{c}\right)^{p(T)-2}.$$

The critical point of h equals (recall $p(t) \le p(T)$, and without loss of generality, assume $p(t) < p(T)$)

$$x_c = \frac{c}{r}\left(1 - \left(\frac{p(t)}{p(T)}\right)^{\frac{1}{p(T)-p(t)}}\right),$$

and

$$h''(x_c) \propto p(t) - p(T) < 0.$$

So $h(x_c)$ is the maximum, and

$$\begin{aligned} h(x) \le h(x_c) &= \left(\frac{p(t)}{p(T)}\right)^{\frac{p(t)}{p(T)-p(t)}} - \left(\frac{p(t)}{p(T)}\right)^{\frac{p(T)}{p(T)-p(t)}} \\ &\le 1 - \frac{p(t)}{p(T)}, \end{aligned}$$

in which the second inequality follows from calculus. Indeed, let

$$z = \frac{p(t)}{p(T) - p(t)},$$

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then the second inequality becomes

$$\left(\frac{z}{1+z}\right)^z - \left(\frac{z}{1+z}\right)^{1+z} \le 1 - \frac{z}{1+z},$$

or equivalently,

$$z^z \le (1+z)^z,$$

which clearly holds for all $z \ge 0$. Thus, we have proved the third inequality in (3.21). □

4. Examples

In this section, we analyze two examples. In the first example, we suppose $T(y)$ is uniformly distributed, that is, the future lifetime random variable follows DeMoivre's law. Recall that the individual is age y at time $t = 0$.

Example 4.1. Suppose $T(y) \sim \mathcal{U}(0, \omega)$; then, $t + T(y+t) \sim \mathcal{U}(t, \omega)$. Also,

$$\lambda(t) = \frac{1}{\omega - t},$$

and one can show that

$$\mathbb{E}\left(\ln \frac{\lambda(t + T(y+t))}{\lambda(t)} \right) = 1,$$

which implies

$$0 \le \tilde{\psi}(x, t) - \psi(x, t) \le \frac{1}{\mathrm{e}},$$

for all $(x, t) \in [0, c/r] \times [0, \omega]$. Moreover, $g(t)$ increases from $g(0) < 1/\mathrm{e}$ to $1/\mathrm{e}$ as t increases from 0 to ω, as we prove next.

Proof. We first prove that g increases with respect to t. Under DeMoivre's law, we have

$$g(t) = \frac{1}{\mathrm{e}} \int_t^\omega \frac{p'(s)}{p(s)} \cdot \frac{\omega - s}{\omega - t} \, \mathrm{d}s,$$

which implies

$$g'(t) = -\frac{1}{\mathrm{e}} \cdot \frac{p'(t)}{p(t)} + \frac{1}{\mathrm{e}(\omega - t)^2} \int_t^\omega \frac{p'(s)}{p(s)} (\omega - s) \mathrm{d}s.$$

Let f denote the integrand in the second term, that is, $f(t) = \frac{p'(t)}{p(t)}(\omega - t)$. We claim that f increases with t; if so, then

$$g'(t) \ge -\frac{1}{\mathrm{e}} \cdot \frac{p'(t)}{p(t)} + \frac{1}{\mathrm{e}(\omega - t)^2} \int_t^\omega f(t) \mathrm{d}s = -\frac{1}{\mathrm{e}} \cdot \frac{p'(t)}{p(t)} + \frac{1}{\mathrm{e}(\omega - t)^2} \frac{p'(t)}{p(t)} (\omega - t)^2 = 0,$$

which implies g increases with respect to t.

We now prove our assertion that f increases with t. To that end, rewrite f as follows:

$$f(t) = \frac{\lambda'(t)(p(t)-1)}{(r+\lambda(r)+\delta)p(t)-2\lambda(t)}(\omega - t) = \frac{\lambda(t)(p(t)-1)}{(r+\lambda(r)+\delta)p(t)-2\lambda(t)}.$$

Because $\lambda(t)$ increases with t, if we are able to prove that f increases with $\lambda(t)$, then we can deduce that f increases with t. To show $\partial f/\partial\lambda > 0$, we compute

$$\begin{aligned}\frac{\partial f(t)}{\partial\lambda(t)} &\propto \left((p(t)-1)+\lambda(t)\frac{\partial p(t)}{\partial\lambda(t)}\right)((r+\lambda(t)+\delta)\,p(t)-2\lambda(t)) \\ &\propto (p(t)-1)(r+\delta)+\lambda(t)(r+\delta-\lambda(t))\frac{1}{p(t)}\frac{\partial p(t)}{\partial\lambda(t)}. \qquad (4.1)\end{aligned}$$

By substituting (3.17) into (4.1), we have

$$\begin{aligned}\frac{\partial f(t)}{\partial\lambda(t)} &\propto (p(t)-1)(r+\delta)+\lambda(t)(r+\delta-\lambda(t))\frac{p(t)-1}{(r+\lambda(t)+\delta)p(t)-2\lambda(t)} \\ &\propto -\lambda^2(t)+(r+\delta)((r+\delta+\lambda(t))p(t)-\lambda(t)) \\ &= -\lambda^2(t)+(r+\delta)rp^2(t),\end{aligned}$$

in which the last equality follows from (3.16). Hence, $\partial f/\partial\lambda > 0$ is equivalent to

$$\sqrt{r(r+\delta)}p(t) > \lambda(t),$$

which is true by a straightforward calculation. Moreover, by L'Hôspital's rule, one can show that

$$\lim_{t\to\omega^-} g(t) = \frac{1}{\mathrm{e}}.$$

Hence, we have proved that $g(t)$ for DeMoivre's law increases from $g(0) < 1/\mathrm{e}$ to $1/\mathrm{e}$ as t increases from 0 to ω. □

Under reasonable parameter values, $\tilde{\psi}$ is quite close to ψ, closer than the bound $1/\mathrm{e}$. Indeed, suppose we use the same parameters as in Moore and Young (2006), that is, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.20$, and $c = 1$. Assume $y = 50$ and $\omega = 70$, so the maximum attainable age is 120 and $\mathbb{E}(T(y)) = 35$. In Figures 1 and 2, we plot the probabilities of lifetime ruin ψ and $\tilde{\psi}$ at time $t = 0$, and we plot the corresponding investment strategies π^* and $\tilde{\pi}$, respectively. In other words, π^* is the optimal investment strategy to minimize the probability of lifetime ruin under DeMoivre's law, and $\tilde{\pi}$ is the proposed suboptimal investment strategy from (3.4). (See Appendix B for a brief description of the numerical scheme we use to compute ψ, π^*, and $\tilde{\psi}$.) From these graphs, we see that the nearly optimal probability of lifetime ruin $\tilde{\psi}$ under the simple investment strategy $\tilde{\pi}$ is quite close to the minimum probability of ruin ψ. However the investment strategies are markedly different.

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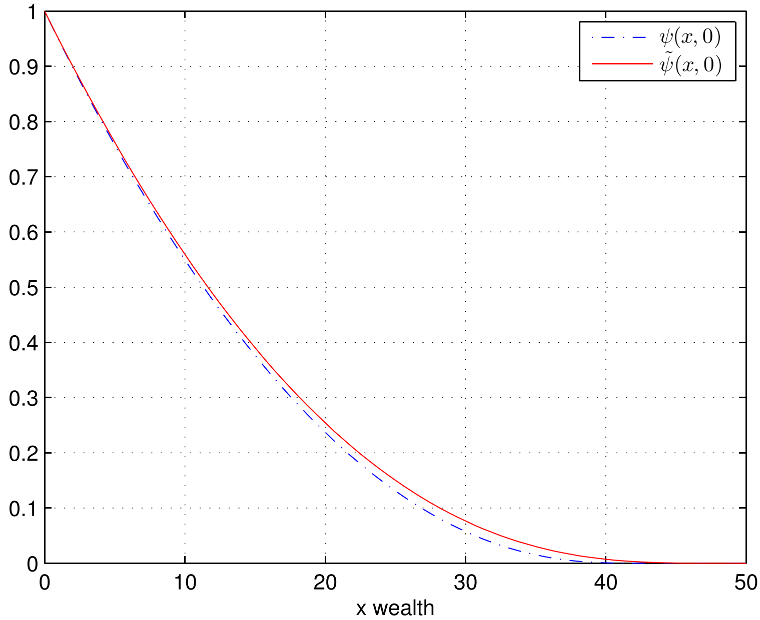

FIGURE 1. Probabilities of lifetime ruin under DeMoivre's law.

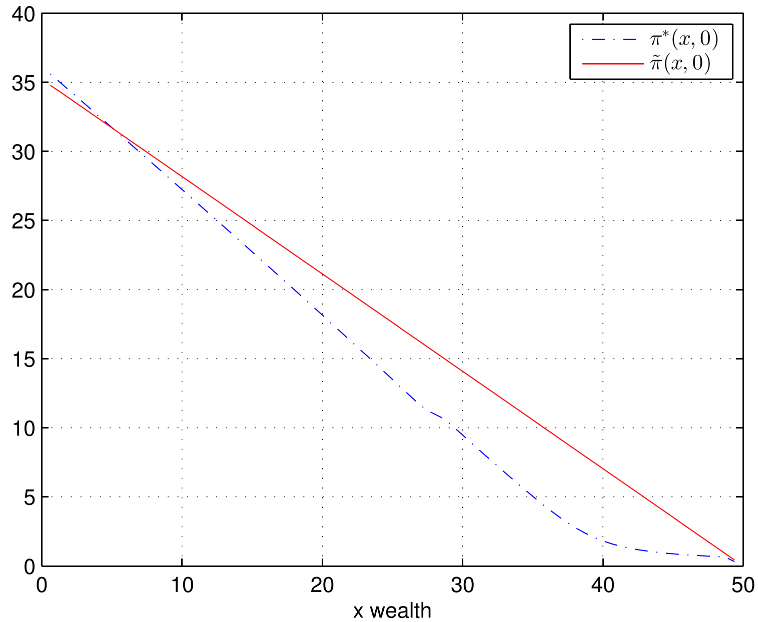

FIGURE 2. Investment strategies under DeMoivre's law.

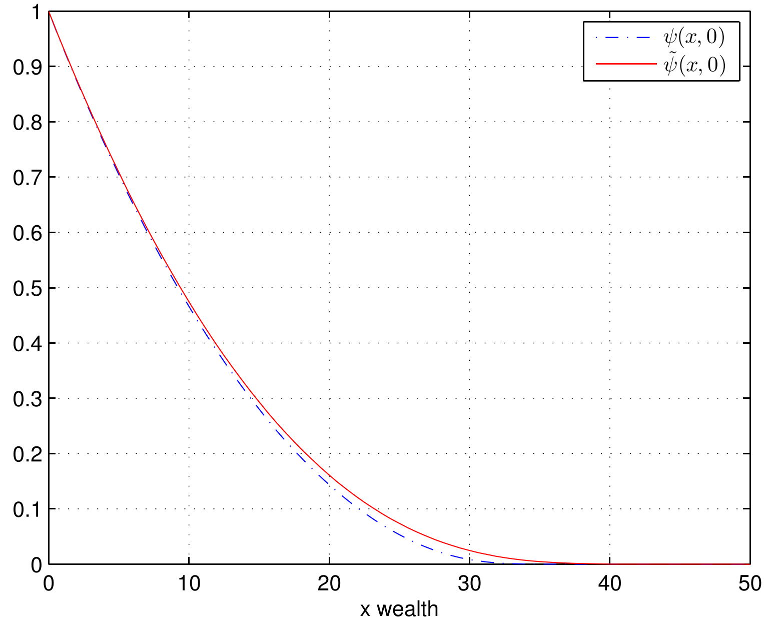

FIGURE 3. Probabilities of lifetime ruin under Makeham's law.

In the second example, we suppose $T(y)$ follows Makeham's law.

Example 4.2. Suppose $\lambda(t) = a + Be^{bt}$ for some $a, b, B > 0$. Then, Lemma 3.1 implies

$$g(t) \leq \frac{1}{\mathrm{e}}\, \mathbb{E}\left(\ln \frac{\lambda(t + T(y+t))}{\lambda(t)} \right) \leq \frac{1}{\mathrm{e}}\, \mathbb{E}\big(\ln \mathrm{e}^{bT(y+t)} \big) = \frac{b}{\mathrm{e}}\, \mathbb{E}(T(y+t)),$$

and $\mathbb{E}(T(y+t))$ decreases with respect to $t > 0$. Thus, we have the following constant upper bound for g, namely,

$$g(t) \leq \frac{b\,\mathbb{E}(T(y))}{\mathrm{e}} \leq \frac{b\,\mathbb{E}(T(y)_a)}{\mathrm{e}} = \frac{b}{a\mathrm{e}},$$

for $t \geq 0$, which is small if the parameter b is close to zero. In the above expression, $T(y)_a$ denotes the future lifetime random variable under the constant hazard rate a. It follows that

$$\psi(x,t) = \tilde{\psi}(x,t) + \mathcal{O}(b),$$

from which we deduce that if b is small in Makeham's law, then using the investment strategy in (3.4) is nearly optimal to order $\mathcal{O}(b)$ as $b \to 0^+$ uniformly for all $(x,t) \in [0, c/r] \times \mathbb{R}^+$.

As for the previous example, consider the parameter values $r = 0.02$, $\mu = 0.06$, $\sigma = 0.20$, and $c = 1$. For Makeham's law, we assume $a = 0.03$, $B = 0.001$, and $b = 0.01$, with $y = 50$, which leads to an expected future lifetime of $\mathbb{E}(T(y)) = 31.80$ years. See Figures 3 and 4 for plots of the probabilities of

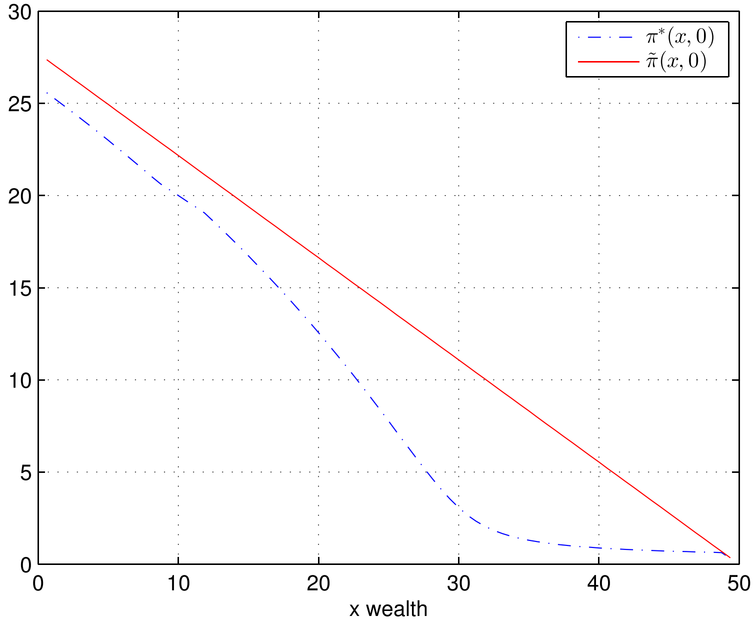

FIGURE 4. Investment strategies under Makeham's law.

lifetime ruin ψ and $\tilde{\psi}$ at time $t=0$, and the corresponding investment strategies π^* and $\tilde{\pi}$, respectively. In other words, π^* is the optimal investment strategy to minimize the probability of lifetime ruin under Makeham's law, and $\tilde{\pi}$ is the proposed suboptimal investment strategy from (3.4). As before, we see that the nearly optimal probability of lifetime ruin $\tilde{\psi}$ under the simple investment strategy $\tilde{\pi}$ is quite close to the minimum probability of ruin ψ, but the investment strategies are markedly different.

5. CONCLUSION

In this article, we reconsidered the problem of minimizing the probability of lifetime ruin. As compared to much of the existing literature, we studied the problem under a general mortality hazard rate. Inspired by Young (2004), we constructed a simple investment strategy by replacing the constant mortality hazard rate by a time-dependent mortality rate. We explored the error between the minimum probability of lifetime ruin ψ (under the general mortality hazard rate) and the achieved probability of lifetime ruin $\tilde{\psi}$ if one follows the simple investment strategy.

For example, under Makeham's law, the approximation will be poor for large values of the exponent b, which we proved in Example 4.2. Specifically, we showed that $|\tilde{\psi}-\psi| \sim \mathcal{O}(b)$ uniformly for all $(x, t) \in [0, c/r] \times \mathbb{R}^+$. As for the error function $g(t)$ more generally, consider the following inequality that we obtained in the proof of Lemma 3.1:

$$g(t) = \frac{1}{\mathrm{e}} \int_t^{\infty} \frac{p'(s)}{p(s)} \, \mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v} \mathrm{d}s \le \frac{1}{\mathrm{e}} \int_t^{\infty} \frac{\lambda'(s)}{\lambda(s)} \, \mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v} \mathrm{d}s.$$

If $\lambda(t)$ is increasing quickly, then the ratio $\lambda'(s)/\lambda(s)$ is large. But, if $\lambda(t)$ is increasing quickly, then the survival function $\mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v}$ will be small. It is not obvious to us which term will dominate in the bound for $g(t)$, $\lambda'(s)/\lambda(s)$ or the survival function $\mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v}$.

We also included numerical examples to illustrate the the errors between the two probabilities of ruin and the corresponding investment strategies. We found that the nearly optimal probability of lifetime ruin is quite close to the original minimum probability of lifetime ruin, but the associated investment strategies are quite different.

In the future, we plan to further consider asymptotic problems under general mortality hazard rates and simplified investment strategies, such as the bequest-goal problem and the utility-of-bequest problem, along with the relationship between those objectives.

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A. OUTLINE OF THE PROOF THAT ψ IS THE UNIQUE, CLASSICAL SOLUTION OF (2.3)

In this appendix, we outline the steps that show that ψ is the unique, classical, decreasing-in-x, convex-in-x solution of the BVP in (2.3).

1. We, first, define the *discounted* minimum probability of lifetime ruin ϕ by

$$\phi(x, t) = {}_tp_y \psi(x, t),$$

in which we discount for mortality. Then, ϕ satisfies the following BVP:

$$\begin{cases} \phi_t + (rx - c)\phi_x + \inf_\pi \left[(\mu - r)\pi \phi_x + \frac{1}{2} \sigma^2 \pi^2 \phi_{xx} \right] = 0, & 0 < x < \frac{c}{r}, t \geq 0, \\ \phi(0, t) = {}_tp_y, \qquad \phi(c/r, t) = 0, \qquad \lim_{t \to \infty} \phi(x, t) = 0. \end{cases} \tag{A.1}$$

2. By "working backwards" from what we know about ϕ, we define an optimal stopping problem whose value function equals the (concave) Legendre transform of ϕ. To that end, define a stochastic process Z_s by

$$dZ_s = -rZ_s ds + \frac{\mu - r}{\sigma} Z_s d\hat{B}_s,$$

in which $\hat{B}_s$ is a standard Brownian motion on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, and consider the following optimal stopping problem:

$$\begin{aligned} \hat{\phi}(z, t) &= \inf_{\tau \in [t, \infty)} \hat{\mathbb{E}} \left[\int_t^\tau cZ_s \mathrm{d}s + u(Z_\tau, \tau) \,\middle|\, Z_t = z \right] \\ &= \inf_{\tau \in [0, \infty)} \hat{\mathbb{E}} \left[\int_0^\tau cZ_s \mathrm{d}s + u(Z_\tau, \tau + t) \,\middle|\, Z_0 = z \right], \end{aligned} \tag{A.2}$$

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in which $u(z, t) = \min({}_tp_y,\ cz/r)$. For any value of $T > 0$, also define a related optimal stopping problem with finite time horizon:

$$\begin{aligned}\hat{\phi}_T(z, t) &= \inf_{\tau \in [t,T)} \hat{\mathbb{E}}\left[\int_t^\tau cZ_s \mathrm{d}s + u(Z_\tau, \tau) \,\middle|\, Z_t = z \right] \\ &= \inf_{\tau \in [0,T-t)} \hat{\mathbb{E}}\left[\int_0^\tau cZ_s \mathrm{d}s + u(Z_\tau, \tau + t) \,\middle|\, Z_0 = z \right].\end{aligned}$$

From page 275 in Section 4 of Krylov (1980), we deduce

$$\begin{aligned}\hat{\phi}(z, t) &= \uparrow \lim_{T\to\infty} \hat{\phi}_T(z, t) \\ &= \sup_{T>T'} \hat{\phi}_T(z, t), \qquad \text{for any } T' > t.\end{aligned}$$

By classical results of optimal stopping problems, one can verify that $\hat{\phi}$ is a unique viscosity solution of

$$\min\left[f_t - rzf_z + \delta z^2 f_{zz} + cz,\ u - f \right] = 0,$$

in which δ is given in (2.17). Also, $\hat{\phi}$ is continuous on $[0, c/r] \times \mathbb{R}^+$, concave and Lipschitz in z, uniformly in t. Moreover, there exist functions z_c and z_0 such that $0 \le z_c(t) < z_0(t) \le c/r$ for all $t \ge 0$, for which the optimal stopping problem in (A.2) has continuation region

$$D = \left\{ (z, t) \in [0, c/r] \times \mathbb{R}^+ : \hat{\phi}(z, t) < u(z, t) \right\} = (z_c(t), z_0(t)) \,.$$

Then, by classical results of second-order parabolic equations, $\hat{\phi}$ is the unique classical solution of the following boundary-value problem:

$$\begin{cases} f_t - rzf_z + \delta z^2 f_{zz} + cz = 0, & \text{on } D, \\ f(z_0(t), t) = {}_tp_y, \qquad f(z_c(t), t) = \dfrac{c}{r}\, z_c(t), & \lim_{t\to\infty} f(z, t) = 0. \end{cases}$$

3. The convex Legendre transform of $\hat{\phi}$ solves the BVP in (A.1) on $[0, c/r] \times \mathbb{R}^+$ and, therefore, equals the discounted minimum probability of lifetime ruin ϕ. Additionally, ϕ is strictly decreasing and strictly convex in x and nonincreasing in t. Finally, the optimal investment strategy is given in feedback form by

$$\pi_t^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{\phi_x(X_t^*, t)}{\phi_{xx}(X_t^*, t)},$$

in which $\{X_t^*\}_{t\geq 0}$ is the optimally controlled wealth process.

B. Numerical scheme we used in Section 4

In this section, we give a brief description of the numerical scheme we use in Examples 4.1 and 4.2. For simplicity, we only show the numerical scheme in Example 4.1, for which the future lifetime follows DeMoivre's law. For Example 4.2 under Makeham's law, the methodology is similar.

We apply the finite-difference method to find the numerical solution of $\tilde{\psi}$ given in (3.7), with the investment strategy $\tilde{\pi}$ given in (3.4). Note that $\tilde{\pi}$ is linear with x. Let u denote the function

$$u(x, z) = \frac{z}{\omega}\, \tilde{\psi}(x, \omega - z), \tag{B.1}$$

for $(x, z) \in [0, c/r] \times [0, \omega]$, in which ω is the maximum future lifetime of the individual. By using the expression for u in (B.1) and $\tilde{\psi}$'s BVP in (3.7), we transform (3.7) into a standard second-order parabolic BVP as follows:

$$\begin{cases} u_z(x, z) - (rx - c)u_x(x, z) - \left[(\mu - r)\hat{\pi}(x, z)u_x(x, z) + \dfrac{1}{2}\sigma^2\hat{\pi}^2(x, z)u_{xx}(x, z)\right] = 0, \\ u(0, z) = \dfrac{z}{\omega}, \qquad u(c/r, z) = 0, \qquad u(x, 0) = 0, \end{cases} \tag{B.2}$$

in which

$$\hat{\pi}(x, z) = \frac{\mu - r}{\sigma^2} \cdot \frac{c/r - x}{p(z) - 1}, \tag{B.3}$$

and

$$p(z) = \frac{1}{2r}\left[(r + \lambda(z) + \delta) + \sqrt{(r + \lambda(z) + \delta)^2 - 4r\lambda(z)}\right] > 1,$$

with $\lambda(z) = 1/z$, reflecting DeMoivre's law.

To use the finite-difference method to solve (B.2), first divide the spatial domain $[0, c/r]$ into M segments, each of length $c/(rM) =: \Delta x$; then divide the time domain $[0, T]$ into N segments, each of duration $T/N =: \Delta t$. Let (x_i, z_k) denote the grid points $(i\Delta x, k\Delta t)$, and let u_i^k denote $u(x_i, z_k)$, for $i = 1, \ldots, M+1$, and $k = 0, \ldots, N$. Then, by replacing the second partial derivative and the first partial derivatives with respect to z and x in Equation

(B.2) with the central-difference approximation and the forward-difference approximation, respectively, we have

$$\begin{cases} \dfrac{u_i^{k+1} - u_i^k}{\Delta t} - \left(rx_i - c + (\mu - r)\hat{\pi}_i^k\right) \dfrac{u_{i+1}^k - u_i^k}{\Delta x} - \dfrac{1}{2}\sigma^2 \left(\hat{\pi}_i^k\right)^2 \dfrac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2} = 0, \\ u_1^k = k\Delta t/\omega, \qquad u_{M+1}^k = 0, \qquad u_i^0 = 0. \end{cases}$$

Finally, by iterating, we obtain the numerical solution of $u(x, z)$, from which $\tilde{\psi}(x, t)$ follows via (B.1).

We also use the finite-difference method to find the numerical solution of ψ given in (2.3), with the investment strategy π^* given by the first-order necessary condition, that is,

$$\pi^*(x, t) = -\frac{\mu - r}{\sigma^2} \cdot \frac{\psi_x(x, t)}{\psi_{xx}(x, t)}. \tag{B.4}$$

Let v denote the function

$$v(x, z) = \frac{z}{\omega}\, \psi(x, \omega - z), \tag{B.5}$$

for $(x, z) \in [0, c/r] \times [0, \omega]$. By using the expression for v in (B.5) and ψ's BVP in (2.3), we transform (2.3) into a standard second-order parabolic BVP as follows:

$$\begin{cases} v_z(x, z) - \inf\limits_{\pi} \left\{ (rx - c + (\mu - r)\pi) v_x(x, z) + \dfrac{1}{2}\sigma^2 \pi^2 v_{xx}(x, z) \right\} = 0, \\ v(0, z) = \frac{z}{\omega}, \qquad v(c/r, z) = 0, \qquad v(x, 0) = 0. \end{cases} \tag{B.6}$$

To apply finite-difference method directly to the Bellman Equation (B.6), subdivide the intervals as before, and let v_i^k denote $v(x_i, z_k)$. Choose the forward-difference or backward-difference approximation of the first derivative v_x, depending on the sign of the coefficients to guarantee the method's convergence to obtain the following system of equations:

$$\begin{cases} \dfrac{v_i^{k+1} - v_i^k}{\Delta t} - \inf\limits_{\pi} \left\{ (rx_i - c + (\mu - r)\pi)_+ \cdot \dfrac{v_{i+1}^k - v_i^k}{\Delta x} - (rx_i - c + (\mu - r)\pi)_- \right. \\ \qquad\qquad \left. \cdot \dfrac{v_i^k - v_{i-1}^k}{\Delta x} + \dfrac{1}{2}\sigma^2\pi^2 \dfrac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{(\Delta x)^2} \right\} = 0, \\ v_1^k = k\Delta t/\omega, \qquad v_{M+1}^k = 0, \qquad v_i^0 = 0. \end{cases}$$

By iterating, we obtain the numerical values of v_i^k and π_i^k for all i and k, from which $\psi(x, t)$ and $\pi^*(x, t)$ follow via (B.5) and (B.4), respectively.