An improved Reynolds technique for approximate solution of linear stochastic differential equations

J. STAHLBERG
Astrophysikalisches Institut Potsdam
Rosa Luxemburg Str.17a, D-O-1590 Potsdam, Germany

Our starting point is a formal linear stochastic differential equation of first order (higher order equations can be transformed to systems of these)

$$\frac{dI(x,\omega)}{dx} = a(x,\omega)I(x,\omega) - W(x,\omega;I), \tag{1}$$

where I, a, W are stochastic functions $I = \bar{I} + I'$, $a = \bar{a} + a'$, $W = \bar{W} + W'$, with $< I >= \bar{I}, < I' >= 0$, and analogously for a and W. I, a, and W are allowed to depend on the element ω of a set Ω in which a probability measure is defined in the usual way (see e.g. Doob, 1953; de Witt-Morette, 1981). To get a solution of eq.(1) for the mean intensity $\bar{I}$ we treat the problem according the Reynolds averaging technique in the usual manner : The stochastic equation is changed into an infinte hierarchical system of equations for the correlations. At first we take the mean of eq.(1)

$$\frac{d < I(x) >}{dx} = \bar{a}(x)\bar{I}(x) + < a'(x)I(x) > - \bar{W}(x). \tag{2}$$

Multiplying eq.(1) with $a'(x')$ resp. with $a'(x')a'(x'')$ and so on we get by taking the mean

$$\frac{d}{dx} < a'(x')I(x) > = < a'(x')(a(x)I(x) - W(x)) >$$

$$\dots$$

$$\frac{d}{dx} < a'(x^{(n)})...a'(x')I(x) > = < a'(x^{(n)})...a'(x')(a(x)I(x) - W(x)) > \tag{3}$$

$$\dots,$$

where $x^{(n)}$ is a n-dashed x (dependence on ω is omitted). The eqs.(2) and (3) form an infinite hierarchical system of differential equations. A (calculable) solution can be obtained by truncating the infinite system. A simple cut off, often used in physics, consists in neglecting higher order correlations ($n > 2$). Thus we get for n $= 2$:

$$< a'(x')a'(x)I(x) > = < a'(x')a'(x) > \bar{I}(x) \tag{4}$$

($< a'a'I' >= 0$). For small deviations from the solution of the completely random (uncorrelated) case, the cut off according to eq.(4) is a valid approximation. **However**, especially for larger fluctuations far away from the totally random case the cut off according to eq.(4) becomes incorrect: It completely neglect higher order correlations. To take into account these higher order correlations via 2- point correlations (as does e.g. a Markow process) resp. via n-point correlations, we approximate the left side of eq.(5) in such a way that the two known limiting cases of

F. Krause et al. (eds.), The Cosmic Dynamo, 251–252.

the stochastic process, the solution of the completely random case and the solution $< \mathbf{I}_\infty >$ of a totally correlated stochastic process, become exactly included:

$$< \mathbf{a}'(\mathbf{x}')\mathbf{a}'(\mathbf{x})\mathbf{I}(\mathbf{x}) >=< \mathbf{a}'(\mathbf{x}')\mathbf{a}'(\mathbf{x})\mathbf{I}_\infty(\mathbf{x}) >, \qquad (5a)$$

where $\mathbf{I}_\infty$ can be obtained by including ω in the nonstochastic(!) solution I_o of eq.(1): $I_o(x) \rightarrow I_o(x,\omega)$(*solution of eq.*(1)*in the totally correlated case only*!) $\equiv I_\infty$. In the n-th order we get the generalised form

$$< \mathbf{a}'(\mathbf{x}^{(\mathbf{n})})...\mathbf{a}'(\mathbf{x})\mathbf{I}(\mathbf{x}) >=< \mathbf{a}'(\mathbf{x}^{(\mathbf{n})})...\mathbf{a}'(\mathbf{x})\mathbf{I}_\infty(\mathbf{x}) >. \qquad (5b)$$

The eqs.(5) are the simplest ansatz to fulfil the above made conditions. A more complex ansatz could give better results.
Cutting off the system of the eqs.(2) and (3) according to eq.(5b) we get

$$\bar{I}(x) = \mathbf{\Lambda} < W(x') > - \mathbf{\Lambda}^2 < a'(x')W(x") > + \ldots + - \mathbf{\Lambda}^n < a'(x')...a'(x^{(n-1)}) \Big(W(x^{(n)}) - a'(x^{(n)})I_\infty(x^{(n)})\Big) > + \Delta_n \qquad (6)$$

with

$$\mathbf{\Lambda}X(x) = exp(\int_0^x \bar{a}(x')dx') \int_x^\infty dx' exp(-\int_0^{x'} \bar{a}(x^*)dx^*)X(x')$$

$$\mathbf{\Lambda}^2X(x') = \mathbf{\Lambda}(exp(\int_0^{x'} \bar{a}(x^*)dx^*) \int_{x'}^\infty dx'' exp(-\int_0^{x''} a(x^\$)dx^\$)X(x")$$

$$\ldots,$$

and the error term $\Delta_n = \mathbf{\Lambda}^n < a'(x')...a'(x^{(n)})(I - I_\infty) >$.
If for a special stochastic process any $\Delta_n = 0$ we have found an exact solution. In general there are no closed systems of equations and a Δ_n is assumed to be sufficiently small to cut off the infinite system. From eq.(6) some special cases follow:
a) $\Delta_1 = 0$: On the one hand it is: $\bar{I} = \mathbf{\Lambda}(\bar{W} - < a'I_\infty >)$, a totally correlated stochastic process and on the other hand $\bar{I} = \mathbf{\Lambda}\bar{W}$, a completely randomly stochastic process.
b) $\Delta_2 = 0$: $\bar{I}(x) = \mathbf{\Lambda}\bar{W}(x') - \mathbf{\Lambda}^2 < a'(x')(W(x") - a'(x")I_\infty(x") >$. (7)

Eq.(7) is (by including of finite correlations) the simplest solution of eq.(1) in the frame of the proposed cut off. It is comparable with solutions in the frame of a Markov process (MP) but is not restricted to special velocity fields as does the MP. Eq.(6) is not only applicable to small perturbations caused by a random process. By taking into account even higher order momentum the accuracy increases like a power law : $D^n/n!$, where $D = (f(v_{mi}) - f(v_{ma}))/f(v_o)$. If $D > 1$ then the validity decreases up to a limited n and then increases! For $n \geq D$ this series expansion is absolutly convergent. Eq.(7) as well as eq.(6) become exact for completely random and totally correlated stochastic processes.

References

De Witt-Morette,C., Elworthy, K.D.: 1981, *Stochastic Diffential Equations, Proceedings of the "5-Tage-Kurs", Bielefeld, University* , .

Doob,J.L.: 1953, *Stochastic Processes, Wiley, New York* , .

https://doi.org/10.1017/S0074180900174224 Published online by Cambridge University Press