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The onset of superconductivity at a superconducting/normal interface

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We analyze the onset of superconductivity, in a type II superconductor adjacent to a normal material, via a generalized Ginzburg–Landau energy functional, which models the effects of superconducting electron pairs diffusing into the normal part. We consider a superconductor and a normal material, each filling a half-space, in the presence of a constant magnetic field parallel to their interface. Among other results, we show that if the normal state conductivity of the superconductor is less than or equal to the conductivity of the normal material, then normal states are the only global minimizers down to the second critical field H_{c_2}. Hence, we analytically confirm experimental predictions that surface superconductivity may be suppressed by coating a superconductor with a normal metal.

1 Introduction

Experimental [5, 6] and theoretical [13, 10] investigations presented in the physics literature argue that when a superconductor is in contact with a normal metal the surface superconducting sheath, usually observed in type II superconductors, may be absent. The determining factor appears to be the relation between the normal state conductivities of the two materials; with the suppression happening if the normal state conductivity of the superconductor, σ_s, is less than or equal to the conductivity of the normal metal, σ_n.

For a type II superconductor placed in a vacuum one can define three critical fields, commonly denoted by H_{c1}, H_{c2} and H_{c3}, with $H_{c1} < H_{c2} < H_{c3}$. Above H_{c3}, the sample is in the so-called normal state, in which no superconductivity is present; as the applied field is decreased superconductivity starts to nucleate at the boundary in the form of a superconducting sheath. The superconductor remains in a surface superconducting state down to H_{c2}, where superconductivity starts appearing in the bulk of the material. Below H_{c1} the material is almost a perfect superconductor [9].

The papers mentioned above suggest that when the superconductor is adjacent to a metal, the nucleation field (which we will denote by $H^n_{c_3}$ to distinguish it from the one in a vacuum) depends on the specific materials used, with $H^n_{c_3} = H^n_{c_3}(\sigma_s/\sigma_n)$. Coating the sample with a normal metal for which $\sigma_s \leqslant \sigma_n$ should block surface nucleation, and bring the upper critical field down to H_{c_2}, that is $H^n_{c_3} = H_{c_2}$. While if

$\sigma_s > \sigma_n$, then $H_{c_2} < H^n_{c_3}(\sigma_s/\sigma_n) < H_{c_3}$ and the classical surface superconductivity should be observed; with $H^n_{c_3}(\sigma_s/\sigma_n) \to H_{c_2}$ when $\sigma_s/\sigma_n \to 1^+$, and $H^n_{c_3}(\sigma_s/\sigma_n) \to H_{c_3}$ for $\sigma_s/\sigma_n \to \infty$.

We are interested in rigorously analysing the phenomenon in the setting of the Ginzburg–Landau theory. Classically, the presence of a normal material is modelled via the so-called de Gennes boundary condition: a Robin-type boundary condition depending on the so-called extrapolation length parameter, $d > 0$. This boundary condition maintains the gauge invariance of the model and the physical intuition that the supercurrent flows tangentially at the boundary of the superconductor (i.e. with zero normal component) [9]. The limiting case, $d = \infty$ corresponds to the Neumann boundary condition and represents the case of a vacuum or insulator. The other limiting case $d = 0$ corresponds to the Dirichlet boundary condition, that is to the physical assumption that the density of the superelectrons is zero at the boundary of the superconductor. Therefore, the Dirichlet boundary condition gives zero supercurrent at the boundary rather than just zero normal component, this might have unexpected mathematical consequences as illustrated, for example, by the very interesting Theorem 4 in Lu & Pan [14]. Motivated by the work of Chapman *et al.* [8] and Hurault [13] (see also Giorgi [11]), and the results of Chapman [7], rather than using the de Gennes boundary condition, we start from a physically-sound generalization of the Ginzburg–Landau free energy density, which models the effects of superconducting electron pairs diffusing into the normal material. Present in the model, see Section 2, there are two parameters m_n and m_s whose ratio m_n/m_s is equal to σ_s/σ_n.

Following the physics literature (see for example [16]), to study the onset of superconductivity we consider a semi-infinite superconductor filling the half-space $x < 0$ in contact with a normal material, which in turn fills the half-space $x > 0$, and a constant applied magnetic field parallel to the surface, i.e. parallel to the z axis. This situation can be analyzed using a one-dimensional version of the generalized Ginzburg–Landau energy functional [13, 10].

Nucleation of superconductivity for the classical one-dimensional Ginzburg–Landau energy has been studied rigorously in the fundamental works of Bolley & Helffer [4, 3] for a superconductor in a vacuum, i.e $d = \infty$, and formally by Chapman [7] for the more general case of the de Gennes boundary condition. Bolley and Helffer, among other results, obtain a precise picture of the stability of normal states, and estimates on the value of H_{c_3}.

Chapman [7] shows that the upper critical field $H^n_{c_3}$ indeed depends on d, that is the presence of a normal material influences its value. According to Chapman surface nucleation occurs for all $d > 0$, with the upper critical field increasing and approaching H_{c3} as d increases, and decreasing to H_{c_2} as d goes to zero. Thus, the de Gennes boundary condition does not account for the situation $\sigma_s < \sigma_n$, and the parameter d, appears to be in the correspondence $d \to 0^+$, $\sigma_s/\sigma_n \to 1$ and $d \to \infty$, $\sigma_s/\sigma_n \to \infty$. Deutscher & de Gennes [10, pg 1028-1029] note that the influence of the normal material in high fields can be modelled by this boundary condition if the extrapolation length is field-independent, an assumption that can be made “only if the normal-state conductivity of S is much larger than that of N”. The model we use is based on the one presented by Hurault [13], which includes the effects of the applied magnetic field on the superconducting order parameter on the normal part, and should be more appropriate to describe the situation $\sigma_s < \sigma_n$, as mentioned by Deutscher & de Gennes [10, pg 1031].

We recover the experimental and theoretical results presented by others [5, 6, 13, 10]. We have rigorous results analogous to the picture described by Chapman for type II superconductors and normal materials which satisfy $m_n/m_s > 1$, see Theorems 3.3 and 5.1, and Lemma 5.6. For $m_n/m_s \leqslant 1$, we obtain that normal states are the only global minimizers down to H_{c_2}, see Theorem 3.3, and that for fields close to but less than H_{c_2}, there are normal states which are not local minimizers (Theorem 5.1) i.e. $H^n_{c_3} \equiv H_{c_2}$. To connect our results to the situation of a vacuum, we prove that as $m_n/m_s \to \infty$ any normal state is a local minimizer down to H_{c_3}, see Theorem 5.7. Finally, Lemma 5.6 and Theorem 5.1 imply the other limiting behaviour i.e. $H^n_{c_3} \to H_{c_2}$ for $m_n/m_s \to 1^+$.

The paper is organized as follows. In § 2, we introduce the energy in a non-dimensional form, and present the one-dimensional set-up. In § 3, we show that in the range of fields of interest the energy is bounded below, and we look at existence of global minimizers (Theorem 3.1 and Theorem 3.3). In § 4, we study under which condition a fixed normal state is a local minimizer. In particular, this leads to the analysis of an eigenvalue problem (Remark 4.4 and Theorem 4.2). In § 5 we show the existence of a transition field above which every normal state is a local minimizer, and below which some normal states are not local minimizers. We prove that this transition field is exactly H_{c2} if $m_n/m_s \leqslant 1$ (Theorem 5.1, and Remarks 5.2 and 5.8). To conclude, we study the behaviour of normal state solutions when $m_n/m_s \to \infty$ (Theorem 5.7).

2 Energy derivation

The Ginzburg–Landau theory describes the state of a superconducting body via a free energy density involving a complex valued order parameter ψ, defined in the superconducting body and a vector field $\mathbf{A}$, the magnetic potential, defined in the whole space. To model the normal part together with the superconductor, we consider the following non-dimensionalized Ginzburg–Landau free energy density [8, 13, 11]:

$$\begin{aligned}\mathscr{F}(\psi, \mathbf{A}, \mathbf{H_a}) := \frac{|a_s|^2}{b_s}\{\mathscr{F}_s - |\psi|^2 + \frac{1}{2}|\psi|^4 + |\,(i\nabla + \mathbf{A})\ \psi\,|^2 \\ + \kappa^2|\mathbf{H}|^2 - 2\kappa^2\,\mathbf{H}\cdot\mathbf{H_a}\}\,\chi_{\mathscr{D}_s} + \frac{|a_s|^2}{b_s}\{\mathscr{F}_n + \frac{a_n}{|a_s|}|\psi|^2 \\ + \frac{m_s}{m_n}\,|\,(i\nabla + \mathbf{A})\ \psi\,|^2 + \frac{\mu_n}{\mu_s}\kappa^2|\mathbf{H}|^2 - 2\,\frac{\mu_n}{\mu_s}\,\kappa^2\,\mathbf{H}\cdot\mathbf{H_a}\}\,\chi_{\mathscr{D}_n},\end{aligned} \tag{2.1}$$

where $\chi_{\mathscr{D}_s}$ and $\chi_{\mathscr{D}_n}$ denote the characteristic functions for the superconducting region $\mathscr{D}_s$ and the normal region $\mathscr{D}_n$ respectively.

For fixed temperature T, we have $a_s \equiv a_s(T) = a_s(0)\left(1 - T/T_{cs}\right)$ and $a_n \equiv a_n(T) = a_n(0)\left(1 - T/T_{cn}\right)$, where $a_s(0) < 0, a_n(0) < 0$ are constants, and T_{cs}, T_{cn} are the zero-field critical temperatures of the superconductor and the normal material, respectively. We consider materials for which $T_{cn} < T_{cs}$ and we are interested in the range of temperatures $T_{cn} < T < T_{cs}$, so that $a_s < 0$ and $a_n > 0$. The positive constants m_s, m_n depend on the physical parameters of the superconducting and normal materials, respectively; and their ratio, m_n/m_s, is proportional to the ratio, σ_s/σ_n, of the normal conductivity of the superconducting region to the conductivity of the normal region. The constant $b_s > 0$ is characteristic of the superconducting body, and $\mu_s, \mu_n > 0$ denote the permeability densities of the superconducting and the normal part, respectively. The material constant κ is the Ginzburg–Landau parameter of the superconducting material, $\mathbf{H_a}$ is the external applied magnetic field and $\mathbf{H} = \frac{1}{\mu}\mathrm{curl}\mathbf{A}$ is the induced magnetic field. We define the piecewise function μ as $\mu = 1$ in $\mathscr{D}_s$, and $\mu = \mu_n/\mu_s$ in $\mathscr{D}_n$. If we introduce the piecewise function m defined as $m = 1$ in $\mathscr{D}_s$, and $m = m_n/m_s$ in $\mathscr{D}_n$, we have that $\mathbf{j} := \frac{1}{m}\left(-\frac{i}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \mathbf{A}|\psi|^2\right)$ is the supercurrent density.

Remark 2.1 The non-dimensionalization used to derive (2.1) is temperature-dependent. Lengths are non-dimensionalized with respect to the coherence length at temperature T of the superconducting material, $\xi_s(T)$. The order parameter is non-dimensionalized with respect to the ratio $|a_s(T)|/b_s$, and magnetic fields with respect to the second critical field of the superconducting material, $H_{c2} = H_{c2}(T)$. We are then interested in fields $h \geqslant h_{c2} \equiv 1$. In particular, by comparing (2.1) with equation (4) in Hurault [13] we see that $m_n/m_s = \sigma_s/\sigma_n$, and we expect the normal state to become unstable at a field $h^n_{c_3}(m_n/m_s) > 1$ when $m_n/m_s > 1$, and to remain stable down to $h_{c2} \equiv 1$ when $m_n/m_s \leqslant 1$. From previous works (see for example [4, 7, 2, 15]), we know that the nucleation field in a vacuum satisfies $H_{c_3} = H_{c_2}/\beta_0^*$, with β_0^* defined as in Lemma 5.2 in [14]. Therefore, we also expect $h^n_{c_3}(m_n/m_s) < h_{c_3} \equiv 1/\beta_0^*$ for every m_n/m_s, with $h_{c_3}(m_n/m_s) \to 1/\beta_0^*$ for $m_n/m_s \to \infty$ and $h^n_{c_3}(m_n/m_s)$ increasing in $m_n/m_s > 1$.

To investigate the influence of the normal part, we take $\mathscr{D}_s = \{(x,y,z), x < 0\}$ and $\mathscr{D}_n = \{(x,y,z), x > 0\}$, and assume the applied field is constant and parallel to the surface of the superconductor, i.e. $\mathbf{H_a} = h\mathbf{e}_3$ with $h > 0$. In this situation, one expects that for fields close to the nucleation field the induced magnetic field will be parallel to the applied field, and the modulus of the order parameter will depend only on x [16]. Hence, one can make the simplified assumption of considering only states of the form:

$$\mathbf{A}(x,y,z) = (0, A(x), 0) \qquad \psi(x,y,z) = f(x)\, e^{i k_2 y}\, e^{i k_3 z}, \tag{2.2}$$

with $f(x)$ real-valued, and k_2, k_3 constants to be determined.

In this set-up the physically relevant quantities can be recovered as follows: $|f|^2$ is the superelectron density, $-1/m\,(A - k_2)\, f^2$ is the supercurrent and $1/\mu(0,0,A')$ is the induced magnetic field. Note that $A - k_2$ is the physically significant quantity, while individually A and k_2 are determined up to a constant. We will take the point of view of [4], and use this degree of freedom in order to have a real order parameter, that is we will consider $k_2 = 0$. Therefore, if we substitute (2.2) in (2.1), set $a = a_n/|a_s| \geqslant 0$ and notice that for f and A fixed the choice $k_3 = 0$ lowers the energy, we are led to the energy

$$\begin{aligned}\mathscr{G}(f,A,h) = \int_{-\infty}^{\infty} [\frac{1}{m}(f')^2 - f^2\chi_- + \frac{1}{2}f^4\chi_- + a f^2\chi_+ \\ + \frac{1}{m}A^2 f^2 + \mu\kappa^2\left(\frac{1}{\mu}A' - h\right)^2]\, dx.\end{aligned} \tag{2.3}$$

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Here, χ_- denotes the characteristic function of the interval $(-\infty, 0)$ and χ_+ the one of $(0, \infty)$.

States can be assumed of the form (2.2) if h is close to or greater than $h_{c_2} \equiv 1$. As a consequence, we should not expect to be able to use (2.3) for the all possible ranges of h. However, for the range of fields in which we are interested, we expect our system to be in a state (f, A) which is a finite-energy global or local minimizer of (2.3). Moreover, since we look at situations where the order parameter decays to zero at infinity, we will consider $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$.

We define a *normal (non-superconducting) state* to be any state of the form $(0, A)$ which is a global minimizer of (2.3). It is easy to see that normal states are pairs of form $(0, h(\mu x - \omega))$ for $\omega \in \mathscr{R}$ fixed.

3 Global minimizers

The energy $\mathscr{G}(f, A, h)$ is not bounded below for every choice of h, even if $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$, as one can see by taking $h = 0$ and following an idea presented in Bolley & Helffer [4]. Consider the sequence $(f_n, 0)$ with

$$f_n(x) = \begin{cases} c(x+n+1) & -n-1 < x < -n, \\ c & -n < x < 0, \\ -c(x-1) & 0 \leqslant x < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

then $\mathscr{G}(f_n, 0, 0) = c^2\left(1 + m_s/m_n\right) - c^2 n - c^2/3 + n c^4/2 + c^4/5 + a c^2/3$, and for any choice of $|c| < 1$ it follows $\lim_{n\to\infty} \mathscr{G}(f_n, 0) = -\infty$. However, we can use a modification of the argument of Bolley & Helffer [4, Proposition 14, p. 22] to show that our energy is bounded below if h is large enough.

Theorem 3.1 *There exists h_0, with $\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{\mu_s}{\mu_s+\mu_n}} < h_0 < \frac{4}{\kappa}$, such that for every $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$, the energy functional $\mathscr{G}(f, A, h)$ is not bounded below for $h < h_0$, and is bounded below for $h > h_0$,*

Proof The function $h \in [0, \infty) \longrightarrow \inf_{(f,A)} \mathscr{G}(f, A, h)$, where the inf is taken over $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$, is monotonically increasing with values in $[-\infty, 0]$. In fact, we already know that the functional is unbounded below, while $\inf_{(f,A)} \mathscr{G}(f, A, h) \leqslant \mathscr{G}(0, h(\mu x - \omega)) = 0$. For h fixed, given any $A \in H^1_{loc}(\mathscr{R})$ we can find a $B \in H^1_{loc}(\mathscr{R})$ with $A = hB$ and vice versa, thus we have that $\inf_{(f,A)} \mathscr{G}(f, A, h) = \inf_{(f,B)} \mathscr{G}(f, hB, h)$, and clearly the right hand side is a monotonically increasing function of h.

Fix $h \neq 0$, consider (f_n, A_n) where f_n is as in (3.1) with $c = 1$, and

$$A_n(x) = \begin{cases} h(x+n+1) & x < -n-1, \\ \frac{\mu_n}{\mu_s} h(x-n-1) & x > n+1, \\ 0 & \text{otherwise}. \end{cases} \tag{3.2}$$

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A direct computation then yields

$$\mathcal{G}(f_n, A_n, h) = \left(1 + \frac{m_s}{m_n}\right) - n - \frac{1}{3} + \frac{1}{2}n + \frac{1}{5} + \frac{a}{3} + \kappa^2 (1 + \frac{\mu_n}{\mu_s})h^2 (n+1),$$

hence if $h < \frac{1}{\sqrt{2}\kappa}\sqrt{\frac{\mu_s}{\mu_s+\mu_n}}$, we see that $\lim_{n\to\infty} \mathcal{G}(f_n, A_n, h) = -\infty$.

Fix $\delta \geqslant 0$, for any $(f, A) \in H^1(\mathcal{R}) \times H^1_{loc}(\mathcal{R})$ either $|A(x)|^2 \geqslant 1+\delta$ for all $x \leqslant 0$ which gives $\mathcal{G}(f, A, h) \geqslant 0$ or there exists an $x_0 \leqslant 0$ such that $|A(x_0)|^2 < 1+\delta$. In the latter case, since $A \in H^1_{loc}(\mathcal{R})$, using the elementary inequality $(\alpha-\beta)^2 \geqslant \frac{1}{2}\alpha^2 - \beta^2$, we derive

$$\frac{1}{2}h^2 (x-x_0)^2 - 2(1+\delta) - 2|x-x_0| \int_{-\infty}^{0} |A'(t)-h|^2 \, dt \leqslant |A(x)|^2. \tag{3.3}$$

The left-hand side of (3.3) is greater than or equal to $(1+\delta)$ if $|x-x_0| > \gamma_2$, where

$$\gamma_2 = \frac{2}{h^2}\left[\int_{-\infty}^{0} |A'(t)-h|^2 \, dt + \sqrt{\left|\int_{-\infty}^{0} |A'(t)-h|^2 \, dt\right|^2 + 3(1+\delta)\frac{h^2}{2}}\right]. \tag{3.4}$$

Thus, we can conclude

$$\int_{\{x\leqslant 0\}\cap\{|f|<\sqrt{2}\}\cap\{|A(x)|^2<1+\delta\}} f^2 \leqslant \frac{16}{h^2}\int_{-\infty}^{0} |A'(t)-h|^2 \, dt + \frac{4\sqrt{6(1+\delta)}}{h}. \tag{3.5}$$

We find a lower bound for $\mathcal{G}(f, A, h)$ by noticing that if $\alpha \geqslant \sqrt{2}$ then $-\alpha^2 + \frac{1}{2}\alpha^4 \geqslant 0$, so that

$$\begin{aligned}\mathcal{G}(f, A, h) \geqslant &\int_{-\infty}^{\infty}\left[\frac{1}{m}(f')^2 + a f^2 \chi_+ + \mu\kappa^2\left(\frac{1}{\mu}A' - h\right)^2\right] dx \\ &+ \int_0^{\infty} \frac{1}{m}A^2 f^2 \, dx + \int_{\{x\leqslant 0:|f|>\sqrt{2}\}}\left(-f^2 + \frac{1}{2}f^4\right) dx \\ &+ \int_{\{x\leqslant 0:|f|<\sqrt{2}\}} \frac{1}{2}f^4 \, dx + \int_{\{x\leqslant 0:|f|>\sqrt{2}\}} A^2 f^2 \, dx \\ &+ \int_{A_\delta^+\cap\{|f|<\sqrt{2}\}} (-f^2 + A^2 f^2) \, dx + \int_{A_\delta^-\cap\{|f|<\sqrt{2}\}} A^2 f^2 \, dx \\ &- \frac{16}{h^2}\int_{-\infty}^{0} |A'(t)-h|^2 \, dt - \frac{4\sqrt{6(1+\delta)}}{h},\end{aligned} \tag{3.6}$$

where $A_\delta^+ = \{x \leqslant 0 : |A(x)|^2 > 1+\delta\}$ and $A_\delta^- = \{x \leqslant 0 : |A(x)|^2 < 1+\delta\}$. It follows that

$$\mathcal{G}(f, A, h) \geqslant \int_{-\infty}^{0}\left(\kappa^2 - \frac{16}{h^2}\right)|A'(x)-h|^2 \, dx - \frac{4\sqrt{6(1+\delta)}}{h}, \tag{3.7}$$

which if $h \geqslant 4/\kappa$ gives $\mathcal{G}(f, A, h) \geqslant -4\sqrt{6(1+\delta)}/h$, for any $\delta \geqslant 0$. □

Remark 3.2 In our non-dimensionalization magnetic fields are measured in units of H_{c_2}, Theorem 3.1 tells that there exists a value $H_0 \equiv h_0 H_{c_2}$ of the applied magnetic field,

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with $\frac{1}{\sqrt{2}\kappa}\sqrt{\frac{\mu_s}{\mu_s+\mu_n}}\,H_{c_2} < H_0 < \frac{4}{\kappa}\,H_{c_2}$, above which our energy functional is bounded below. Therefore, if $\kappa > 4$ we have that $H_0 < H_{c_2}$, and our energy is bounded below in the range of fields of interest.

In the literature $H_c = H_c(T)$ is defined as the field at which the energy of the normal state and the energy of the *purely superconducting state* (i.e. in non-dimensional units, the state $|\psi| \equiv 1$ and $\mathbf{A} \equiv \mathbf{0}$) are equal, with $H_{c_2} = \sqrt{2}\,\kappa H_c$. Since $H_0 > H_c\sqrt{\frac{\mu_s}{\mu_s+\mu_n}}$, our bound is consistent with this definition.

We will work with the weighted Sobolev space:

$$W^{1,2}_{0,0}(\mathscr{R}) = \left\{ u \in \mathscr{D}'(\mathscr{R});\ \frac{u}{\left(1+x^2\right)^{1/2}} \in L^2(\mathscr{R}); u' \in L^2(\mathscr{R}) \right\};$$

here $\mathscr{D}'(\mathscr{R})$ denotes the dual space of $C_0^\infty(\mathscr{R})$. (see [1] for details)

The space $W^{1,2}_{0,0}(\mathscr{R})$ is a reflexive Banach space with respect to the norm

$$||u||_{W^{1,2}_{0,0}(\mathscr{R})} \equiv \left\{ \left|\left| \frac{u}{\left(1+x^2\right)^{1/2}} \right|\right|^2_{L^2(\mathscr{R})} + ||u'||^2_{L^2(\mathscr{R})} \right\}^{\frac{1}{2}},$$

and a Hilbert space with respect to the induced scalar product. Also, by Theorem 330 in [12], one has for any $u \in H^1_{loc}(\mathscr{R})$ with $u' \in L^2(\mathscr{R})$ that

$$\int_{-\infty}^{\infty} \frac{|u(x)-u(0)|^2}{\left(1+x^2\right)}\,dx \leqslant 4\int_{-\infty}^{\infty} (u')^2\,dx. \tag{3.8}$$

Theorem 3.3 *Let $\kappa \geqslant \sqrt{2}/2$ be fixed. If $\frac{m_n}{m_s} \leqslant 1$ and $h > 1$ or if $\frac{m_n}{m_s} > 1$ and $h > \frac{m_n}{m_s}$ then normal states are the only global minimizers in the class of function $(f, A) \in H^1(\mathscr{R})\times H^1_{loc}(\mathscr{R})$, with $\frac{1}{\mu}\,(A(x) - A(0)) - hx \in W^{1,2}_{0,0}(\mathscr{R})$.*

Let $\kappa \geqslant 4$ be fixed. If $\frac{m_n}{m_s} > 1$ and $1 \leqslant h \leqslant \frac{m_n}{m_s}$, there exists a global minimizer $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$, with $\frac{1}{\mu}\,(A(x) - A(0)) - hx \in W^{1,2}_{0,0}(\mathscr{R})$.

Proof Fix $h \geqslant 1$ and consider $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$. Note that, if we set $\hat{f} = f$ when $|f| < \sqrt{2}$ and $\hat{f} = \sqrt{2}$ otherwise, then $\hat{f} \in H^1(\mathscr{R})$, and from

$$\int_{(-\infty,0)\,\cap\,\{|f|\,\geqslant\,\sqrt{2}\}} \left(-f^2 + \frac{1}{2}f^4\right)dx \geqslant 0,$$

one has $\mathscr{G}(f, A, h) \geqslant \mathscr{G}(\hat{f}, A, h)$. Therefore, without loss of generality we will assume $|f| \leqslant \sqrt{2}$.

For a fixed $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$ we have that either $\mathscr{G}(f, A, h) = \infty$, and so trivially $\mathscr{G}(f, A, h) \geqslant 0$, or $-\infty < \mathscr{G}(f, A, h) < \infty$. Hence, $f\,f'\,A \in L^1(\mathscr{R})$, $f^2 A' \in L^1(\mathscr{R})$, and $f^2 A \in W^{1,1}(\mathscr{R})$, which implies $\lim\limits_{|t|\to\infty} f^2(t)\,A(t) = 0$.

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If $m_n/m_s \leqslant 1$, using integration by parts, we obtain

$$\mathscr{G}(f,A,h) \geqslant \int_0^\infty \left[f^2\,(A' - \frac{\mu_n}{\mu_s}\,h) + \frac{\kappa^2\,\mu_s}{\mu_n}\left(A' - \frac{\mu_n}{\mu_s}\,h\right)^2 + \frac{\mu_n}{\mu_s}\,hf^2 \right] dx$$
$$+ \int_{-\infty}^0 \left[f^2(A' - h) + \kappa^2\,(A' - h)^2 + \frac{1}{2} f^4 \right] dx$$
$$+ \int_0^\infty af^2\,dx + \int_{-\infty}^0 (h-1)\,f^2\,dx \geqslant \min\{a, h-1\}\,||f||^2_{L^2(\mathscr{R})},$$

where we observe that the first two integrals are positive for the range of values h and κ considered and for $|f| \leqslant \sqrt{2}$. The theorem follows since the energy is lowest for states satisfying $f \equiv 0$.

If $m_n/m_s > 1$, when $h > m_n/m_s$ we can repeat the argument to find

$$\mathscr{G}(f,A,h) \geqslant \min\{a, \frac{m_s}{m_n}\,h - 1\}\,||f||^2_{L^2(\mathscr{R})}.$$

We consider next the case $m_n/m_s > 1$, $1 \leqslant h \leqslant m_n/m_s$ and $\kappa > 4$. Set $I = \inf \mathscr{G}(f,A,h)$ for $(f,A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$, from Theorem 3.1, we know

$$I \geqslant -4/h\,\sqrt{6\,(1+\delta)}$$

for any $\delta \geqslant 0$, so either $I = 0$, and we are done, or we can assume $-4/h\,\sqrt{6\,(1+\delta)} \leqslant I < 0$. Fix a $\delta > 0$, and consider a minimizing sequence (f_n, A_n). We can assume $|f_n| \leqslant \sqrt{2}$, as well as $-4/h\,\sqrt{6\,(1+\delta)} \leqslant \mathscr{G}(f_n, A_n, h) \leqslant -|I|/2$. Under these conditions, inequality (3.6) gives

$$-\frac{|I|}{2} \geqslant \mathscr{G}(f_n, A_n, h) \geqslant \int_{-\infty}^\infty \left[\frac{1}{m}\,(f_n')^2 + a\,f_n^2\chi_+ + \frac{1}{2}\,f_n^4\chi_- \right] dx$$
$$+ \int_0^\infty \frac{1}{m}\,A_n^2\,f_n^2\,dx + \int_0^\infty \frac{\mu_n}{\mu_s}\kappa^2 \left(\frac{\mu_s}{\mu_n}A_n' - h\right)^2 dx$$
$$+ \int_{A^+_{n,\delta}} (-f_n^2 + A_n^2\,f_n^2)\,dx + \int_{A^-_{n,\delta}} A_n^2\,f_n^2\,dx$$
$$+ \int_{-\infty}^0 \left(\kappa^2 - \frac{16}{h^2}\right) |A_n' - h|^2\;dx - \frac{4\,\sqrt{6\,(1+\delta)}}{h}, \tag{3.9}$$

from which we obtain that $\{1/\mu A_n' - h\}$ and $\{1/m f_n'\}$ are uniformly bounded in $L^2(\mathscr{R})$, and $\left\{\int_{A^+_{n,\delta}} \left(-f_n^2 + A_n^2\,f_n^2\right)\,dx\right\}$ is uniformly bounded.

The above bounds, together with inequality (3.5) and the fact that for $\delta > 0$ one has

$$\int_{A^+_{n,\delta}} \left(-f_n^2 + A_n^2\,f_n^2\right)\,dx > \delta \int_{A^+_{n,\delta}} f_n^2\;dx,$$

imply that $\{f_n\}$ is uniformly bounded in $H^1(\mathscr{R})$. By inequality (3.8) the sequence $\left\{\frac{1}{\mu}\,(A_n - A_n(0)) - h\,x\right\}$ is uniformly bounded in $W^{1,2}_{0,0}(\mathscr{R})$. We conclude $f_n \rightharpoonup f$ weakly in $H^1(\mathscr{R})$ and $\frac{1}{\mu}(A_n(x) - A_n(0)) - h\,x \rightharpoonup \frac{1}{\mu}\,B(x) - h\,x$ weakly in $W^{1,2}_{0,0}(\mathscr{R})$. By a diagonal

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argument (up to subsequences) we can also assume $f_n \to f$ in L^2 and uniformly on bounded sets, and $\frac{1}{\mu}(A_n - A_n(0)) \to \frac{1}{\mu} B$ uniformly on bounded sets, with $B(0) = 0$. Note that due to Sobolev Embeddings f_n, f, A_n, B are continuous functions.

Set $A(0) \equiv \liminf_{n\to\infty} A_n(0)$, we have either $|A(0)| < \infty$ or $|A(0)| = \infty$. In fact, we will assume again up to a subsequence that $A(0) = \lim_{n\to\infty} A_n(0)$.

We first consider the case $|A(0)| < \infty$. For $x < 0$ fixed, we can find a constant C independent of n such that for any $x < 0$:

$$h\,x - C\,|x|^{\frac{1}{2}} \leqslant A_n(x) - A_n(0) \leqslant h\,x + C\,|x|^{\frac{1}{2}}. \tag{3.10}$$

Fix $\epsilon > 0$, we choose an $\hat{x} < 0$ large and negative, depending on $A(0), h, C, \epsilon$ but not on $n > n_\epsilon$, for which $|A_n(x)| - 1 \geqslant 0$, for any $x < \hat{x}$ and n large, and rewrite the energy in the form

$$\begin{aligned} \mathscr{G}(f_n, A_n, h) = & \int_{-\infty}^{\infty} \left[\frac{1}{m} (f_n')^2 + a\, f_n^2 \chi_+ + \frac{1}{2} f_n^4 \chi_- \right] dx \\ & + \int_0^{\infty} \frac{1}{m} A_n^2 f_n^2\, dx + \int_{-\infty}^{\infty} \mu\kappa^2 \left(\frac{1}{\mu} A_n' - h \right)^2 dx \\ & + \int_{-\infty}^{\hat{x}} (-f_n^2 + A_n^2 f_n^2)\, dx + \int_{\hat{x}}^{0} (-f_n^2 + A_n^2 f_n^2)\, dx. \end{aligned} \tag{3.11}$$

We then set $A = B + A(0)$ and obtain $\mathscr{G}(f, A, h) \leqslant \lim_{n\to\infty} \mathscr{G}(f_n, A_n, h) = I$, with $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$, $\frac{1}{\mu}(A(x) - A(0)) - hx \in W^{1,2}_{0,0}(\mathscr{R})$.

Assume next $|A(0)| = \infty$, and set $g(x) = \liminf_{n\to\infty} 1/m\,|A_n(x)|\,|f_n(x)|$ a.e. By equation (3.9), $g \in L^2(\mathscr{R})$ so that $g(x) < \infty$ a.e. For x fixed we know that $A_n(x) - A_n(0) \to B(x)$ and $f_n(x) \to f(x)$, where $B(x), f(x)$ are finite numbers. Therefore, $\lim_{n\to\infty} |A_n(0)|\,|f_n(x)| < \infty$ for any x fixed, which is possible only if $\lim_{n\to\infty} |f_n(x)| = 0$ i.e. $f \equiv 0$, and which in turn gives $\lim_{n\to\infty} |A_n(0)|\,|f_n^2(x)| = 0$ for any $x \in \mathscr{R}$. Using integration by parts, for our range of values, we then have $\mathscr{G}(f_n, A_n, \omega_n, h) \geqslant \left(1 - m_s/m_n\right) A_n(0)\, f_n^2(0)$, that is $I \geqslant 0$, which is a contradiction. □

Remark 3.4 For h fixed, consider $(f, A) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$, with f not identically equal to zero. If $-\infty < \mathscr{G}(f, A, h) < \infty$, define

$$\omega(f, A) = \frac{\int_{-\infty}^{\infty} \frac{1}{m} A\, f^2\, dx}{\int_{-\infty}^{\infty} \frac{1}{m} f^2\, dx},$$

and $\hat{A} = A - \omega(f, A)$. Clearly, $\mathscr{G}(f, \hat{A}, h) \leqslant \mathscr{G}(f, A, h)$, with strict inequality if $\omega(f, A) \neq 0$, and $\int_{-\infty}^{\infty} \frac{1}{m} \hat{A}\, f^2\, dx = 0$. In other words, superconducting global minimizers of our energy functional satisfy the so-called *zero-current condition.*

4 Normal states as local minimizers

We define, following physical intuition, the *nucleation field* $h^n_{c_3}(m_n/m_s)$ as the field below which there are normal states that are not local minimizers, and above which every normal state is a local minimizer. Theorem 3.3 provides an upper bound for $h^n_{c_3}(m_n/m_s)$. We need an estimate of the value of h at which normal states cease to be global as well as local minimizers.

We fix ω, and we study for which values of h the normal state $(0, h(\mu x - \omega))$ is a local minimizer with respect to small perturbations in the class of admissible states. Pick any $(g, Q) \in H^1(\mathscr{R}) \times H^1_{loc}(\mathscr{R})$, with $\frac{1}{\mu}(Q - Q(0)) \in W^{1,2}_{0,0}(\mathscr{R})$, and consider for $t \in \mathscr{R}$ the energy of the normal state perturbed in the direction given by (g, Q):

$$\begin{aligned}
&\mathscr{G}(0 + t\,g, h(\mu x - \omega) + t\,Q, h) \\
&= t^2 \int_{-\infty}^{\infty} \left[\frac{1}{m} (g')^2 + (a+1)\, g^2 \chi_+ + \frac{h^2}{m} (\mu x - \omega)^2 g^2 - g^2 \right] dx \\
&\quad + t^2 \int_{-\infty}^{\infty} \frac{\kappa^2}{\mu} (Q')^2\, dx + t^3 \int_{-\infty}^{\infty} \frac{2h}{m} (\mu x - \omega)\, Q\, g^2\, dx \\
&\quad + t^4 \int_{-\infty}^{\infty} \left[\frac{1}{2} g^4 \chi_- + \frac{1}{m} Q^2 g^2 \right] dx. \qquad (4.1)
\end{aligned}$$

Since the energy of any normal state is zero, and $Q \equiv 0$ is an admissible choice, the higher order term in (4.1) leads to the eigenvalue problem

$$\begin{aligned}
&\tau\left(a, \frac{m_n}{m_s}, \frac{\mu_n}{\mu_s}, h, \omega\right) \\
&= \inf_{\substack{\int_{-\infty}^{\infty} g^2\,dx = 1 \\ g \in H^1(\mathscr{R})}} \int_{-\infty}^{\infty} \left[\frac{1}{m} (g')^2 + \frac{h^2}{m} (\mu x - \omega)^2 g^2 + (a+1)\, g^2 \chi_+ \right] dx. \qquad (4.2)
\end{aligned}$$

Remark 4.1 Following the general idea of Proposition 0.1 in Bolley & Helffer [3], with due modifications, one can show that if $\tau > 1$ then there is a t_0 independent of (g, Q) such that for any $t < t_0$ one has $\mathscr{G}(0 + t\,g, h(\mu x - \omega) + t\,Q, h) > 0$ for $(g, Q) \neq (0, 0)$, and $||g||_{H^1(\mathscr{R})} + ||\frac{1}{\mu}(Q - Q(0))||_{W^{1,2}_{0,0}(\mathscr{R})} = 1$, that is if $\tau > 1$ the energy of the normal state can not be lowered using small perturbations.

Theorem 4.2 *Let $h \geqslant 0$, $\omega \in \mathscr{R}$, and $m_n, m_s, a, \mu_n, \mu_s$ be fixed. For $g \in H^1(\mathscr{R})$, we define the functional*

$$F(h, \omega, g) = \int_{-\infty}^{\infty} \left[\frac{1}{m} (g')^2 + \frac{h^2}{m} (\mu x - \omega)^2 g^2 + (a+1)\, g^2 \chi_+ \right] dx,$$

and consider $\tau(h, \omega) = \inf_{\{g \in H^1(\mathscr{R}), \int_{-\infty}^{\infty} g^2\,dx = 1\}} F(h, \omega, g).$

If $h \neq 0$, there is $g \in H^1(\mathscr{R})$ with $\int_{-\infty}^{\infty} g^2\,dx = 1$ and $\tau(h, \omega) = F(h, \omega, g)$.

Proof Let $v(x) = 1/(\sqrt[4]{\pi})\, e^{-x^2/2}$, then $F(h, \omega, v) < \infty$ and so $0 \leqslant \tau(h, \omega) < \infty$. Set $\tau \equiv \tau(h, \omega)$, $F(\cdot) \equiv F(h, \omega, \cdot)$, and take a sequence $\{g_n\} \subset H^1(\mathscr{R})$, $\int_{-\infty}^{\infty} g_n^2\,dx = 1$, with $F(g_n) \to \tau$ as $n \to \infty$. Since, $\{g_n\}$ is uniformly bounded in $H^1(\mathscr{R})$, up to subsequences using a diagonal argument, we can find a function $\hat{g} \in H^1(\mathscr{R})$ with g_n converging to $\hat{g}$ pointwise, weakly in $H^1(\mathscr{R})$ and uniformly and in L^2 on bounded sets. By Fatou's Lemma we conclude $0 \leqslant F(\hat{g}) \leqslant \tau$.

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Since we are working on $\mathscr{R}$, we need to exclude the possibility $F(\hat{g}) = 0$. If so, the definition of F would imply $\hat{g}' \equiv 0$ in $L^2(\mathscr{R})$, i.e. $\hat{g}$ ≡constant, and $\int_0^\infty (a+1)\,\hat{g}^2\,dx = 0$, which gives $\hat{g} \equiv 0$, so that $g_n \to 0$ uniformly on bounded sets. We then choose $c > 0$ and $C > 0$, which depend possibly on h, ω, m and μ, such that $\frac{h^2}{m}\,(\mu x - \omega)^2 > c\,(x^2+1)$ for $|x| > C$, and, due to the uniform convergence in $[-N, N]$, pick an $N > C$ for which there exists $n(N)$ such that for any $n > n(N)$, it holds $|g_n| < 1/(\sqrt{2}N)$ in $[-N, N]$. This yields $\tau \geqslant \lim_{n\to\infty} \int_{-\infty}^{\infty} h^2 m\,(\mu x - \omega)^2\, g_n^2\,dx > c\,(N^2+1)\,(1 - \frac{1}{N})$, for any $N > C$, which is a contradiction since τ is bounded. We then have $F(\hat{g}) > 0$ and as a consequence $\hat{g} \neq 0$, $\tau \neq 0$.

We know $\lim_{n\to\infty} ||g_n - \hat{g}||^2_{L^2(\mathscr{R})} = 1 - ||\hat{g}||^2_{L^2(\mathscr{R})}$ by classical analysis, since $||g_n||^2_{L^2(\mathscr{R})} = 1$. If $\lim_{n\to\infty} ||g_n - \hat{g}||^2_{L^2(\mathscr{R})} = 0$, then $||\hat{g}||^2_{L^2(\mathscr{R})} = 1$ and we are done as we can take $g = \hat{g}$. Otherwise, since by the definition of F and weak convergence we have $F(g_n - \hat{g}) + F(\hat{g}) - F(g_n) \to 0$ as $n \to \infty$, for $\epsilon > 0$ we can consider n_ϵ such that for $n > n_\epsilon$ the following inequalities are true: $\tau - \epsilon < F(g_n) < \tau + \epsilon$, $1 - \epsilon < ||g_n - \hat{g}||^2_{L^2(\mathscr{R})} + ||\hat{g}||^2_{L^2(\mathscr{R})} < 1 + \epsilon$, and $F(g_n) - \epsilon < F(g_n - \hat{g}) + F(\hat{g}) < F(g_n) + \epsilon$. We multiply the second inequality by τ, combine it with the first and substitute the result in the third, to obtain

$$0 < \tau\,||\hat{g}||^2_{L^2(\mathscr{R})} \leqslant F(\hat{g}) \leqslant \tau\,\epsilon + 2\epsilon + \tau\,||\hat{g}||^2_{L^2(\mathscr{R})} \qquad \text{for any } \epsilon > 0,$$

where we took in to account that by definition $\tau\,||u||^2_{L^2(\mathscr{R})} \leqslant F(u)$ for any $u \in H^1(\mathscr{R})$. Define $g = \hat{g}/||\hat{g}||_{L^2(\mathscr{R})}$ and the theorem is proven. □

The following lemma tells us that a fixed normal state, i.e. for fixed ω, besides being a global minimizers for $h \geqslant \max\{1, m_n/m_s\}$ it remains stable even below such a field, note the strict inequalities. It will allow us as well to consider only $\omega < 0$, when looking at the supremum of fields at which normal states become unstable.

Lemma 4.3 *Let ω and $\frac{m_n}{m_s}$ be fixed. Under the hypothesis of Theorem 4.2, the following holds:*

(1) *There exists a unique $h = h(\omega, \frac{m_n}{m_s})$ such that $\tau(h(\omega, \frac{m_n}{m_s}), \omega) = 1$;*

(2) *If $\frac{m_n}{m_s} \leqslant 1$, then $h(\omega, \frac{m_n}{m_s}) < 1$. Moreover, if $\omega < 0$ then $\frac{1}{3} < h(\omega, \frac{m_n}{m_s})$;*

(3) *If $\frac{m_n}{m_s} > 1$ and $\omega < 0$, then $\frac{1}{3} < h(\omega, \frac{m_n}{m_s}) < \frac{m_n}{m_s}$, while if $\omega \geqslant 0$ then $h(\omega, \frac{m_n}{m_s}) < 1$.*

Proof In our proof, we use eigenvalue estimates obtained by Lu & Pan [14]. Following their notation, we let $\lambda(z)$ and $\beta_\gamma(z)$ denote

$$\lambda(z) = \inf_{u \in H_0^1([0,\infty))} \frac{\int_0^\infty \{|u'|^2 + (z+t)^2\,|u|^2\}\,dt}{\int_0^\infty |u|^2\,dt}, \tag{4.3}$$

and

$$\beta_\gamma(z) = \inf_{u \in H^1([0,\infty))} \frac{\int_0^\infty \{|u'|^2 + (z+t)^2\,|u|^2\}\,dt + \gamma\,|u(0)|}{\int_0^\infty |u|^2\,dt}, \tag{4.4}$$

where both infima are attained. Since $\tau(h, \omega)$ is also attained, the change of variables

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$x = -h^{-1/2}\, t$ for $x < 0$, $x = (\frac{\mu_n}{\mu_s} h)^{-1/2}\, t$ for $x > 0$, and straightforward computations yield

$$\tau(h,\omega) \leqslant h\, \lambda(h^{1/2}\omega), \tag{4.5}$$

as well as, for any eigenfunction g_h associated with $\tau(h,\omega)$,

$$\tau(h,\omega) \int_{-\infty}^{\infty} g_h^2\, dx \geqslant h \quad \beta_0(h^{1/2}\omega) \int_{-\infty}^{0} g_h^2\, dx \\ + h\, \frac{\mu_n\, m_s}{\mu_s\, m_n} \quad \beta_0\left(-\left(\frac{\mu_s\, h}{\mu_n}\right)^{1/2} \omega\right) \int_0^{\infty} g_h^2\, dx + (a+1) \int_0^{\infty} g_h^2\, dx. \tag{4.6}$$

Also, it is easy to check that

$$\tau(h,\omega) = h\ \lambda(h^{1/2}\omega) \qquad \text{if} \qquad \int_0^{\infty} g_h^2\, dx = 0. \tag{4.7}$$

We find another simple but helpful lower bound for $F(h,\omega,f)$ with $f \in H^1(\mathscr{R})$, by following the argument used to derive (3.9):

$$F(h,\omega,f) \geqslant h\, \min(1, \frac{m_s}{m_n}) \left[\int_{-\infty}^{0} f^2\, dx + \frac{\mu_n}{\mu_s} \int_0^{\infty} f^2\, dx\right] + (a+1) \int_0^{\infty} f^2\, dx. \tag{4.8}$$

Proof of 1 For ω fixed, a direct computation shows that $\tau(h,\omega)$ is a continuous strictly increasing function of $h > 0$. Therefore, it is enough to show that for h small $\tau(h,\omega) < 1$, while if h is large then $\tau(h,\omega) > 1$. The later is an easy consequence of (4.8), as for $h > (a+1)/\min(1, \frac{m_s}{m_n})$ it implies $F(h,\omega,f) > (a+1) \int_{-\infty}^{\infty} f^2\, dx$ for any $f \in H^1(\mathscr{R})$. On the other hand, we fix $L > 0$ and pick the test function defined as $g_L(x) = \sin(\frac{\pi x}{L})$ for $-L < x < 0$, and $g_L(x) = 0$ otherwise, to obtain

$$\tau(h,\omega) \leqslant \frac{\pi^2}{L^2} + h^2\, \frac{\int_{-L}^{0} (\mu x - \omega)^2\, \sin^2(\frac{\pi x}{L})\, dx}{\int_{-L}^{0} \sin^2(\frac{\pi x}{L})\, dx} \qquad \text{any } L > 0. \tag{4.9}$$

Which leads to the desired estimate, by choosing L large enough and h small enough.

Proof of 2 If $\frac{m_n}{m_s} \leqslant 1$, inequality (4.8) implies that when $h \geqslant 1$ it holds $\tau(h,\omega) \geqslant 1 + \frac{\mu_n}{\mu_s} \int_0^{\infty} g_h^2\, dx + a \int_0^{\infty} g_h^2\, dx$, from which we deduce $h(\omega, \frac{m_n}{m_s}) < 1$ whenever $\int_0^{\infty} g_h^2\, dx \neq 0$. Strict inequality for the case $\int_0^{\infty} g_h^2\, dx = 0$ is consequence of (4.7) due to Lemma 6.2 pg 1266 in [14] which tells us that $\lambda(z) > 1$ for any $z \in \mathscr{R}$. The same lemma gives the lower bound for $\omega \leqslant 0$ thanks to (4.5), as $\lambda(z)$ is a strictly increasing continuous function with $\lambda(0) = 3$.

Proof of 3 If $\frac{m_n}{m_s} > 1$ as above we have $h(\omega, \frac{m_n}{m_s}) < \frac{m_n}{m_s}$ for every $\omega \in \mathscr{R}$. When $\omega \geqslant 0$, Lemma 7.6 pg 1273 in [14] tells us that $\beta_0(z) \geqslant 1$ for $z \geqslant 0$, thus inequality (4.6) implies $h(\omega, \frac{m_n}{m_s}) < 1$ if $\int_0^{\infty} g_h^2\, dx \neq 0$. Strict inequality when $\int_0^{\infty} g_h^2\, dx = 0$ follows as in part *2.*, and so does the lower bound for $\omega < 0$. □

Remark 4.4 For ω fixed, we denote by g_ω the eigenfunction associated with the eigenvalue $\tau(h(\omega, \frac{m_n}{m_s}), \omega) = 1$. Remark 4.1 implies that if $h > h(\omega, \frac{m_n}{m_s})$ the normal state $(0, h(\mu\, x - \omega))$

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is a local minimizer, while if $h < h(\omega, \frac{m_n}{m_s})$ it is not. On the other hand, (4.1) shows that at $h = h(\omega, \frac{m_n}{m_s})$ it is a local minimizer if g_ω verifies the *zero-current condition*: $\int_{-\infty}^{\infty} \frac{1}{m}(\mu x - \omega) g_\omega^2 \, dx = 0$.

5 Onset of superconductivity

This final section is dedicated to the determination of the value and properties of the nucleation field $h_{c_3}^n(m_n/m_s)$. We are able to recover the experimental picture found in Burger *et al.* [5, 6].

Theorem 5.1 *For $\omega \in \mathscr{R}$, let $h(\omega, \frac{m_n}{m_s})$ defined as in Lemma 4.3. We have that*

(1) *If $\frac{m_n}{m_s} \leqslant 1$, then*

$$h^*\left(\frac{m_n}{m_s}\right) \equiv \sup_{\omega \in \mathscr{R}} h(\omega, \frac{m_n}{m_s}) = 1, \tag{5.1}$$

the value $h^*(\frac{m_n}{m_s})$ is not attained, and for every $h > h^*(\frac{m_n}{m_s})$ every normal state is a local minimizer, while for $h \leqslant h^*(\frac{m_n}{m_s})$ there are normal states which are not local minimizers;

(2) *If $\frac{m_n}{m_s} > 1$, then there exists an $\omega^* < 0$ such that*

$$h(\omega^*, \frac{m_n}{m_s}) = h^*\left(\frac{m_n}{m_s}\right) \equiv \sup_{\omega \in \mathscr{R}} h(\omega, \frac{m_n}{m_s}) > 1. \tag{5.2}$$

For every $h \geqslant h^*(\frac{m_n}{m_s})$ every normal state is a local minimizer, while for $h < h^*(\frac{m_n}{m_s})$ there are normal states which are not local minimizers. The eigenfunction g_{ω^*} associated with $\omega^*, h^*(\frac{m_n}{m_s})$ satisfies the zero-current condition.

Proof of 1 If $\frac{m_n}{m_s} \leqslant 1$, by Lemma 4.3 we know that $h(\omega, \frac{m_n}{m_s}) < 1$, hence it is enough to show that $\lim_{\omega \to -\infty} h(\omega, m_n/m_s) = 1$. For $\omega < 0$, Lemma 4.3 implies $\frac{1}{3} < h(\omega, \frac{m_n}{m_s}) < 1$. Hence, from (4.5) and (4.6), which tell us that

$$\frac{1}{\lambda(h^{1/2}(\omega, \frac{m_n}{m_s})\,\omega)} \leqslant h(\omega, \frac{m_n}{m_s}) \leqslant \frac{1}{\beta_0(h^{1/2}(\omega, \frac{m_n}{m_s})\,\omega)}, \tag{5.3}$$

and Lemmas 5.2, 6.2 in [14], which give $\lim_{z \to -\infty} \beta_0(z) = \lim_{z \to -\infty} \lambda(z) = 1$, we conclude $\lim_{\omega \to -\infty} h(\omega, m_n/m_s) = 1$. Note that (5.3) holds for $m_n/m_s > 1$, as well.

Proof of 2 We start by proving that the sup is strictly greater than 1. We use the fact that for ω fixed τ is strictly increasing in h, and show that if $\frac{m_n}{m_s} > 1$ there exists an $\omega < 0$ such that $\tau(1, \omega) < 1$.

Since $m_n/m_s > 1$, we can find a $\delta > 0$ with $2m_n/m_s - 1 = (1+\delta)^2 + \delta$. Additionally, we can pick an $\omega_1 < 0$ such that for any $\omega < \omega_1$ and $x \geqslant 0$ we have $\frac{m_n}{m_s}(a+1) < \delta\,\omega^2 \leqslant \delta\,(x-\omega)^2$.

For ω fixed, consider the test function $f_\omega(x) = 1/(\sqrt[4]{\pi})\, e^{-\frac{(x-\omega)^2}{2}}$, and notice that from Lemma 5.4 below, if $\omega < \omega_2$, it holds

$$\int_0^\infty (\frac{\mu_n}{\mu_s} x - \omega)^2 f_\omega^2 \, dx < (1+\delta)^2 \int_0^\infty (x-\omega)^2 f_\omega^2 \, dx, \tag{5.4}$$

the inequality is trivially true for every $\omega < 0$ if $\mu_n/\mu_s < 1$. For $\omega < \min(\omega_1, \omega_2)$, we then have

$$\int_0^\infty \left[\left(\frac{\mu_n}{\mu_s}x - \omega\right)^2 + \frac{m_n}{m_s}(a+1)\right] f_\omega^2\, dx < \left(2\frac{m_n}{m_s} - 1\right)\int_0^\infty (x-\omega)^2 f_\omega^2\, dx, \tag{5.5}$$

and we conclude $\tau(1,\omega) < 2\int_{-\infty}^{\infty}(x-\omega)^2 f_\omega^2\, dx = 1$.

To show that the sup is obtained we use Lemma 5.5, and consider $K = [\omega_0, 0]$, where $\omega_0 < 0$ is such that $h(\omega, m_n/m_s) < \sup_{\omega\in\mathscr{R}} h(\omega, m_n/m_s)$ for every $\omega < \omega_0$. That such ω_0 exists is due to the fact that $\lim_{\omega\to-\infty} h(\omega, m_n/m_s) = 1$. This limit is obtained as in part *1.* since for $\omega < 0$ Lemma 4.3 implies $1/3 < h(\omega, m_n/m_s) < m_n/m_s$.

Let $h^*(m_n/m_s) = \sup_{\omega\in\mathscr{R}} h(\omega, m_n/m_s)$, we know from Lemma 4.3 that $h(\omega, m_n/m_s) < 1$ for $\omega \geqslant 0$, thus we can find a sequence $\{\omega_l\}$ with $\omega_l \in K$, such that $h(\omega_l, m_n/m_s) \to h^*(m_n/m_s)$. Since K is compact we can assume (up to a subsequence) that $\omega_l \to \omega^*$. But, then

$$\left|\tau\left(h^*\left(\frac{m_n}{m_s}\right), \omega^*\right) - 1\right| \leqslant \left|\tau\left(h^*\left(\frac{m_n}{m_s}\right), \omega^*\right) - \tau\left(h^*\left(\frac{m_n}{m_s}\right), \omega_l\right)\right| + \left|\tau\left(h^*\left(\frac{m_n}{m_s}\right), \omega_l\right) - \tau\left(h\left(\omega_l, \frac{m_n}{m_s}\right), \omega_l\right)\right|,$$

and as $l \to \infty$ the terms in the left hand side tend to zero: the first because τ is continuous in ω for h fixed, the second due to Lemma 5.5 as by Lemma 4.3 we have $h(\omega_l, m_n/m_s) > 1/3$, and so $h^*(m_n/m_s), \omega_l) \geqslant 1/3$. Therefore, $\tau(\omega^*, h^*(m_n/m_s)) = 1$ and since for ω fixed τ is strictly increasing in h, this implies $h^*(m_n/m_s) = h(\omega^*, m_n/m_s)$.

Set $\hat{\omega}^* = \left(\int_{-\infty}^{\infty}\frac{1}{m}\mu x\, g_{\omega^*}^2\, dx\right) / \left(\int_{-\infty}^{\infty}\frac{1}{m} g_{\omega^*}^2\, dx\right)$ and assume $\omega^* \neq \hat{\omega}^*$. A direct computation yields $\int_{-\infty}^{\infty}\frac{1}{m}(\mu x - \hat{\omega}^*)^2 g_{\omega^*}^2\, dx < \int_{-\infty}^{\infty}\frac{1}{m}(\mu x - \omega^*)^2 g_{\omega^*}^2\, dx$, which translates to $\tau(h(\omega^*, m_n/m_s), \hat{\omega}^*) < \tau(h(\omega^*, m_n/m_s), \omega^*) = 1$, that is $h(\hat{\omega}^*, m_n/m_s) > h(\omega^*, m_n/m_s)$. This is in contradiction with the definition of $h^*\left(m_n/m_s\right) = h(\omega^*, m_n/m_s)$. Hence, $\omega^* = \hat{\omega}^*$ and g_{ω^*} satisfies the zero-current condition. □

Remark 5.2 In physical terms, the previous theorem establishes that the nucleation field H_{c3}^n depends on σ_s/σ_n, with $H_{c3}^n = H_{c2}$ when $\sigma_s/\sigma_n \leqslant 1$, and $H_{c3}^n > H_{c2}$ otherwise. Note that $h^* = h_{c_3}^n$.

Remark 5.3 In the case of a bounded domain, and in the absence of the jump discontinuity at the interface, standard arguments imply that $h(\omega, m_n/m_s)$ is analytic in ω. In here, although a typical direct computation does imply that $\tau(h,\omega)$ is continuous and strictly increasing in $h > 0$, the traditional line of arguments do not facilely lead to the assertions that $\partial\tau/\partial h$ is continuous, strictly positive, and that $\partial\tau/\partial\omega$ is continuous, from which analyticity of h follows via the implicit function theorem.

Lemma 5.4 *For ω fixed, consider the test function $f_\omega(x) = \frac{1}{\sqrt[4]{\pi}}e^{-\frac{(x-\omega)^2}{2}}$, then*

$$\lim_{\omega\to-\infty}\frac{\int_0^\infty(\frac{\mu_n}{\mu_s}x - \omega)^2 f_\omega^2\, dx}{\int_0^\infty (x-\omega)^2 f_\omega^2\, dx} = 1.$$

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In particular, for any $\delta > 0$ *and* $\frac{\mu_n}{\mu_s} \geqslant 1$, *there exists an* $\omega_2 < 0$ *such that for* $\omega < \omega_2$ *we have*

$$1 \leqslant \frac{\int_0^\infty (\frac{\mu_n}{\mu_s} x - \omega)^2 f_\omega^2 \, dx}{\int_0^\infty (x-\omega)^2 f_\omega^2 \, dx} \leqslant \left(1 + \frac{\delta}{2}\right)^2. \tag{5.6}$$

Proof We make the change of variable $s = -\omega$ and the substitution $t = x + s$, we then expand the square at the numerator and apply L'Hôpital's Rule to gather

$$I = \lim_{s \to \infty} \frac{\int_s^\infty \left[2\frac{\mu_n}{\mu_s} t(1 - \frac{\mu_n}{\mu_s}) + 2s(1 - \frac{\mu_n}{\mu_s})^2\right] e^{-t^2} \, dt}{-s^2 \, e^{-s^2}} + 1.$$

We apply L'Hôpital's Rule again to the new limit in the right hand side:

$$I = 1 + 2\left(1 - \frac{\mu_n}{\mu_s}\right)^2 \lim_{s \to \infty} \frac{\int_s^\infty e^{-t^2} \, dt}{2(s^3 - s) \, e^{-s^2}}. \tag{5.7}$$

The lemma follows after applying L'Hôpital's Rule on the limit in (5.7). □

Lemma 5.5 *Let* $\tau(h, \omega)$ *be defined as in Theorem 4.2 and* $c > 0$ *fixed. If* $\omega \in \mathscr{R}^-$ *then* τ *is uniformly continuous in* $h > c$.

Proof Let $\omega \in \mathscr{R}^-$, $h > 0$ and g_h denote the eigenfunction associated to $\tau(h, \omega)$ with L^2−norm equal to 1. Consider $\Delta h > 0$, since τ is strictly increasing in h for ω fixed, we have $\tau(h, \omega) < \tau(h + \Delta h, \omega) \leqslant \tau(h, \omega) + (2\,h + \Delta h)\, \frac{\Delta h}{c^2}\, \tau(h, \omega)$.

We take as test function $g(x) = \sin(x - \omega + \pi)$ if $\omega - \pi < x < \omega$ and $g(x) = 0$ otherwise, to find (here recall that we are looking only at $\omega \leqslant 0$) the bound

$$\tau(h, \omega) \leqslant 1 + h^2 \, \frac{2}{\pi} \int_{\omega - \pi}^{\omega} (x - \omega)^2 \, \sin^2(x - \omega + \pi) \, dx \leqslant 1 + \frac{2}{3} \, h^2 \, \pi^2, \tag{5.8}$$

which together with the previous inequality gives

$$0 \leqslant \tau(h + \Delta h, \omega) - \tau(h, \omega) \leqslant (2\,h + \Delta h) \; \frac{\Delta h}{c^2} \, (1 + \frac{2}{3} \, h^2 \, \pi^2). \tag{5.9}$$

Similarly, if $\Delta h < 0$ is small enough so that $h + \Delta h > c$ we derive

$$0 \leqslant \tau(h, \omega) - \tau(h + \Delta h, \omega) \leqslant (2\,h + \Delta h) \; \frac{|\Delta h|}{c^2} \left(1 + \frac{2}{3} \, (h + \Delta h)^2 \, \pi^2\right), \tag{5.10}$$

and the lemma follows. □

Due to the observations of Hurault [13] and Chapman [7], one expects $h_{c_3}^n$ to be a strictly increasing function of its argument for values greater than one, which is what we prove in the following lemma.

Lemma 5.6 *Let* a, μ_n, μ_s *be fixed and* h^* *defined as in Theorem 5.1. If* $1 < m_1 < m_2$ *then*

$$h^*(m_1) < h^*(m_2). \tag{5.11}$$

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Proof Using the notation and results of Theorem 5.1, we know that there exists an $\omega_1^* < 0$ such that $h^*(m_1) = h(\omega_1^*, m_1)$ with $\tau(h(\omega_1^*, m_1), \omega_1^*) = 1$.

From $m_1 < m_2$ and Theorem 4.2, it is easy to see that

$$\tau\left(a, m_2, \frac{\mu_n}{\mu_s}, h, \omega\right) < \tau\left(a, m_1, \frac{\mu_n}{\mu_s}, h, \omega\right) \text{ for any } h > 0,\ \omega \in \mathscr{R}.$$

Hence, $\tau(a, m_2, \frac{\mu_n}{\mu_s}, h(\omega_1^*, m_1), \omega_1^*) < \tau(a, m_1, \frac{\mu_n}{\mu_s}, h(\omega_1^*, m_1), \omega_1^*) = 1$, and using once again that τ is strictly increasing in h for all the other variables fixed, we can say that $h(\omega_1^*, m_2) > h(\omega_1^*, m_1)$, which implies $h^*(m_2) \equiv \sup_{\omega \in \mathscr{R}} h(\omega, m_2) \geqslant h(\omega_1^*, m_2) > h(\omega_1^*, m_1) = h^*(m_1)$. □

We conclude our study, by examining the behaviour of $h_{c_3}^n$ as we let our model approximate the case of the superconductor in a vacuum, i.e. as we consider m_n/m_s increasing toward infinity. Again, what we expect and we are able to prove is that in this case $h_{c_3}^n$ approaches the third upper critical field, which in our units is $h_{c_3} = 1/\beta_0^*$, see Remark 2.1.

Theorem 5.7 *Let a, μ_n, μ_s be fixed and h^* defined as in Theorem 5.1. For β_0^* as in Lu & Pan* [14, Lemma 5.2], *it holds*

$$\lim_{\frac{m_n}{m_s} \to \infty} h^*\left(\frac{m_n}{m_s}\right) = \frac{1}{\beta_0^*}. \tag{5.12}$$

Proof Consider the eigenvalue $\beta_0(z)$ given in (4.4), Lu & Pan [14] show that there exists $z_0 < 0$ and β_0^* which verify $\beta_0(z_0) = \beta_0^* = \inf_{z \in \mathscr{R}} \beta_0(z)$. Inequality (5.3) then implies $h(\omega, m_n/m_s) \leqslant 1/\beta_0^*$, for every $\omega < 0$, and by Theorem 5.1: $h^*\left(m_n/m_s\right) \leqslant 1/\beta_0^*$, for every $m_n/m_s > 1$. Therefore, thanks to Lemma 5.6 it is enough to prove that for every $\epsilon > 0$ small there exist $m_\epsilon \equiv m_n^\epsilon/m_s^\epsilon$ and ω_ϵ such that $h(\omega_\epsilon, m_\epsilon) > 1/\beta_0^* - \epsilon$.

Denote by u the eigenfunction corresponding to $\beta_0(z_0)$ with $\int_0^\infty u^2(t)\, dt = 1$, and recall that $u'(0) = 0$ and $u(0) \neq 0$.

For $0 < 2\epsilon < 1/\beta_0^*$ and $m_\epsilon > 1$ to be chosen later, we pick

$$\omega_\epsilon = \frac{z_0}{\left(\frac{1}{\beta_0^*} - \epsilon\right)^{1/2}} < 0 \qquad h_\epsilon = \frac{1}{\beta_0^*} - \epsilon, \tag{5.13}$$

and consider the test function

$$f(x) = \begin{cases} u(-\sqrt{h_\epsilon} x) & x < 0, \\ u(0) & 0 < x < \frac{1}{2\sqrt{m_\epsilon}}, \\ 2\sqrt{m_\epsilon}\, u(0) \left(\frac{1}{\sqrt{m_\epsilon}} - x\right) & \frac{1}{2\sqrt{m_\epsilon}} \leqslant x \leqslant \frac{1}{\sqrt{m_\epsilon}}, \\ 0 & x > \frac{1}{\sqrt{m_\epsilon}}. \end{cases} \tag{5.14}$$

Direct computations yield

$$\int_{-\infty}^{\infty} f^2(x)\,dx = \frac{1}{\sqrt{h_\epsilon}} + \frac{2}{3}\,\frac{u^2(0)}{\sqrt{m_\epsilon}},$$

$$\int_{-\infty}^{0} \left\{(f'(x))^2 + h_\epsilon^2\,(x-\omega_\epsilon)^2 f^2(x)\right\}\,dx = \sqrt{h_\epsilon}\;\beta_0^*,$$

and

$$\frac{1}{m_\epsilon}\int_0^\infty \left[(f'(x))^2 + h_\epsilon^2\left(\frac{\mu_n}{\mu_s}x-\omega_\epsilon\right)^2 f^2(x) + (a+1)\,f^2(x)\right]dx$$
$$\leqslant \frac{u^2(0)}{\sqrt{m_\epsilon}}\left(2+\frac{1}{m_\epsilon}\left[\frac{1}{(\beta_0^*)^2}\left(\frac{\mu_n}{\mu_s}-2\,z_0\,\beta_0^*\right)^2+a+1\right]\right).$$

Then, according to the definition of τ, the previous calculations imply

$$\tau(a,m_\epsilon,\frac{\mu_n}{\mu_s},h_\epsilon,\omega_\epsilon)\left(\frac{1}{\sqrt{h_\epsilon}}+\frac{2}{3}\,\frac{u^2(0)}{\sqrt{m_\epsilon}}\right)\leqslant\sqrt{h_\epsilon}\;\beta_0^*+\frac{C(z_0,\beta_0^*,a,u^2(0))}{\sqrt{m_\epsilon}},$$

that is $\tau(a,m_\epsilon,\frac{\mu_n}{\mu_s},h_\epsilon,\omega_\epsilon)\leqslant 1-\,\beta_0^*+C(z_0,\beta_0^*,a,u^2(0))/\left(m_\epsilon\,\beta_0^*\right)^{1/2}$.

Choose $m_\epsilon>1$, so to have $2\,C(z_0,\beta_0^*,a,u^2(0))/\left(m_\epsilon\,\beta_0^*\right)^{1/2}<\beta_0^*\epsilon$, then

$$\tau\left(a,m_\epsilon,\frac{\mu_n}{\mu_s},h_\epsilon,\omega_\epsilon\right)<1-\beta_0^*\epsilon/2<1,$$

and again since τ is increasing in h for the other parameters fixed, we conclude $h^*(m_\epsilon)>h(\omega_\epsilon,m_\epsilon)>h_\epsilon=1/\beta_0^*-\epsilon$. □

Remark 5.8 Theorem 5.7 and Lemma 5.6 imply that for $m_n/m_s>1$ the nucleation field $H_{c_3}^n$ is strictly less than H_{c_3}, which is the last result we needed to completely recover the experimental picture.

6 Conclusion

We studied a one-dimensional superconducting/normal system, in the framework of the Ginzburg–Landau theory. We explored the dependence of the nucleation field H_{c_3} on the ratio of the normal state conductivities of the two materials. Through the analysis of an eigenvalue problem defined on the whole real line and with discontinuous coefficients, we confirm experimental and theoretical predictions. We prove the existence of a critical value of the conductivities ratio above which the nucleation field is an increasing function, with the vacuum nucleation field as upper limit, and the second critical field H_{c_2} as lower limit. For values of the ratio below the critical one, we conclude all normal states are local (global) minimizers down to H_{c_2} and start losing stability at H_{c_2}.

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