Canad. Math. Bull. Vol. **64** (3), 2021, pp. 687–696
http://dx.doi.org/10.4153/S0008439520000752

Dyson's rank, overpartitions, and universal mock theta functions

Helen W. J. Zhang

Abstract. In this paper, we decompose $\overline{D}(a, M)$ into modular and mock modular parts, so that it gives as a straightforward consequencethe celebrated results of Bringmann and Lovejoy on Maass forms. Let $\overline{p}(n)$ be the number of partitions of n and $\overline{N}(a, M, n)$ be the number of overpartitions of n with rank congruent to a modulo M. Motivated by Hickerson and Mortenson, we find and prove a general formula for Dyson's ranks by considering the deviation of the ranks from the average:

$$\overline{D}(a, M) = \sum_{n=0}^{\infty} \Big(\overline{N}(a, M, n) - \frac{\overline{p}(n)}{M}\Big) q^n.$$

Based on Appell–Lerch sum properties and universal mock theta functions, we obtain the stronger version of the results of Bringmann and Lovejoy.

1 Introduction

Here and throughout the paper, we adopt the following common notation:

$$(x_1, x_2, \dots, x_k; q)_\infty = \prod_{n=0}^{\infty} (1 - x_1 q^n)(1 - x_2 q^n)\cdots(1 - x_k q^n),$$
$$j(z; qs) = (z; q)_\infty (q/z; q)_\infty (q; q)_\infty,$$
$$J_{a,m} = j(q^a, q^m), \quad J_m = (q^m; q^m)_\infty, \quad \overline{J}_{a,m} = j(-q^a, q^m),$$

where we assume that $|q| < 1$.

The rank of a partition was introduced by Dyson [8] as the largest part of the partition minus the number of parts. Let $N(s, \ell, n)$ denote the number of partitions of n with rank congruent to s modulo ℓ. Atkin and Swinnerton-Dyer [2] obtained generating functions for rank differences $N(s, \ell, \ell n + d) - N(t, \ell, \ell n + d)$ with $\ell = 5$ or 7 and $0 \le d, s, t < \ell$, which lead to combinatorial interpretations of Ramanujan's congruences modulo 5 and 7. The connection between classical mock theta functions and the generating functions of rank differences of partitions have been extensively

Received by the editors July 18, 2020; revised August 28, 2020.
Published online on Cambridge Core September 7, 2020.
The author would like to thank the referees for their valuable comments and suggestions. This work was supported by the National Science Foundation of China (Grant Nos. 11871370 and 12001182) and the Fundamental Research Funds for the Central Universities (Grant No. 531118010411).
AMS subject classification: **11P84, 11B65, 11F11, 11F27**.
Keywords: Overpartition, Dyson's rank, Appell-Lerch sum, universal mock theta functions.

studied. For example, Andrews and Garvan [1] found that the fifth order mock theta functions $\chi_0(q)$ and $\chi_1(q)$ can be expressed in terms of the rank differences of partitions modulo 5, which was later proved by Hickerson [10]; that is,

$$2+\sum_{n=0}^{\infty}\big(N(1,5,5n)-N(0,5,5n)\big)q^n=\chi_0(q),$$

$$\sum_{n=0}^{\infty}\big(2N(2,5,5n+3)-N(0,5,5n+3)-N(1,5,5n+3)\big)q^n=\chi_1(q).$$

Subsequently, Hickerson [11] showed that the seventh order mock theta functions $\mathfrak{F}_0(q)$, $\mathfrak{F}_1(q)$, and $\mathfrak{F}_2(q)$ are related to the rank differences of partitions modulo 7. Since then, the work on Dyson's rank and mock theta functions have been extensively studied; see, for example, [3, 4, 13, 18].

Dyson's rank can be extended to overpartitions in the obvious way. Recall that an overpartition [7] is a partition in which the first occurrence of a part may be overlined. The rank of an overpartition is defined to be the largest part of an overpartition minus its number of parts. Similarly, let $\overline{N}(s,\ell,n)$ denote the number of overpartitions of n with rank congruent to s modulo ℓ. We define the general rank difference for overpartitions

$$\overline{R}(a,b,M,c,m)=\sum_{n=0}^{\infty}\big(\overline{N}(a,M,mn+c)-\overline{N}(b,M,mn+c)\big)q^n, \tag{1.1}$$

where a, b, c, m, and M are integers with $0\le a,b<M$, and $0\le c<m$.

The rank differences of overpartitions are also related to mock theta functions. For third order mock theta function $\omega(q)$, the author and Wei [17] established the relation between $\omega(q)$ and the ranks of overpartitions modulo 6:

$$\omega(q)=\frac{1}{2}\overline{R}(0,1,6,2,3).$$

As an example of Lovejoy and Osburn's results, they proved for modulus 3 [16, Theorem 1.1], slightly written,

$$\overline{R}(0,1,3,0,3)=-1+\frac{J_6^3J_{3,6}}{J_{1,6}^2J_2},$$

$$\overline{R}(0,1,3,1,3)=\frac{2J_6^3}{J_{1,6}J_2},$$

$$\overline{R}(0,1,3,2,3)=\frac{4J_6^3}{J_2J_{3,6}}-6h(q,q^3),$$

where

$$h(x,q)=\frac{1}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(n+1)}}{1-xq^n} \tag{1.2}$$

is a universal mock theta function that appeared in [6].

Bringmann and Lovejoy [5] show that $\overline{R}(a,b,t,d,t)$ is a weakly holomorphic modular form when $\left(\frac{d}{t}\right) = -\left(\frac{-1}{t}\right)$. Moreover, they also mentioned that

$$\sum_{n=0}^{\infty}\left(\overline{N}(a,M,n) - \frac{\overline{p}(n)}{M}\right)q^n$$

is the holomorphic part of a weak Maass form.

For integers $0 \le a \le M$, define

$$\overline{D}(a,M) = \overline{D}(a,M,q) = \sum_{n=0}^{\infty}\left(\overline{N}(a,M,n) - \frac{\overline{p}(n)}{M}\right)q^n. \tag{1.3}$$

In terms of (1.1), we obtain

$$\overline{R}(a,b,M,0,1) = \overline{D}(a,M) - \overline{D}(b,M). \tag{1.4}$$

Theorem 1.1 *If $0 \le a < M$, then*

$$\overline{D}(a,M,q) = \overline{d}(a,M,q) + \overline{T}_{a,M}(q),$$

where

$$\overline{d}(a,M,q) = \begin{cases} h\left(-1, (-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\right) + (-1)^{\frac{M}{2}-1}q^{\frac{M}{2}-1}h\left(-q^{\frac{M^2}{4}-\frac{M}{2}}, (-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\right), \\ \qquad\qquad \text{if } a = 0 \text{ and } M \equiv 0 \pmod 2 \\ (-1)^{\frac{M}{2}-\frac{a}{2}+1}q^{\frac{aM}{4}-\frac{a^2}{4}}h\left(q^{\frac{M^2}{4}-\frac{aM}{4}}, (-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\right) \\ \quad +(-1)^{\frac{M}{2}-\frac{a}{2}}q^{\frac{aM}{4}+\frac{M}{2}-\frac{(a+2)^2}{4}}h\left(q^{\frac{M^2}{4}-\frac{aM}{4}-\frac{M}{2}}, (-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\right), \\ \qquad\qquad \text{if } a \ne 0 \text{ and } a \equiv M \equiv 0 \pmod 2 \\ 2(-1)^{\frac{M}{2}-\frac{a}{2}+\frac{1}{2}}q^{\frac{aM}{4}+\frac{M}{4}-\frac{(a-1)^2}{4}}h\left(-q^{\frac{M^2}{4}-\frac{aM}{4}-\frac{M}{4}}, (-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\right), \\ \qquad\qquad \text{if } a \equiv 1 \pmod 2 \text{ and } M \equiv 0 \pmod 2 \\ \hline 0, \qquad\qquad \text{if } a = 0 \text{ and } M = 1 \\ 2(-1)^{\frac{M-1}{2}}q^{\frac{M^2-1}{4}}h\left(q^{M(\frac{M-1}{2})}, q^{M^2}\right) - q^{M-1}h\left(q^{M^2-M}, q^{M^2}\right), \\ \qquad\qquad \text{if } a = 0 \text{ and } M \equiv 1 \pmod 2 \\ 2(-1)^{\frac{M+1}{2}-\frac{a}{2}}q^{\frac{M^2-1}{4}-\frac{a^2+2a}{4}}h\left(-q^{M(\frac{M-1}{2})-\frac{Ma}{2}}, q^{M^2}\right) \\ \quad +(-1)^{M-\frac{a}{2}}q^{\frac{a}{2}(M-\frac{a}{2})}h\left(-q^{M^2-\frac{aM}{2}}, q^{M^2}\right) \\ \quad +(-1)^{M-\frac{a}{2}+1}q^{(\frac{a}{2}+1)(M-\frac{a}{2}-1)}h\left(-q^{M(M-\frac{a}{2}-1)}, q^{M^2}\right), \\ \qquad\qquad \text{if } a \ne 0,\ M \ne 1,\ a \equiv 0 \pmod 2 \\ \qquad\qquad \text{and } M \equiv 1 \pmod 2 \\ 2(-1)^{M-\frac{a-1}{2}}q^{(\frac{a+1}{2})(M-\frac{a+1}{2})}h\left(-q^{M(M-\frac{a+1}{2})}, q^{M^2}\right) \\ \quad +(-1)^{\frac{M-a}{2}+1}q^{\frac{M^2-a^2}{4}}h\left(q^{M(\frac{M-a}{2})}, q^{M^2}\right) \\ \quad +(-1)^{\frac{M-a}{2}+1}q^{\frac{M^2-(a+2)^2}{4}}h\left(-q^{M(\frac{M-a}{2}-1)}, q^{M^2}\right), \\ \qquad\qquad \text{if } M \ne 1 \text{ and } a \equiv M \equiv 1 \pmod 2 \end{cases}$$

and $\overline{T}_{a,M}(q)$ is a theta function.

https://doi.org/10.4153/S0008439520000752 Published online by Cambridge University Press

In Section 2, we prove the main theorem using classical methods. In Section 3, we demonstrate how the main theorem yields the results of Bringmann and Lovejoy on Dyson's ranks and Maass form.

2 Proof of Theorem 1.1

This section is devoted to the decomposition of $\overline{D}(a, M)$ as stated in Theorem 1.1. It is known that mock theta functions can be expressed in terms of the Appell–Lerch sum $m(x, q, z)$. Recall that the Appell–Lerch sum is defined by

$$m(x,q,z) = \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1 - q^{r-1} xz}, \tag{2.1}$$

where $x, z \in \mathbb{C}^*$ with neither z nor xz an integral power of q.

First, we recall the universal mock theta function $h(x, q)$ defined by Gordon and McIntosh [9]

$$h(x,q) = \frac{1}{J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1 - xq^n}. \tag{2.2}$$

Hickerson and Mortenson [12] showed that $h(x, q)$ and $m(x, q, z)$ have the following relation:

$$h(x,q) = -x^{-1} m(x^{-2}q, q^2, x^2 z) + \frac{J_1^3 j(xz;q) j(z;q^2)}{j(q;q^2) j(x;q) j(z;q) j(x^2 z;q^2)}, \tag{2.3}$$

where the right-hand side is actually z-independent.

We note that the $z = x^{-1}$ and $z = -x^{-2}$ specialisations of (2.3):

$$h(x,q) = -x^{-1} m(x^{-2}q, q^2, x), \tag{2.4}$$

$$h(x,q) = -x^{-1} m(x^{-2}q, q^2, -1) + \Delta(x;q), \tag{2.5}$$

where

$$\Delta(x;q) = \frac{x^{-1} J_1^3 j(-x;q) j(-x^2;q^2)}{j(q;q^2) j(x;q) j(-x^2;q) j(-1;q^2)}. \tag{2.6}$$

The following lemmas play central roles in the proof of Theorem 1.1.

Lemma 2.1 ([12, Theorem 3.9]) *Let n and k be integers with $0 \le k < n$. Let ω be a primitive n-th root of unity. Then*

$$\sum_{t=0}^{n-1} \omega^{-kt} m(\omega^t x, q, z) = n q^{-\binom{k+1}{2}} (-x)^k m\big(-q^{\binom{n}{2}-nk}(-x)^n, q^{n^2}, z'\big) + n\Psi_k^n(x, z, z'; q),$$

where

$$\Psi_k^n(x,z,z';q) = -\frac{x^k z^{k+1} J_{n^2}^3}{j(z;q) j(z';q^{n^2})} \times \sum_{t=0}^{n-1} \frac{q^{\binom{t+1}{2}+kt}(-z)^t j\big(-q^{\binom{n+1}{2}+nk+nt}(-z)^n/z', q^{nt}x^n z^n z'; q^{n^2}\big)}{j\big(-q^{\binom{n}{2}-nk}(-x)^n z', q^{nt}x^n z^n; q^{n^2}\big)}.$$

https://doi.org/10.4153/S0008439520000752 Published online by Cambridge University Press

Lemma 2.2 *If $0 \le r < t$ are integers, then we have*

$$\sum_{n=0}^{\infty} \overline{N}(r,t,n)q^n = \frac{1}{t}\sum_{n=0}^{\infty} \overline{p}(n)q^n + \frac{1}{t}\sum_{j=1}^{t-1} \zeta_t^{-rj}\overline{R}(\zeta_t^j;q). \tag{2.7}$$

Proof There is just one overpartition of 0, the empty overpartition. We define its rank to be 0. Let $\overline{N}(m,n)$ denote the number of overpartitions of n with rank m. Lovejoy [15] obtained the following generating function for $\overline{N}(m,n)$,

$$\overline{R}(z;q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{N}(m,n)z^m q^n$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}. \tag{2.8}$$

Since we have

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \overline{R}(1;q),$$

it follows that the right-hand side of (2.7) is

$$\frac{1}{t}\sum_{j=0}^{t-1} \zeta_t^{-rj}\overline{R}(\zeta_t^j;q).$$

Therefore, the n-th coefficient of this series, say $a(n)$, is given by

$$a(n) = \frac{1}{t}\sum_{j=0}^{t-1}\zeta_t^{-rj}\sum_{m=-\infty}^{\infty}\zeta_t^{mj}\overline{N}(m,n) = \frac{1}{t}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)\sum_{j=0}^{t-1}\zeta_t^{(m-r)j}.$$

Equation (2.7) follows, since the inner sum is t if $m \equiv r \pmod{t}$, and is 0 otherwise. ∎

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. The proofs of the seven cases are similar, so we give only a few as examples. For the sake of brevity, we do not write out an explicit theta function for each $\overline{T}_{a,M}(q)$. However, we point out that our arguments are effective in that we can easily keep track of the various summands of quotients of theta functions that arise from using Lemma 2.1.

Let $\zeta_M = e^{2\pi i/M}$. Substituting (2.7) into (1.3), we find that

$$\overline{D}(a,M) = \frac{1}{M}\sum_{j=1}^{M-1}\zeta_M^{-aj}\overline{R}(\zeta_M^j;q). \tag{2.9}$$

Applying the following identity [14, p. 251]:

$$\overline{R}(\zeta;q) = (1-\zeta)(1-\zeta^{-1})h(\zeta,q)$$

https://doi.org/10.4153/S0008439520000752 Published online by Cambridge University Press

to (2.9), we get

$$\overline{D}(a,M)=\frac{1}{M}\sum_{j=1}^{M-1}\zeta_M^{-aj}(1-\zeta_M^j)(1-\zeta_M^{-j})h(\zeta_M^j,q).$$

By (2.5), we have

(2.10)

$$\begin{aligned}\overline{D}(a,M)&=-\frac{1}{M}\sum_{j=1}^{M-1}\zeta_M^{-(a+1)j}\left(2-\zeta_M^j-\zeta_M^{-j}\right)m(\zeta_M^{-2j}q,q^2,-1)\\&\quad+\frac{1}{M}\sum_{j=1}^{M-1}\zeta_M^{-aj}\left(2-\zeta_M^j-\zeta_M^{-j}\right)\Delta(\zeta_M^j;q)\\&=-\frac{2}{M}\sum_{j=0}^{M-1}\zeta_M^{-(a+1)j}m(\zeta_M^{-2j}q,q^2,-1)+\frac{1}{M}\sum_{j=0}^{M-1}\zeta_M^{-aj}m(\zeta_M^{-2j}q,q^2,-1)\\&\quad+\frac{1}{M}\sum_{j=0}^{M-1}\zeta_M^{-(a+2)j}m(\zeta_M^{-2j}q,q^2,-1)+\frac{1}{M}\sum_{j=1}^{M-1}\zeta_M^{-aj}\left(2-\zeta_M^j-\zeta_M^{-j}\right)\Delta(\zeta_M^j;q).\end{aligned}$$

We first consider the case $a=0$, $M\equiv 0\pmod 2$ and recall that $\overline{T}_{a,m}(q)$ will depend on an arbitrary z. We write $j=\frac{M}{2}t+r$, where $0\le t\le 1$ and $0\le r\le\frac{M}{2}-1$. The first summation in (2.10) is then

$$\begin{aligned}-\frac{2}{M}\sum_{j=0}^{M-1}\zeta_M^{-j}m(\zeta_M^{-2j}q,q^2,-1)&=-\frac{2}{M}\sum_{t=0}^{1}\sum_{r=0}^{(M/2)-1}\zeta_M^{-\frac{M}{2}t-r}m\left(q\zeta_{\frac{M}{2}}^{-\frac{M}{2}t-r},q^2,-1\right)\\&=-\frac{2}{M}\sum_{t=0}^{1}\zeta_2^{-t}\sum_{r=0}^{(M/2)-1}\zeta_M^{-r}m\left(q\zeta_{\frac{M}{2}}^{-\frac{M}{2}t-r},q^2,-1\right)\\&=0.\end{aligned}$$

For the second summation,

$$\begin{aligned}\frac{1}{M}\sum_{j=0}^{M-1}m(\zeta_M^{-2j}q,q^2,-1)&=\frac{1}{M}\sum_{t=0}^{1}\sum_{r=0}^{(M/2)-1}m\left(q\zeta_{\frac{M}{2}}^{-\frac{M}{2}t-r},q^2,-1\right)\\&=\frac{2}{M}\sum_{r=0}^{(M/2)-1}m\left(q\zeta_{\frac{M}{2}}^{-r},q^2,-1\right).\end{aligned}\tag{2.11}$$

Substituting k, n, z, x and q by 0, $M/2$, -1, q and q^2, respectively, in Lemma 2.1, we find that

$$\sum_{r=0}^{(M/2)-1}m\left(q\zeta_{\frac{M}{2}}^{-r},q^2,-1\right)=\frac{M}{2}m\left((-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}},q^{\frac{M^2}{2}},z'\right)\tag{2.12}$$

$$+\frac{M}{2}\Psi_0^{\frac{M}{2}}(q,-1,z';q^2).\tag{2.13}$$

https://doi.org/10.4153/S0008439520000752 Published online by Cambridge University Press

Combining (2.12) and (2.11), we are led to

$$\frac{1}{M}\sum_{j=0}^{M-1} m(\zeta_M^{-2j}q, q^2, -1) = m\Big((-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}, q^{\frac{M^2}{2}}, z'\Big) + \Psi_0^{\frac{M}{2}}(q, -1, z'; q^2).$$

Similarly, the third summation can be rewritten as

$$\frac{1}{M}\sum_{j=0}^{M-1} \zeta_M^{-2j} m(\zeta_M^{-2j}q, q^2, -1) = \frac{1}{M}\sum_{t=0}^{1}\sum_{r=0}^{(M/2)-1} \zeta_{\frac{M}{2}}^{-(\frac{M}{2}t+r)} m\Big(\zeta_{\frac{M}{2}}^{-(\frac{M}{2}t+r)}, q^2, -1\Big)$$
$$= \frac{2}{M}\sum_{r=0}^{(M/2)-1} \zeta_{\frac{M}{2}}^{-r} m\Big(\zeta_{\frac{M}{2}}^{-r}, q^2, -1\Big).$$

Substituting k, n, z, x, and q by $(M/2)-1$, $M/2$, -1, q, and q^2, respectively, in Lemma 2.1, we find that

$$\frac{2}{M}\sum_{r=0}^{(M/2)-1} \zeta_{\frac{M}{2}}^{-r} m\Big(\zeta_{\frac{M}{2}}^{-r}, q^2, -1\Big) = (-1)^{\frac{M}{2}-1}q^{-\frac{(M-2)^2}{4}} m\Big((-1)^{\frac{M}{2}+1}q^{-\frac{M^2}{4}+M}, q^{\frac{M^2}{2}}, z'\Big)$$
$$+ \Psi_{\frac{M}{2}-1}^{\frac{M}{2}}(q, -1, z'; q^2).$$

So for $a = 0$ and $M \equiv 0 \pmod 2$,

$$\overline{D}(0, M) = m\Big((-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}, q^{\frac{M^2}{2}}, z'\Big) + (-1)^{\frac{M}{2}-1}q^{-\frac{(M-2)^2}{4}} m\Big((-1)^{\frac{M}{2}+1}q^{-\frac{M^2}{4}+M}, q^{\frac{M^2}{2}}, z'\Big)$$
$$+ \Psi_0^{\frac{M}{2}}(q, -1, z'; q^2) + \Psi_{\frac{M}{2}-1}^{\frac{M}{2}}(q, -1, z'; q^2) + \frac{1}{M}\sum_{j=1}^{M-1}\Big(2 - \zeta_M^{j} - \zeta_M^{-j}\Big)\Delta(\zeta_M^{j}; q).$$

Set up (2.4) instead of (2.5) in the above equation, and we have

$$\overline{D}(0, M) = h\Big(-1, (-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\Big) + (-1)^{\frac{M}{2}-1}q^{\frac{M}{2}-1}h\Big(-q^{\frac{M^2}{4}-\frac{M}{2}}, (-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\Big)$$
$$+ \Psi_0^{\frac{M}{2}}\Big(q, -1, -1; q^2\Big) + \Psi_{\frac{M}{2}-1}^{\frac{M}{2}}\Big(q, -1, -q^{\frac{M^2}{4}-\frac{M}{2}}; q^2\Big)$$
$$+ \frac{1}{M}\sum_{j=1}^{M-1}\Big(2 - \zeta_M^{j} - \zeta_M^{-j}\Big)\Delta(\zeta_M^{j}; q).$$

Then we consider the case $a \equiv M \equiv 0 \pmod 2$ and $a \neq 0$. We write $j = tM/2 + r$, where $0 \le t \le 1$ and $0 \le r \le M/2 - 1$. The first summation in (2.10) is then

$$-\frac{2}{M}\sum_{j=0}^{M-1} \zeta_M^{-(a+1)j} m(\zeta_M^{-2j}q, q^2, -1)$$
$$= -\frac{2}{M}\sum_{t=0}^{1}\sum_{r=0}^{(M/2)-1} \zeta_M^{-(a+1)(\frac{M}{2}t+r)} m\Big(q\zeta_{\frac{M}{2}}^{-\frac{M}{2}t-r}, q^2, -1\Big)$$
$$= -\frac{2}{M}\sum_{t=0}^{1}\zeta_2^{-(a+1)t}\sum_{r=0}^{(M/2)-1} \zeta_M^{-(a+1)r} m\Big(q\zeta_{\frac{M}{2}}^{-\frac{M}{2}t-r}, q^2, -1\Big)$$
$$= 0.$$

https://doi.org/10.4153/S0008439520000752 Published online by Cambridge University Press

The second summation in (2.10) is then

$$\frac{1}{M}\sum_{j=0}^{M-1}\zeta_M^{-aj}m(\zeta_M^{-2j}q,q^2,-1)=\frac{2}{M}\sum_{r=0}^{(M/2)-1}\zeta_{\frac{M}{2}}^{-\frac{a}{2}j}m\Big(q\zeta_{\frac{M}{2}}^{-r},q^2,-1\Big)$$
$$=(-1)^{\frac{M}{2}-\frac{a}{2}}q^{-\frac{(M-a)^2}{4}}m\Big((-1)^{\frac{M}{2}+1}q^{-\frac{M^2}{4}+\frac{aM}{2}},q^{\frac{M^2}{2}},z'\Big)$$
$$+\Psi^{\frac{M}{2}}_{\frac{M}{2}-\frac{a}{2}}(q,-1,z';q^2),$$

where the second equality follows from Lemma 2.1 with $k=(M-a)/2$, $n=M/2$, $z=-1$, $x=q$, and $q=q^2$.

The third summation in (2.10) is then

$$\frac{1}{M}\sum_{j=0}^{M-1}\zeta_M^{-(a+2)j}m(\zeta_M^{-2j}q,q^2,-1)$$
$$=\frac{2}{M}\sum_{r=0}^{(M/2)-1}\zeta_{\frac{M}{2}}^{-j(\frac{a}{2}+1)}m\Big(q\zeta_{\frac{M}{2}}^{-r},q^2,-1\Big)$$
$$=(-1)^{\frac{M}{2}-\frac{a}{2}-1}q^{-\frac{(M-a-2)^2}{4}}m\Big((-1)^{\frac{M}{2}+1}q^{-\frac{M^2}{4}+\frac{aM}{2}+M},q^{\frac{M^2}{2}},z'\Big)$$
$$+\Psi^{\frac{M}{2}}_{\frac{M}{2}-\frac{a}{2}-1}(q,-1,z';q^2),$$

where the second equality follows from Lemma 2.1 with $k=(M-a-2)/2$, $n=M/2$, $z=-1$, $x=q$, and $q=q^2$.

So for $a\equiv M\equiv 0 \pmod 2$ with $a\neq 0$,

$$\overline{D}(a,M)=(-1)^{\frac{M}{2}-\frac{a}{2}+1}q^{\frac{aM}{4}-\frac{a^2}{4}}h\Big(q^{\frac{M^2}{4}-\frac{aM}{4}},(-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\Big)$$
$$+(-1)^{\frac{M}{2}-\frac{a}{2}}q^{\frac{aM}{4}+\frac{M}{2}-\frac{(a+2)^2}{4}}h\Big(q^{\frac{M^2}{4}-\frac{aM}{4}-\frac{M}{2}},(-1)^{\frac{M}{2}+1}q^{\frac{M^2}{4}}\Big)$$
$$+\Psi^{\frac{M}{2}}_{\frac{M}{2}-\frac{a}{2}}\Big(q,-1,q^{\frac{M^2}{4}-\frac{aM}{4}};q^2\Big)+\Psi^{\frac{M}{2}}_{\frac{M}{2}-\frac{a}{2}-1}\Big(q,-1,q^{\frac{M^2}{4}-\frac{aM}{4}-\frac{M}{2}};q^2\Big)$$
$$+\frac{1}{M}\sum_{j=1}^{M-1}\zeta_M^{-aj}(2-\zeta_M^{j}-\zeta_M^{-j})\Delta(\zeta_M^{j};q).$$

This completes the proof. ∎

3 On Generalisations of Dyson's Rank Differences for Overpartitions

In this section, we will reprove the result of Bringmann and Lovejoy in [5]. They put identities involving rank differences for overpartitions in the framework of weak Maass forms.

https://doi.org/10.4153/S0008439520000752 Published online by Cambridge University Press

Theorem 3.1 ([5, Theorem 1.6]) *Suppose that* $\ell \geq 3$ *is prime,* $0 \leq s_1, s_2 < \ell$, *and* $0 \leq d < \ell$. *If* $\left(\frac{d}{\ell}\right) = -\left(\frac{-1}{\ell}\right)$, *then*

$$\sum_{n=0}^{\infty} \left(\overline{N}(s_1, \ell, \ell n + d) - \overline{N}(s_2, \ell, \ell n + d)\right) q^{\ell n + d}$$

is a weakly holomorphic modular form on $\Gamma_1(16\ell^4)$.

We find that for a given summand $q^m h(q^{Mk}, q^{M^2})$ in Theorem 1.1, if $m \equiv d \pmod{M}$, it can be determined that $(a+b)^2 \equiv -4d \pmod{M}$ where $b = 0, 1, 2$. For example, taking the case $M \neq 1$ and $a \equiv M \equiv 1 \pmod{2}$ in Theorem 1.1,

$$\begin{aligned}
\overline{D}(a, M) &= 2(-1)^{M - \frac{a-1}{2}} q^{\left(\frac{a+1}{2}\right)\left(M - \frac{a+1}{2}\right)} h\left(-q^{M\left(M - \frac{a+1}{2}\right)}, q^{M^2}\right) \\
&\quad + (-1)^{\frac{M-a}{2}+1} q^{\frac{M^2-a^2}{4}} h\left(q^{M\left(\frac{M-a}{2}\right)}, q^{M^2}\right) \\
&\quad + (-1)^{\frac{M-a}{2}+1} q^{\frac{M^2-(a+2)^2}{4}} h\left(-q^{M\left(\frac{M-a}{2}-1\right)}, q^{M^2}\right) + \overline{T}_{a,M}(q),
\end{aligned} \tag{3.1}$$

For the first summand in (3.1), the condition

$$\left(\frac{a+1}{2}\right)\left(M - \frac{a+1}{2}\right) \equiv d \pmod{M}$$

implies

$$(a+1)^2 \equiv -4d \pmod{M}.$$

Combining the properties of Legendre symbol and the condition $\left(\frac{d}{M}\right) = -\left(\frac{-1}{M}\right)$, we have

$$\left(\frac{-4d}{M}\right) = \left(\frac{-1}{M}\right)\left(\frac{2}{M}\right)^2\left(\frac{d}{M}\right) = -\left(\frac{-1}{M}\right)^2\left(\frac{2}{M}\right)^2 = -1.$$

If $\left(\frac{-4d}{M}\right) = -1$, we will never have an $m \equiv d \pmod{M}$, so the certain rank difference is a weakly holomorphic modular form, which implies Theorem 3.1.

References

[1] G. E. Andrews and F. G. Garvan, *Ramanujan's "lost" notebook VI: The mock theta conjectures.* Adv. Math. **73**(1989), 245–255. http://dx.doi.org/10.1016/0001-8708(89)90070-4

[2] A. O. L. Atkin and P. Swinnerton-Dyer, *Some properties of partitions.* Proc. Lond. Math. Soc. **4**(1954), 84–106. http://dx.doi.org/10.1112/plms/s3-4.1.84

[3] K. Bringmann and K. Ono, *The* $f(q)$ *mock theta function conjecture and partition ranks.* Invent. Math. **165**(2006), 243–266. http://dx.doi.org/10.1007/s00222-005-0493-5

[4] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms.* Ann. of Math. (2) **171**(2010), 419–449. http://dx.doi.org/10.4007/annals.2010.171.419

[5] K. Bringmann and J. Lovejoy, *Dyson's rank, overpartitions, and weak Maass forms.* Int. Math. Res. Not. **2007**(2007), no. 19, rnm063. http://dx.doi.org/10.1093/imrn/rnm063

[6] Y.-S. Choi, *Tenth order mock theta functions in Ramanujan's lost notebook.* Invent. Math. **136**(1999), 497–569. http://dx.doi.org/10.1007/s002220050318

[7] S. Corteel and J. Lovejoy, *Overpartitions.* Trans. Amer. Math. Soc. **356**(2004) 1623–1635. http://dx.doi.org/10.1090/S0002-9947-03-03328-2

[8] F. J. Dyson, *Some guesses in the theory of partitions.* Eureka **8**(1944), 10–15.

[9] B. Gordon and R. J. McIntosh, *A survey of classical mock theta functions*. In: Partitions, q-series, and modular forms, Dev. Math. 23, Springer, New York, 2012, pp. 95–144. http://dx.doi.org/10.1007/978-1-4614-0028-8_9
[10] D. Hickerson, *A proof of the mock theta conjectures*. Invent. Math. **94**(1988), 639–660. http://dx.doi.org/10.1007/BF01394279
[11] D. Hickerson, *On the seventh order mock theta functions.* Invent. Math. **94**(1988), 661–677. http://dx.doi.org/10.1007/BF01394280
[12] D. Hickerson and E. Mortenson, *Hecke-type double sums, Appell-Lerch sums, and mock theta functions I*. Proc. Lond. Math. Soc. (3) **109**(2014), 382–422. http://dx.doi.org/10.1112/plms/pdu007
[13] D. Hickerson and E. Mortenson, *Dyson's ranks and Appell-Lerch sums*. Math. Ann. **367**(2017), 373–395. http://dx.doi.org/10.1007/s00208-016-1390-5
[14] C. Jennings-Shaffer, *The generating function of theM_2-rank of partitions without repeated odd parts as a mock modular form.* Trans. Amer. Math. Soc. **371**(2019), 249–277. http://dx.doi.org/10.1090/tran/7212
[15] J. Lovejoy, *Rank and conjugation for the Frobenius representation of an overpartition.* Ann. Comb. **9**(2005), 321–334. http://dx.doi.org/10.1007/s00026-005-0260-8
[16] J. Lovejoy and R. Osburn, *Rank differences for overpartitions.* Q. J. Math. **59**(2008), 257–273. http://dx.doi.org/10.1093/qmath/ham031
[17] B. Wei and H. W. J. Zhang, *Rank differences for overpartitions modulo* 6. Proc. Amer. Math. Soc. **148**(2020), no. 10, 4333–4349. http://dx.doi.org/10.1090/proc/15075
[18] D. Zagier, *Ramanujan's mock theta functions and their applications. Séminaire Bourbaki*. Vol. 2007/2008. Astérisque No. 326 (2009), Exp. No. 986, vii–viii, 143–164 (2010).

School of Mathematics, Hunan University, Changsha 410082, People's Republic of China
e-mail: helenzhang@hnu.edu.cn