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On non-existence of steady periodic solutions of the Prandtl equations

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We prove that periodic solutions of the steady Prandtl equations do not exist on a stationary boundary. On a moving boundary, there are no solutions with a monotone velocity profile.

Key words: boundary layers, boundary layer structure, general fluid mechanics, mathematical foundations

1. Introduction

Consider a bounded two-dimensional region bounded by a number of closed curves. Then steady flows of the Euler equations are highly non-unique. Prandtl (1904) and Batchelor (1956) have raised the question of which of these many flows can be realized as limits of steady flows of the Navier–Stokes equations in the limit of infinite Reynolds number. An important result (Prandtl 1904; Batchelor 1956) is that, in any region of closed streamlines that is free of singularities, the vorticity must be constant. Further characterization of these flows raises the question of boundary layers, in particular, what steady flow solutions of the Prandtl equations exist on closed boundaries and to which exterior flows can they be matched? In general, of course, one does not expect steady flows to be stable at high Reynolds numbers. However, there are some flows like rigid-body motion that are always stable, and the theory has received impetus from the hope that there may be flows stable at sufficiently high Reynolds number to make asymptotic theory applicable.

We shall show below that stationary solutions of the Prandtl equations with periodic boundary conditions do not exist on a stationary boundary. On the other hand, solutions on moving boundaries do exist. The only case that seems to be well understood is that where the difference between the wall velocity and the velocity of the outer flow is small. Wood (1957) derived a formal series expansion. It shows that, for given wall velocity, the velocity in the outer flow is restricted to a manifold of codimension one. At the leading order, the average of the outer flow velocity is equal to the velocity of the moving wall. Further work (Feynman & Lagerstrom

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1956; Moore 1957; Kim 2000, 1998; Edwards 1997; Van Wijngaarden 2007) has applied Wood's result to specific flow situations and provided a rigorous mathematical justification for the series expansion.

Well-posedness of the time-dependent Prandtl equations is assured only when the velocity in the boundary layer is monotone (Oleinik & Samokhin 1999). Recent work of Gérard-Varet & Dormy (2010) shows that without this assumption we cannot expect well-posedness. The argument in Gérard-Varet & Dormy (2010) is based on a shear flow instability first considered in Cowley, Hocking & Tutty (1985), and it appears to be 'localizable', so that ill-posedness should be expected whenever there is a non-degenerate velocity extremum in the boundary layer. Indeed, a result along these lines is given by Ding (2012).

The solutions found by Wood clearly do not have a monotone velocity profile. Indeed, we shall prove that solutions with a monotone velocity profile do not exist.

These results show that we cannot expect to find physically relevant stationary flows on the basis of an asymptotics which combines Euler and Prandtl equations. If any component of the boundary is stationary, such solutions do not exist at all. On moving boundaries, the boundary layer solutions would be in a regime where the Prandtl equations are dynamically ill-posed. For the full Navier–Stokes equations, this would still imply instabilities with a growth rate that increases with Reynolds number (see the remarks in Cowley, Hocking & Tutty 1985).

2. Non-existence on stationary boundaries

We consider the Prandtl system

$$uu_x + vu_y = u_{yy} - p_x, \tag{2.1a}$$

$$p_y = 0, \tag{2.1b}$$

$$u_x + v_y = 0, \tag{2.1c}$$

in the half-space $y > 0$. We seek solutions which are smooth, satisfy the boundary conditions

$$u(x, 0) = v(x, 0) = 0, \tag{2.2}$$

periodicity in the x-direction,

$$u(x + \lambda, y) = u(x, y), \quad v(x + \lambda, y) = v(x, y), \quad p(x + \lambda, y) = p(x, y), \tag{2.3}$$

and the following conditions at infinity:

(*a*) $u(x, y) \to u_0(x)$ as $y \to \infty$ uniformly in x;
(*b*) $p(x, y) \to -u_0(x)^2/2$ as $y \to \infty$ uniformly in x;
(*c*) $u_y(x, y) \to 0$ as $y \to \infty$ uniformly in x;
(*d*) u_{yy} is integrable on $(0, \lambda) \times (0, \infty)$.

We shall prove that, with the obvious exception of $u = 0$, no solutions with these properties exist.

We multiply (2.1*a*) by u. The resulting equation can be put in the form

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{u^2}{2} + p\right) = uu_{yy}. \tag{2.4}$$

Let L denote the left-hand side and R the right-hand side of this equation. Since u is bounded and u_{yy} is integrable, it follows that R is integrable. Moreover, an integration by parts yields

$$\int_0^\infty \int_0^\lambda R \, \mathrm{d}x \, \mathrm{d}y = -\int_0^\infty \int_0^\lambda u_y^2 \, \mathrm{d}y. \tag{2.5}$$

This is strictly negative unless u is identically zero.

Now we estimate the integral of L. Since R is integrable and $L = R$, L must be integrable, too. We can therefore evaluate the integral of L as

$$\int_0^\infty \int_0^\lambda L \, \mathrm{d}x \, \mathrm{d}y = \lim_{M\to\infty} \int_{\Omega(M)} L \, \mathrm{d}x \, \mathrm{d}y, \tag{2.6}$$

where $\Omega(M)$ is any sequence of domains which exhausts the flow domain as $M \to \infty$. To construct $\Omega(M)$, we proceed as follows: We start with the strip $0 < y < M$. Now we consider all streamlines which pass through this strip. We take $\Omega(M)$ to be the union of all these streamlines, truncated at $y = Y(M)$, where $Y(M)$ shall be chosen large relative to M. The boundary of $\Omega(M)$ consists of the wall $y = 0$, segments of streamlines and segments of the line $y = Y(M)$. Noting that L has divergence form, we find that

$$\int_{\Omega(M)} L \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial\Omega(M)} (\boldsymbol{u} \cdot \boldsymbol{n}) \left(\frac{u^2}{2} + p \right) \mathrm{d}s. \tag{2.7}$$

Here $\boldsymbol{u} = (u, v)$, and $\boldsymbol{n}$ is the outward unit normal. The contributions from the boundary $y = 0$ and from streamlines are zero. Thus, if $\Sigma(M)$ is the part of the boundary $y = Y(M)$ that belongs to $\partial\Omega(M)$, we have

$$\int_{\Omega(M)} L \, \mathrm{d}x \, \mathrm{d}y = -\int_{\Sigma(M)} v \left(\frac{u^2}{2} + p \right) \mathrm{d}x. \tag{2.8}$$

Now introduce the streamfunction ψ by $u = \psi_y$, $v = -\psi_x$. We find

$$\int_{\Omega(M)} L \, \mathrm{d}x \, \mathrm{d}y = \int_{\Sigma(M)} \left(\frac{u^2}{2} + p \right) \mathrm{d}\psi \tag{2.9}$$

and hence

$$\left| \int_{\Omega(M)} L \, \mathrm{d}x \, \mathrm{d}y \right| \leqslant \max_{y=Y(M)} \left| \frac{u^2}{2} + p \right| \int_{\Sigma(M)} |\mathrm{d}\psi| \leqslant \max_{y=Y(M)} \left| \frac{u^2}{2} + p \right| \int_{y=M} |\mathrm{d}\psi|. \tag{2.10}$$

Here we have used the fact that all streamlines crossing $y = Y(M)$ through $\Sigma(M)$ must come from $y = M$. Since $u^2/2 + p$ tends to zero at infinity, we can make the right-hand side of (2.10) small by choosing $Y(M)$ large relative to M. Consequently, the integral of L must be zero.

In essence, the argument given here is an energy balance. Viscous dissipation in the boundary layer requires energy, and this energy cannot be extracted from the outer flow. If the boundary is stationary, no work is done at the boundary, so there is no source of energy. The situation is different for moving boundaries, which we shall consider in the next section.

3. Non-existence of monotone solutions

We now allow a moving boundary. Only the behaviour at infinity will be relevant in the following. We first consider the case where the limiting velocity u_0 at infinity is unidirectional, say $u_0 > 0$. We can then use von Mises variables, in which the Prandtl equations take the form (see e.g. Wood 1957)

$$\frac{\partial}{\partial x}(u^2 - u_0^2) = u\frac{\partial^2 u^2}{\partial \psi^2}. \tag{3.1}$$

We set $w = u^2 - u_0^2$ and obtain

$$\frac{1}{\sqrt{u_0^2 + w}}\frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial \psi^2}. \tag{3.2}$$

If the velocity in the boundary layer is strictly monotone, then w and w_ψ do not change sign and $w_\psi / w < 0$. We divide (3.2) by w and substitute $w = \pm \mathrm{e}^a$. This leads to the equation

$$a_x = \sqrt{u_0^2 \pm \mathrm{e}^a}(a_{\psi\psi} + a_\psi^2). \tag{3.3}$$

Let $b = \sqrt{u_0^2 \pm \mathrm{e}^a} a_\psi$. Then we can write the preceding equation in the form

$$a_x = b_\psi \mp \frac{\mathrm{e}^a}{2\,(u_0^2 \pm \mathrm{e}^a)^{3/2}}b^2 + \frac{1}{\sqrt{u_0^2 \pm \mathrm{e}^a}}b^2. \tag{3.4}$$

Since $\mathrm{e}^a \to 0$ for $\psi \to \infty$, the coefficient of b^2 on the right-hand side is bounded below by some positive constant C if ψ is large enough. We now integrate over x and find

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\int_0^\lambda b\,\mathrm{d}x\right) \leqslant -C\int_0^\lambda b^2\,\mathrm{d}x \leqslant -\frac{C}{\lambda}\left(\int_0^\lambda b\,\mathrm{d}x\right)^2. \tag{3.5}$$

By assumption b is negative. But then (3.5) implies that $\int_0^\lambda b\,\mathrm{d}x$ approaches $-\infty$ at a finite value of ψ, and we have a contradiction. Hence monotone solutions with a unidirectional outer flow cannot exist.

Now suppose u_0 is not unidirectional, but u is strictly monotone, say $u_y > 0$. Let $[a, b]$ be an interval on which $u_0 > 0$, with $u_0(a) = u_0(b) = 0$. Pick $x_0 \in (a, b)$, and consider streamlines passing through a point (x_0, Y). Since $u(x_0, y)$ is positive and monotone for large y, we conclude that the streamline through (x_0, Y) can cross the line $x = x_0$ only once if Y is large enough. Moreover, by our assumption that u_0 is not unidirectional, there exists another point x_1 where $u_0(x_1) < 0$. It follows that $\psi(x_1, y)$ tends to $-\infty$ for $y \to \infty$. Hence, if Y is large enough, the streamline through (x_0, Y) cannot cross the line $x = x_1$. In conclusion, the streamline through (x_0, Y) can neither cross the line $x = x_0$ a second time, nor cross the line $x = x_1$. Thus it cannot be closed or periodic, leaving only the possibilities that at either end it must approach a stagnation point or go to infinity. Sard's lemma states that the set of critical values of a smooth function (i.e. the set of values corresponding to points where the gradient is zero) has zero measure. Hence the values of the streamfunction associated with stagnation points have measure zero. Hence there are values of Y with Y large such that the streamline $\varGamma$ through (x_0, Y) goes to infinity at both ends. Along $\varGamma$, the horizontal flow cannot reverse, since there cannot be a point with $u < 0$ directly above another point where $u > 0$. Hence $\varGamma$ is described by a function $y = \phi(x)$, where $\lim_{x\to a} = \lim_{x\to b}\phi(x) = \infty$. Now consider the domain $\varOmega = \{(x, y) \mid x \in (a, b), y > \phi(x)\}$, and integrate (2.4) over $\varOmega$. As in the preceding section, we argue that the integral of L must be zero. On the other hand, we find that

$$\int_a^b \int_{\phi(x)}^{\infty} R \,\mathrm{d}y\,\mathrm{d}x = -\int_a^b \int_{\phi(x)}^{\infty} u_y^2 \,\mathrm{d}y\,\mathrm{d}x - \int_a^b u(x, \phi(x)) u_y(x, \phi(x)) \,\mathrm{d}x < 0. \tag{3.6}$$

This is a contradiction.

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