Macroeconomic Dynamics, **12**, 2008, 445–462. Printed in the United States of America.
doi:10.1017/S1365100508070235

ARTICLES

DEBT POLICY RULE, PRODUCTIVE GOVERNMENT SPENDING, AND MULTIPLE GROWTH PATHS

KOICHI FUTAGAMI
Osaka University

TATSURO IWAISAKO
Ritsumeikan University

RYOJI OHDOI
Osaka City University

This paper constructs an endogenous growth model with productive government spending. In this model, the government can finance its costs through income tax and government debt and has a target level of government debt relative to the size of the economy. We show that there are two steady states. One is associated with high growth and the other with low growth. It is also shown that whether the government uses income taxes or government bonds makes the results differ significantly. In particular, an increase in government bonds reduces the growth rate in the high-growth steady state and raises the growth rate in the low-growth steady state. Conversely, an increase in the income tax rate reduces the growth rate in the low-growth steady state and there exists some tax rate that maximizes the growth rate in the high-growth steady state. Finally, the level of welfare in the low-growth steady state is lower than that in the high-growth steady state.

Keywords: Public Expenditure, Debt Policy, Endogenous Growth, Multiple Equilibria

1. INTRODUCTION

An important growth engine made known by Barro (1990) is productive government spending. In fact, for developing countries, a deficiency in infrastructure constitutes one of the more serious problems for development. A well-organized police system, established courts, and many other publicly provided services are good examples. But exactly how governments in less-developed countries finance these services is a serious problem, because they may be difficult to finance by

The authors would like to thank two anonymous referees for their helpful comments on an earlier version of this paper. They also gratefully acknowledge Ryo Horii, Alexandru Minea, Kazuo Mino, Akira Momota, Yasutomo Murasawa, Katsuya Takii, Akira Yakita, seminar participants at Osaka Prefecture University and the OSIPP International Public Policy research conference, and conference attendees at the Japanese Economic Association Meeting (June 2005) for their useful suggestions. This work was partly supported by MEXT.OPENRESEARCH(2004–2008). Any remaining errors are, of course, the authors' responsibility. Address correspondence to: Tatsuro Iwaisako, 1-1-1, Noji-higashi, Kusatsu, Shiga 525-8577, Japan; e-mail: tiwai@ec.ritsumei.ac.jp.

 1365-1005/08

any means other than taxation. Even in developed countries, it has become more difficult recently to collect funds to finance public services, as many developed countries currently suffer high government debt. Almost all countries must then rely on the issue of government bonds to finance the cost of essential infrastructure.

However, if there is no restriction on issuing government debt, the cumulative increase of government bonds may result. In fact, the debt–GDP ratio in Japan climbed from 68% to 113.1% during the 1990s. Therefore, many countries have set various constraints on budget deficits or the number of government bonds. We find one such constraint in the Maastricht treaty. The Maastricht criterion states that member countries of the European Union (EU) and countries that hope to join the EU must maintain a government debt–GDP ratio of less than 60%. A similar rule applies in the United Kingdom, where under the Code of Fiscal Stability the net government debt–GDP ratio must be kept under 30%.[1]

Accordingly, it is important to investigate just how such debt policy rules affect growth paths and welfare. In this paper, we show that when the government uses a debt policy rule that forces the government to keep the amount of government bonds at a certain level, there are two equilibria: one exhibits low-growth performance and the other high-growth performance. Whether an increase in the target level of government bonds enhances economic growth depends on which of these two equilibria the economy is in. If the government raises its target level of government bonds, the growth rate decreases in the high-growth equilibrium. On the other hand, the growth rate increases in the low-growth equilibrium. Moreover, if the government raises funds through income taxes, this leads to a sharp contrast with the use of bond finance. If the government raises the tax rate, the growth rate decreases in the low-growth equilibrium. However, there exists a tax rate that maximizes the growth rate in the high-growth equilibrium, as in Barro's model. These results have important policy implications for countries in the EU. Less developed EU countries are allowed to issue government bonds to finance the costs of productive government spending in order to promote economic growth. On the other hand, more developed EU countries should choose tax financing rather than bond financing but must be cautious when choosing the tax rate.[2]

There is a substantial amount of research based on endogenous growth models that takes into account productive government spending and debt. Bruce and Turnovsky (1999) investigate how a reduction in the role of government through a tax cut, or a tax cut with expenditure cuts, influences the long-run government fiscal balance. Greiner and Semmler (1999, 2000) examine how governmental financing methods affect economies. They show that a less strict budget policy does not necessarily promote economic growth. By introducing government bonds, Turnovsky (1997; 2000, Ch.14) examines how the government's expenditure policy affects economic growth independently of tax policy. Furthermore, Ghosh and Mourmouras (2004) assume that the benevolent government maximizes the welfare of households and endogenously derive the fiscal rule under the standard government budget constraint and the golden rule of public finance. The golden rule constrains the government to use the revenue raised by issuing government bonds only to finance public investment. Ghosh and Mourmouras compare the optimal fiscal policy under these different regimes and show that the golden rule can be an effective constraint.[3] Although these studies explore the policy implications of budgetary policies, there are few studies concerning the debt policy rule found in the Maastricht criterion.[4]

The remainder of this paper is structured as follows. Section 2 provides the model to be examined. Section 3 establishes the equilibria of the economy. Section 4 conducts comparative static analyses of the equilibria and a welfare comparison of the equilibria. Section 5 concludes the paper.

2. THE MODEL

2.1. Households and Firms

Without any loss of generality, we assume that the population is constant over time and normalized to be unity. The utility of the representative household endowed with infinite lifetime and perfect foresight is given by

$$U_0 = \int_0^{\infty} \ln C_t \ \exp(-\rho t)\, dt, \tag{1}$$

where C and ρ stand for consumption and the rate of time preference, respectively. The household seeks to maximize (1) subject to the budget constraint

$$\dot{W}_t = (1-\tau)(r_t W_t + w_t) - C_t,$$

where r, w, and W denote the interest rate, the wage rate, and the household's nonhuman wealth, respectively. τ is the income tax rate imposed by the government, which is assumed to be time-invariant. The intertemporal maximization yields the following condition:

$$\gamma_t \equiv \frac{\dot{C}_t}{C_t} = (1-\tau)r_t - \rho. \tag{2}$$

In addition, the following transversality condition must hold:

$$\lim_{t\uparrow\infty} (C_t)^{-1} W_t \exp(-\rho t) = 0.$$

Following Barro (1990), the government provides productive public services. The production function takes the Cobb–Douglas form

$$Q_t = A K_t^{\alpha} (G_t L_t)^{1-\alpha}, \tag{3}$$

where Q, K, L, and G are output, private capital, labor input, and productive public services. The first-order conditions for profit maximization are given by

$$r_t = A\alpha\,(G_t L_t / K_t)^{1-\alpha}, \quad w_t = A(1-\alpha) G_t\,(G_t L_t / K_t)^{-\alpha}. \tag{4}$$

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2.2. Government

The government finances its expenditure by two methods: one is by levying income tax as mentioned in the previous section, and the other is by issuing bonds. Thus, the government is allowed to run budget deficits. The government's budget constraint is then

$$\dot{B}_t = r_t B_t - \tau\,(r_t W_t + w_t) + G_t, \tag{5}$$

where B_t stands for government bonds.

We suppose that the government has a target level of circulated government bonds in the market. More concretely, the government attempts to maintain the ratio of government bonds to the size of the economy as a constant. In this paper, we gauge the size of the economy by the level of private capital, K_t.[5] We assume that the government adjusts $b_t \equiv B_t/K_t$ gradually so that it equals a target level in the long run. In particular, we assume the adjustment rule

$$\dot{b}_t = -\phi(b_t - \overline{b}), \tag{6}$$

where $\overline{b}$ and $\phi(>0)$ stand for the target level of government bonds and the adjustment coefficient of the rule, respectively.[6] Given the rate of capital accumulation, (6) sets down the movements of B_t. Therefore, given such a bond-issuance rule and the tax revenue, the government must adjust the level of government spending at every point in time.

3. EQUILIBRIUM

Using market equilibrium conditions, we derive the equilibrium paths of the economy. Because the population size is unity and each household supplies one unit of labor inelastically, the labor market clears as $L_t = 1$ for all $t \geq 0$. On the other hand, the asset market equilibrium implies $W_t = K_t + B_t$. Substituting these into (5) and applying $Q_t = r_t K_t + w_t$ to the result, we have

$$\dot{B}_t = (1-\tau) r_t B_t - (\tau Q_t - G_t). \tag{7}$$

On the right-hand side (RHS) of (7), the first term, $(1-\tau)r_t B_t$ is the net value of the government's interest payment, and the second and third terms in the parentheses, $\tau Q_t - G_t$, represent the primary surplus.[7] From the definition of b, (7) is rearranged as

$$\frac{\dot{b}_t}{b_t} = (1-\tau) r_t + \frac{G_t}{B_t} - \frac{\tau Q_t}{B_t} - \frac{\dot{K}_t}{K_t},$$

which is combined with (3), (4), and (6) to give

$$-\phi\left(1 - \frac{\overline{b}}{b_t}\right) = (1-\tau) A\alpha \left(\frac{G_t}{K_t}\right)^{1-\alpha} + \frac{G_t}{B_t} - \frac{\tau A K_t^{\alpha} G_t^{1-\alpha}}{B_t} - \frac{\dot{K}_t}{K_t}. \tag{8}$$

The good market equilibrium condition is given by

$$\dot{K}_t = A K_t^{\alpha} G_t^{1-\alpha} - C_t - G_t. \tag{9}$$

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Substituting (9) into (8), we obtain

$$x_t = \psi(y_t, b_t)$$
$$\equiv \left[1 - \alpha(1-\tau) + \frac{\tau}{b_t}\right] A y_t^{1-\alpha} - \left(\frac{1}{b_t} + 1\right) y_t - \phi\left(1 - \frac{\bar{b}}{b_t}\right), \quad \textbf{(10)}$$

where $x_t \equiv C_t/K_t$ and $y_t \equiv G_t/K_t$. Equation (10) constitutes the relation that x_t and y_t must follow for all $t \geq 0$. On the other hand, by using (2), (4), and (9), we obtain the dynamics with respect to x_t:

$$\frac{\dot{x}_t}{x_t} = x_t - [1 - \alpha(1-\tau)] A y_t^{1-\alpha} + y_t - \rho. \quad \textbf{(11)}$$

Equations (6), (10), and (11) formulate an autonomous dynamic system with respect to x_t, y_t, and b_t. Once the time paths of these variables are determined, we can obtain the entire time paths of all endogenous variables for predetermined K_0 and B_0.[8]

3.1. Steady States

Now we examine the steady states of the economy where x_t, y_t, and b_t become constant over time. Imposing $\dot{x} = 0$ in (11) results in[9]

$$x = [1 - \alpha(1-\tau)] A y^{1-\alpha} - y + \rho. \quad \textbf{(12)}$$

Substituting (10) into (12), we obtain the following equation for the steady-state values of y:

$$\bar{b} = \zeta(y; \tau)/\rho, \quad \textbf{(13)}$$

where

$$\zeta(y; \tau) \equiv \tau A y^{1-\alpha} - y.$$

By looking closely at (13), we find that this equation represents the intertemporal budget constraint for the government at the steady states. The LHS and RHS of (13) are the long-run ratios of government debt and the (discounted present) value of primary surplus to capital stock, respectively.[10] Hence, given the value of $\bar{b}$ and the movement of capital stock, the government must determine the level of spending such that y satisfies the equality (13) in the long run.

Differentiating the RHS of (13) with respect to y, we obtain

$$\zeta_y(y; \tau) = \tau A(1-\alpha) y^{-\alpha} - 1 \gtreqless 0 \Longleftrightarrow y \lesseqgtr \hat{y}(\tau) \equiv [\tau A(1-\alpha)]^{1/\alpha}. \quad \textbf{(14)}$$

Therefore we can depict the representative shape of (13) as Figure 1. As shown by (14), the primary surplus and government spending follow an inverted U-shape.

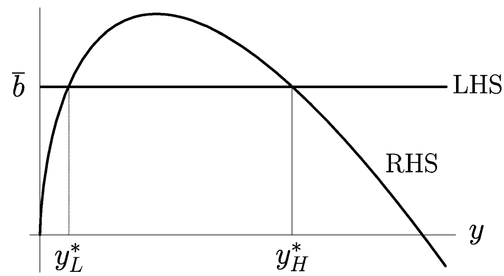

FIGURE 1. Multiple steady states ($\bar{b} < \zeta(\hat{y}(\tau); \tau)/\rho$).

The reasoning is as follows. Since government spending works as the productive input, an increase in spending expands the output of the economy, which in turn boosts tax revenue. Then, if the level of government spending is small relative to the capital stock, the increase in tax revenue exceeds government spending itself, which results in an increase in the primary surplus. The reverse is true if the level of government spending is large relative to the capital stock.

From Figure 1, a number of properties can be observed. First, steady states exist in the economy if and only if the target ratio of the government debt to the capital stock is small enough so that $\bar{b} \leq \zeta(\hat{y}(\tau); \tau)/\rho \equiv \alpha(1-\alpha)^{(1-\alpha)/\alpha}(\tau A)^{1/\alpha}/\rho$. To grasp the intuition behind this result, note that $\zeta(\hat{y}(\tau); \tau)$ is the primary surplus that the government can earn at most, given the capital stock. Therefore, if $\bar{b} > \zeta(\hat{y}(\tau); \tau)/\rho$, the government cannot repay its debt and eventually ends up in default; the no-Ponzi-game condition is violated.[11] Therefore, we hereafter assume away this case. Second, if $\bar{b}$ is strictly smaller than $\zeta(\hat{y}(\tau); \tau)/\rho$, the economy has two steady states $(x_{\mathrm{L}}^*, y_{\mathrm{L}}^*, \bar{b})$ and $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, \bar{b})$, where x_i^* and y_i^* $(i \in \{L, H\})$ are the values that solve (10) and (13). This implies that the following two policy schemes are available to the government in order to keep the ratio of its debt to capital at its target value: (i) the government determines its spending so that its ratio to capital stock eventually becomes y_{L}^*, where both the spending and tax revenue become small, and (ii) it determines its spending so that the ratio eventually becomes y_{H}^*, where both the spending and tax revenue become large. Third, the growth rate at the steady state with y_{H}^* is higher than that at the steady state with y_{L}^*, because the growth rate at each state is given by

$$\gamma_i = (1-\tau)\alpha A y_i^{*1-\alpha} - \rho \text{ for } i \in \{\mathrm{L}, \mathrm{H}\}. \quad \mathbf{(15)}$$

In sum, we can state the following proposition:

PROPOSITION 1. *(a) There exist steady states in this economy if and only if* $\bar{b} \leq \zeta(\hat{y}(\tau); \tau)/\rho$; *(b) two steady states,* $(x_{\mathrm{L}}^*, y_{\mathrm{L}}^*, \bar{b})$ *and* $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, \bar{b})$, *emerge if and only if* $\bar{b} < \zeta(\hat{y}(\tau); \tau)/\rho$; *and (c) the growth rate at* $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, \bar{b})$ *is higher than that at* $(x_{\mathrm{L}}^*, y_{\mathrm{L}}^*, \bar{b})$.

In addition, we immediately arrive at the next lemma, which is useful for the following analysis.

LEMMA 1. $x_{\mathrm{H}}^* > x_{\mathrm{L}}^*$*; the ratio of consumption to capital at the high-growth steady state is higher than that at the low-growth steady state.*

Proof. From (12) and (13), we obtain

$$\begin{aligned} x &= [1-\alpha(1-\tau)]\, y^{1-\alpha} - \left(\tau A y^{1-\alpha} - \rho\overline{b}\right) + \rho \\ &= (1-\alpha)(1-\tau)Ay^{1-\alpha} + \rho(1+\overline{b}). \end{aligned}$$

Therefore x_{L}^* is smaller than x_{H}^* because y_{L}^* is smaller than y_{H}^*. ■

Before concluding this section, we briefly discuss the mechanism generating the multiple steady states. First, according to the Euler equation (2), the growth rate, γ, and the value of the government spending–capital stock ratio, y, relate as follows:

$$\left(\frac{\dot{C}_t}{C_t} =\right) \gamma = (1-\tau)\alpha A y^{1-\alpha} - \rho. \tag{16}$$

This shows that the growth rate increases as y gets larger, because this implies the capital stock becomes relatively scarce. On the other hand, by using (3), (4), and the fact that $\dot{B}_t/B_t = \gamma$ in the long run, we can rewrite the government's budget constraint, (5), as follows:

$$y = \tau A y^{1-\alpha} + \overline{b}\left[\gamma - (1-\tau)\alpha A y^{1-\alpha}\right]. \tag{17}$$

This shows that the spending–capital stock ratio, y, increases if the growth rate of the government's outstanding bonds gets larger, because this allows the government to spend more resources on productive government spending. Equations (16) and (17) show that in this model there exists a complementarity between the growth rate and the government spending. This complementarity generates the multiplicity of steady states.

3.2. Stability

We next examine the dynamic stability of the steady states. The dynamic adjustment of b_t is autonomously given by (6) and it is always stable, whereas (x_t, y_t) evolves over time according to (10) and (11). However, we cannot simply reduce these two equations into one differential equation to eliminate x_t or y_t, because function $\psi(y, b)$ is not monotone with respect to y:

$$\psi_y(y,b) \gtreqless 0 \iff y \lesseqgtr \widetilde{y}(b) \equiv \left(\frac{A(1-\alpha)\left\{[1-\alpha(1-\tau)]\,b+\tau\right\}}{b+1}\right)^{1/\alpha}. \tag{18}$$

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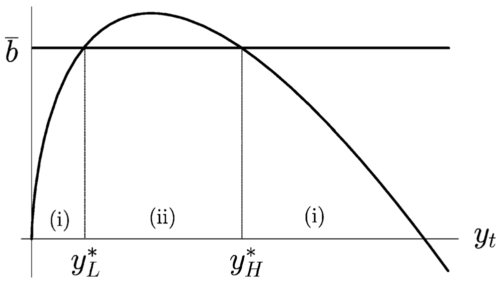

FIGURE 2. Local dynamics of x_t (given y_t). $\dot{x}_t < 0$ if y_t belongs to region (i), and $\dot{x}_t > 0$ if y_t belongs to region (ii).

Equation (18) means that y is set-valued for a value of x. Thus, y_t can evolve over time with a perpetual discontinuous jump, which in turn generates a discontinuous change in $\dot{x}_t$ over time.[12] To avoid this problem, we assume the following:

Assumption 1. *y_t is continuous with respect to t for all $t \in (0, \infty)$.*

In this model, as already shown, the government must earn a primary surplus sufficient to repay the loan and to eventually keep b_t at $\overline{b}$. In addition, it has already been shown that two policy schemes are available to the government in order to do so: namely, it can become either a large or a small government. Assumption 1 means that the government continues to apply one policy scheme over time once it chooses a scheme from the two available at the initial time.[13]

To obtain the local property of the dynamics, it is sufficient to examine whether the following dynamics are stable:

$$\dot{x}_t = \left\{x_t - [1 - \alpha(1 - \tau)]\, A y_t^{1-\alpha} + y_t - \rho\right\} x_t, \tag{19}$$

$$x_t = \psi(y_t, \overline{b}), \tag{20}$$

because b_t eventually converges to $\overline{b}$, irrespective of the movements of (x_t, y_t).[14] From (19) and (20), we obtain

$$\dot{x}_t \gtreqless 0 \iff (\rho\overline{b})^{-1}[\zeta(y_t; \tau)/\rho - \overline{b}] \gtreqless 0. \tag{21}$$

From (21), we find that $\dot{x}_t < 0$ if $y_t < y_{\mathrm{L}}^*$ or $y_t > y_{\mathrm{H}}^*$, whereas $\dot{x}_t > 0$ if $y_{\mathrm{L}}^* < y_t < y_{\mathrm{H}}^*$. This result is summarized by Figure 2.

Utilizing this figure and (20), we can obtain the movement of (x_t, y_t). Figure 3, which depicts the graph of ψ in (x_t, y_t) space, suggests that the movements of (x_t, y_t) are classified into two types. As shown in panel a, both $(x_{\mathrm{L}}^*, y_{\mathrm{L}}^*)$ and $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*)$ become unstable if $y_{\mathrm{L}}^* < \widetilde{y}(\overline{b}) < y_{\mathrm{H}}^*$. On the other hand, if $y_{\mathrm{H}}^* < \widetilde{y}(\overline{b})$, $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*)$ becomes stable, whereas $(x_{\mathrm{L}}^*, y_{\mathrm{L}}^*)$ is unstable, as depicted in panel b.[15] Because b_t monotonically converges to $\overline{b}$ and both of x_0 and y_0 are jumpable,[16] this implies that given $b_0 \simeq \overline{b}$, the pair of (x_0, y_0, b_0) that converges to $(x_{\mathrm{L}}^*, y_{\mathrm{L}}^*, \overline{b})$ is

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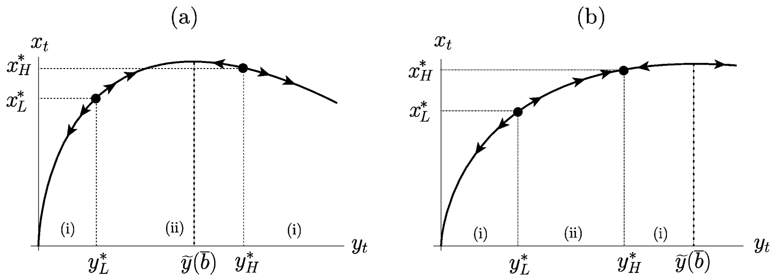

FIGURE 3. Local dynamics of (x_t, y_t).

determinate, whereas the pair that converges to $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, \overline{b})$ can be either determinate or indeterminate.

Thus, as long as Assumption 1 holds, we can show the following fact as a corollary of Proposition 1:[17]

COROLLARY (Local Stability). *(a) The low-growth steady state* $(x_{\mathrm{L}}^*, y_{\mathrm{L}}^*, \overline{b})$ *is locally saddle-point stable, whereas (b) the high-growth steady state* $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, \overline{b})$ *is locally saddle-point stable if* $y_{\mathrm{L}}^* < \widetilde{y}(\overline{b}) < y_{\mathrm{H}}^*$, *but locally stable if* $y_{\mathrm{H}}^* < \widetilde{y}(\overline{b})$.

Consequently, both steady states are economically meaningful in the sense that both are accessible.[18]

Proposition 1, especially part c, together with the above-mentioned corollary indicates that the economy can be stuck in the steady state of a growth trap with insufficient supply of productive government services. Therefore, if this is the case, some may wonder if it can be true that the government purposely continues to choose y_{L}^*, even though it recognizes that the economy is now stuck in the growth trap. The result obtained in this section, however, shows the difficulties the government faces in helping the economy escape from this trap. In this model, the economy falls into the growth trap, not only because of the insufficient supply of productive government services, but also from households' holding pessimistic expectations. To see this more concretely, suppose that at date 0 the economy is trapped in a low-growth steady state $(x_{\mathrm{L}}^*, y_{\mathrm{L}}^*, \overline{b})$, and the government changes its policy scheme so that $y_t =$ (or $\rightarrow$) y_{H}^*. However, as shown in Figure 3, such policy reform is not feasible unless households *simultaneously* change their consumption path so that $x_t =$ (or $\rightarrow$)x_{H}^*, because x_t and y_t must satisfy $x_t = \psi(y_t, \overline{b})$ for all $t \geq 0$ and $x_{\mathrm{L}}^* \neq x_{\mathrm{H}}^*$, as shown in Lemma 1. Thus, unless the government can achieve both an expansion of spending and fine control of households' expectations, the latter of which is considered to be difficult, even for the government, it cannot induce the economy to escape from the growth trap.

4. CHARACTERS OF STEADY STATES

In this section, we examine how changes in the policy variables affect the steady states of the economy.[19] As shown, whether the government uses income tax or

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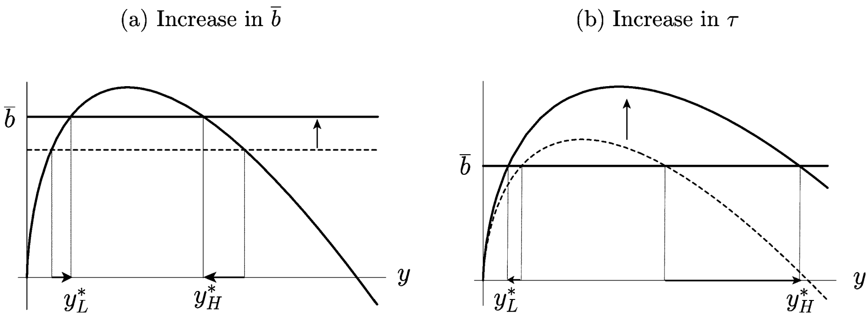

FIGURE 4. Effects of $\bar{b}$ and τ on y_i^*.

government bonds to finance the cost of expenditure makes the results differ significantly.

We first examine the effects of changes in the target level of government bonds, $\bar{b}$, on the steady states. These effects can easily be examined by using (13). When the target level $\bar{b}$ increases, the horizontal line goes up, as shown in panel (a) of Figure 4. This implies that the government is confronted by a more severe constraint; it must achieve an enlargement of the primary surplus without changing the tax rate. Therefore, the ratio of public services to private capital in the low-growth steady state, y_{L}^*, rises, because in this state public service is productive, and by increasing government spending, tax revenue increases more than government spending itself. On the other hand, the ratio in the high-growth steady state, y_{H}^*, is reduced, because in this state public service is less productive and hence the government must cut off expenditure in order to expand the primary surplus.

Next, we examine the effects of changes in the tax rate τ on the steady states. Similarly to the case of $\bar{b}$, it is useful to use (13). When the tax rate τ increases, the graph of the RHS of (13) goes up (see panel b of Figure 4). This implies that the government is confronted by a looser constraint, in the sense that it can achieve an enlargement of the primary surplus without changing the level of spending. However, the long-run present value of the primary surplus must be at the same level, since the target level $\bar{b}$ does not change. Then, in the low-growth steady state, the government must cut expenditure in order to pull back the level of the primary surplus to $\bar{b}$: that is, y_{L}^* is reduced. In contrast, in the high-growth steady state the government can increase spending: that is, y_{H}^* rises.

We summarize these results in the following lemma:

LEMMA 2. *The effects of changes in policy variables $\bar{b}$ and τ are the following:*

$$\frac{\partial y_{\mathrm{L}}^*}{\partial \bar{b}} > 0, \quad \frac{\partial y_{\mathrm{H}}^*}{\partial \bar{b}} < 0,$$

$$\frac{\partial y_{\mathrm{L}}^*}{\partial \tau} < 0, \quad \frac{\partial y_{\mathrm{H}}^*}{\partial \tau} > 0.$$

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This lemma indicates the importance of the method of financing for both developing and developed countries, because these produce significant differences in growth rates. We next examine how the growth rates of the steady states are affected by these changes.

As can be easily seen from (15), we can state the following with respect to the effects of changes in the target level $\overline{b}$ on the growth rates of the steady states:

PROPOSITION 2. *(a) An increase in the target level raises the growth rate at the low-growth steady state; that is,* $\frac{\partial \gamma_L}{\partial \overline{b}} > 0$. *In contrast, (b) an increase in the target level reduces the growth rate at the high-growth steady state; that is,* $\frac{\partial \gamma_H}{\partial \overline{b}} < 0$.

We next turn to the effects of income tax. As can be easily seen from (15), we can state the following with respect to the effects of changes in the tax rate τ on the growth rates of the steady states:

PROPOSITION 3. *(a) An increase in the income tax rate reduces the growth rate at the low-growth steady state; that is,* $\frac{\partial \gamma_L}{\partial \tau} < 0$. *On the other hand, (b) there is a tax rate that maximizes the growth rate at the high-growth steady state.*

Proof. Part a of this proposition is obvious. The proof of part b can be found in Appendix B.

Proposition 3 presents a sharp contrast to Proposition 2. In the low-growth steady state, collecting funds for productive government services through government bonds can promote economic growth. In contrast, in the high-growth steady state, collecting funds for productive government services through government bonds can deteriorate economic growth. This has important policy implications, because developing countries that grow slowly usually face difficulties when collecting funds for government spending. Because of this, it can be rational to raise funds by issuing bonds. On the other hand, developed countries that grow relatively faster should be cautious in using bonds to raise money. Proposition 3 implies that developed countries should use income tax instead of bonds, even though an increase in income tax can reduce the growth rate.

Finally, we compare the welfare level of the low-growth steady state with that of the high-growth steady state. Based on (1), the welfare level at a steady state is calculated by

$$
\begin{aligned}
U_i &= \int_0^\infty \ln C_t \;\exp(-\rho t)\, dt \\
&= \int_0^\infty (\ln C_0 + \gamma_i t)\;\exp(-\rho t)\, dt \\
&= \frac{1}{\rho}(\ln x_i^* + \ln K_0) + \frac{1}{\rho^2}\gamma_i, \;\text{for}\; i \in \{\text{L}, \text{H}\}.
\end{aligned}
$$

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Then, from part c of Proposition 1 and Lemma 1, we can state the following proposition:

PROPOSITION 4. *Suppose that the initial levels of private capital are the same in both of two steady states. Then the welfare level in the high-growth steady state is higher than that in the low-growth steady state.*

This indicates that the low-growth steady state falls not only into a growth trap but also into a poverty trap.

5. CONCLUSION

This paper examines a simple endogenous growth model with productive government spending. When a government relies not just on income taxes, but also on government bonds, two steady states exist under constant debt policies such as the Maastricht criterion. One is associated with low growth and the other is associated with high growth. The low-growth steady state contrasts significantly with the high-growth steady state. Bond finance results in a different result than tax finance in both steady states. The results imply that under such a policy, less developed countries would do better to use bond finance rather than tax finance to raise the growth rate. On the other hand, developed countries should use tax finance rather than bond finance.

NOTES

1. In Japan, the government has attempted to reduce the number of issued government bonds by implementing the Reform of Fiscal Structure law in 1997. One of its main purposes is to reduce the budget deficit to less than 3%.
2. See Ghosh and Mourmouras (2004) for alternative rules.
3. Greiner and Semmler (1999), Turnovsky (2000, Ch.14), and Ghosh and Mourmouras (2004) treat productive government spending as public investment that formulates public capital, that is, as a stock variable. In contrast to these studies, this analysis treats productive government spending as public services instead of public investment. First, although public capital, such as highways or railways, plays an important role, productive public services also play an important role in developing processes. Second, a model using public capital, that is, as a stock variable, makes analyses more difficult because it raises the dimension of the dynamic system of models. Third, although welfare differences exist, as Futagami et al. (1993) show, there is not much difference between public services and public capital when only steady states are investigated.
4. Minea and Villieu (2006) also allow governments to issue government bonds in an endogenous growth model with productive government spending and examine how deficit and debt affect economic growth. However, in contrast to the present paper, in which the ratio of debt to capital stock is constant in the long run, they assume that the ratio of deficit to output is constant over time and examine how a change in this ratio affects economic growth and welfare.
5. De la Croix and Michel (2002) define a constant debt policy as follows: the government maintains the ratio of bonds to population size as a constant.
6. We can set the ratio of bonds to GDP as the target. However, this results in a rule similar to that in the text, because $B_t/Q_t = (1/A)(K_t/G_t)^{1-\alpha}(B_t/K_t)$.
7. Because the government levies a tax on all household assets and because these assets include government bonds, $\tau r_t B_t$ is cancelled out by an accounting gimmick, as if the government were exempt

from interest payments in the form of τr_t units of interest waiver. Note that even when we assume that the government subtracts the tax on household income from the interest receipts of government bonds, the results in this paper are essentially the same, because the interest rate on government bonds and the after-tax rent of physical capital are equated through arbitrage.

8. Once $\{x_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=0}^{\infty}$ are determined, $\{C_t\}_{t=0}^{\infty}$, $\{K_t\}_{t=0}^{\infty}$, and $\{r_t\}_{t=0}^{\infty}$ are determined from (2), (4), and (9). Then $\{K_t\}_{t=0}^{\infty}$ and $\{b_t\}_{t=0}^{\infty}$ determine $\{B_t\}_{t=0}^{\infty}$. On the other hand, $\{K_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=0}^{\infty}$ determine $\{G_t\}_{t=0}^{\infty}$, which in turn determines $\{w_t\}_{t=0}^{\infty}$ from (4).

9. Needless to say, it is obvious that $x_t = 0$ satisfies $\dot{x}_t = 0$. We exclude this case in order to focus on the steady states that make sense from an economic point of view in the main text. We discuss the steady state $x = 0,\ y = 0,\ b = \bar{b}$ in some detail in Appendix A.

10. To see why, let us integrate (7) forward and impose the no-Ponzi-game condition of the government given by $\lim_{t\to\infty} B_t \exp\left[-\int_0^t (1-\tau) r_s ds\right] = 0$ in the result. Then we obtain

$$B_t = \int_t^{\infty} (\tau Q_s - G_s) \exp\left[-\int_t^s (1-\tau) r_v dv\right] ds.$$

At the steady states, if they exist, the ratios of government bonds and government spending to capital stock, b and y, are constant. Furthermore, the interest rate, r, is constant and hence C, Q, K, and G grow at constant rate γ. Therefore, we obtain

$$\bar{b} K_t = \frac{1}{(1-\tau) r - \gamma} (\tau A y^{1-\alpha} K_t - y K_t).$$

Because $(1-\tau) r - \gamma = \rho$, we can rewrite this intertemporal budget constraint as (13).

11. Because (13) is the long-run intertemporal budget constraint of the government with a no-Ponzi-game condition, failure of equation (13) directly implies the violation of the no-Ponzi-game condition.

12. This implies that the dynamic system in this model does not satisfy its causality. “The dynamics satisfies its causality” means that in a dynamic system, say, in $\dot{\mathbf{x}}_t = f(\mathbf{x}_t)$, $\dot{\mathbf{x}}_t$ is uniquely determined from a value of $\mathbf{x}_t$. We would like to thank an anonymous referee for pointing this out.

13. This assumption appears to be plausible, because the levels of government spending rarely change rapidly without any great change in the legal or political system. Moreover, such systems rarely change greatly in general.

14. As mentioned above, without Assumption 1, the dynamic system does not satisfy causality. By introducing Assumption 1, we can avoid this problem as follows: once the values of x_0 and y_0 are determined at the initial point of time, due to Assumption 1, y_t must move continuously and thus $\dot{x}_t/x_t$ must move continuously following equation (19). Therefore, at each point of time other than the initial point of time, $\dot{x}_t/x_t$ is uniquely determined by the value of x_t, and thus the dynamic system satisfies causality.

15. Note that $\tilde{y}(\bar{b}) < y_{\mathrm{L}}^*$ never occurs. To see why, suppose otherwise. Then, from the fact that $\tilde{y}(\bar{b}) < y_{\mathrm{L}}^* < y_{\mathrm{H}}^*$ and the fact that $\psi_y(y, b) < 0$ for $y > \tilde{y}(b)$, x_{L}^* become larger than x_{H}^*. This contradicts Lemma 1. Therefore the possible cases are $y_{\mathrm{L}}^* < \tilde{y}(\bar{b}) < y_{\mathrm{H}}^*$ and $y_{\mathrm{H}}^* < \tilde{y}(\bar{b})$.

16. x_0 is jumpable simply because it is a proxy of the households’ consumption. Note that also y_0 is jumpable, although this is linked with x_0 by (10). This is because y_0 has two candidates given a value of x_0. As Assumption 1 guarantees, once y_0 is chosen from the candidates, from then on y_t continuously evolves with x_t according to the directions of the arrows in Figure 3.

17. We present a more formal analysis of the dynamic system, which includes the case where $b_t \neq \bar{b}$, in Appendix A.

18. This result indicates that the high-growth steady state may exhibit indeterminacy. See Benhabib and Farmer (1999) for indeterminacy. Greiner and Semmler (1999) point out the possibility of a multiplicity of equilibrium paths and the indeterminacy of paths based on an endogenous growth

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model similar to the present model. Their model incorporates public capital instead of public services. However, they do not examine the characteristics of the equilibria. Hu et al. (Forthcoming) examine the indeterminacy issue in a two-sector model with productive government spending. Futagami and Mino (1995) incorporate threshold externalities of public capital into the model and show that multiple equilibria exist.

19. As shown in the Corollary of Proposition 1, the economy necessarily converges to one of two steady states. Comparative static analysis that focuses on the steady state therefore provides a means of understanding the long-run effects of the government's budgetary plans.

20. If we do not impose Assumption 1 on y_t, $\dot{y}_t$ do not necessarily follow (A.4) and y_t can jump discontinuously so as to satisfy (A.1), (A.2), and (A.3). Naturally we need to adopt Assumption 1 to examine the dynamic system that is described by b_t and y_t as in Section 3.

21. $(y, b) = (0, \bar{b})$ is a steady state also. Moreover, as we can find from Figure A.2., this steady state is a sink. On the paths converging to the steady state, the values of both x_t and y_t decline to zero, and the growth rate of capital, $\dot{K}_t/K_t = Ay_t{}^{1-\alpha} - y_t - x_t$ approaches zero, and thus the volume of output, $Y_t = Ay_t{}^{1-\alpha}K_t$ and consumption, $C_t = x_t K_t$ are declining to zero. In particular, when $\rho\bar{b} > \zeta(y, \tau)$, that is, when the target level of debt-capital ratio is too high, $(y, b) = (0, \bar{b})$ is the only steady state and there exists no steady state with sustained growth. In such a case, whatever the initial level of debt-capital ratio is, each economy eventually converges to this steady state with no production and no consumption.

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APPENDIX A

THE DYNAMIC SYSTEM WHEN $b_t \neq \bar{b}$

In this Appendix, we examine the dynamic system of the economy when $b_t \neq \bar{b}$ as well as when $b_t = \bar{b}$. The dynamic system is characterized by the following differential equations of b_t and x_t:

$$\dot{b}_t = -\phi(b_t - \bar{b}), \tag{A.1}$$

$$\dot{x}_t = [x_t - \chi(y_t) - \rho]\, x_t, \tag{A.2}$$

$$x_t = \psi(y_t, b_t) \equiv \chi(y_t) + \frac{\zeta(y_t)}{b_t} - \phi\left(1 - \frac{\bar{b}}{b_t}\right), \tag{A.3}$$

where $\chi(y_t) \equiv [1 - (1 - \tau)\alpha]\, Ay_t^{1-\alpha} - y_t$. In this Appendix, the equilibrium paths are characterized by the dynamic system of b_t and y_t on Assumption 1. Once the values of y_t and b_t are given on Assumption 1, (A.3) determines a value of x_t and the change of y_t, $\dot{y}_t$ is determined in such a way as to satisfy (A.1), (A.2), and (A.3).

First, using (A.1), (A.2), and (A.3), we derive $\dot{y}_t$. Differentiating both the sides of equation (A.3) with respect to time t, we obtain

$$\dot{x}_t = \psi_y(y_t, b_t)\dot{y}_t + \psi_b(y_t, b_t)\dot{b}_t,$$

where $\psi_y \equiv \frac{\partial \psi}{\partial y_t}$ and $\psi_b \equiv \frac{\partial \psi}{\partial b_t}$. By rearranging, we obtain[20]

$$\dot{y}_t = \frac{1}{\psi_y(y_t, b_t)}\left[\dot{x}_t - \psi_b(y_t, b_t)\dot{b}_t\right]. \tag{A.4}$$

We focus our analysis on the range of y_t that satisfies $\zeta(y_t) \geq 0$. Then we obtain $\psi_b(y_t, b_t) < 0$ because $\psi_b(y_t, b_t) = -b_t^{-2}(\zeta(y_t) + \phi\bar{b})$. Therefore when $\dot{x}_t < 0$ and $\dot{b}_t < 0$, the sign of the bracketed expression on the right-hand side of (A.4) takes a negative value; on the other hand, when $\dot{x}_t > 0$ and $\dot{b}_t > 0$, the sign of the bracketed term on the right-hand side of (A.4) takes a positive value. From (A.2) and (A.3), the $\dot{x}_t = 0$ locus is given by $b_t = [\zeta(y_t) + \phi\bar{b}]/(\phi + \rho)$. Then we can draw the $\dot{x}_t = 0$ locus as well as the $\dot{b}_t = 0$, as depicted in Figure A.1. In addition, using the definition of $\widetilde{y}(b_t)$, we obtain $\psi_y(y_t, b_t) < (>) 0$ when $y > (<) \widetilde{y}(b_t)$. By summing these facts, we can find whether $\dot{y}_t > 0$ or $\dot{y}_t < 0$ at each point of the plane of (b_t, y_t) except the shaded region as depicted in Figure A.1. Furthermore, we can expect that the $\dot{y}_t = 0$ line is between the $\dot{x}_t = 0$ locus and the $\dot{b}_t = 0$ locus.

Next we consider the borderline that divides the $y_t - b_t$ plane into the region in which $\dot{y}_t$ takes a positive value and the region in which $\dot{y}_t$ takes a negative value. Whether $\dot{y}_t$ is positive or negative depends on the signs of ψ_y and $(\dot{x}_t - \psi_b\dot{b}_t)$. Because we already know the region where the sign of ψ_y is positive or negative (see (18)), what we must examine is only the sign of $(\dot{x}_t - \psi_b\dot{b}_t)$. Substituting (A.1) and (A.2), (A.3) into $(\dot{x}_t - \psi_b\dot{b}_t) \geq 0$ yields

$$\left[\chi(y_t) + \frac{\zeta(y_t)}{b_t} - \phi\left(1 - \frac{\bar{b}}{b_t}\right)\right]\left[\frac{\zeta(y_t)}{b_t} - \phi\left(1 - \frac{\bar{b}}{b_t}\right) - \rho\right] \geq \left[\frac{\zeta(y_t)}{b_t} + \phi\frac{\bar{b}}{b_t}\right]\phi\left(1 - \frac{\bar{b}}{b_t}\right). \tag{A.5}$$

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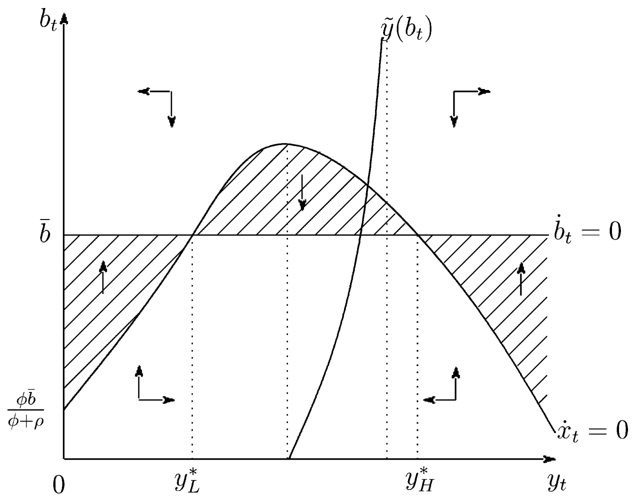

FIGURE A.1. The region where the sign of $\dot{y}_t$ is unknown.

This inequality can be reduced to the following quadratic inequality with respect to $\hat{b}_t \equiv b_t - \bar{b}$:

$$\dot{x}_t - \psi_b \dot{b}_t \geq 0 \Longleftrightarrow a_1(y_t)\hat{b}_t^{\,2} + a_2(y_t)\hat{b}_t + a_3(y_t) \geq 0, \tag{A.6}$$

where

$$a_1(y_t) = (\phi + \rho)\left[\phi - \chi(y_t)\right],$$
$$a_2(y_t) = -[\phi - \chi(y_t)]\left[\zeta(y_t) - \rho\bar{b}\right] - (\phi + \rho)\left[\chi(y_t)\bar{b} + \zeta(y_t)\right] - \phi\left[\zeta(y_t) + \phi\bar{b}\right],$$
$$a_3(y_t) = \left[\chi(y_t)\bar{b} + \zeta(y_t)\right]\left[\zeta(y_t) - \rho\bar{b}\right].$$

When the quadratic inequality holds with equality, the two solutions of the quadratic equation are given by

$$\hat{b}_1(y_t) \equiv \frac{-a_2(y_t) - \sqrt{a_2(y_t)^2 - 4a_1(y_t)a_3(y_t)}}{2a_1(y_t)},$$

$$\hat{b}_2(y_t) \equiv \frac{-a_2(y_t) + \sqrt{a_2(y_t)^2 - 4a_1(y_t)a_3(y_t)}}{2a_1(y_t)}.$$

These two solutions, which depend on y_t, are candidates for the borderline. In what follows, we will show that the $\dot{y}_t = 0$ locus is given by $\hat{b}_t = \hat{b}_1(y_t)$, whether $a_1(y_t)$ is positive or not.

First, when $a_1(y_t) > 0$, that is, $\chi(y_t) < \phi$, the quadratic function is convex; the range of the values of $\hat{b}_t$ that satisfy (A.5) is given by $\hat{b}_t \leq \hat{b}_1(y_t)$ or $\hat{b}_t \geq \hat{b}_2(y_t)$. In addition, the values of $(\hat{b}_t, y_t)$ must satisfy $x_t (\equiv C_t/K_t) \geq 0$ also. From (A.3), $x_t \geq 0$ if the following inequality holds:

$$[\phi - \chi(y_t)]\, b_t \leq \zeta(y_t) + \phi\bar{b}.$$

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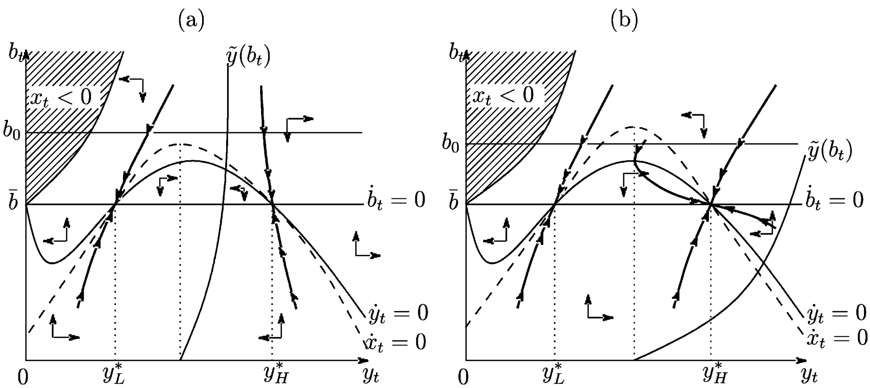

FIGURE A.2. The dynamics of the economies when $b_t \neq \bar{b}$.

As we can find easily, when $\phi < \chi(y_t)$, this inequality necessarily holds. When $\phi > \chi(y_t)$, we obtain $x_t \geq 0$ if and only if the following inequality holds:

$$b_t \leq b_{\max}(y_t) \equiv \frac{\zeta(y_t) + \phi\bar{b}}{\phi - \chi(y_t)}.$$

The region where $x_t \leq 0$ is given by the shaded one in Figure A.2. We examine the relation among $\hat{b}_1(y_t)$, $\hat{b}_2(y_t)$, and $b_{\max}(y_t)$. According to the definition of $b_{\max}(y_t)$ and (A.3), if $b_t = b_{\max}(y_t)$, $x_t = 0$ and thus $\dot{x}_t = 0$. Moreover, because $b_{\max}(y_t) > \bar{b}$, we obtain $\dot{b}_t < 0$. Thus $(\dot{x}_t - \psi_b \dot{b}_t) = -\psi_b \dot{b}_t < 0$ when $b_t = b_{\max}(y_t)$; that is, the left-hand side (LHS) of inequality (A.6) is negative. We find that the larger root of the quadratic equation, $\hat{b}_2(y_t)$, is larger than $(b_{\max}(y_t) - \bar{b})$. Therefore $\hat{b}_2(y_t)$ has no economic meaning and consequently the range of the values of $(\hat{b}_t, y_t)$ that satisfies (A.5) is given by $\hat{b}_t \leq \hat{b}_1(y_t)$.

Next we consider the case when $a_1(y_t) < 0$; that is, $\chi(y_t) > \phi$. In this case, the quadratic function is concave and thus the range that the values of $\hat{b}_t$ must satisfy is given by $\hat{b}_2(y_t) \leq \hat{b}_t \leq \hat{b}_1(y_t)$. In addition, when $b_t = 0$, $\dot{b}_t > 0$ and $\dot{x}_t > 0$ from (A.3). Then we can easily find that $(\dot{x}_t - \psi_b \dot{b}_t) > 0$ when $b_t = 0$, and $\hat{b} = -\bar{b}$ is larger than the smaller root of the quadratic equation (A.6), $\hat{b}_2(y_t)$. Therefore, focusing our analysis on the case when $b_t \geq 0$, the region where $(\dot{x}_t - \psi_b \dot{b}_t) \geq 0$ becomes $(-\bar{b} \leq)\hat{b}_t \leq \hat{b}_1(y_t)$.

To sum up, the $\dot{y}_t = 0$ locus is given by $b_t = \bar{b} + \hat{b}_1(y_t)$, whether $a_1(y_t)$ is positive or not. This locus intersects the $\dot{b}_t = 0$ locus at $y_t = 0$, y_L^*, and y_H^* as depicted in Figure A.2.[21] Furthermore, if $y_t < \tilde{y}(b_t)$, in the region above this locus $\dot{y}_t < 0$ and $\dot{y}_t > 0$ in the region under this locus.

We can draw the phase diagram of the economy as depicted in panels (a) and (b) in Figure A.2 when $b_t \neq \bar{b}$. According to Assumption 1, y_t cannot jump at any point of time other than the initial point of time. On the assumption, once the value of b_t, b_0 is given at the initial point of time as depicted in Figure A.2, the initial value of y_t, y_0 is determined, and after this the motion of y_t follows the phase diagram of (b_t, y_t) without any discontinuous change. As a result, as we can see from Figure A.2, there exists a saddle path converging to the low-growth steady state y_L^*, and there exists a saddle path converging to the high-growth steady state y_H^* or there exist infinite paths converging to this: panel (a) corresponds to the case where the steady state $y_t = y_H^*$ is a saddle point and panel (b) corresponds to the case

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where the steady state $y_t = y_H^*$ is a sink. As mentioned in the Corollary in Section 3, if $y_H^* > \tilde{y}(\bar{b})$, the equilibrium paths are given by panel (a), while if $y_H^* < \tilde{y}(\bar{b})$, the equilibrium paths are given by panel (b). In either case, for a given b_t, there are multiple equilibrium paths for y_t that converging to y_L^* or y_H^*.

APPENDIX B

PROOF OF PROPOSITION 3

In this Appendix, we prove part (b) of Proposition 3. Differentiating (15) and (13) with respect to y and τ and combining them, we have

$$\frac{d\gamma_i^*}{d\tau} = -A\alpha y_i^{*-\alpha}\left[y_i^* + (1-\tau)(1-\alpha)\frac{\zeta_\tau(y_i^*;\tau)}{\zeta_y(y_i^*;\tau)}\right], \tag{B.1}$$

where $\zeta_\tau(y;\tau) = Ay^{1-\alpha} > 0$. From (14),

$$\lim_{y\downarrow\hat{y}(\tau)} 1/\zeta_y(y,\tau) = -\infty \quad \text{for all} \quad \tau \in (0,1). \tag{B.2}$$

Let $\hat{\tau}$ denote the tax rate that satisfies $\overline{b} = \zeta(\hat{y}(\tau),\tau)$. Then $\lim_{\tau\downarrow\hat{\tau}} y_i^* = \hat{y}(\hat{\tau})$. Applying this into (B.2),

$$\lim_{\tau\downarrow\hat{\tau}} 1/\zeta_y(y_i^*,\tau) = -\infty. \tag{B.3}$$

From (B.1) and (B.3), we obtain

$$\begin{aligned}\lim_{\tau\downarrow\hat{\tau}} \frac{d\gamma_H^*}{d\tau} &= -A\alpha\hat{y}(\hat{\tau})^{-\alpha}\left[\hat{y}(\hat{\tau}) + (1-\tau)(1-\alpha)\frac{\zeta_\tau(\hat{y}(\hat{\tau});\tau)}{\zeta_y(\hat{y}(\hat{\tau});\hat{\tau})}\right]\\ &= \infty.\end{aligned}$$

On the other hand, from (B.1) we immediately obtain

$$\lim_{\tau\uparrow 1} \frac{d\gamma_H^*}{d\tau} = -A\alpha y_H^{*1-\alpha} < 0.$$

Because γ_H is a continuous function of τ for $\tau \in [\hat{\tau}, 1]$, there exists at least one value of the tax rate at which $d\gamma_H^*/d\tau = 0$ where the growth rate takes a maximum value. ■