The Journal of Symbolic Logic
Volume 86, Number 4, December 2021

THE ORDER OF REFLECTION

JUAN P. AGUILERA

Abstract. Extending Aanderaa's classical result that $\pi^1_1 < \sigma^1_1$, we determine the order between any two patterns of iterated Σ^1_1- and Π^1_1-reflection on ordinals. We show that this *order of linear reflection* is a prewellordering of length ω^ω. This requires considering the relationship between linear and some *non-linear* reflection patterns, such as $\sigma \wedge \pi$, the pattern of simultaneous Σ^1_1- and Π^1_1-reflection. The proofs involve linking the lengths of α-recursive wellorderings to various forms of stability and reflection properties satisfied by ordinals α within standard and non-standard models of set theory.

§1. Introduction. Let L_α denote the αth level of Gödel's constructible hierarchy, given by $L_0 = \varnothing$, $L_{\alpha+1} =$ all sets definable over L_α with parameters, and $L_\eta = \bigcup_{\alpha<\eta} L_\alpha$ at limit stages. In α-recursion theory, one lifts the usual notion of "computation" over the natural numbers (or, equivalently, over L_ω) to L_α, for sufficiently closed α. As became evident from early work by Kreisel, Kripke, Platek, Sacks, Takeuti, and others (see e.g., Simpson [11]), facts about recursion on L_α can be translated into facts about recursion on L_ω in various ways. In particular, the termination of simple inductive definitions of sets of natural numbers is deeply connected with the reflecting structure of L (see e.g., Cenzer [7] or Aczel and Richter [3]). The purpose of this article is to study the order in which various reflecting properties given in terms of iterated Σ^1_1- and Π^1_1-reflection first occur in the constructible hierarchy.

A formula in the language of set theory is Σ^1_1 if it contains only existential second-order quantifiers (i.e., ranging over classes) followed by arbitrary first-order quantifiers. An ordinal α is said to be Σ^1_1-*reflecting* if whenever ϕ is a Σ^1_1 formula in the language of set theory and $a_1, \dots, a_n$ are finitely many elements of L_α, then

$$L_\alpha \models \phi(a_1, \dots, a_n) \text{ implies } \exists \beta < \alpha\, (a_1, \dots, a_n \in L_\beta \wedge L_\beta \models \phi(a_1, \dots, a_n)).$$

Given a class of ordinals X, an ordinal α is said to be Σ^1_1-reflecting *on* X if one can additionally demand that the ordinal β above belong to X. The least Σ^1_1-reflecting ordinal is denoted by σ^1_1, and π^1_1 is defined dually.

An ordinal α is said to be β-*stable* if L_α is a Σ_1-elementary substructure of L_β; in symbols:

$$L_\alpha \prec_1 L_\beta.$$

Received May 6, 2019.
2020 *Mathematics Subject Classification*. Primary 03D60, 03E45, Secondary 03D40, 03E10.
Key words and phrases. reflecting ordinal, stability, α-recursion, Gandy ordinal, pattern of reflection.

0022-4812/21/8604-0011
DOI:10.1017/jsl.2021.3

Given an ordinal α, write α^+ for the smallest admissible ordinal greater than α. Aczel and Richter [3] showed that $\pi^1_1 \neq \sigma^1_1$ and that a countable ordinal α is Π^1_1-reflecting if, and only if, it is α^+-stable. Afterwards, Aanderaa [1] showed that $\pi^1_1 < \sigma^1_1$. Gostanian [8] showed that σ^1_1 is smaller than the least α which is $(\alpha^+ + 1)$-stable; moreover, he showed that any α which is both $(\alpha^+ + 1)$-stable and locally countable is also Σ^1_1-reflecting. Later Gostanian and Hrbacek [9] employed Gostanian's method to give a new proof of Aanderaa's theorem. A third, apparently folklore proof appears in Simpson [11]. Aanderaa's theorem is also an immediate consequence of Proposition 16 below.

Let us now generalize the definitions of σ^1_1 and π^1_1 as follows:

DEFINITION 1. The notion of a *reflection pattern* is given inductively: the empty set is a reflection pattern; if s and t are reflection patterns, then so too are $s \wedge t$, σs, and πs.

Most of the time, we omit writing $\varnothing$ and instead write e.g., σ for $\sigma\varnothing$ and π for $\pi\varnothing$.

DEFINITION 2. A reflection pattern is *linear* if it contains no conjunctions, and *non-linear* otherwise.

DEFINITION 3. An ordinal is said to be $\varnothing$-reflecting if it is admissible. Let s and t be reflection patterns. Inductively, an ordinal α is said to be σs-reflecting if it reflects Σ^1_1 statements on s-reflecting ordinals; it is said to be πs-reflecting if it reflects Π^1_1 statements on s-reflecting ordinals; it is said to be $s \wedge t$-reflecting if it is both s-reflecting and t-reflecting.

Thus, being σ-reflecting is the same as being Σ^1_1-reflecting, and we might use these two terms interchangeably; however, the reader will soon realize that the shorthand notation just introduced is immensely more convenient for longer reflection patterns.

The main problem of concern in this article is the *ordering problem*: given two reflection patterns s and t, determine whether the least s-reflecting ordinal is smaller than the least t-reflecting ordinal.

DEFINITION 4. Let s and t be reflection patterns. We write $s < t$ if the least s-reflecting ordinal is smaller than the least t-reflecting ordinal. We write $s \leq t$ if the least s-reflecting ordinal is smaller than or equal to the least t-reflecting ordinal.

Thus, instances of the ordering problem are e.g., determining whether

$$\sigma\sigma < \sigma\pi(\sigma \wedge \pi)$$

or whether

$$\pi\sigma < \pi\sigma \wedge \pi\pi.$$

Other related problems emerge. For instance, one may ask whether the least $\sigma\sigma$-reflecting ordinal is the least σ-reflecting ordinal which is also a limit of σ-reflecting ordinals. (Incidentally, the answer to all three questions is "no.")

DEFINITION 5. The *order of reflection* is the set of all reflection patterns, prewellordered by $\leq$. The *order of linear reflection* is the subordering of the order of reflection comprised of linear reflection patterns.

In this article, we solve the ordering problem for linear reflection patterns: we exhibit a way of assigning ordinals to linear patterns in a way that respects their ordering; in particular, we show:

THEOREM 6. *The order of linear reflection is a prewellordering of length* ω^ω.

The proof requires analyzing the structure of the (full) order of reflection to a certain extent. We shall see that all reflection patterns are witnessed for the first time by ordinals between the least α which is α^+-stable and the least α which is $(\alpha^+ + 1)$-stable. In addition, we show:

THEOREM 7. *The order of linear reflection is cofinal in the order of reflection.*

This raises the question of whether the (full) order of reflection also has length ω^ω. This turns out to be false:

THEOREM 8. *The pattern* $\sigma\sigma$ *has rank* ω^ω *in the order of reflection.*

As part of the proof of Theorem 8, we compute the rank of every pattern below $\sigma\sigma$ in the order of reflection.

In the course of proving these theorems, we find various easier results which we believe to be of independent interest; these are labelled "propositions."

CONVENTION. Even if not mentioned explicitly, every ordinal in this article is assumed to be both countable and locally countable (i.e., for all $\beta < \alpha$, there is a surjection from ω to β in L_α). These are the hypotheses for the theorems of Gostanian and Aczel–Richter mentioned above, respectively.

§2. Stability and Gandy ordinals.

For an admissible ordinal α, write

$$\delta_\alpha = \sup\{\delta : \delta \text{ is the length of an } \alpha\text{-recursive wellordering of a subset of } \alpha\},$$

where a subset of α is said to be α-*recursive* if it is Δ_1-definable over L_α with parameters. The value of δ_α remains unchanged if one replaces " α-recursive" by " α-r.e." in the definition. For every admissible α, δ_α is easily seen to be a limit and e.g., additively indecomposable. We always have $\delta_\alpha \leq \alpha^+$; an ordinal α is *Gandy* if $\delta_\alpha = \alpha^+$. Gostanian [8] showed that σ^1_1 is the smallest ordinal which is not Gandy. In fact, he showed that a locally countable ordinal is not Gandy if, and only if, it is Σ^1_1-reflecting. Abramson and Sacks [2] showed that $(\aleph^L_\omega)^+$ is Gandy, so not every Gandy ordinal is locally countable.

The purpose of this section is to derive connections between reflecting properties of ordinals and their degrees of stability. We begin with the following elementary fact:

LEMMA 9. *Suppose* α *is* $(\delta_\alpha + 1)$-*stable. Then* α *is* Σ^1_1-*reflecting.*

PROOF. Otherwise, $\delta_\alpha = \alpha^+$ by Gostanian's characterization of Σ^1_1-reflection, so α is $(\alpha^+ + 1)$-stable. By Gostanian's result mentioned in the introduction, if α is $(\alpha^+ + 1)$-stable, then it is Σ^1_1-reflecting. ⊣

Since we know what the degree of stability of π^1_1 is, *viz*. $(\pi^1_1)^+$, a possible first question is that of the degree of stability of σ^1_1.

PROPOSITION 10. *σ^1_1 is not $(\delta_{\sigma^1_1}+1)$-stable.*

PROOF. Let $\delta = \delta_{\sigma^1_1}$. Since $\delta < (\sigma^1_1)^+$, it is not admissible. As we observed before, δ is a limit ordinal; thus, the failure of admissibility must be due to an instance of collection. Choose some Δ_0 formula ψ such that for some $\vec{a} \in L_\delta$, $L_\delta \not\models \psi(\vec{a})$-collection. To see that σ^1_1 is not $(\delta+1)$-stable, consider the formula ϕ in the language of set theory asserting that there are sets A and B such that:

1. A and B are transitive sets satisfying $V = L$, A is admissible, $A \in B$, and there is $\vec{a} \in B$ such that B does not satisfy $\psi(\vec{a})$-collection;
2. for each $(\mathsf{Ord} \cap A)$-recursive linear ordering $R \in B$, either there is an infinite descending sequence b through R with $b \in A$, or there is an ordinal $\beta \in B$ and an isomorphism $f \in B$ from R to β;
3. for each $\beta \in B$, there is an $(\mathsf{Ord} \cap A)$-recursive linear ordering $R \in B$ and an isomorphism $f \in B$ from R to β.

Notice that ϕ is a Σ_1 formula, since the only unbounded quantifier is the one on B. Moreover, it does not hold in $L_{\sigma^1_1}$, for the sets A and B would need to be of the form L_α and L_β, with $\alpha < \beta < \sigma^1_1$. Conditions (2) and (3) together imply that $\beta = \delta_\alpha$, but Gostanian's characterization of σ^1_1 then implies $\beta = \alpha^+$, contradicting condition (1). Finally, it does hold in $L_{\delta+1}$, as witnessed by $A = L_{\sigma^1_1}$ and $B = L_\delta$. To see that (2) holds, recall a theorem of Gostanian [8, Theorem 3.2] by which if α is Σ^1_1-reflecting, then every α-recursive linear ordering which is not a wellordering has an infinite descending sequence in L_α. Thus, every σ^1_1-recursive linear ordering R either has an infinite descending sequence in $L_{\sigma^1_1}$, or else is isomorphic to some ordinal $\beta < \delta$. One can construct an isomorphism witnessing this by transfinite recursion: at stage $\gamma < \beta$, one has defined $f \restriction \gamma$ and sets $f(\gamma)$ equal to the R-least element not in the range of $f \restriction \gamma$. Since this process takes β-many stages and $R \in L_{\sigma^1_1+1}$, such an isomorphism belongs to $L_{\sigma^1_1+\delta}$. Since δ is additively indecomposable, it belongs to L_δ. The proof that (3) holds is similar. ⊣

In the proof of Proposition 10, one could also extend conditions (1)–(3) by demanding that the set A satisfy any first-order property. This shows:

PROPOSITION 11. *Suppose α is the least Σ^1_1-reflecting ordinal satisfying some first-order property ϕ with parameters in α. Then, α is not $(\delta_\alpha+1)$-stable.*

The proof of the proposition concludes by arriving at the contradiction that there is a smaller Σ^1_1-reflecting ordinal satisfying the property ϕ. Thus, one can rephrase the result as:

PROPOSITION 12. *Suppose α is $(\delta_\alpha+1)$-stable and satisfies some first-order property ϕ. Then, α is a limit of Σ^1_1-reflecting ordinals satisfying ϕ.*

PROOF. The proof of the proposition is as in the comment directly preceding its statement. The only additional observation needed is that the hypothesis implies that α is Σ^1_1-reflecting, by Lemma 9. ⊣

DEFINITION 13. We denote by $\sigma^{1,\ell}_1$ the least Σ^1_1-reflecting ordinal which is a limit of Σ^1_1-reflecting ordinals.

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

PROPOSITION 14. *$\sigma_1^{1,\ell}$ is smaller than the least α which is $(\delta_\alpha + 1)$-stable.*

PROOF. Let α be as in the statement. By Proposition 12, it is a limit of Σ^1_1-reflecting ordinals. This fact is first-order expressible over α, since it is recursively inaccessible (and thus Σ^1_1-correct). By Proposition 12 again, α is a limit of Σ^1_1-reflecting ordinals which are limits of Σ^1_1-reflecting ordinals. ⊣

One cannot improve the conclusion of Proposition 10 by replacing $\delta_{\sigma^1_1} + 1$ by $\delta_{\sigma^1_1}$—every Σ^1_1-reflecting ordinal is stable to the supremum of its recursive wellorderings:

PROPOSITION 15. *Suppose α is Σ^1_1-reflecting. Then α is δ_α-stable.*

PROOF. Write $\delta = \delta_\alpha$. Since δ is a limit ordinal, it suffices to consider arbitrary $\gamma < \delta$ and show that

$$L_\alpha \prec_1 L_\gamma.$$

Let $a \in L_\alpha$ and ϕ be a Σ_1-formula such that $L_\gamma \models \phi(a)$. Without loss of generality, assume that a is an ordinal. Let R be a α-recursive wellordering of length γ. In particular, R is α-r.e., so there is a Σ_1 formula ψ, such that for all $x, y \in L_\alpha$,

$$xRy \leftrightarrow L_\alpha \models \psi(x, y).$$

Let us assume for notational simplicity that ψ is defined without parameters. Given an ordinal α', let $R_{\alpha'}$ be the binary relation given by

$$xR_{\alpha'}y \leftrightarrow L_{\alpha'} \models \psi(x, y).$$

Since ψ is Σ_1, we have $R_{\alpha'} \subset R_\tau$ whenever $\alpha' \leq \tau \leq \alpha$. In particular, $R_{\alpha'}$ is wellfounded for all $\alpha' < \alpha$.

Because $L_\gamma \models \phi(a)$, there is a subset A of L_α such that

1. A codes a model (M, E) of $\mathsf{KP} + V = L$;
2. M has a largest admissible ordinal τ and (τ, E) is isomorphic to $(\alpha, \in)$;
3. there is an ordinal β of M and a function $f \in M$ which is an isomorphism between R^M_τ (i.e., R_τ computed within M) and β, and $L^M_\beta \models \phi(a)$.

The existence of such an A can be expressed by a set-theoretic Σ^1_1 formula over L_α with parameter a (as well as any other parameters involved in the definition of R). In particular, reference to α can be made over L_α, since $\alpha = \mathsf{Ord}^{L_\alpha}$. Thus, clause (2) can be expressed in a Σ^1_1 way over L_α by asserting the existence of a bijection h with domain α and range (the set of codes of elements of) τ such that for all $\beta_0, \beta_1 < \alpha$, $\beta_0 \in \beta_1$ if, and only if, $h(\beta_0)Eh(\beta_1)$. Note that quantification over α is first-order over L_α.

By Σ^1_1-reflection, there is some $\alpha' < \alpha$ and some $A_{\alpha'} \subset L_{\alpha'}$ such that $a \in L_{\alpha'}$ and

4. $A_{\alpha'}$ codes a model (N, F) of $\mathsf{KP} + V = L$;
5. N has a largest admissible ordinal η and (η, F) is isomorphic to $(\alpha', \in)$;
6. there is an ordinal b of N and a function $g \in N$ which is an isomorphism between R^N_η and b, and $L^N_b \models \phi(a)$.

Here and for the rest of our lives, let us identify the wellfounded part of N with its transitive collapse. Condition (5) implies that $L_{\alpha'} \in N$. By (6), there is an ordinal b of N and an isomorphism $g \in N$ from $R_{\alpha'}$ to b. Because $R_{\alpha'} \subset R$, it is wellfounded,

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

and so b really is an ordinal. Now, $L_b \models \phi(a)$, and N has no admissible ordinals above α', so $b < (\alpha')^+ < \alpha$. Since ϕ is Σ_1, we conclude that $L_\alpha \models \phi(a)$, as was to be shown. ⊣

We have shown that σ^1_1 is $\delta_{\sigma^1_1}$-stable and not $(\delta_{\sigma^1_1} + 1)$-stable. The proof of Proposition 15 illustrates how one derives consequences of an ordinal being Σ^1_1-reflecting. We shall carry out many similar arguments in the future, perhaps omitting some of the details that show up repeatedly. We note the following consequence of Proposition 15:

PROPOSITION 16. *There is a Σ^1_1-sentence ϕ such that $L_\alpha \models \phi$ if, and only if, α is Σ^1_1-reflecting or Π^1_1-reflecting.*

PROOF. Let ϕ be the sentence that asserts the existence of some $A \subset L_\alpha$ coding a model (M, E) of $\mathsf{KP} + V = L$ containing α and such that

$$M \models L_\alpha \prec_1 L_{\delta_\alpha}.$$

Clearly, every Π^1_1-reflecting ordinal satisfies this sentence, as does every Σ^1_1-reflecting ordinal, by Proposition 15.

Suppose that $L_\alpha \models \phi$, as witnessed by (M, E). Suppose moreover that α is not Σ^1_1-reflecting, so that $\delta_\alpha = \alpha^+$ by Gostanian's characterization. Since $\alpha \in M$, a well-known theorem of F. Ville (see e.g., Barwise [5] for a proof) implies that $L_{\alpha^+} \subset M$. Given an arbitrary $\beta < \alpha^+$, we then have $\beta \in M$ and $\beta < \delta^M_\alpha$, for otherwise $\delta^M_\alpha < \alpha^+$, which is impossible, since any α-recursive wellordering of a subset of α of length δ^M_α would belong to M. By choice of M,

$$M \models L_\alpha \prec_1 L_{\delta_\alpha},$$

and so $M \models L_\alpha \prec_1 L_\beta$. However, being Σ_1-elementary is absolute, so we really do have $L_\alpha \prec_1 L_\beta$ and, since β was arbitrary, we have $L_\alpha \prec_1 L_{\alpha^+}$, so α is Π^1_1-reflecting. ⊣

An immediate consequence is Aanderaa's classical result:

COROLLARY 17 (Aanderaa). $\pi^1_1 < \sigma^1_1$.

Corollary 17 holds in a strong form:

COROLLARY 18. *σ^1_1 reflects Σ^1_1 sentences on Π^1_1-reflecting ordinals.*

PROOF. Let ϕ be the sentence from Proposition 16. Then, if ψ is another Σ^1_1 sentence, so is the conjunction $\phi \wedge \psi$. ⊣

Corollary 18 is not new; it also follows from the proof of Corollary 17 written down in Simpson [11]. Our method for analyzing the order of reflection is to prove results akin to Corollary 18. Now that we know the degree of stability of σ^1_1, it is natural to ask what the least ordinal α which is $(\delta_\alpha + 1)$-stable is. We shall eventually see that it is rather small and in fact smaller than the successor of σ^1_1 in the order of reflection. We finish this section with some related results that will not be used in future sections.

PROPOSITION 19. *Suppose α is $\sigma\sigma$-reflecting. Then α is $(\delta_\alpha + 1)$-stable.*

$$L_\alpha \prec_1 L_{\alpha^+} \to \sigma_1^1 \to \sigma_1^{1,\ell} \to L_\alpha \prec_1 L_{\delta_\alpha+1} \to L_\alpha \prec_1 L_{\delta_\alpha+2} \to \cdots \to \pi_1^1(\sigma_1^1)$$

Figure 1. Ordinals below the least $\pi\sigma$-reflecting ordinal. The arrow from left to right represents the "less than" relation.

Proof. This is similar to the proof of Proposition 15. Again, it is easy to see that δ_α is a limit. Let $\gamma = \delta_\alpha + 1$ and $a \in L_\alpha$ be such that $L_\gamma \models \phi(a)$, for some Σ_1 formula ϕ. Let ψ be the Σ_1^1 formula expressing that α is locally countable and there is a set $A \subset L_\alpha$ coding a model (M, E) of $\mathsf{KP} + V = L$ with $\alpha \in M$ and such that

$$M \models \text{“}L_{\delta_\alpha+1} \models \phi(a).\text{”}$$

Then $L_\alpha \models \psi$. By hypothesis, there is a Σ_1^1-reflecting $\tau < \alpha$ such that $L_\tau \models \psi$. Thus, τ is locally countable and there is a model (M, E) of $\mathsf{KP} + V = L$ with $\tau \in M$ and such that

$$M \models \text{“}L_{\delta_\tau+1} \models \phi(a).\text{”}$$

By Ville's theorem, $L_{\tau^+} \subset M$ and, since τ is Σ_1^1-reflecting, $\delta_\tau < \tau^+$. Hence, M computes δ_τ and $\delta_\tau + 1$ correctly and so we really have $L_{\delta_\tau+1} \models \phi(a)$. Since ϕ is Σ_1, we conclude $L_\alpha \models \phi(a)$, as desired. ⊣

The preceding proof shows that if α is as in Proposition 19, then α is $(\delta_\alpha + 2)$-stable, $(\delta_\alpha)^\omega$-stable, etc. It shows that if f is a function on ordinals which is uniformly Σ_1-definable (with parameters in L_α) on e.g., multiplicatively indecomposable levels of L containing all parameters, then α is $f(\delta_\alpha)$-stable.

As a consequence of Propositions 14 and 19, we obtain a negative answer to one of the questions posed in the introduction.

Corollary 20. *$\sigma_1^{1,\ell}$ is not $\sigma\sigma$-reflecting.*

We state without proof a result implying that $\sigma_1^{1,\ell}$ is smaller than the least $\pi\sigma$-reflecting ordinal. Its proof is similar to that of Theorem 32 below.

Proposition 21. *Let τ denote the least $\pi\sigma$-reflecting ordinal. For every $\beta < \tau$, there is some $\alpha < \tau$ which is both Σ_1^1-reflecting and $(\delta_\alpha + \beta)$-stable.*

Figure 1 summarizes the relationships between the ordinals considered so far. We shall also see that

$$\pi\sigma < \sigma\sigma.$$

§3. Reflection transfer theorems.

Definition 22. Let s and t be reflection patterns. We write $s \to t$ if every (countable and locally countable) s-reflecting ordinal is t-reflecting. We write $s \equiv t$ if $s \to t$ and $t \to s$.

In this section, we will present some results on the transfer of reflection properties, i.e., results of the form

$$s \to t,$$

where s and t are reflection patterns. Some properties of the relation $\to$ follow trivially from simple propositional reasoning. For instance,

$$\begin{aligned} \sigma s &\to \sigma, \\ \pi s &\to \pi, \\ s &\to s \wedge s, \text{ and} \\ s \wedge t &\to s. \end{aligned}$$

It is also obvious that the collection of reflection patterns forms a directed preorder under the relation $\to$. Moreover, if $s \to t$, then $\sigma s \to \sigma t$ and $\pi s \to \pi t$. These facts, as well as generalizations thereof obtained by straightforward propositional reasoning we shall use in the future without mention.

In the next lemmata we present a few reflection transfer results whose proofs, although possibly not straightforward propositional reasoning, are still rather elementary:

LEMMA 23. *Let s be a reflection pattern.*

1. $\sigma\sigma s \to \sigma s$;
2. $\pi\pi s \to \pi s$.

PROOF. If α is $\sigma\sigma s$-reflecting and L_α satisfies a Σ^1_1 sentence ϕ, then, by definition, there is a σs-reflecting $\tau < \alpha$ such that $L_\tau \models \phi$. By σs-reflection, there is an s-reflecting $\eta < \tau$ such that $L_\tau \models \phi$. Hence, α is σs-reflecting. The argument for $\pi\pi s$-reflection is similar. ⊣

For the next reflection transfer results, we need the following observation on the definability of reflection patterns.

LEMMA 24. *Let s be a reflection pattern. Then, the property*

$$\alpha \textit{ is } \pi s\textit{-reflecting}$$

is Σ^1_1-definable over L_α. Similarly, the property

$$\alpha \textit{ is } \sigma s\textit{-reflecting}$$

is Π^1_1-definable over L_α.

PROOF. We prove the first claim; the second one is similar. An admissible α being πs-reflecting means that for every Π^1_1 sentence ϕ with a parameter $\gamma < \alpha$, if $L_\alpha \models \phi(\gamma)$, then there is β with $\gamma < \beta < \alpha$ such that β is s-reflecting and $L_\beta \models \phi(\gamma)$. By choosing a suitably universal Π^1_1 formula, we can reduce πs-reflection to a single sentence: it suffices to show that for a fixed Π^1_1 sentence ϕ and a fixed parameter γ, the sentence

$$\big(L_\alpha \models \phi(\gamma)\big) \to \exists\beta\,\big(\gamma < \beta < \alpha \wedge \text{“}\beta \text{is } s\text{-reflecting”} \wedge L_\beta \models \phi(\gamma)\big) \tag{1}$$

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

is uniformly equivalent to a Σ^1_1 sentence, for then we can quantify over all $\gamma < \alpha$ and the resulting sentence is still Σ^1_1.

The key observation for this is the fact that Π^1_1 (and thus also Σ^1_1) sentences with parameters $a_0, \dots, a_n$ about a structure of the form L_ξ are equivalent to first-order sentences with parameters $a_0, \dots, a_n, \xi$ over any admissible L_ζ, with $\xi < \zeta$, and the equivalence is uniform (see Aczel–Richter [3, Theorem 6.2] for a proof of this fact). Hence, every recursively inaccessible ordinal is Σ^1_1-correct and a straightforward induction shows that whether an ordinal ξ is t-reflecting is computed correctly by any L_ζ whenever $\xi < \zeta$ and ζ is recursively inaccessible, for any reflection pattern t. Since every Σ^1_1- or Π^1_1-reflecting ordinal is recursively inaccessible, (1) is equivalent to the conjunction of

2. α is recursively inaccessible, and
3. $\big(L_\alpha \models \phi(\gamma)\big) \rightarrow \exists\beta\,\big(\gamma < \beta < \alpha \wedge L_\alpha \models$“ β is s-reflecting and $L_\beta \models \phi(\gamma)$” $\big)$,

and both of these conjuncts are easily seen to be Σ^1_1. ⊣

Lemma 25. *Let s and t be reflection patterns.*

1. $\sigma s \wedge \pi t \rightarrow \pi(\sigma s \wedge t)$;
2. $\sigma s \wedge \pi t \rightarrow \sigma(s \wedge \pi t)$;
3. $\sigma s \rightarrow \sigma(s \wedge \pi)$.

Proof. Recall that if an ordinal α is s-reflecting, for any nontrivial reflection pattern s, then it is recursively inaccessible and, in fact, a limit of recursively inaccessible ordinals. (1) then follows from the fact that being σs-reflecting is expressible by a Π^1_1 sentence ψ. Thus, if α is $\sigma s \wedge \pi t$-reflecting and satisfies some Π^1_1-sentence ϕ, then the conjunction $\phi \wedge \psi$ is also Π^1_1, and any ordinal satisfying it must be σs-reflecting.

(2) is similar. For (3), there are two cases: if α is π-reflecting, then the result follows from (2). If α is not π-reflecting, it is not α^+-stable. Hence, there is a least $\gamma < \alpha^+$ such that α is not γ-stable, i.e., there is a Σ_1-formula ψ and some parameter $\beta < \alpha$ such that $L_\gamma \models \psi(\beta)$, but $L_\alpha \not\models \psi(\beta)$. The remainder of the proof is an adaptation of the proof of Corollary 17 presented in Simpson [11]:

Let ϕ be the Σ^1_1 statement expressing that there is a model (M, E) of $\mathsf{KP} + V = L$ end-extending $L_{\alpha+1}$ such that for some $\gamma' \in M$ with $\gamma' < \alpha^{+M}$, $L^M_{\gamma'} \models \psi(\beta)$ and, moreover, if γ' is least such, then $M \models$“ α is $(<\gamma')$-stable.” Then $L_\alpha \models \phi$. By choice of α, there is an s-reflecting ordinal $\tau < \alpha$ such that $L_\tau \models \phi$. This means that there is a model (M, E) of $\mathsf{KP} + V = L$ end-extending $L_{\tau+1}$ such that for some $\gamma' \in M$ with $\gamma' < \tau^{+M}$, $L^M_{\gamma'} \models \psi(\beta)$ and, for the least such γ', we have $M \models$“ τ is $(<\gamma')$-stable.” Since ψ is Σ_1 and $L_\alpha \not\models \psi(\beta)$, γ' must belong to the illfounded part of M. So τ is s-reflecting and, as in the proof of Proposition 16, τ is π-reflecting. By taking conjunctions as before, one sees that every Σ^1_1 sentence satisfied by L_α is satisfied by some $(s \wedge \pi)$-reflecting $\tau < \alpha$, as was to be shown. ⊣

Example 26. We claim that

$$\sigma\pi\sigma < \sigma\sigma.$$

To see this, notice that Lemma 25(1) and (2) implies that

$$\sigma \wedge \pi \equiv \sigma \wedge \pi\sigma \equiv \sigma\pi\sigma \wedge \pi \equiv \sigma \wedge \pi\sigma\pi\sigma,$$

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

and thus that

$$\sigma\pi\sigma < \sigma \wedge \pi.$$

On the other hand, Lemma 25(3) implies that

$$\sigma\sigma \equiv \sigma(\sigma \wedge \pi),$$

and so

$$\sigma\pi\sigma < \sigma \wedge \pi < \sigma\sigma,$$

as claimed.

A natural question is whether one can strengthen π in the statement of Lemma 25(3) and, in particular, whether σ_1^1 is $\sigma\pi\pi$-reflecting. By generalizing the proof of Lemma 25(3), we will soon see that the answer is "yes."

DEFINITION 27. Let s be a reflection pattern. An ordinal α is β *-stable on* s if whenever L_β satisfies a Σ_1 sentence $\phi(L_\alpha)$ with additional parameters in L_α, there is an s-reflecting $\gamma < \alpha$ such that $L_{\gamma^+} \models \phi(L_\gamma)$.

Stability to the next admissible on a reflection pattern can be characterized in terms of reflection as follows:

LEMMA 28. *Let s be a reflection pattern. The following are equivalent*:

1. *α is α^+-stable on s*;
2. *α is πs-reflecting.*

We omit the proof of Lemma 28, which is a simple adaptation of Aczel and Richter's characterization of π-reflection (see Aczel–Richter [3, Theorem 6.2]).

THEOREM 29. *Let s be a reflection pattern. Then $\sigma s \to \sigma(s \wedge \pi s)$.*

PROOF. The conclusion of the theorem follows from Lemma 25 if α is πs-reflecting, so we may assume that it is not.

Since α is not πs-reflecting, it is not α^+-stable on s, so there is a least $\beta < \alpha^+$ and a Σ_1-formula $\exists x\, \phi(y, x)$ such that $L_\beta \models \exists x\, \phi(L_\alpha, x)$ and whenever $\gamma < \alpha$ and γ is s-reflecting, then $L_{\gamma^+} \not\models \exists x\, \phi(L_\gamma, x)$. Let ψ be the formula expressing that there is a model M of $\mathsf{KP} + V = L$ such that

1. M contains α.
2. $M \models$ " $\exists x\, \phi(L_\alpha, x)$ and, letting β be least such that $M \models \phi(L_\alpha, a)$ for some $a \in L_\gamma$, α is $<\beta$-stable on s."

Since α is σs-reflecting, there is an s-reflecting $\tau < \alpha$ with $L_\tau \models \psi$, as witnessed by some model N which end-extends L_{τ^+}. Let γ be N-least such that $N \models \exists x \in L_\gamma\, \phi(L_\tau, x)$. Then, we cannot have $\gamma < \tau^+$, for otherwise τ is an s-reflecting ordinal such that $L_{\tau^+} \models \exists x\, \phi(L_\tau, x)$, contradicting the choice of ϕ. Thus, γ belongs to the illfounded part of N and, in N, τ is $<\gamma$-stable on s. Since τ is recursively inaccessible (this can be assumed also if $s = \varnothing$), N is correct about s-reflection below τ, so an argument as before shows that τ is τ^+-stable on s and thus πs-reflecting. ⊣

EXAMPLE 30. By repeatedly applying Theorem 29, we obtain

$$\sigma \equiv \sigma\pi \equiv \sigma\pi\pi \equiv \cdots .$$

This implies the inequality

$$\pi^n < \sigma,$$

for every $n \in \mathbb{N}$, which strengthens Corollary 17.

The following strengthening of Proposition 15 is proved similarly:

LEMMA 31. *Suppose α is σs-reflecting. Then, it is δ_α-stable on s.*

PROOF. Let $\theta(L_\alpha)$ be a Σ_1 sentence with parameters in L_α, say, of the form $\exists x\, \theta_0(x, L_\alpha)$. Let $\eta < \delta_\alpha$ and $b \in L_\eta$ be such that $L_{\delta_\alpha} \models \theta_0(b, L_\alpha)$. Since $\eta < \delta_\alpha$, there is an α-recursive wellorder R of length η. Let ψ be the sentence asserting the existence of a model M of KP such that

1. M end-extends $L_{\alpha+1}$ and $M \models$ " α^+ exists";
2. in M, R is isomorphic to an ordinal η_0 and there is $b_0 \in L^M_{\eta_0}$ such that $L^M_{\eta_0} \models \theta_0(b_0, L_\alpha)$.

Then $L_\alpha \models \psi$. Moreover, ψ is Σ^1_1 so, by reflection, there is an s-reflecting $\tau < \alpha$ such that $L_\tau \models \psi$, as witnessed by some model N which end-extends L_{τ^+}. Now, in N, $L^N_{\tau^+} \models \theta_0(b_0, L_\tau)$ for some $b_0 \in L^N_{\eta_0}$, where η_0 is some N-ordinal isomorphic to R_τ. However, $R_\tau \subset R$, since R is α-recursive, and θ_0 is Σ_0, so we really have $L_{\tau^+} \models \theta(L_\tau)$, as desired. ⊣

The following theorem, although perhaps odd-looking at first, is crucial for our analysis of the order of reflection.

THEOREM 32. *Let s be a reflection pattern. Suppose α is $\pi\sigma s$-reflecting but not σ-reflecting. Then α is πs-reflecting.*

PROOF. Suppose α is π-reflecting on σs-reflecting ordinals but not σ-reflecting. Let ψ be the statement expressing that whenever (M, E) is an end-extension of $L_{\alpha+1}$ satisfying KPi,[1] then $M \models$" α is not σs-reflecting." This sentence is Π^1_1 and thus cannot be satisfied by L_α, for otherwise it would be reflected to a σs-reflecting ordinal. But clearly L_β cannot satisfy ψ if β is σs-reflecting.

Thus, $L_\alpha \not\models \psi$, so there is a model M of KPi end-extending $L_{\alpha+1}$ such that

$$M \models \text{"}\alpha \text{ is } \sigma s\text{-reflecting."}$$

For ordinals $\tau < \alpha$, whether τ is t-reflecting is computed correctly by M, for any reflection pattern t. By Lemma 31 applied within M,

$$M \models \text{"}\alpha \text{ is } \delta_\alpha\text{-stable on } s.\text{"}$$

Let ϕ be a Π^1_1 statement, and $a \in L_\alpha$ be a parameter such that $L_\alpha \models \phi(a)$. By Barwise–Gandy–Moschovakis [6], there is a Σ_1 formula ϕ^* such that for all admissible β with $a \in L_\beta$, $L_\beta \models \phi(a)$ if, and only if, $L_{\beta^+} \models \phi^*(a, L_\beta)$; thus, $L_{\alpha^+} \models \phi^*(a, L_\alpha)$. Let $b \in L_{\alpha^+}$ be a witness for ϕ^* and let $\gamma < \alpha^+$ be large enough so that

[1] KPi is the extension of KP by an axiom asserting that every set is contained in an admissible set.

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

$b \in L_\gamma$. Since α is not σ^1_1-reflecting (in the real world), $\delta_\alpha = \alpha^+$, and thus

$$\gamma < \delta^M_\alpha.$$

Since ϕ^* is Σ_1,

$$M \models \text{“}L_{\delta_\alpha} \models \phi^*(a, L_\alpha),\text{”}$$

so by the δ_α-stability of α on s within M, there is an s-reflecting $\tau < \alpha$ such that

$$M \models \text{“}L_{\tau^+} \models \phi^*(a, L_\tau).\text{”}$$

Since $\tau^+ < \alpha$, we really do have

$$L_{\tau^+} \models \phi^*(a, L_\tau),$$

and so $L_\tau \models \phi(a)$. This completes the proof of the theorem. ⊣

Remark 33. Theorem 32 essentially states that

$$\pi\sigma s \to \sigma \vee \pi s.$$

Indeed, this is how we would have stated it had we included disjunctions in the recursive definition of “reflection pattern.” We have chosen not to do so, however.

Remark 34. The assumption that α is not σ-reflecting cannot be removed from the statement of Theorem 32. To see this, let s be the pattern π. By Lemma 25(1), $\sigma \wedge \pi \to \pi\sigma\pi$. However, the least $\sigma \wedge \pi$-reflecting ordinal is not $\pi\pi$-reflecting, for

$$\sigma \wedge \pi < \sigma \wedge \pi(\sigma \wedge \pi) \equiv \sigma \wedge \pi\pi,$$

where the equivalence follows from Lemma 25(1).

Remark 35. By analogy with Theorem 29, one might expect that Theorem 32 can be improved to conclude, under the same assumptions, that α is $\pi(\sigma s \wedge s)$-reflecting. However, this is not the case, for let $s = \pi$, so the hypothesis yields that α is $\pi\sigma\pi$-reflecting. However, by Lemma 25(3),

$$\pi\sigma\pi \equiv \pi\sigma,$$

and we have seen in Example 26 that

$$\pi\sigma < \sigma\pi\sigma < \sigma \wedge \pi,$$

so we cannot conclude that α is $\pi(\sigma \wedge \pi)$-reflecting.

Example 36. By Lemma 25(1), the least $\pi\sigma$ reflecting ordinal is not σ-reflecting. Hence, by combining Theorems 29 and 32, one sees that the least $\pi\sigma$ reflecting ordinal is $\pi\sigma \wedge \pi\pi$-reflecting, $\pi\sigma \wedge \pi\pi\pi$-reflecting, and so on. As a useful exercise, the reader might want to verify that $\pi\sigma$ is smaller than both $\sigma\sigma$ and $\sigma \wedge \pi$ and conclude that σ and $\pi\sigma$ have order-types ω and $\omega + 1$ in the order of reflection.

Example 37. Let us present a proof of the inequality

$$\pi(\sigma \wedge \pi(\sigma \wedge \pi)) < \sigma\sigma.$$

We apply Theorem 29 three times:

$$\begin{aligned}\sigma\sigma &\equiv \sigma(\sigma \wedge \pi)\\ &\equiv \sigma(\sigma \wedge \pi(\sigma \wedge \pi))\\ &\equiv \sigma(\sigma \wedge \pi(\sigma \wedge \pi) \wedge \pi(\sigma \wedge \pi(\sigma \wedge \pi))),\end{aligned}$$

so $\sigma\sigma$ is a limit of $\pi(\sigma \wedge \pi(\sigma \wedge \pi))$-reflecting ordinals.

The preceding example involved an equivalence of reflection patterns obtained by applying Theorem 29 multiple times. This type of computation will occur frequently in latter sections.

We finish this section with a final reflection transfer theorem. It is a strengthening of Theorem 32 which clarifies the hypothesis on α not being σ-reflecting. We state it separately, however, since the proof is longer and the result is not used afterwards.

THEOREM 38. *Suppose α is $\pi\sigma(t \wedge \pi s)$-reflecting. Then, one of the following holds*:

1. *α is πs-reflecting*; *or*
2. *α is $\sigma(t \wedge \pi s)$-reflecting.*

PROOF. Suppose α is $\pi\sigma(t \wedge \pi s)$-reflecting but not πs-reflecting. Let θ be a Σ^1_1 sentence with parameters in L_α such that

$$L_\alpha \models \theta;$$

we need to find a $(t \wedge \pi s)$-reflecting $\tau < \alpha$ such that

$$L_\tau \models \theta.$$

By Barwise–Gandy–Moschovakis [6], there is a Π_1 formula $\theta^*(a)$ such that for every admissible β containing the parameters of θ, $L_\beta \models \theta$ if, and only if, $L_{\beta^+} \models \theta^*(L_\beta)$. In particular,

$$L_{\alpha^+} \models \theta^*(L_\alpha).$$

Since α is not πs-reflecting, there is a least $\gamma < \alpha^+$ such that α is not γ-stable on s. Because θ^* is Π_1,

$$L_\gamma \models \theta^*(L_\alpha).$$

Let ϕ be the sentence asserting the non-existence of a model M of $\mathsf{KPi} + V = L$ end-extending $L_{\alpha+1}$ in which α is $\sigma(t \wedge \pi s)$-reflecting. This is a Π^1_1 sentence and thus cannot be satisfied by any $\pi\sigma(t \wedge \pi s)$-reflecting ordinal and, in particular, by α. Thus, there is a model M of $\mathsf{KPi} + V = L$ end-extending $L_{\alpha+1}$ and such that

$$M \models \text{“}\alpha \text{ is } \sigma(t \wedge \pi s)\text{-reflecting.”}$$

Let χ be the sentence asserting the existence of a model N of $\mathsf{KP} + V = L$ such that

1. N contains α;
2. in N, letting γ be least such that α is not γ-stable on s, we have

$$N \models \text{“}L_\gamma \models \theta^*(L_\alpha)\text{.”}$$

Since $\gamma < \alpha^+$ and M must end-extend L_{α^+}, $\gamma \in M$ and M is correct about γ being the least ordinal at which α fails to be stable on s. Thus, we have

$$M \models \text{“}L_\alpha \models \chi,\text{”}$$

as witnessed, say, by $L^M_{\alpha^+}$. Within M, α is $\sigma(t \wedge \pi s)$-reflecting and thus there is some $\tau < \alpha$ such that

$$M \models \text{“}L_\tau \models \chi,\text{”}$$

and so we really do have

$$L_\tau \models \chi.$$

Moreover, M is correct about reflection below α, so we may assume that τ is $(t \wedge \pi s)$-reflecting. By the definition of χ, there is a model N of $\mathsf{KP} + V = L$ such that

1. N contains τ;
2. in N, letting γ be least such that τ is not γ-stable on s, we have

$$N \models \text{“}L_\gamma \models \theta^*(L_\tau).\text{”}$$

Since τ is πs-reflecting, it is τ^+-stable on s, and thus γ cannot be a true ordinal smaller than τ^+. By Ville's Theorem, N must end-extend L_{τ^+}. Because θ^* is Π_1, it follows that

$$L_{\tau^+} \models \theta^*(L_\tau),$$

and thus, that

$$L_\tau \models \theta,$$

as was to be shown. ⊣

REMARK 39. Theorem 38 could alternatively be stated as

$$\pi\sigma(t \wedge \pi s) \rightarrow \pi s \vee \sigma(t \wedge \pi s).$$

§4. Linear patterns. The purpose of this section is to prove some results concerning the order of linear reflection. Let us begin with the following “contraction” lemma, which will be crucial. It will be used frequently throughout the remainder of the article, sometimes without mention.

LEMMA 40 (Contraction). *Suppose s is a reflection pattern. Then,*

$$\sigma\pi\sigma\pi s \rightarrow \sigma\pi s.$$

PROOF. Suppose α is $\sigma\pi\sigma\pi s$-reflecting. For every Σ^1_1-sentence ϕ satisfied by L_α, one can find some $\pi\sigma\pi s$-reflecting $\beta < \alpha$ such that $L_\beta \models \phi$ and β is not σ-reflecting. (To see this, use $\sigma\pi\sigma s$-reflection to find some $\pi\sigma\pi s$-reflecting ordinal β_0 such that $L_{\beta_0} \models \phi$. If β_0 is not σ-reflecting, then we are done. Otherwise, β_0 is σ-reflecting and $\pi\sigma\pi s$-reflecting, thus $\sigma\pi\sigma\pi s$-reflecting by Lemma 25. Hence, there is some $\pi\sigma\pi s$-reflecting $\beta_1 < \beta_0$ such that $L_{\beta_1} \models \phi$. Proceeding this way, one eventually finds some $\pi\sigma\pi s$-reflecting β_n such that $L_{\beta_n} \models \phi$ and β_n is not σ-reflecting. Then $\beta = \beta_n$ is as

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

desired.) By Theorem 32, β is $\pi\pi s$-reflecting. By Lemma 23, β is πs-reflecting, as desired. ⊣

REMARK 41. Although this issue shall not arise again in this article, we caution the reader that, in contrast to Lemma 40, in general not every $\pi\sigma\pi\sigma s$-reflecting ordinal is $\pi\sigma s$-reflecting. For example, the least ordinal which is

$$\sigma \wedge \pi\sigma\pi\sigma\sigma\text{-reflecting}$$

is not $\pi\sigma\sigma$-reflecting. A proof of this fact would require digressing significantly, so we omit it.

Our first result in this section concerns the length of the order of linear reflection.

THEOREM 42. *The length of the order of linear reflection is ω^ω.*

PROOF. Recursively, we assign ordinals to reflection patterns without conjunction: we assign the ordinal ω^n to the pattern

$$\sigma^n.$$

In particular, the ordinal 1 is assigned to the empty pattern. If s and t are (possibly empty) patterns to which ordinals $o(s)$ and $o(t)$ have been assigned, we assign the ordinal

$$o(s) + o(t)$$

to the pattern

$$t\pi s.$$

We have to check that this assignment is well defined, in the sense that $s \equiv t$ if and only if $o(s) = o(t)$. Note that if $0 < n < m$, then, on the one hand,

$$\alpha + \omega^n + \omega^m = \alpha + \omega^m,$$

while, on the other,

$$\begin{aligned}
\sigma^m\pi s &\to \sigma^{m-n}\sigma^n\pi s \\
&\to \sigma^{m-n}(\sigma^n\pi s \wedge \pi) && \text{by Lemma 25(3)} \\
&\to \sigma^{m-n}(\sigma^n\pi s \wedge \pi\sigma^n\pi s) && \text{by Lemma 25(1)} \\
&\to \sigma^{m-n}(\sigma^n(\pi s \wedge \pi\sigma^n\pi s) \wedge \pi\sigma^n\pi s) && \text{by Lemma 25(2)} \\
&\to \sigma^m(\pi s \wedge \pi\sigma^n\pi s) \\
&\to \sigma^m\pi s \wedge \sigma^m\pi\sigma^n\pi s \\
&\to \sigma^m\pi s.
\end{aligned}$$

Thus, every $\sigma^m\pi s$-reflecting ordinal is also $\sigma^m\pi\sigma^n\pi s$-reflecting when $n < m$; the converse is also true, by Lemmas 23 and 40. By Lemma 23 and Theorem 29, we have

$$\sigma\pi^n\sigma \equiv \sigma\pi^m\sigma,$$

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

for any pair of nonzero numbers n and m. It follows that every linear pattern of reflection is equivalent to one of the form

$$\pi^m \sigma^{k_0} \pi \sigma^{k_1} \pi \cdots \pi \sigma^{k_l}, \tag{2}$$

with $0 < k_0 \leq k_1 \leq \cdots \leq k_l$, and moreover there is a unique choice of such $m, l, k_0, \dots, k_l$. Observe that transforming a linear reflection pattern into one of the form (2) preserves the assigned ordinal. We conclude that if s and t are two linear patterns of reflection and $s \equiv t$, then $o(s) = o(t)$. Similarly, if $o(s) = o(t)$, then s and t are each equivalent to the same pattern of the form (2) and hence to each other, thus proving the claim that the assignment of ordinals is well defined.

Now that we know that the assignment is well defined, we can prove that for all linear patterns s and t, we have $s < t$ if, and only if, $o(s) < o(t)$. If $o(s) < o(t)$, let γ be so that $o(s) + \gamma = o(t)$ and let r be a pattern so that $o(r) = \gamma$. Then, $o(r\pi s) = o(t)$, so that $r\pi s \equiv t$. It follows that every t-reflecting ordinal is a limit of s-reflecting ordinals.

Conversely, suppose that $s < t$. We cannot have $o(s) = o(t)$, as this would contradict well-definedness; we cannot have $o(t) < o(s)$, as this would contradict the conclusion of the previous paragraph. Therefore, we must have $o(s) < o(t)$. ⊣

By comparing the ordinals associated to linear reflection patterns in the proof of Theorem 42, we can determine which one is greater.

Example 43. Let

$$\begin{aligned} s &= \sigma^4 \pi^2 \sigma \pi^2 = \sigma^4 \pi \sigma^0 \pi \sigma^1 \pi \sigma^0 \pi \sigma^0, \\ t &= \pi^3 \sigma \pi^3 \sigma^5 = \sigma^0 \pi \sigma^0 \pi \sigma^0 \pi \sigma^1 \pi \sigma^0 \pi \sigma^0 \pi \sigma^5, \\ r &= \sigma^4 \pi \sigma^3. \end{aligned}$$

Then

$$\begin{aligned} o(s) &= 1 + 1 + \omega + 1 + \omega^4 = \omega^4, \\ o(t) &= \omega^5 + 1 + 1 + \omega + 1 + 1 + 1 = \omega^5 + \omega + 3, \\ o(r) &= \omega^3 + \omega^4 = \omega^4, \end{aligned}$$

so $s < t$ and $s \equiv r$.

The second result of this section is that the linear patterns are cofinal in the order of reflection.

Theorem 44. *The sequence* $\{\sigma^n : n \in \mathbb{N}\}$ *is cofinal in the order of reflection.*

Proof. To prove the theorem, we shall prove that for every reflection pattern s there is some $n \in \mathbb{N}$ such that whenever $n \leq k$, for every reflection pattern t, every $\sigma(\sigma^k \wedge t)$-reflecting ordinal is also $\sigma(\sigma^k \wedge t \wedge s)$-reflecting. If s is a reflection pattern, then for such an n we have $s < \sigma^{n+1}$ (this follows by taking t to be trivial), so the theorem will follow.

This is done by induction on the construction of s. The case that s is of the form $\pi s'$ is immediate from Theorem 29. Suppose s is of the form $s_1 \wedge s_2$. Use the induction hypothesis to find $n_1, n_2 \in \mathbb{N}$ such that whenever $n_1 \leq k_1$ and $n_2 \leq k_2$, for every reflection pattern t,

$$\sigma(\sigma^{k_1} \wedge t) \equiv \sigma(\sigma^{k_1} \wedge t \wedge s_1),$$

and

$$\sigma(\sigma^{k_2} \wedge t) \equiv \sigma(\sigma^{k_2} \wedge t \wedge s_2).$$

Let $n = \max\{n_1, n_2\}$ and $n \leq k$. Then, we have

$$\sigma(\sigma^k \wedge t) \equiv \sigma(\sigma^k \wedge t \wedge s_1) \equiv \sigma(\sigma^k \wedge t \wedge s_1 \wedge s_2).$$

Finally, suppose that $s = \sigma s'$. Use the induction hypothesis to find $n \in \mathbb{N}$ such that whenever $n \leq k$, for every reflection pattern t, every $\sigma(\sigma^k \wedge t)$-reflecting ordinal is also $\sigma(\sigma^k \wedge t \wedge s')$-reflecting. Then,

$$\sigma(\sigma^{k+1} \wedge t) \equiv \sigma(\sigma\sigma^k \wedge t) \equiv \sigma\big(\sigma(\sigma^k \wedge s') \wedge t\big) \equiv \sigma(\sigma^{k+1} \wedge \sigma s' \wedge t) \equiv \sigma(\sigma^{k+1} \wedge t \wedge s),$$

as desired. ⊣

§5. The reflection order below σ^2. In this section, we describe the order of reflection below σ^2. Below, we express concatenation of patterns by direct juxtaposition, so that e.g., if $s = \sigma \wedge \pi$, then

$$ss = \sigma \wedge \pi(\sigma \wedge \pi).$$

(We emphasize that ss does not mean $(\sigma \wedge \pi)(\sigma \wedge \pi)$, among other reasons, because this expression has not been defined.) Clearly, if $t_0 \to t_1$, then $st_0 \to st_1$ for any s.

Definition 45. Let $k \in \mathbb{N}$ and s be a reflection pattern. We write $c^k_0 s = s$; inductively,

$$c^k_{n+1} s = \sigma \wedge \pi^k \sigma\pi c^k_n s.$$

We write c^k_n for $c^k_n s$, where s is the empty pattern.

We remark that, in particular, $c^0_n = (\sigma\pi)^n$.

Lemma 46. *For every $n, k \in \mathbb{N}$ and every reflection pattern s, we have*

$$\sigma \wedge \pi^{k+1} s \to \pi c^k_n s.$$

Proof. By Lemma 25,

$$\sigma \wedge \pi^{k+1} s \equiv \sigma \wedge \pi(\sigma \wedge \pi^k s) \equiv \sigma \wedge \pi(\sigma\pi s \wedge \pi^k s) \equiv \sigma \wedge \pi(\sigma \wedge \pi^k (s \wedge \sigma\pi s)).$$

We show by induction on n that

$$\sigma \wedge \pi c^k_1 s \equiv \sigma \wedge \pi c^k_1 c^k_n s.$$

Suppose that

$$\sigma \wedge \pi c^k_1 s \equiv \sigma \wedge \pi c^k_1 c^k_n s.$$

After some applications of Lemma 25, we have

$$\begin{aligned}\sigma \wedge \pi c^k_1 c^k_n s &\equiv \sigma \wedge \pi(\sigma \wedge \pi^k \sigma\pi c^k_n s)\\ &\equiv \sigma\pi(\sigma \wedge \pi^k \sigma\pi c^k_n s) \wedge \pi(\sigma \wedge \pi^k \sigma\pi c^k_n s)\end{aligned}$$

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

$$\equiv \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s) \wedge \pi(\sigma \wedge \pi^k \sigma\pi c_n^k s \wedge \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s))$$
$$\equiv \sigma \wedge \pi(\sigma \wedge \pi^k \sigma\pi c_n^k s \wedge \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s))$$
$$\equiv \sigma \wedge \pi(\sigma \wedge \pi^k (\sigma\pi c_n^k s \wedge \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s))).$$

After some additional manipulations, we have

$$\begin{aligned}\sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s) &\equiv \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s) \wedge \sigma\pi(\pi^k \sigma\pi c_n^k s)\\ &\equiv \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s) \wedge \sigma\pi\sigma\pi c_n^k s\\ &\equiv \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s) \wedge \sigma\pi c_n^k s\\ &\equiv \sigma\pi c_n^k s \wedge \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s),\end{aligned}$$

where the second equivalence follows from Lemma 23 and the third follows from Lemma 40. Putting this together with the previous computation,

$$\begin{aligned}\sigma \wedge \pi c_1^k c_n^k s &\equiv \sigma \wedge \pi(\sigma \wedge \pi^k (\sigma\pi c_n^k s \wedge \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s)))\\ &\equiv \sigma \wedge \pi(\sigma \wedge \pi^k \sigma\pi(\sigma \wedge \pi^k \sigma\pi c_n^k s))\\ &\equiv \sigma \wedge \pi(\sigma \wedge \pi^k \sigma\pi c_{n+1}^k s)\\ &\equiv \sigma \wedge \pi(c_1^k c_{n+1}^k s),\end{aligned}$$

as desired. The chain of equivalences at the beginning of the proof shows that every $\sigma \wedge \pi^{k+1} s$-reflecting ordinal is also $\pi c_1^k s$-reflecting (the converse is not true in general), and the equivalence just proved by induction shows that it is therefore $\pi c_n^k s$-reflecting for any n, as desired. ⊣

Corollary 47. *For every $n, k \in \mathbb{N}$, every $l \leq k$, and every reflection pattern s, we have*

$$\sigma \wedge \pi^{k+1} s \to \pi^{k+1} \sigma\pi c_n^l s.$$

Proof. By Lemma 46,

$$\sigma \wedge \pi^{k+1} s \to \pi c_{n+1}^k s.$$

By definition,

$$\pi c_{n+1}^k s \equiv \pi(\sigma \wedge \pi^k \sigma\pi c_n^k s);$$

in particular,

$$\sigma \wedge \pi^{k+1} s \to \pi^{k+1} \sigma\pi c_n^k s.$$

Applying Lemma 23 n times to see that

$$c_n^k s \to c_n^l s,$$

from which the result follows. ⊣

Lemma 48. *For every $k \in \mathbb{N}$, every $n_0, \dots, n_k \in \mathbb{N}$, and every reflection pattern s, we have*

$$\sigma \wedge \pi^{k+1} s \to \pi^{k+1} \sigma\pi c_{n_0}^0 c_{n_1}^1 \dots c_{n_k}^k s.$$

PROOF. Observe that every reflection pattern of the form $\sigma\pi c^l_m$, for $l, m \in \mathbb{N}$, is of the form $t'\sigma\pi$. Thus, Lemma 40 implies that

$$\sigma\pi c^l_m \sigma\pi s \equiv \sigma\pi c^l_m s.$$

The result then follows from applying Corollary 47 repeatedly. ⊣

DEFINITION 49. A reflection pattern is 2 *-normal* or simply *normal* [2] if it is of the form

$$\pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k},$$

for some natural numbers $k, n, n_0, n_1, \dots, n_k$. If w is the reflection pattern above, we define

$$o(w) = \omega^{k+1} \cdot n_k + \omega^k \cdot n_{k-1} + \dots + \omega \cdot n_0 + n.$$

We shall see that normal patterns have very nice properties.

LEMMA 50. *Suppose s is a normal reflection pattern. Then,*

$$\sigma\sigma \to \sigma s.$$

PROOF. Let

$$s = \pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k}.$$

Suppose first $n_0 \neq 0$. By Theorem 29,

$$\sigma\sigma \to \sigma(\sigma \wedge \pi^{\max\{n,k+1\}}).$$

By Lemmas 23 and 48,

$$\sigma(\sigma \wedge \pi^{\max\{n,k+1\}}) \to \sigma(\sigma \wedge \pi^n \sigma\pi c^0_{n_0-1} c^1_{n_1} \dots c^k_{n_k}),$$

as desired. Suppose now that $n_0 = 0$ and let l be least such that $n_l \neq 0$ (if there is no such l, the lemma follows from Theorem 29). Thus,

$$s = \pi^n c^l_{n_l} \dots c^k_{n_k} = \pi^n(\sigma \wedge \pi^l \sigma\pi c^l_{n_l-1} \dots c^k_{n_k}).$$

As before,

$$\sigma\sigma \to \sigma(\sigma \wedge \pi^l \sigma\pi c^l_{n_l-1} c^1_{n_1} \dots c^k_{n_k}).$$

By Theorem 29,

$$\sigma(\sigma \wedge \pi^l \sigma\pi c^l_{n_l-1} c^1_{n_1} \dots c^k_{n_k}) \to \sigma\pi^n(\sigma \wedge \pi^l \sigma\pi c^l_{n_l-1} c^1_{n_1} \dots c^k_{n_k}).$$

as desired. ⊣

LEMMA 51. *Suppose πt and πs are normal reflection patterns such that $o(\pi t) \leq o(\pi s)$. Then, every πs-reflecting ordinal is either πt-reflecting or σ-reflecting.*

PROOF. Let

$$\pi t = \pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k},$$

[2] The " 2" refers to the fact that—as we shall see—restricting to these suffices for computing the order of reflection below σ^2.

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

and

$$\pi s = \pi^m c^0_{m_0} c^1_{m_1} \dots c^l_{m_l},$$

where n and m are nonzero. Without loss of generality, we assume that n_k and m_l are also nonzero. It follows that $k \leq l$. Suppose that $o(\pi t) < o(\pi s)$. It will be convenient, for illustrative purposes, to consider the case that $k < l$ first. If so, it suffices to show that every $\pi(\sigma \wedge \pi^l)$-reflecting ordinal which is not σ-reflecting is (πt)-reflecting, for then the result follows from Theorem 32 and contraction. By Lemma 48,

$$\pi(\sigma \wedge \pi^l) \to \pi(\sigma \wedge \pi^l \sigma\pi c^0_{n_0} c^1_{n_1} \dots c^k_{n_k}).$$

By Lemma 25,

$$\pi(\sigma \wedge \pi^l \sigma\pi c^0_{n_0} c^1_{n_1} \dots c^k_{n_k}) \to \pi(\sigma\pi c^0_{n_0} c^1_{n_1} \dots c^k_{n_k}),$$

and, by Theorem 29,

$$\pi(\sigma\pi c^0_{n_0} c^1_{n_1} \dots c^k_{n_k}) \to \pi(\sigma\pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k}),$$

so that, if a $\pi(\sigma \wedge \pi^l)$-reflecting ordinal is not σ-reflecting, then it is

$$\pi(\pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k})\text{-reflecting,}$$

by Theorem 32, i.e., πt-reflecting.

The case $k = l$ is not very different: let $i \leq k$ be greatest such that $n_i < m_i$ and notice that

$$c^i_{m_i} = c^i_{m_i - n_i} c^i_{n_i}.$$

Thus,

$$\pi t = \pi^n c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t',$$

where $t' = c^i_{n_i} \dots c^k_{n_k}$; and

$$\pi s = \pi^m c^0_{m_0} c^1_{m_1} \dots c^i_{m_i - n_i} t'.$$

It suffices to show that every $\pi(\sigma \wedge \pi^i \sigma\pi t')$-reflecting ordinal which is not σ-reflecting is πt-reflecting, for then the result follows from Theorem 32 and contraction. Lemma 48 (with $\sigma\pi t'$ being the s in the statement) shows that

$$\pi(\sigma \wedge \pi^i \sigma\pi t') \to \pi(\sigma \wedge \pi^i \sigma\pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma\pi t').$$

By Lemma 25,

$$\pi(\sigma \wedge \pi^i \sigma\pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma\pi t') \to \pi(\sigma\pi^i \sigma\pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma\pi t').$$

By contraction,

$$\begin{aligned} \pi(\sigma\pi^i \sigma\pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma\pi t') &\to \pi(\sigma\pi\sigma\pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma\pi t') \\ &\to \pi(\sigma\pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma\pi t') \\ &\to \pi(\sigma\pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t'). \end{aligned}$$

By Theorem 29,

$$\pi(\sigma\pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t') \to \pi(\sigma\pi^n c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t'),$$

so that if a $\pi(\sigma \wedge \pi^i \sigma\pi t')$-reflecting ordinal is not σ-reflecting, then it is

$$\pi(\pi^n c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t')\text{-reflecting},$$

by Theorem 32, as desired. ⊣

Although Lemma 51 does not directly provide any equivalences between reflection patterns, it leads to some when coupled with previous techniques.

LEMMA 52. *Suppose $s_0, s_1, \dots, s_n$ are normal reflection patterns, all of the form πv. Suppose moreover that $o(s_i) \leq o(s_n)$ for all $i \leq n$. Then,*

$$\sigma s_n \equiv \sigma(s_0 \wedge s_1 \wedge \cdots \wedge s_n) \equiv \sigma s_0 \wedge \sigma s_1 \wedge \cdots \wedge \sigma s_n.$$

PROOF. We restrict to the case $n = 1$ and let $s_0 = t$ and $s_1 = s$, since the general case is similar. We prove $\sigma s \equiv \sigma(s \wedge t)$, since clearly every $\sigma(s \wedge t)$-reflecting ordinal is $(\sigma s \wedge \sigma t)$-reflecting, and every $(\sigma s \wedge \sigma t)$-reflecting ordinal is σs-reflecting. Suppose α is σs-reflecting and let ϕ be a Σ^1_1-sentence satisfied by L_α. We must find some $(s \wedge t)$-reflecting $\beta < \alpha$ such that $L_\beta \models \phi$. By Lemma 51, it suffices that β be s-reflecting and not σ-reflecting. β is found as in the proof of Lemma 40:

By σs-reflection, there is some s-reflecting ordinal β_0 such that $L_{\beta_0} \models \phi$. If β_0 is not σ-reflecting, then we are done. Otherwise, β_0 is σ-reflecting and s-reflecting, thus (because s is of the form πv) σs-reflecting by Lemma 25. Hence, there is some s-reflecting $\beta_1 < \beta_0$ such that $L_{\beta_1} \models \phi$. Proceeding this way, one eventually finds some s-reflecting β_n such that $L_{\beta_n} \models \phi$ and β_n is not σ-reflecting. Then $\beta = \beta_n$ is as desired. ⊣

LEMMA 53. *Suppose t is a normal reflection pattern. Then $\sigma \wedge t$ is equivalent to a normal reflection pattern.*

PROOF. Put $t = \pi^k c^0_{m_0} c^1_{m_1} \dots c^l_{m_l}$. The lemma is immediate unless $k \neq 0$ and there is some least $i \leq l$ such that $m_i \neq 0$. Thus,

$$\begin{aligned} t &= \pi^k c^i_{m_i} c^{i+1}_{m_{i+1}} \dots c^l_{m_l} \\ &= \pi^k(\sigma \wedge \pi^i \sigma\pi c^i_{m_i-1} c^{i+1}_{m_{i+1}} \dots c^l_{m_l}). \end{aligned}$$

Let

$$s = c^{k+i}_{m_{k+i}} c^{k+i+1}_{m_{k+i+1}} \dots c^l_{m_l},$$

so that

$$t = \pi^k(\sigma \wedge \pi^i \sigma\pi c^i_{m_i-1} c^{i+1}_{m_{i+1}} \dots c^{k+i-1}_{m_{k+i-1}} s).$$

If all the displayed m_j are zero, then the result follows easily; otherwise, by Lemma 25 and contraction,

$$t \equiv \pi^k(\sigma \wedge \pi^i \sigma\pi c^i_{m_i-1} c^{i+1}_{m_{i+1}} \dots c^{k+i-1}_{m_{k+i-1}} \sigma\pi s).$$

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

By Lemma 25,

$$\begin{aligned}\sigma \wedge t &\equiv \sigma \wedge \pi^k(\sigma \wedge \pi^i \sigma\pi c^i_{m_i-1} c^{i+1}_{m_{i+1}} \dots c^{k+i-1}_{m_{k+i-1}} \sigma\pi s)\\ &\equiv \sigma \wedge \pi^k(\pi^i \sigma\pi c^i_{m_i-1} c^{i+1}_{m_{i+1}} \dots c^{k+i-1}_{m_{k+i-1}} \sigma\pi s)\\ &\equiv \sigma \wedge \pi^{k+i} \sigma\pi c^i_{m_i-1} c^{i+1}_{m_{i+1}} \dots c^{k+i-1}_{m_{k+i-1}} \sigma\pi s).\end{aligned}$$

By Lemma 48 on the one hand and Lemma 23 and contraction on the other,

$$\sigma \wedge \pi^{k+i} \sigma\pi s \equiv \sigma \wedge t, \tag{3}$$

but the reflection pattern on the left-hand side is readily seen to be equivalent to a normal one. ⊣

Theorem 55 below is our main technical result. Before stating it, we introduce the notion of a pattern being *quasi-normal*.

DEFINITION 54. The notion of a reflection pattern being *quasi-normal* is defined recursively: every normal reflection pattern is quasi-normal. Suppose that

1. s is a quasi-normal reflection pattern which is not normal, or s is a normal reflection pattern of the form $\pi s'$, and
2. t is a quasi-normal reflection pattern which is not normal, or t is a normal reflection pattern of the form $\pi t'$.

Then $s \wedge t$ and πt are quasi-normal reflection patterns.

THEOREM 55. *Let s be a reflection pattern in which the string $\sigma\sigma$ does not occur. Then, s is equivalent to a quasi-normal reflection pattern.*

PROOF. We shall prove the following claims:

1. If s is a normal reflection pattern, then so too is πs and, if s is of the form $\pi s'$, σs is also equivalent to a normal reflection pattern.
2. If s and t are normal reflection patterns, and at least one of them is not of the form πv, then $s \wedge t$ is equivalent to a normal reflection pattern.
3. If s is a quasi-normal reflection pattern which is not normal, then σs and $\sigma \wedge s$ are equivalent to normal reflection patterns, provided the string $\sigma\sigma$ does not occur in them.

Let us derive the conclusion of the theorem from (1)–(3) by induction on the construction of s. Suppose s is quasi-normal. Then so too is πs by definition. Suppose s is quasi-normal and the string $\sigma\sigma$ does not occur in σs. If s is not normal, then σs is normal by (3). Otherwise, if s is normal, then, since $\sigma\sigma$ does not occur in s, s must be of the form $\pi s'$, and thus σs is normal by (1). Lastly, suppose that s and t are quasi-normal reflection patterns. If none of them is normal, then $s \wedge t$ is quasi-normal by definition. Otherwise, say t is normal and s is not. If t is of the form $\pi t'$, then $s \wedge t$ is quasi-normal by definition. Otherwise, $t \equiv \sigma \wedge t$, so that

$$s \wedge t \equiv \sigma \wedge s \wedge t.$$

$\sigma \wedge s$ is normal by (3) and thus $s \wedge t$ is normal by (2). Finally, if both s and t are normal, then either they are both of the form πu, so that $s \wedge t$ is quasi-normal by definition, or $s \wedge t \equiv \sigma \wedge s \wedge t$ is normal by (2).

We begin with the proof of (1). Clearly, if s is equivalent to a normal reflection pattern, then so too is πs. We only need to consider the string σs in the case that s is of the form

$$\pi^m c^0_{m_0} c^1_{m_1} \dots c^l_{m_l},$$

where m is nonzero. Then, it is easy to see that

$$\begin{aligned}\sigma\pi^m c^0_{m_0} c^1_{m_1} \dots c^l_{m_l} &\equiv \sigma\pi c^0_{m_0} c^1_{m_1} \dots c^l_{m_l}\\ &\equiv c^0_{m_0+1} c^1_{m_1} \dots c^l_{m_l}.\end{aligned}$$

This proves (1).

To prove (2), let

$$s = \pi^m c^0_{m_0} c^1_{m_1} \dots c^l_{m_l}$$

be as above, and let

$$t = \pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k}.$$

We need to show that $s \wedge t$ is equivalent to a normal reflection pattern. By the case hypothesis, at least one of m and n is zero, so that

$$s \wedge t \equiv \sigma \wedge s \wedge t.$$

Write $n = n_{-1}$ and $m = m_{-1}$ and let i and j be least such that n_i and m_j are nonzero, respectively. There are four cases to consider. The first one is that in which both i and j are equal to 0. Then, there are reflection patterns u and v, both normal, such that

$$t = \sigma\pi u$$

and

$$s = \sigma\pi v.$$

The conclusion then follows from Lemma 52.

We now consider the case that both i and j are nonzero. The remaining cases, in which one of them is 0 and the other one is not, are treated similarly, and we leave them to the reader. We need to show that the pattern

$$s \wedge t \equiv \sigma \wedge s \wedge t \equiv (\sigma \wedge s) \wedge (\sigma \wedge t)$$

is normal. It was shown as part of Lemma 53 (see Equation (3)) that if s is normal, then $\sigma \wedge s$ has the form

$$\sigma \wedge \pi^p \sigma\pi c^p_{m_p-1} \dots c^l_{m_l},$$

for some suitably chosen $p \in \mathbb{N}$. Similarly, $\sigma \wedge t$ has the form

$$\sigma \wedge \pi^q \sigma\pi c^q_{n_q-1} \dots c^k_{n_k},$$

for some suitably chosen $q \in \mathbb{N}$. Put $u = c^p_{m_p-1} \dots c^l_{m_l}$ and $v = c^q_{n_q-1} \dots c^k_{n_k}$, so

$$\sigma \wedge s \equiv \sigma \wedge \pi^p \sigma\pi u$$

and

$$\sigma \wedge t \equiv \sigma \wedge \pi^q \sigma\pi v.$$

By Lemma 52 one of u and v, say w, has the property that

$$\sigma\pi w \equiv \sigma(\pi u \wedge \pi v).$$

Let $r = \max\{p, q\}$. ⊣

CLAIM 56. $\sigma \wedge \pi^r \sigma\pi w \equiv s \wedge t$.

PROOF. We have to show that

$$\sigma \wedge \pi^r \sigma\pi w \equiv \sigma \wedge \pi^p \sigma\pi u \wedge \pi^q \sigma\pi v.$$

By direct computation, using the usual tools:

$$\begin{aligned}
\sigma \wedge \pi^r \sigma\pi w &\equiv \sigma\pi^r \sigma\pi w \wedge \pi^r \sigma\pi w \\
&\equiv \sigma\pi\sigma\pi w \wedge \pi^r \sigma\pi w \\
&\equiv \sigma\pi w \wedge \pi^r \sigma\pi w \\
&\equiv \sigma(\pi u \wedge \pi v) \wedge \pi^r \sigma\pi w \\
&\equiv \sigma \wedge \pi^r(\sigma\pi w \wedge \sigma(\pi u \wedge \pi v)) \\
&\equiv \sigma \wedge \pi^r \sigma\pi w \wedge \pi^r \sigma\pi u \wedge \pi^r \sigma\pi v.
\end{aligned}$$

Since w is either u or v, we have

$$\sigma \wedge \pi^r \sigma\pi w \wedge \pi^r \sigma\pi u \wedge \pi^r \sigma\pi v \equiv \sigma \wedge \pi^r \sigma\pi u \wedge \pi^r \sigma\pi v.$$

Now, on the one hand, $r = \max\{p, q\}$ and this implies that

$$\sigma \wedge \pi^r \sigma\pi u \wedge \pi^r \sigma\pi v \to \sigma \wedge \pi^p \sigma\pi u \wedge \pi^q \sigma\pi v.$$

On the other hand,

$$\begin{aligned}
\sigma \wedge \pi^p \sigma\pi u \wedge \pi^q \sigma\pi v &\to \sigma(\pi^p \sigma\pi u \wedge \pi^q \sigma\pi v) \\
&\to \sigma(\pi^r \sigma\pi u \wedge \pi^r \sigma\pi v),
\end{aligned}$$

where the last implication follows by Theorem 29. Since r is one of p and q, we have

$$\sigma \wedge \pi^p \sigma\pi u \wedge \pi^q \sigma\pi v \to (\sigma(\pi^r \sigma\pi u \wedge \pi^r \sigma\pi v) \wedge \pi^r).$$

With one last computation, we obtain:

$$\begin{aligned}
\sigma(\pi^r \sigma\pi u \wedge \pi^r \sigma\pi v) \wedge \pi^r &\to \sigma(\pi^r \sigma\pi u \wedge \pi^r \sigma\pi v) \wedge \pi^r \sigma(\pi^r \sigma\pi u \wedge \pi^r \sigma\pi v) \\
&\to \sigma \wedge \pi^r \sigma(\pi^r \sigma\pi u \wedge \pi^r \sigma\pi v) \\
&\to \sigma \wedge \pi^r \sigma\pi^r \sigma\pi u \wedge \pi^r \sigma\pi^r \sigma\pi v \\
&\to \sigma \wedge \pi^r \sigma\pi u \wedge \pi^r \sigma\pi v \\
&\to \sigma(\pi^r \sigma\pi u \wedge \pi^r \sigma\pi v) \wedge \pi^r,
\end{aligned}$$

so that

$$\sigma(\pi^r \sigma\pi u \wedge \pi^r \sigma\pi v) \wedge \pi^r \equiv \sigma \wedge \pi^r \sigma\pi u \wedge \pi^r \sigma\pi v \equiv \sigma \wedge \pi^p \sigma\pi u \wedge \pi^q \sigma\pi v.$$

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

We have shown that

$$\begin{aligned}\sigma \wedge \pi^r \sigma\pi w &\equiv \sigma \wedge \pi^r \sigma\pi u \wedge \pi^r \sigma\pi v\\ &\equiv \sigma \wedge \pi^p \sigma\pi u \wedge \pi^q \sigma\pi v,\end{aligned}$$

as had been claimed. ⊣

Since w is a normal reflection pattern by assumption, Lemma 53 implies that $\sigma \wedge \pi^r \sigma\pi w$ is equivalent to a normal reflection pattern. This proves (2).

It remains to prove claim (3). Let s be a quasi-normal reflection pattern which is not normal. We prove that $\sigma \wedge s$ is equivalent to a normal reflection pattern by induction on the construction on s. Suppose that $s = \pi t$ and $\sigma \wedge t$ is equivalent to a normal reflection pattern. Since

$$\sigma \wedge s \equiv \sigma \wedge \pi(\sigma \wedge t)$$

and $\sigma \wedge t$ is equivalent to a normal reflection pattern, $\pi(\sigma \wedge s)$ is too, so the result follows from (2). Suppose that $s = t_0 \wedge t_1$, and that $\sigma \wedge t_0$ and $\sigma \wedge t_1$ are equivalent to normal reflection pattern. Then

$$\sigma \wedge s \equiv (\sigma \wedge t_0) \wedge (\sigma \wedge t_1),$$

so the result follows again from (2).

The proof that σs is equivalent to a normal reflection pattern is not quite by induction, but it is similar. By the definition of "quasi-normal," s is of the form

$$\pi^k(s_0 \wedge s_1 \wedge \cdots \wedge s_n),$$

for some k (possibly zero) and some $s_0, s_1, \dots, s_n$, each of which is either normal or quasi-normal but not normal. We may assume without loss of generality that none of the s_i is a conjunction (otherwise we could have split it into s_i^0 and s_i^1). None of the s_i can be of the form $\sigma s_i'$, because the definition of "quasi-normal pattern," does not allow taking conjunctions of patterns of the form $\sigma s'$, so if s_i were of the form $\sigma s_i'$, then the conjunction

$$s_0 \wedge s_1 \wedge \cdots \wedge s_n$$

would have to be trivial, and thus we would have

$$s = \pi^k s_i,$$

contradicting the hypothesis that s is not normal. We conclude that each s_i is necessarily of the form $\pi s_i'$.

With an application of Theorem 29, we see that

$$\begin{aligned}\sigma(s_0 \wedge s_1 \wedge \cdots \wedge s_n) &\to \sigma s\\ &\to \sigma(\pi^k s_0 \wedge \pi^k s_1 \wedge \cdots \wedge \pi^k s_n)\\ &\to \sigma(s_0 \wedge s_1 \wedge \cdots \wedge s_n),\end{aligned}$$

where the last implication follows from the fact that each s_i is of the form $\pi s_i'$. Hence,

$$\sigma s \equiv \sigma(s_0 \wedge s_1 \wedge \cdots \wedge s_n).$$

If each s_i is normal, then σs is equivalent to a normal reflection pattern by Lemma 52. Otherwise, say s_i is not normal. Certainly it is quasi-normal, so—by the same argument—it must be of the form $\pi^{k_i}(s_i^0 \wedge s_i^1 \dots s_i^{n_i})$ for some k_i and some patterns $s_i^0, s_i^1, \dots, s_i^{n_i}$, all of the form $\pi s'$. As before, we obtain

$$\sigma s \equiv \sigma(s_0 \wedge \cdots \wedge s_{i-1} \wedge s_i^0 \wedge s_i^1 \wedge \cdots \wedge s_i^{n_i} \wedge s_{i+1} \wedge \cdots \wedge s_n).$$

Continuing this way, we eventually see that σs is equivalent to a reflection pattern of the form $\sigma(t_0 \wedge t_1 \wedge \cdots \wedge t_n)$ for some n, where each t_i is a normal reflection pattern of the form $\pi t'$, so that the result follows from Lemma 52. This proves (3) and completes the proof of the theorem.

With little more work, we can show that quasi-normal patterns which are not normal play no role in computing the rank of a normal pattern in the order of reflection. To do this, we assign ordinals to these patterns:

Definition 57. Recursively, we extend $o(\cdot)$ to quasi-normal reflection patterns which are not normal.

1. Let s and t be normal patterns, both of the form πu. Then $o(s \wedge t)$ is defined to be $\max\{o(s), o(t)\}$.
2. Suppose s is quasi-normal but not normal. Then $o(\pi s) = o(s) + 1$.
3. Suppose t is quasi-normal but not normal and s is quasi-normal and of the form $\pi s'$. Then $o(s \wedge t) = \max\{o(s), o(t)\}$.

Lemma 58. *Suppose s is a quasi-normal reflection pattern which is not normal. Then, $o(s)$ is a successor ordinal.*

Proof. This is immediate from the definition. ⊣

Lemma 59. *Suppose s is a quasi-normal reflection pattern and t is a normal reflection pattern such that $o(s) = o(t)$. Then, every t-reflecting ordinal is either σ-reflecting or s-reflecting.*

Proof. This is proved by induction on the construction of s. If s is a conjunction of two normal reflection patterns, both of the form πu, then the lemma follows from Lemma 51. If s is of the form $s_0 \wedge s_1$, where, say, s_1 is not normal, then $o(s) = \max\{o(s_0), o(s_1)\}$. Both $o(s_0)$ and $o(s_1)$ are successor ordinals, so the normal reflection patterns t_0 and t_1 such that $o(t_0) = o(s_0)$ and $o(t_1) = o(s_1)$ are both of the form $\pi t'$. Assume without loss of generality that $o(s_0) < o(s_1)$. By Lemma 51, every t_1-reflecting ordinal is either σ-reflecting or t_0-reflecting. By the induction hypothesis applied to each of s_0 and s_1, every t_1-reflecting ordinal is either σ-reflecting or both s_0-reflecting and s_1-reflecting.

Finally, if s is of the form $\pi s'$, for some s' which is quasi-normal but not normal, then t is of the form $\pi t'$, for some t'. By induction hypothesis, every t'-reflecting ordinal is either σ-reflecting or s'-reflecting. Suppose α is t-reflecting but not σ-reflecting. There are two cases: if α is π-reflecting on ordinals which are t'-reflecting but not σ-reflecting, then α is π-reflecting on ordinals which are s'-reflecting, by induction hypothesis, so α is $\pi s'$-reflecting.

The other case is a bit subtler: suppose α is $\pi(\sigma \wedge t')$-reflecting. s' is quasi-normal but not normal, so $o(s')$ is a successor ordinal, and thus t' is of the form $\pi t''$. It follows that α is $\pi(\sigma t' \wedge t')$-reflecting and in particular $\pi\sigma t'$-reflecting. We claim that

α is π-reflecting on ordinals which are σ-reflecting on ordinals which are t'-reflecting but not σ-reflecting. To see this, let ϕ be any Π^1_1 sentence satisfied by L_α and find a $\sigma t'$-reflecting ordinal $\beta < \alpha$ such that L_β satisfies ϕ. Now, let ψ be a Σ^1_1 sentence satisfied by L_β. We must find some $\gamma < \beta$ which is t'-reflecting and not σ-reflecting and such that $L_\gamma \models \psi$. By $\sigma t'$-reflection, there is some $\gamma_0 < \beta$ which is t'-reflecting and such that $L_{\gamma_0} \models \psi$. If γ_0 is not σ-reflecting, then we are done. Otherwise, γ_0 is σ-reflecting and t'-reflecting, thus $\sigma t'$-reflecting. Hence, there is $\gamma_1 < \gamma_0$ which is t'-reflecting and such that $L_{\gamma_1} \models \psi$. Since there cannot be an infinite descending sequence of ordinals, this procedure eventually produces a γ as desired. Thus, α is π-reflecting on ordinals which are σ-reflecting on ordinals which are s'-reflecting (by induction hypothesis), i.e., α is $\pi\sigma s'$-reflecting. Since α is not σ-reflecting (by assumption), it is $\pi s'$-reflecting, by Theorem 32, as desired. ⊣

Corollary 60. *Suppose s is a quasi-normal reflection pattern and t is a normal reflection pattern such that $o(s) = o(t)$. Then, the least s-reflecting ordinal is the least t-reflecting ordinal.*

Proof. Let us assume that s is not normal, for otherwise the result is trivial. That the least t-reflecting ordinal is s-reflecting follows from Lemma 59 and the observation that the least t-reflecting ordinal is not σ-reflecting, by Lemmas 25 and 58. That the least s-reflecting ordinal is t-reflecting is immediate from the definition of $o(s)$. ⊣

Theorem 61. *$\sigma\sigma$ has order-type ω^ω in the order of reflection.*

Proof. If a reflection pattern contains the string $\sigma\sigma$, then naturally, it cannot be strictly smaller than $\sigma\sigma$. If it does not contain the string $\sigma\sigma$, then it is equivalent to a quasi-normal reflection pattern by Theorem 55. By Corollary 60, the least ordinal which reflects according to a quasi-normal reflection pattern is the least ordinal which reflects according to some normal reflection pattern, and these ordinals are all smaller than the least $\sigma\sigma$-reflecting ordinal, by Lemma 50. Thus, the rank of $\sigma\sigma$ in the order of reflection is the order-type of the set of normal reflection patterns. An easy induction using Lemma 48 shows that, for normal reflection patterns u and v, $u < v$ if, and only if, $o(u) < o(v)$, so the result follows. ⊣

§6. Concluding remarks and questions. The most obvious open problem is that of the length of the order of reflection:

Question 62. What is the length of the order of reflection?

While we do not have an answer, the following proposition, which contrasts with the fact that π^1_1 is the least ordinal α which is α^+-stable, provides a (non-recursive) upper bound:

Proposition 63. *Let s be a reflection pattern and suppose α is an ordinal such that*

$$L_\alpha \prec_1 L_{\alpha^+ + 1}.$$

Then, α is s-reflecting.

https://doi.org/10.1017/jsl.2021.3 Published online by Cambridge University Press

PROOF. Since

$$L_\alpha \prec_1 L_{\alpha^+},$$

it follows that α is π-reflecting. By Gostanian's theorem [8] mentioned in the introduction, α is σ-reflecting. Inductively, suppose α is s-reflecting and let ψ be a Σ^1_1 sentence such that

$$L_\alpha \models \psi.$$

Choose a Π_1 sentence ψ^* such that for all admissible β containing all relevant parameters,

$$L_\beta \models \psi,$$

if, and only if,

$$L_{\beta^+} \models \psi^*(L_\beta),$$

so that, in particular,

$$L_{\alpha^+} \models \psi^*(L_\alpha).$$

Then, from the point of view of $L_{\alpha^+ +1}$, there are admissible sets L_α and L_{α^+} such that

1. α is s-reflecting (this is a first-order statement about L_{α^+}),
2. $L_{\alpha^+} \models \psi^*(L_\alpha)$.

Thus, by stability, there are admissible sets L_β and L_{β^+} in L_α such that β is s-reflecting and $L_{\beta^+} \models \psi^*(L_\beta)$. Hence, α is σs-reflecting. A similar argument shows that α is πs-reflecting. A simple induction thus shows that α is s-reflecting for every reflection pattern. ⊣

The length of the order of reflection is thus at most the least ordinal α such that

$$L_\alpha \prec_1 L_{\alpha^+ +1}.$$

Moreover, surely each inequality between reflection patterns is provable in any theory that proves the existence of the corresponding ordinals. This suggests strongly that the length of the order of reflection is smaller than the proof-theoretic ordinal of the subsystem $\mathbf{\Pi}^1_2$-CA $_0$ of analysis and in fact smaller than the ordinal described in Rathjen [10], though we do not have a proof of this.

In regard to lower bounds, the reader may consult [4] for an example of a chain of length ε_0 in the order of reflection. The construction in [4] uses Theorem 38. The facts that σ and σ^2 have ranks ω and ω^ω in the order of reflection suggest that perhaps the bound is optimal.

QUESTION 64. Is the rank of σ^3 in the order of reflection ω^{ω^ω}?

It was shown in [4] that the rank of σ^3 is at least ω^{ω^ω}. A related project concerns studying reflection patterns with focus on the implication ordering:

QUESTION 65. What is the structure of the set of reflection patterns under the ordering $\to$?

Acknowledgments. We would like to thank David Schrittesser and Lev Beklemishev for fruitful conversations during the 2019 Arctic Set Theory meeting and the 2019 Barcelona Wormshop, respectively. This work was partially supported by FWO grant 3E017319, by FWF-FWO grant I-4513, and by FWF-RSF grant I-4427.

REFERENCES

[1] S. Aanderaa, *Inductive definitions and their closure ordinals*, ***Generalized Recursion Theory*** (J. E. Fenstad and P. G. Hinman, editors), North-Holland, Amsterdam, 1974, pp. 207–220.

[2] F. G. Abramson and G. E. Sacks, *Uncountable Gandy ordinals*. ***Journal of the London Mathematical Society***, vol. 14 (1976), no. 2, pp. 387–392.

[3] P. Aczel and W. Richter, *Inductive definitions and reflecting properties of admissible ordinals*, ***Generalized Recursion Theory*** (J. E. Fenstad and P. G. Hinman, editors), 1974, pp. 301–381.

[4] J. P. Aguilera, ***Between the finite and the infinite***, Ph.D. thesis, Vienna University of Technology, 2019.

[5] J. Barwise, ***Admissible Sets and Structures***, Perspectives of Logic, vol. 7, Springer-Verlag, Berlin, 1975.

[6] K. J. Barwise, R. O. Gandy, and Y. N. Moschovakis, *The next admissible set*, this Journal, vol. 36 (1971), pp. 108–120.

[7] D. Cenzer, *Ordinal recursion and inductive definitions*, ***Generalized Recursion Theory*** (J. E. Fenstad and P. G. Hinman, editors), 1974, pp. 221–264.

[8] R. Gostanian, *The next admissible ordinal*. ***Annals of Mathematical Logic***, vol. 17 (1979), pp. 171–203.

[9] R. Gostanian and K. Hrbacek, *A new proof that* $\pi^1_1 < \sigma^1_1$. ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 25 (1979), pp. 407–408.

[10] M. Rathjen, *An ordinal analysis of parameter-free* Π^1_2*-comprehension*. ***Archive for Mathematical Logic***, vol. 44 (2005), pp. 263–362.

[11] S. G. Simpson, *Short course on admissible recursion theory*, ***Generalized Recursion Theory, II*** (J. E. Fenstad, R. O. Gandy, and G. E. Sacks, editors), North-Holland, Amsterdam, 1978, pp. 355–390.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GHENT
KRIJGSLAAN 281-S8, 9000 GHENT, BELGIUM

and

INSTITUTE OF DISCRETE MATHEMATICS AND GEOMETRY
VIENNA UNIVERSITY OF TECHNOLOGY
WIEDNER HAUPTSTRASSE 8–10, 1040 VIENNA, AUSTRIA
E-mail: aguilera@logic.at