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ON THE LOCAL LANGLANDS CORRESPONDENCE FOR SPLIT CLASSICAL GROUPS OVER LOCAL FUNCTION FIELDS

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Abstract We prove certain depth bounds for Arthur's endoscopic transfer of representations from classical groups to the corresponding general linear groups over local fields of characteristic 0, with some restrictions on the residue characteristic. We then use these results and the method of Deligne and Kazhdan of studying representation theory over close local fields to obtain, under some restrictions on the characteristic, the local Langlands correspondence for split classical groups over local function fields from the corresponding result of Arthur in characteristic 0.

Keywords: local Langlands correspondence; close local fields; endoscopy

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1. Introduction

The purpose of this article is to prove the local Langlands correspondence for split classical groups over local function fields (with some restrictions on the characteristic) using the work of Arthur [7] in characteristic 0 and the Deligne–Kazhdan philosophy. The Deligne–Kazhdan correspondence can be summarized as follows.

(a) Given a local field F' of characteristic p and an integer $m \geqslant 1$, there exists a local field F of characteristic 0 such that F' is m-close to F, i.e., $\mathfrak{O}_F/\mathfrak{p}_F^m \cong \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$.

(b) In [22], Deligne proved that, if F and F' are m-close, then

$$\mathrm{Gal}(\bar{F}/F)/I_F^m \cong \mathrm{Gal}(\bar{F}'/F')/I_{F'}^m,$$

where $\bar{F}$ is a separable algebraic closure of F, I_F is the inertia subgroup, and I_F^m denotes the mth higher ramification subgroup of I_F with upper numbering. This gives a

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bijection

$$\{\text{Continuous, complex, finite dimensional representations of } \mathrm{Gal}(\bar{F}/F) \text{ trivial on } I_F^m\}$$
$$\longleftrightarrow \{\text{Cont., complex, f.d. representations of } \mathrm{Gal}(\bar{F}'/F') \text{ trivial on } I_{F'}^m\}.$$

Moreover, all of the above holds when $\mathrm{Gal}(\bar{F}/F)$ is replaced by W_F, the Weil group of F.
(c) Let $\underline{\mathrm{G}}$ be a split connected reductive group defined over $\mathbb{Z}$. For an object X associated to the field F, we will use the notation X' to denote the corresponding object over F'. In [41], Kazhdan proved that, given $m \geqslant 1$, there exists $l \geqslant m$ such that, if F and F' are l-close, then there is an algebra isomorphism $\mathrm{Kaz}_m : \mathcal{H}(\underline{\mathrm{G}}(F), K_m) \to \mathcal{H}(\underline{\mathrm{G}}(F'), K'_m)$, where K_m is the mth usual congruence subgroup of $\underline{\mathrm{G}}(\mathfrak{O}_F)$. Hence, when the fields F and F' are sufficiently close, we have a bijection

$$\{\text{Irreducible admissible representations } (\sigma, V) \text{ of } \underline{\mathrm{G}}(F) \text{ such that } \sigma^{K_m} \neq 0\}$$
$$\longleftrightarrow \{\text{Irreducible admissible representations } (\sigma', V') \text{ of } \underline{\mathrm{G}}(F') \text{ such that } \sigma'^{K'_m} \neq 0\}.$$

These results suggest that, if one understands the representation theory of $\mathrm{Gal}(\bar{F}/F)$ for all local fields F of characteristic 0, then one can use it to understand the representation theory of $\mathrm{Gal}(\bar{F}'/F')$ for a local field F' of characteristic p, and similarly, with an understanding of the representation theory of $\underline{\mathrm{G}}(F)$ for all local fields F of characteristic 0, one can study the representation theory of $\underline{\mathrm{G}}(F')$, for F' of characteristic p. Such applications of this philosophy can be found in [8, 9, 29, 49].

For the rest of the introduction, $\underline{\mathrm{H}}$ will denote one of the split classical groups Sp_{2n}, SO_{2n+1}, or SO_{2n} and N will denote the rank of the group Langlands dual to $\underline{\mathrm{H}}$. These groups can be realized as endoscopic groups of $\widetilde{\mathrm{GL}}_N$, where $\widetilde{\mathrm{GL}}_N$ is the twisted space of GL_N with respect to a nontrivial outer automorphism θ. In [7], Arthur, among many things, defined and characterized the local Langlands correspondence for $H := \underline{\mathrm{H}}(F)$, where F is a non-Archimedean local field of characteristic 0, via endoscopic character identities. Let $\Pi_{\mathrm{temp}}(H)$ denote the set of tempered representations of H, and let $\tilde{\Pi}_{\mathrm{temp}}(H)$ denote the $\mathrm{Out}(\underline{\mathrm{H}})$-orbits of representations of H, where $\mathrm{Out}(\underline{\mathrm{H}})$ is the set of outer automorphisms of $\underline{\mathrm{H}}$ over F (note that $\Pi_{\mathrm{temp}}(H) = \tilde{\Pi}_{\mathrm{temp}}(H)$ when $\underline{\mathrm{H}}$ equals Sp_{2n} or SO_{2n+1}). Similarly, we let $\tilde{\Phi}_{\mathrm{bdd}}(H)$ denote the set of $\hat{\underline{\mathrm{H}}}$-conjugacy classes of tempered parameters of H when $\underline{\mathrm{H}}$ equals Sp_{2n} or SO_{2n+1}, and the $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy classes of parameters when $\underline{\mathrm{H}} = \mathrm{SO}_{2n}$. Arthur's Langlands parameterization includes a surjective finite-to-one map from $\tilde{\Pi}_{\mathrm{temp}}(H)$ to $\tilde{\Phi}_{\mathrm{bdd}}(H)$. In addition, for $\phi \in \tilde{\Phi}_{\mathrm{bdd}}(H)$ it parameterizes the set $\tilde{\Pi}_\phi$, that is the fiber of the above map over ϕ, by establishing a bijection between $\tilde{\Pi}_\phi$ and $\hat{\mathcal{S}}_\phi$, where $\mathcal{S}_\phi$ is the component group of ϕ (see § 2.2). Let F' be a non-Archimedean local field of characteristic p. It is natural to ask if one can obtain the Langlands correspondence for $\underline{\mathrm{H}}(F')$ using the results of Arthur [7] in characteristic 0 and the Deligne–Kazhdan philosophy. More precisely, let π' be an irreducible admissible representation of $\underline{\mathrm{H}}(F')$ with $\mathrm{depth}(\pi') \leqslant m$. Here, the notion of depth is defined via the Moy–Prasad filtration subgroups (see [62, 63]). We want to attach a Langlands parameter ϕ' to π', by choosing an l-close local field F of characteristic 0, where l is an integer that depends only on m and perhaps N, and force the following diagram to be commutative.

$$\begin{array}{ccccc} F' & \qquad & \pi' & \dashrightarrow & \phi' \\ l\text{-close} \, \vdots & & \text{Kazhdan} \downarrow & & \uparrow \text{Deligne} \\ F & & \pi & \xrightarrow{\text{Arthur}} & \phi \end{array} \tag{1.0.1}$$

The following questions naturally arise in this setting.

(1) It is easy to show that the Kazhdan isomorphism preserves depth, so the irreducible representation π of $\underline{\mathrm{H}}(F)$ has depth at most m. Now, to obtain the ϕ' from ϕ via the Deligne isomorphism, we need to know that $\phi|_{I_F^l} = 1$, or, in other words, that $\mathrm{depth}(\phi) < l$. In summary, we need to know that, if $\mathrm{depth}(\pi) \leqslant m$, then $\mathrm{depth}(\phi) < l$, where l is an integer that depends only on m and N.

(2) We want to know that the ϕ' obtained as in (1.0.1) is independent of this local field F of characteristic 0 used to define it. In addition, we also need that the Langlands parameter defined in this manner satisfies the expected properties, and is uniquely characterized by those properties.

(3) As in characteristic 0, we need to establish a bijection between $\tilde{\Pi}_{\phi'}$ and the characters of the component group $\mathcal{S}_{\phi'}$. Furthermore, we want this bijection to be also compatible with the corresponding bijection in characteristic 0 via the Deligne–Kazhdan theory.

The main purpose of this article is to provide a complete answer to (1) and (2) when p is odd. We also illustrate how the forthcoming work of Mœglin [58] gives a solution to (3) by describing this solution for discrete series L-packets of $\underline{\mathrm{H}} = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}$ when p is sufficiently large. A careful study of [58] and [7, § 2.4] should be enough to give a complete picture on (3), but we do not pursue it here. Let us summarize the results of this paper.

As for Question (1), it is expected that the local Langlands correspondence (LLC) preserves depth at least when the residue characteristic is large enough. Such a result is already known for the LLC for tori [87], GL_n [86], and for GSp_4 in odd residue characteristic [29, 67]. In this article, we show that, when the residue characteristic of F is greater than 2,

$$\mathrm{depth}(\pi) \leqslant m \implies \mathrm{depth}(\phi) \leqslant m + 1. \tag{1.0.2}$$

The proof of this result occupies a large part of this paper, and the idea is as follows. Since the LLC for GL_N preserves depth, it is enough to show that, if $\mathrm{depth}(\pi) \leqslant m$, then the functorial lift π^{GL} of π to $\mathrm{GL}_N(F)$ has a nonzero K_{m+2}^{GL}-fixed vector, where K_m^{GL} denotes the mth usual congruence subgroup in $\mathrm{GL}_N(F)$. This statement will be a consequence of endoscopic character identities that were proved for classical groups by Arthur [7], as soon as we know that, for some $a \neq 0$, $a \cdot \mathbb{1}_{K_m}$ is a 'transfer' of the characteristic function $\mathbb{1}_{K_m^{\widetilde{\mathrm{GL}}}}$ of $K_m^{\mathrm{GL}}\theta \subset \widetilde{\mathrm{GL}}_N(F)$, i.e., that $a \cdot \mathbb{1}_{K_m}$ and $\mathbb{1}_{K_m^{\widetilde{\mathrm{GL}}}}$ have matching orbital integrals. This is achieved using the fundamental lemma for Lie algebras as follows. Using semisimple descent to deal with twisted endoscopy, the Cayley transform $\mathfrak{c}$ to pass between the set of topologically nilpotent elements of the Lie algebra, and that of the topologically unipotent elements of the group, we show that this matching question can reduced to an analogous question for Lie algebras. Here we note that we have used the Cayley transform and not the usual exponential map to reduce ourselves to the Lie algebra situation. The reason for this is that, although the exponential map is a diffeomorphism between the set of topologically nilpotent elements of the Lie algebra and the set of topologically unipotent elements of the group when p is large (that is essential to prove the aforementioned matching), how large p needs to be depends on the ramification index of F over $\mathbb{Q}_p$. However, to apply the philosophy explained in Diagram (1.0.1) for a fixed local field F' of positive characteristic p, we need the depth bound in equation (1.0.2) to hold for all local fields F of characteristic 0 with residue characteristic p. The Cayley transform has this feature for $p \neq 2$. However, the Cayley transform is not compatible with matching of semisimple elements in the context of nonstandard endoscopy. Therefore we use the Cayley transform $\mathfrak{c}$ together with a variant $\mathfrak{c}'$ (see Lemma 6.3.1 for details). Now, the matching of the relevant characteristic functions at the level of Lie algebras is done using the fundamental lemma (standard or nonstandard, depending on the case) for Lie algebras and a shrinking argument (that uses the homogeneity of nilpotent orbital integrals, as in, for example, [74, pp. 323–327]). The last point that remains to be explained concerns why the fundamental lemma for Lie algebras holds for $p \neq 2$. The work of Ngô [65] on the fundamental lemma for Lie algebras in positive characteristic (combined with [85] or [15]) gives the fundamental lemma for Lie algebras in characteristic 0, provided the residue characteristic p is a sufficiently large integer determined by the absolute root datum of $\underline{\mathrm{H}}$. Waldspurger [82] proved that the fundamental lemma for (unit elements in the spherical Hecke algebras of) groups follows from the one for Lie algebras (for sufficiently large p).

In the situation of interest to us, Lemaire, Mœglin, and Waldspurger have shown that the same in fact holds without any restriction on p; see [48, Proposition 4.13] (see also [7, pp. 412–413]. We also note that an analogous result in the case of standard endoscopy holds quite generally; cf. [32]). Using a suggestion of Mœglin and Waldspurger, we show that the fundamental lemma for Lie algebras can be deduced from the one for groups provided $p \neq 2$. Combining this with the above results, we obtain that the fundamental lemma for Lie algebras holds for $p \neq 2$, and hence equation (1.0.2) holds for $p \neq 2$.

To address (2), we have to understand a set of properties that characterize the LLC over close local fields. Consider two approaches to obtaining a characterization of the LLC: one by using the theory of L- and γ-factors and Plancherel measures à la [27], and the other via the theory of endoscopy; cf. [7]. In [7], as mentioned earlier, Arthur defined and characterized the correspondence for classical groups via certain endoscopic character identities. At present, we do not know how to prove that these endoscopic character identities are compatible with the Deligne–Kazhdan theory. However, the former approach, namely a characterization in terms of local L- and ϵ-factors, would put Question (2) for split classical groups in the setting of [29, §§ 5 and 6]. These sections of [29] study the compatibility of the Langlands–Shahidi local coefficient, which gives rise to the theory of L- and γ-factors, and Plancherel measures, over close local fields. Using several important results of [7], along with [16, 60], we can reconcile the two characterizations and obtain that Arthur's Langlands correspondence also matches the

L- and γ-factors (respectively, the Plancherel measures) of pairs of generic (respectively, nongeneric) representations $\pi \times \sigma$ of $H \times \mathrm{GL}_r(F)$, $r \leqslant N-1$ with the corresponding Artin factors (respectively, a suitable product of Artin factors). In addition, we show that the Langlands parameter of a representation a discrete series representation of H of depth at most m is uniquely determined by these properties using only the representations σ of depth at most l, where l depends only on m and N; this restriction on the depth of σ is crucial for our purposes (see Theorem 12.8.1). We consequently obtain that the Langlands parameter defined in positive characteristic via (1.0.1) is independent of this choice of the field F of characteristic 0, and that the map $\mathrm{LLC}' : \tilde{\Pi}_{\mathrm{temp}}(H') \to \tilde{\Phi}_{\mathrm{bdd}}(H')$ preserves invariants such as L- and γ-factors and Plancherel measures (see § 13.6). These results hold provided $\mathrm{char}(F') > 2$.

Next, we turn our attention to Question (3) on understanding the surjectivity of the map LLC' and its fibers. In characteristic 0, for the groups $\underline{\mathrm{H}} = \mathrm{Sp}_{2n}$ and SO_{2n+1} and for any $\phi \in \Phi_2(H)$, a forthcoming work of Mœglin [58] explicitly describes the character of the component group $\epsilon_\pi : \mathcal{S}_\phi \to \{\pm 1\}$, attached to $\pi \in \Pi_\phi$ in terms of certain normalized intertwining operators using certain results from [7, Ch. 2]. It was shown in [29] that intertwining operators are compatible with the Deligne–Kazhdan theory. Hence this puts Question (3) in the setting of the Deligne–Kazhdan theory. However, the hurdle in establishing this bijection in positive characteristic is the following. Suppose that $\phi' \in \Phi_{\mathrm{bdd}}(H')$ with $\mathrm{depth}(\phi') \leqslant m$. For an integer l that depends only on m and N, and a field F of characteristic 0 that is l-close to F', consider the following diagram (with self-explanatory notation):

$$\begin{array}{ccccc} F' & \qquad & \phi' & \cdots\cdots\cdots\to & \Pi_{\phi'} \\ l\text{-close} \,\vdots & & \text{Deligne} \downarrow & & \uparrow\vdots\, \text{Kazhdan} \\ F & & \phi & \xrightarrow{\quad\text{Arthur}\quad} & \Pi_\phi \end{array} \tag{1.0.3}$$

It is clear that $\mathrm{depth}(\phi) \leqslant m$. But in order to choose l so as to force the Kazhdan isomorphism to be defined on Π_ϕ, we need that

$$\mathrm{depth}(\phi) \leqslant m \implies \mathrm{depth}(\pi) \leqslant l' \qquad \forall \pi \in \Pi_\phi, \tag{1.0.4}$$

where l' depends only on m and N. Note that the above would give a bijection between $\Pi_{\phi'}$ and Π_ϕ. In fact, without the above, we do not even know that $\Pi_{\phi'} \neq \emptyset$ (that is, that the LLC' is surjective). In this article, we prove that equation (1.0.4) holds with $l' = m$ provided p is a large enough integer completely determined by the absolute root datum of $\underline{\mathrm{H}}$ (this result includes the case when $\underline{\mathrm{H}}$ is the quasi-split SO_{2n}; see § 10 for details). Let us briefly explain the idea of the proof. This is not as straightforward as (1.0.2), since we cannot determine whether an irreducible admissible representation π^{GL} of GL_N has a K_m^{GL}-fixed vector using twisted trace (recall that Arthur's endoscopic character identities involve the twisted trace, and not the usual trace, on general linear groups). We are not able to come up with reasonable compact open subgroups the existence of fixed vectors with respect to which can be detected using the twisted trace. Instead, our approach is to use the endoscopic character identities to show that the range of

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validity of the Harish-Chandra–Howe character expansion behaves well with respect to endoscopic transfer. This helps because the range of validity of the character expansion of an irreducible admissible representation is closely related to its depth. While it is a result of Waldspurger [20, 80] that (under suitable hypotheses) the character expansion is valid on a certain range determined by the depth of the representation, what we need is something like a converse. We prove something like a converse (under the same hypotheses as in [20]), using a certain function introduced by Waldspurger and DeBacker, featuring in the latter's classification of nilpotent orbits on a p-adic group [21] (see Corollary 10.6.4). If $\underline{H} = SO_{2n+1}$, we give a crude depth bound for the generic member of the packet using the work [79], but without any assumptions on the residue characteristic. Using this, and the description of ϵ_π of Mœglin, we obtain a bijective map $\Pi_{\phi'} \to \hat{\mathcal{S}}_{\phi'}$ for $\phi' \in \Phi_2(H')$, where $\underline{H} = Sp_{2n}, SO_{2n+1}$, that is also compatible with the theory in characteristic 0.

Finally, we wish to bring to the reader's attention the recent results of Gan and Lomelí on the LLC for supercuspidal representations of quasi-split classical groups over local function fields (see [26]). Further, the work in progress of A. Genestier and V. Lafforgue will obtain, among other things, the LLC for arbitrary reductive groups over these fields.

2. Notation and review

Let F be a non-Archimedean local field with residue characteristic $\neq 2$. Let $\mathfrak{O}_F$ denote its ring of integers and $\mathfrak{p}_F$ its maximal ideal. We let κ denote the residue field of F and q the cardinality of κ.

2.1. The groups

In this article, we will write $\underline{G}$ to denote a general algebraic group over F, and $\mathfrak{g}$ for its Lie algebra. We will often write G to denote the F-points of $\underline{G}$. For an automorphism θ of $\underline{G}$, we will write $\underline{G}^\theta$ for the θ-fixed subgroup of $\underline{G}$, and $\underline{G}_\theta$ for the identity component $(\underline{G}^\theta)^0$ of $\underline{G}^\theta$. We will reserve $\underline{\mathsf{G}}$ to denote $\mathrm{GL}(V)$ for a finite-dimensional vector space V over F, and $\mathsf{g} = \mathrm{End}(V)$ for its Lie algebra. We will sometimes write G for the group of F-points of $\underline{\mathsf{G}}$.

2.1.A. The twisted space. Consider the 'twisted space' $\tilde{\underline{\mathsf{G}}}$ over $\underline{\mathsf{G}}$ of nondegenerate bilinear forms on $V \times V$ (cf. [83, pp. 42–43], 'Le cas du group linéare tordu', which we follow closely). This variety is a $\underline{\mathsf{G}}$-bitorsor under $g\tilde{x}g'(v, v') = \tilde{x}(g^{-1}v, g'v')$. Fix $\tilde{\theta} \in \tilde{\underline{\mathsf{G}}}$.

Remark 2.1.1. Of particular interest to us will be the case when $\tilde{\theta}$ is obtained by fixing an ordered basis $e_1, \ldots, e_n$ of V and an element $\nu \in F^\times$, and setting $\tilde{\theta}(e_k, e_l) = \nu(-1)^k \delta_{k,d+1-l}$.

We will write $\underline{H}$ to denote one of the following groups for some $n \in \mathbb{N}$.

(a) $\underline{H} = \mathrm{Sp}(W, q_W)/F$, $\dim(V) = 2n + 1$, $\dim W = 2n$.

(b) $\underline{H} = \mathrm{SO}(W, q_W)/F$, $\dim(V) = 2n$, $\dim W = 2n + 1$, such that $\mathrm{SO}(W)$ is split.

(c) $\underline{H} = \mathrm{SO}(W, q_W)/F$, $\dim(V) = \dim W = 2n$, such that $\mathrm{SO}(W)$ is quasi-split (not necessarily split or even unramified). If $\underline{H}$ is split, we require that $2n > 2$.

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In each case, $\underline{\mathrm{H}}$ can be realized as a twisted endoscopic group of $\underline{\mathsf{G}} = \mathrm{GL}(V)$ with $\dim(V)$ as prescribed in the cases above (see § 5 for more details). We will write $\mathfrak{h}$ and $\mathfrak{g}_\theta$ to denote the Lie algebras of $\underline{\mathrm{H}}$ and $\underline{\mathsf{G}}_\theta$, respectively.

2.2. Work of Arthur

The central result of [7] is the classification of automorphic representations of quasi-split orthogonal and symplectic groups in terms of those of general linear groups. In this subsection, we recall results from [7] concerning the local classification of representations of H in terms of those of $\underline{\mathsf{G}}(F)$, where F is a non-Archimedean local field of characteristic 0.

Arthur defined and characterized the LLC for $\underline{\mathrm{H}}(F)$ via endoscopic character identities (in case (c), the parameters are determined only up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy), conditional on the stabilization of the twisted trace formulas that have recently been made available in a series of papers by Waldspurger and by Mœglin and Waldspurger (cf. [57]). We will be using these results of Arthur on endoscopic classification via character identities, but for only tempered representations of $\underline{\mathrm{H}}(F)$. The referee has informed us that these results for tempered representations are not conditional on the stabilization of the twisted trace formula, but only need a generalization of [Art96] to the twisted case (see [55, 59]).

Following [7], we let $\mathrm{Out}(\underline{\mathrm{H}})$ denote the group of outer automorphisms of $\underline{\mathrm{H}}$ over F. This group is trivial when $\underline{\mathrm{H}}$ is as in case (a) or case (b), and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ when $\underline{\mathrm{H}}$ is as in case (c). Let W_F be the Weil group of F and WD_F the Weil–Deligne group of F. Let $\tilde{\Phi}(H)$ be the set of homomorphisms $\phi : \mathrm{WD}_F \to {}^L\underline{\mathrm{H}}$, where the homomorphisms are taken up to $\hat{\underline{\mathrm{H}}}$-conjugacy if $\underline{\mathrm{H}}$ is as in case (a) or case (b), and up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy in case (c). We write $\tilde{\Phi}_{\mathrm{bdd}}(H)$ for the subset of $\tilde{\Phi}(H)$ consisting of tempered parameters (i.e., whose image in ${}^L\underline{\mathrm{H}}$ projects onto a bounded subset of $\hat{\underline{\mathrm{H}}}$) and $\tilde{\Phi}_2(H)$ for that consisting of discrete parameters, i.e., whose image does not factor through a parabolic subgroup of ${}^L\underline{\mathrm{H}}$ (cf. [7]).

For $\phi \in \tilde{\Phi}(H)$, let

$$S_\phi = \mathrm{Cent}_{\hat{\underline{\mathrm{H}}}}(\mathrm{Im}(\phi))$$

and

$$\mathcal{S}_\phi = S_\phi / S_\phi^0 Z(\hat{\underline{\mathrm{H}}})^{\mathrm{Gal}(\bar{F}/F)}.$$

Let $\Pi(H)$ be the set of irreducible admissible representations of H, $\Pi_{\mathrm{temp}}(H)$ the set of tempered representations of H, and $\Pi_2(H)$ the set of discrete series representations of H. Let $\tilde{\Pi}(H)$ be the set of $\mathrm{Out}(\underline{\mathrm{H}})$-orbits in $\Pi(H)$, and similarly define $\tilde{\Pi}_{\mathrm{temp}}(H)$ and $\tilde{\Pi}_2(H)$. It is clear that $\tilde{\Pi}_{\mathrm{temp}}(H) = \Pi_{\mathrm{temp}}(H)$ unless we are in case (c), in which case $\tilde{\Pi}_{\mathrm{temp}}(H)$ contains orbits of order 2 and order 1.

Theorem 2.2.1 [7, Theorem 1.5.1]. *For each $\phi \in \tilde{\Phi}_{\mathrm{bdd}}(H)$ there exists a finite set $\tilde{\Pi}_\phi$ of $\tilde{\Pi}_{\mathrm{temp}}(H)$, which is constructed from ϕ via endoscopic transfer and which, for a fixed Whittaker datum, is equipped with a canonical bijective mapping*

$$\pi \to \epsilon_\pi, \quad \pi \in \tilde{\Pi}_\phi$$

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from $\tilde{\Pi}_\phi$ *into the group* $\hat{\mathcal{S}}_\phi$ *of characters of* $\mathcal{S}_\phi$. *Further, every element of* $\tilde{\Pi}_{\text{temp}}(H)$ *lies in exactly one packet* $\tilde{\Pi}_\phi$. *Moreover,*

$$\tilde{\Pi}_{\text{temp}}(H) = \bigsqcup_{\phi \in \tilde{\Phi}_{\text{bdd}}(H)} \tilde{\Pi}_\phi$$

and

$$\tilde{\Pi}_2(H) = \bigsqcup_{\phi \in \tilde{\Phi}_2(H)} \tilde{\Pi}_\phi.$$

When we write $\pi \in \tilde{\Pi}_{\text{temp}}(H)$, we mean that π denotes a tempered representation of H in cases (a) and (b), and the $\mathrm{Out}(\underline{H})$-orbit of a tempered representation of H in case (c). For $\pi \in \tilde{\Pi}_{\text{temp}}(H)$, we write ϕ_π for the Langlands parameter of π as in Theorem 2.2.1. Composing with the standard embedding of $\hat{\underline{H}} \hookrightarrow \hat{\underline{\mathsf{G}}}$ and using the LLC for $\underline{\mathsf{G}}$, we obtain an irreducible self-dual representation of $\underline{\mathsf{G}}(F)$ that we denote as π^{GL}.

For an algebraic group $\underline{G}$ defined over F, let $\mathcal{B}(\underline{G}, F)$ denote the (enlarged) Bruhat–Tits building of G, and, for $x \in \mathcal{B}(\underline{G}, F)$ and $r \geqslant 0$, let $G_{x,r}$ and $G_{x,r+}$ denote the Moy–Prasad filtration subgroups as in [62, 63]. For a representation π of G, the depth $\mathrm{depth}(\pi)$ of π is defined in [62, 63]. It is given as

$$\mathrm{depth}(\pi) := \inf\{r \mid \text{there exists } x \in \mathcal{B}(\underline{G}, F) \text{ with } \pi^{G_{x,r+}} \neq 0\}.$$

The depth of a Langlands parameter $\phi : \mathrm{WD}_F \to {}^L G$ is defined as follows:

$$\mathrm{depth}(\phi) := \inf\{r \mid \phi|_{I_F^{r+}} = 1\},$$

where $I_F \subset W_F \subset \mathrm{WD}_F$ denotes the inertia group and the filtration is the upper numbering filtration of ramification subgroups (see [72, Ch. IV]). It is expected that the LLC will preserve depth, at least when the residue characteristic is sufficiently large.

Let F have characteristic 0 and odd residue characteristic. Let π be a tempered representation of H, and let ϕ_π be as in Theorem 2.2.1. Let $m \geqslant 1$ be such that $\mathrm{depth}(\pi) \leqslant m$. The goal of §§ 3–9 is to prove that $\mathrm{depth}(\phi_\pi) \leqslant m + 1$, where $\underline{H}$ is as in § 2.1 but additionally assumed to be unramified in case (c).

3. A depth bound for endoscopic transfer: preliminaries

3.1. Topological nilpotence and unipotence

Definition 3.1.1. $X \in \mathfrak{g}(F) = \mathrm{End}(V)(F)$ (respectively, $\gamma \in \underline{\mathsf{G}}(F)$) is said to be topologically nilpotent if $\lim_{n\to\infty} X^n = 0$ in $\mathrm{End}(V)$ (respectively, $\lim_{n\to\infty} \gamma^{p^n} = 1$) – of course, in the Hausdorff topology. Write $\mathfrak{g}(F)_{\mathrm{tn}}$ (respectively, $\underline{\mathsf{G}}(F)_{\mathrm{tu}}$) for the set of topologically nilpotent elements in $\mathfrak{g}(F)$ (respectively, topologically unipotent elements in $\underline{\mathsf{G}}(F)$).

Remark 3.1.2. It is easy to see that $X \in \mathfrak{g}(F) = \mathrm{End}(V)(F)$ (respectively, $\gamma \in \underline{\mathsf{G}}(F) = \mathrm{GL}(V)(F)$) is topologically nilpotent (respectively, topologically unipotent) if and only if every generalized eigenvalue λ of X (respectively, γ) satisfies $|\lambda| < 1$ (respectively, $|\lambda - 1| < 1$). This again is equivalent to requiring that the coefficients a_i of the characteristic polynomial of X (respectively, the coefficients a_i of $T \mapsto f(T+1)$, where f is the characteristic polynomial of γ) satisfy $|a_i| < 1$ for all i; equivalently, $|a_i| \leqslant (\#\kappa)^{-1}$

for all i (for the 'if part', for example, if λ is a generalized eigenvalue of X and $|\lambda| \geqslant 1$, then $|\lambda| = |a_1 + a_2\lambda^{-1} + \cdots + a_n\lambda^{-(n-1)}| < 1$, a contradiction.) Hence $\mathsf{g}(F)_{\mathrm{tn}}$ and $\underline{\mathsf{G}}(F)_{\mathrm{tu}}$ are open *and closed* in $\mathsf{g}(F)$ and $\underline{\mathsf{G}}(F)$, respectively. Moreover, $\underline{\mathsf{G}}(F)_{\mathrm{tu}}$ is closed in $\mathrm{End}(V)(F)$ too (not just in $\underline{\mathsf{G}}(F) = \mathrm{GL}(V)(F)$), since the set of elements of $\underline{\mathsf{G}}(F)$ with determinant belonging to $\mathfrak{O}^\times$ is closed in $\mathrm{End}(V)(F)$.

Remark 3.1.3. The prescription of Definition 3.1.1 also defines a subset $\mathsf{g}(\bar{F})_{\mathrm{tn}} \subset \mathsf{g}(\bar{F})$ (respectively, $\underline{\mathsf{G}}(\bar{F})_{\mathrm{tu}} \subset \underline{\mathsf{G}}(\bar{F})$), which is the union of the analogously defined subsets $\mathsf{g}(L)_{\mathrm{tn}} \subset \mathsf{g}(L)$ (respectively, $\underline{\mathsf{G}}(L)_{\mathrm{tu}} \subset \underline{\mathsf{G}}(L)$) as L ranges over the finite extensions of F contained in $\bar{F}$.

3.2. Mock exponential maps

Remark 3.2.1. It is immediate from Remark 3.1.2 that $X \mapsto 1 + X$, a priori a birational map from g to $\underline{\mathsf{G}}$, defines a homeomorphism $\mathsf{g}(F)_{\mathrm{tn}} \to \underline{\mathsf{G}}(F)_{\mathrm{tu}}$.

Definition 3.2.2. By the Cayley transform $\mathfrak{c}$, we refer to the birational map $\mathsf{g} \to \underline{\mathsf{G}}$ defined by

$$\mathfrak{c}(X) = (1 + X/2)(1 - X/2)^{-1}.$$

A restriction of this map will also be referred to as the Cayley transform.

Lemma 3.2.3. $\mathfrak{c}$ *defines a homeomorphism* $\mathsf{g}(F)_{\mathrm{tn}} \to \underline{\mathsf{G}}(F)_{\mathrm{tu}}$.

Proof. By Remark 3.2.1 it suffices to show that $(\gamma \mapsto \gamma - 1) \circ \mathfrak{c}$ defines a homeomorphism $\mathsf{g}(F)_{\mathrm{tn}} \to \mathsf{g}(F)_{\mathrm{tn}}$. For this it suffices to show that $(\gamma \mapsto \gamma - 1) \circ \mathfrak{c}$ and $\mathfrak{c}^{-1} \circ (X \mapsto 1 + X)$ are both well defined on and preserve $\mathsf{g}(F)_{\mathrm{tn}}$. This is easy (and also uses the fact that $p \neq 2$). □

Remark 3.2.4. It is easy to see that

$$\mathfrak{c} \circ (X \mapsto -{}^tX) = (\gamma \mapsto {}^t\gamma^{-1}) \circ \mathfrak{c}, \quad \text{and} \quad \mathrm{Int}(J) \circ \mathfrak{c} = \mathfrak{c} \circ \mathrm{Ad}(J),$$

for all $J \in \underline{\mathsf{G}}(\bar{F})$. (Here the transpose may be taken with respect to any identification of $\underline{\mathsf{G}}$ with a GL_n.) Hence, for any automorphism θ of $\underline{\mathsf{G}}$ of the form $\mathrm{Int}(J) \circ (\gamma \mapsto {}^t\gamma^{-1})$, where $J \in \underline{\mathsf{G}}(F)$ is fixed, we have $\mathfrak{c} \circ d\theta = \theta \circ \mathfrak{c}$.

Definition 3.2.5. Consider an automorphism θ of $\underline{\mathsf{G}}$ as in Remark 3.2.4. Then we define

$$\mathsf{g}_\theta(F)_{\mathrm{tn}} = \mathsf{g}(F)_{\mathrm{tn}} \cap \mathsf{g}_\theta(F) \quad \text{and} \quad \underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}} = \underline{\mathsf{G}}(F)_{\mathrm{tu}} \cap \underline{\mathsf{G}}_\theta(F)$$

to be the set of topologically nilpotent elements of $\mathsf{g}_\theta(F)$ and the set of topologically unipotent elements of $\underline{\mathsf{G}}_\theta(F)$, respectively.

Remark 3.2.6. It follows from Remark 3.2.4 that, for $\underline{\mathsf{G}}_\theta$ as in Definition 3.2.5, $\mathfrak{c}$ defines a homeomorphism $\mathsf{g}_\theta(F)_{\mathrm{tn}} \to \underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$.

Lemma 3.2.7. *Let θ be as in Definition 3.2.5. Then $\gamma \mapsto \gamma^2$ defines a homeomorphism from $\underline{\mathsf{G}}(F)_{\mathrm{tu}}$ to itself, and restricts to a homeomorphism from $\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$ to itself.*

Proof. It is easy to see that the assertion about $\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$ follows once the assertion about $\underline{\mathsf{G}}(F)_{\mathrm{tu}}$ is proven, so we focus on the latter.

Injectivity. If $\gamma, \delta \in \underline{\mathsf{G}}(F)_{\mathrm{tu}}$ are such that $\gamma^2 = \delta^2$, then $\gamma^{p^n+1} = \delta^{p^n+1}$ for all $n \in \mathbb{N}$. Taking the limit as n goes to infinity, $\gamma = \delta$.

Surjectivity. If $\gamma \in \underline{\mathsf{G}}(F)_{\mathrm{tu}}$, then $X = \gamma - 1$ is topologically nilpotent, so $\{X^n\}$ is bounded and hence contained in some lattice in $\mathsf{End}(V)(F)$. Therefore $\{\gamma^{(p^n+1)/2}\}$ has a limit point γ' in $\mathsf{End}(V)(F)$, which necessarily lies in $\underline{\mathsf{G}}(F)_{\mathrm{tu}}$ as $\underline{\mathsf{G}}(F)_{\mathrm{tu}} \subset \mathsf{End}(V)$ is closed by Remark 3.1.2. By continuity, $\gamma'^2 = \gamma$.

Now that $\gamma \mapsto \gamma^2$ is shown to be bijective on $\underline{\mathsf{G}}(F)_{\mathrm{tu}}$, and, since it is obviously continuous, it suffices to show that it is submersive and hence open. The derivative of this map at $x_0 \in \underline{\mathsf{G}}(F)_{\mathrm{tu}}$, under appropriate identifications, equals $B \mapsto \mathrm{Ad}(x_0^{-1})B + B$. This linear map from $\mathsf{End}(V)$ to itself is invertible, as all its generalized eigenvalues λ satisfy $|\lambda - 2| < 1$ ($B \mapsto \mathrm{Ad}(x_0^{-1})B$ being topologically unipotent, as x_0 is). Note that we used $p \neq 2$ in this proof. □

Remark 3.2.8. It follows that the map

$$\mathfrak{c}' : X \mapsto \mathfrak{c}(X/2)^2 = \frac{1 + \frac{X}{2} + \frac{X^2}{16}}{1 - \frac{X}{2} + \frac{X^2}{16}}$$

defines a homeomorphism $\mathsf{g}(F)_{\mathrm{tn}} \to \underline{\mathsf{G}}(F)_{\mathrm{tu}}$, as well as a homeomorphism $\mathsf{g}_\theta(F)_{\mathrm{tn}} \to \underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$ for any θ as in Definition 3.2.5. Clearly the same prescription gives a $\mathrm{Gal}(\bar{F}/F)$-equivariant map $\mathsf{g}(\bar{F})_{\mathrm{tn}} \to \underline{\mathsf{G}}(\bar{F})_{\mathrm{tu}}$ that restricts to homeomorphisms $\mathsf{g}(L)_{\mathrm{tn}} \to \underline{\mathsf{G}}(L)_{\mathrm{tu}}$ and $\mathsf{g}_\theta(L)_{\mathrm{tn}} \to \underline{\mathsf{G}}_\theta(L)_{\mathrm{tu}}$ for any finite extension L of F contained in $\bar{F}$.

4. Semisimple descent

Now consider the twisted space $\underline{\tilde{\mathsf{G}}}$ over $\underline{\mathsf{G}} = \mathrm{GL}(V)$ as in §2.1.A. Let $\tilde{\theta} \in \underline{\tilde{\mathsf{G}}}(F)$. The automorphism θ of $\underline{\mathsf{G}}$ such that $\tilde{\theta} g = \theta(g)\tilde{\theta}$ is of the form discussed in Remark 3.2.4. We restrict now to the case where $\theta^2 = 1$; this condition is satisfied if θ is as in Remark 2.1.1.

Lemma 4.0.1. *Suppose that $m_1, m_2 \in \underline{\mathsf{G}}_\theta(\bar{F})_{\mathrm{tu}}$ and $g \in \underline{\mathsf{G}}(\bar{F})$ satisfy $g^{-1} \cdot m_1\tilde{\theta} \cdot g = m_2\tilde{\theta}$. Then $g \in \underline{\mathsf{G}}^\theta(\bar{F})$.*

Proof. We have $g^{-1}m_1\theta(g) = m_2$. Since $\theta^2 = 1$, we get $(g^{-1}m_1 g)^2 = m_2^2$. Since $g^{-1}m_1 g, m_2 \in \underline{\mathsf{G}}(\bar{F})_{\mathrm{tu}}$, Lemma 3.2.7 applied to a suitable extension of F as explained in Remark 3.2.8 gives $g^{-1}m_1 g = m_2$, which along with $g^{-1}m_1\theta(g) = m_2$ gives $g = \theta(g)$. □

For a set $\mathcal{C}$ in a group or a twisted space or a Lie algebra where the notion of 'strongly regular semisimple' (namely 'belonging to a closed conjugacy class and having abelian centralizer') is clear from the context, we write $\mathcal{C}_{\mathrm{srss}}$ for the set of strongly regular semisimple elements in $\mathcal{C}$.

Notation 4.0.2. We define the twisted conjugation map

$$\mathrm{tc} : \underline{\mathsf{G}}(F) \times \underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}} \to \underline{\tilde{\mathsf{G}}}(F)$$

by $(g, m) \mapsto g^{-1}m\tilde{\theta}g$, and set $\mathcal{U} = \mathrm{tc}(\underline{\mathsf{G}}(F) \times \underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}})$.

Remark 4.0.3. If $\tilde{\theta}$ is as in Remark 2.1.1, every coset of $\underline{\mathsf{G}}_\theta(F)$ (respectively, $\underline{\mathsf{G}}_\theta(\bar{F})$) in $\underline{\mathsf{G}}^\theta(F)$ (respectively, $\underline{\mathsf{G}}^\theta(\bar{F})$) intersects the center of $\underline{\mathsf{G}}$ (in more general situations, one has a variant '$\underline{\mathsf{G}}^\theta = \underline{\mathsf{G}}^1 Z(\underline{\mathsf{G}})^\theta$', as discussed just before § 1.2 in [46]).

Corollary 4.0.4. (a) *The obvious map from $\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$ to $\mathcal{U}$, given by $m \mapsto m\tilde{\theta}$, induces a bijection between the set of $\underline{\mathsf{G}}^\theta(F)$-conjugacy classes in $\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$ and the set of $\underline{\mathsf{G}}(F)$-conjugacy classes in $\mathcal{U}$.*

(b) *In the special case where $\tilde{\theta}$ is as in Remark 2.1.1, $m \mapsto m\tilde{\theta}$ also induces a bijection between the set of equivalence classes in $\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$ for $\underline{\mathsf{G}}_\theta(\bar{F})$-conjugacy and the set of equivalence classes in $\mathcal{U}$ for $\underline{\mathsf{G}}(\bar{F})$-conjugacy.*

These bijections respect semisimplicity and strong regularity.

Proof. (a) follows from Lemma 4.0.1, as also the variant of (b) with $\underline{\mathsf{G}}_\theta$ replaced by $\underline{\mathsf{G}}^\theta$. (b) then follows Remark 4.0.3. □

Lemma 4.0.5. *Let $X \in \mathfrak{g}_\theta(F)_{\mathrm{tn}}$ be semisimple, and let $\tilde{\theta}$ be as in Remark 2.1.1. Then X is regular semisimple if and only $\mathfrak{c}(X) \in \underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$ is strongly regular semisimple, and this is so if and only if $\mathfrak{c}(X)\tilde{\theta} \in \mathcal{U}$ is strongly regular semisimple.*

Proof. This follows from the conjugation invariance of $\mathfrak{c}$, Remark 4.0.3 and Lemma 4.0.1. □

Lemma 4.0.6. *The map* tc *is submersive everywhere (and hence open).*

Proof. It suffices to check the submersivity of the map $(g, m) \mapsto g^{-1}m\theta(g)$ on $\underline{\mathsf{G}}(F) \times \underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$, and that too only at points of the form $(1, m)$. Using the computation

$$(1 - \epsilon X)m(1 + \epsilon Y)\theta(1 + \epsilon X) = m(1 + \epsilon((d\theta - \mathrm{Ad}\, m^{-1})X + Y)),$$

it follows that at the derivative of this map at such an element $(1, m)$, viewed as a map $\mathfrak{g} \times \mathfrak{g}_\theta \to \mathfrak{g}$, is $(X, Y) \mapsto (d\theta - \mathrm{Ad}\, m^{-1})X + Y$. It is easy to see that this map is surjective if and only if $d\theta - \mathrm{Ad}\, m^{-1}$ is invertible as a map on $(1 - d\theta)\mathfrak{g}$. But this follows from the fact that, m and hence $\mathrm{Ad}\, m^{-1}$ being topologically unipotent, all the generalized eigenvalues λ of $d\theta - \mathrm{Ad}\, m^{-1}$ on $(1 - \theta)\mathfrak{g}$ satisfy $|\lambda + 2| < 1$. □

4.1. Discriminant factors

Notation 4.1.1. Let $\delta = g\tilde{\theta} \in \tilde{\underline{\mathsf{G}}}$, where $g \in \underline{\mathsf{G}}(F)$. Then we have an automorphism $\mathrm{Int}\,\delta = \mathrm{Int}(g) \circ \theta$ of $\underline{\mathsf{G}}$ characterized more intrinsically by $\mathrm{Int}\,\delta(g) \cdot \delta = \delta \cdot g$. Thus we may talk of $\mathrm{Ad}\,\delta = \mathrm{Ad}(g) \circ \theta \in \mathrm{GL}(\mathfrak{g})$, the centralizer $\underline{\mathsf{G}}^\delta = \underline{\mathsf{G}}^{\mathrm{Int}\,\delta}$ of δ in $\underline{\mathsf{G}}$ as well as the identity component $\underline{\mathsf{G}}_\delta = \underline{\mathsf{G}}_{\mathrm{Int}\,\delta}$ of $\underline{\mathsf{G}}^\delta$.

Definition 4.1.2. For $\delta = g\tilde{\theta} \in \tilde{\underline{\mathsf{G}}}(F)$ (where $g \in \underline{\mathsf{G}}(F)$), set

$$D_{\tilde{\underline{\mathsf{G}}}}(\delta) = D_{\underline{\mathsf{G}},\theta}(g) = \left|\det\left(\mathrm{Ad}\, g \circ d\theta - 1; \mathfrak{g}/\mathfrak{g}_{\mathrm{Int}(g)\circ\theta}\right)\right| = \left|\det\left(\mathrm{Ad}\,\delta - 1; \mathfrak{g}/\mathfrak{g}_\delta\right)\right|.$$

For $m \in \underline{\mathsf{G}}_\theta(F)$ (respectively, $X \in \mathfrak{g}_\theta(F)$), set

$$D_{\underline{\mathsf{G}}_\theta}(m) = \left|\det\left(1 - \mathrm{Ad}\, m^{-1}; \mathfrak{g}_\theta/\mathfrak{g}_{\theta,m}\right)\right| \ \left(\text{respectively}, D_{\mathfrak{g}_\theta}(X) = \left|\det\left(\mathrm{ad} X; \mathfrak{g}_\theta/\mathfrak{g}_{\theta,m}\right)\right|\right).$$

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Lemma 4.1.3. *Suppose that $m \in \underline{G}_\theta(F)_{\mathrm{tu}}$ is given as $m = \mathfrak{c}(X) = \mathfrak{c}'(X')$, where $X, X' \in \mathfrak{g}_\theta(F)_{\mathrm{tn}}$. Then*

$$D_{\tilde{\underline{G}}}(m\tilde{\theta}) = D_{\underline{G}_\theta}(m) = D_{\mathfrak{g}_\theta}(X) = D_{\mathfrak{g}_\theta}(X').$$

Proof. The equality $D_{\tilde{\underline{G}}}(m\tilde{\theta}) = D_{\underline{G}_\theta}(m)$ follows from the proof of Lemma 4.0.6. For each $2 \neq \lambda \in \bar{F}$, write $\lambda' = (1 + (\lambda/4))^2(1 - (\lambda/4))^{-2}$ and $\lambda'' = 2(\lambda' - 1)(\lambda' + 1)^{-1}$. $\mathfrak{c}, \mathfrak{c}'$ being conjugation equivariant, it is readily checked that the generalized eigenvalues of m and X are given by the λ' and λ'' as λ ranges over the generalized eigenvalues of X'. $D_{\mathfrak{g}_\theta}(X')$ is a product of terms of the form $|\lambda - \mu|$, where (λ, μ) varies over certain pairs of eigenvalues of X'. $D_{\underline{G}_\theta}(m)$ (respectively, $D_{\mathfrak{g}_\theta}(X)$) is a product over the same pairs (λ, μ), of $|1 - \mu'\lambda'^{-1}|$ (respectively, $|\lambda'' - \mu''|$). Thus, we are reduced to showing that, if $\lambda, \mu \in \bar{F}$ with $|\lambda|, |\mu| < 1$, then $|\lambda - \mu| = |1 - \mu'\lambda'^{-1}| = |\lambda'' - \mu''|$, which is straightforward. □

4.2. Normalized orbital integrals and semisimple descent

Notation 4.2.1. Let $f \in C_c^\infty(\tilde{\underline{G}}(F))$, $\delta \in \tilde{\underline{G}}(F)$. Let dg and dt_δ be measures on $\underline{G}(F)$ and $\underline{G}^\delta(F)$, respectively. Then we will denote by

$$I(\delta, f) = I(\delta, f, \underline{G}(F)) = I(\delta, f, \underline{G}(F), dg/dt_\delta)$$

the normalized orbital integral of f at δ with respect to dg/dt_δ, given as a product of the corresponding usual, unnormalized, orbital integral with $|D_{\tilde{\underline{G}}}(\delta)|^{1/2}$. We will use obvious analogs of this notation to denote various other normalized orbital integrals.

Fix measures dg and dm on $\underline{G}(F)$ and $\underline{G}^\theta(F)$, respectively. For each $\gamma \in \underline{G}_\theta(F)_{\mathrm{tu}}$, choose an arbitrary measure dt_γ on $\underline{G}^{\theta,\gamma}(F) = \underline{G}^{\gamma\tilde{\theta}}(F)$ (the equality following from Lemma 4.0.1).

Definition 4.2.2. Suppose that $f \in C_c^\infty(\mathcal{U}) \subset C_c^\infty(\tilde{\underline{G}}(F))$ (cf. Notation 4.0.2). We say that $\phi \in C_c^\infty(\underline{G}_\theta(F)_{\mathrm{tu}})$ can be obtained from f by semisimple descent at $\tilde{\theta}$ (with respect to dg and dm) if for all $\gamma \in \underline{G}_\theta(F)_{\mathrm{tu}}$ we have $I(\gamma, \phi, \underline{G}^\theta(F), dm/dt_\gamma) = I(\gamma\tilde{\theta}, f, \underline{G}(F), dg/dt_\gamma)$.

Clearly the above notion does not depend on the choices of the dt_γ.

Lemma 4.2.3. *Suppose that $L \subset \mathfrak{g}(F)_{\mathrm{tn}}$ is a lattice such that $L \cdot L \subset L$. Then the following hold.*

(a) *$1 + L$ is a compact open subgroup of $\underline{G}(F)$ and $1 + L = \mathfrak{c}(L) = \mathfrak{c}'(L)$.*

(b) *$\mathfrak{c}(L)^\theta = \mathfrak{c}(L^{d\theta}) = \mathfrak{c}'(L^{d\theta}) = \mathfrak{c}'(L)^\theta = (1 + L) \cap \underline{G}_\theta(F)$ is a compact open subgroup of $\underline{G}_\theta(F)$.*

Proof. (b) follows from (a) together with $\mathfrak{c} \circ d\theta = \theta \circ \mathfrak{c}$, $\mathfrak{c}' \circ d\theta = \theta \circ \mathfrak{c}'$, and the irreducibility of $\mathfrak{g}_\theta$. So let us prove (a). $1 + L$ is clearly closed under products. If $X \in L$, then $\sum(-X)^i$, the sum being over $i \in \mathbb{N}$ (for us $0 \notin \mathbb{N}$), is a convergent sum (as X is topologically nilpotent) of elements of L, adding 1 to which furnishes an inverse for $1 + X$ in $1 + L$. Thus, $1 + L$ is a subgroup. Thus, it suffices to show that $\mathfrak{c}(L) = \mathfrak{c}'(L) = 1 + L$.

Since $1+L$ is a subgroup, it is easy to see that $\mathfrak{c}^{-1}\circ(X\mapsto 1+X)$ and $(\gamma\mapsto\gamma-1)\circ\mathfrak{c}$ both preserve L, so $\mathfrak{c}(L)=1+L$. From the proof of Lemma 3.2.7 it is clear that $\gamma\mapsto\gamma^2$ is a self-homeomorphism of $1+L$, from which it follows that $\mathfrak{c}'(L)=\mathfrak{c}(L)$ as well. □

Lemma 4.2.4. *Let $L\subset\mathfrak{g}(F)_{\mathrm{tn}}$ be a $d\theta$-invariant lattice with $L\cdot L\subset\varpi L$. Assume also that $L=L_1\oplus L_2$, where L_1 is a lattice in $\mathfrak{g}_\theta(F)$ and L_2 one in $(1-\theta)\mathfrak{g}(F)$. Let $K_L=\mathfrak{c}(L)$ and $K_{L,\theta}=\mathfrak{c}(L)\cap\underline{\mathsf{G}}_\theta(F)$, so that by Lemma 4.2.3 the subgroup $K_L\subset\underline{\mathsf{G}}(F)_{\mathrm{tu}}$ is a compact open subgroup of $\underline{\mathsf{G}}(F)$ and $K_{L,\theta}=K_L^\theta\subset\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$. Then the following hold.*

(i) *$K_L\tilde\theta=\mathrm{tc}(K_L,K_{L,\theta})$. In particular, every element of $K_L\tilde\theta$ is $\underline{\mathsf{G}}(F)$-conjugate to an element of the form $m\tilde\theta$ for some $m\in K_{L,\theta}$.*

(ii) *Let $C_\theta\subset\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$ be an open compact subset, and let $K'\subset\underline{\mathsf{G}}(F)$ be a compact open subgroup. Assume C_θ to be invariant under conjugation by $K'\cap\underline{\mathsf{G}}^\theta(F)$. Let $C=\mathrm{tc}(K',C_\theta)$, which is compact and also open (by Lemma 4.0.6). Then $(\mathrm{meas}\,K'\cap\underline{\mathsf{G}}^\theta(F))^{-1}\mathbb{1}_{C_\theta}$ can be obtained from $(\mathrm{meas}\,K')^{-1}\mathbb{1}_C$ by semisimple descent. In particular, $(\mathrm{meas}\,K_{L,\theta})^{-1}\mathbb{1}_{K_{L,\theta}}$ can be obtained from $(\mathrm{meas}\,K_L)^{-1}\mathbb{1}_{K_L\tilde\theta}$ by semisimple descent.*

Proof. Let us prove (i) first. It is clear that $\mathrm{tc}(K_L,K_{L,\theta})\subset K_L\tilde\theta$. Let us prove the converse. First note that, by Lemma 4.0.6 and the compactness of L_1, there exists $m_0\in\mathbb{N}$ such that $\mathrm{tc}(K_L,K_{L,\theta})\supset\mathfrak{c}(L_1+\varpi^{m_0}L_2)\tilde\theta$. Hence by 'reverse induction' it suffices to show that, for $m\geqslant 0$, every element of $\mathfrak{c}(L_1+\varpi^m L_2)$ can be written as $g^{-1}\delta\theta(g)$ for some $g\in K_L$, $\delta\in\mathfrak{c}(L_1+\varpi^{m+1}L_2)$.

Thus, let $m\geqslant 0$ and $X=X_1+X_2$ with $X_1\in L_1$, $X_2\in\varpi^m L_2$. Set $Y=(1/2)X_2$. (i) will follow if we show that

$$c(Y)^{-1}(1+X)\theta(c(Y))=c(-Y)(1+X)c(-Y)\in 1+X_1+\varpi^{m+1}L. \tag{4.2.1}$$

Note that, since $Y\in\varpi^m L$,

$$c(-Y)-(1-Y)=\left(1+\frac{Y}{2}\right)^{-1}\frac{Y^2}{2}\in\varpi^{m+1}L,$$

and, using that $L\cdot L\subset\varpi L$,

$$c(Y)^{-1}(1+X)c(-Y)\in(1-Y+\varpi^{m+1}L)(1+X)(1-Y+\varpi^{m+1}L)\subset 1+X-2Y+\varpi^{m+1}L,$$

giving equation (4.2.1) and hence yielding (i).

Now let us prove (ii). Let $\gamma\in\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}$. By Lemma 4.1.3 we have $D_{\underline{\mathsf{G}}_\theta}(\gamma)=D_{\underline{\tilde{\mathsf{G}}}}(\gamma\tilde\theta)$. Hence to show (ii) it suffices to prove that, with any choice of measures as in Definition 4.2.2, writing 'O' in place of 'I' for unnormalized orbital integrals, we have

$$\frac{1}{\mathrm{meas}\,K'}O\left(\gamma\tilde\theta,\mathbb{1}_C,\underline{\mathsf{G}}(F)\right)=\frac{1}{\mathrm{meas}\,K'\cap\underline{\mathsf{G}}^\theta(F)}O\left(\gamma,\mathbb{1}_{C_\theta},\underline{\mathsf{G}}^\theta(F)\right).$$

For this, we may assume that $\gamma\in C_\theta$, and then the desired identity follows from Lemma 4.2.5 below applied with $\underline{\mathsf{G}}(F)$, $\underline{\mathsf{G}}^\theta(F)$, $\underline{\mathsf{G}}^{\gamma\tilde\theta}(F)=\underline{\mathsf{G}}^{\theta,\gamma}(F)$, K', and $g\mapsto\mathbb{1}(g^{-1}\gamma\tilde\theta g)$ taking on the roles of G_1, H_1, I_1, K_1, and f, respectively (using Lemma 4.0.1 to justify the hypotheses). □

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Lemma 4.2.5. *Let G_1 be a locally compact totally disconnected topological group, and $I_1 \subset H_1 \subset G_1$ closed subgroups. Assume that G_1, H_1, and I_1 are unimodular. Let $K_1 \subset G_1$, and let f be a locally constant integrable function on $I_1\backslash G_1$ supported in $H_1 K_1$ and right K_1-invariant. Then*

$$\int_{I_1\backslash G_1} f(g)\,dg/di = \frac{\operatorname{meas} K_1}{\operatorname{meas} K_1 \cap H_1} \int_{I_1\backslash H_1} f(h)\,dh/di.$$

Proof. This is a rather special case of [45, Lemma 2.3]. □

5. The endoscopic data

Let $\underline{H}$ be as in one of the cases (a)–(c) of § 2.1. Henceforth, $\tilde{\theta}$ will be always chosen as in Remark 2.1.1. In each case, we will fix an endoscopic datum realizing $\underline{H}$ as an endoscopic group of $\underline{G}$. In cases (a) and (b), one chooses the basic endoscopic datum associated to $(\underline{G}, \theta)$ (cf. Shelstad's appendix to [75, pp. 317–318]).

(In case (a), this is equivalent to the endoscopic datum constructed in 'Cas du groupe linéare tordu avec d impair' on [83, page 51], with $d = d^+ = 2n+1$ and $d^- = 0$, and with χ as the trivial (not merely unramified) character. It is in fact possible to accommodate nontrivial unramified χ by being slightly more careful about Lemma 6.5.1 below, provided we choose the 'η' that shows up in that lemma to belong to $\mathfrak{O}^\times$. In case (b), this is equivalent to the endoscopic datum constructed in 'Cas du groupe linéare tordu avec d pair' on [83, page 51], with $d = d^+ = 2n$ and $d^- = 0$.)

For the endoscopic datum in case (c), we refer to § 7 below.

Cases (a) and (b), being similar to each other, will be treated in § 6. Case (c) will require a slightly different kind of treatment and will be handled in § 7.

6. Matching and semisimple descent in cases (a) and (b)

In this section we assume that we are in case (a) or case (b) of § 2.1.

6.1. Nonstandard endoscopic data for cases (a) and (b)

One gets a nonstandard endoscopic triplet $(\underline{H}_{\mathrm{sc}}, \underline{G}_{\theta,\mathrm{sc}}, j_*)$ as follows (cf. [82, § 1.7] for the definition of a nonstandard endoscopic triplet). Fix maximal tori $\underline{T}$ and $\underline{T}_{\underline{H}}$ in $\underline{G}$ and $\underline{H}$, respectively, such that $\underline{T}$ belongs to a θ-stable Borel pair $(\underline{B}, \underline{T})$ in $\underline{G}$ and $\underline{T}_{\underline{H}}$ to a Borel pair $(\underline{B}_{\underline{H}}, \underline{T}_{\underline{H}})$ in $\underline{H}$. The endoscopic datum (which includes the relevant L-group data) then gives a homomorphism $\xi : \underline{T} \to \underline{T}_{\underline{H}}$ that descends to an isomorphism $\tilde{\xi} : \underline{T}/(1-\theta)\underline{T} \to \underline{T}_{\underline{H}}$. Set $\underline{T}_{\underline{G}_\theta} = \underline{T}_\theta$. Let $\underline{T}_{\underline{H}_{\mathrm{sc}}}$ and $\underline{T}_{\underline{G}_{\theta,\mathrm{sc}}}$ be the preimages of $\underline{T}_{\underline{H}}$ and $\underline{T}_{\underline{G}_\theta}$ in $\underline{H}_{\mathrm{sc}}$ and $\underline{G}_{\theta,\mathrm{sc}}$, respectively. Thus, we get a composite isomorphism

$$\begin{aligned} j_* : X_*(\underline{T}_{\underline{H}_{\mathrm{sc}}}) \otimes_{\mathbb{Z}} \mathbb{Q} &\to X_*(\underline{T}_{\underline{H}}) \otimes \mathbb{Q} \to X_*(\underline{T}/(1-\theta)\underline{T}) \otimes_{\mathbb{Z}} \mathbb{Q} \\ &\to X_*(\underline{T}_\theta) \otimes_{\mathbb{Z}} \mathbb{Q} \to X_*(\underline{T}_{\underline{G}_{\theta,\mathrm{sc}}}) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned}$$

We will also use j_* for the induced map

$$\mathfrak{t}_{\underline{H}} = \mathfrak{t}_{\underline{H}_{\mathrm{sc}}} \to \mathfrak{t}_{\underline{G}_\theta} = \mathfrak{t}_{\underline{G}_{\theta,\mathrm{sc}}}.$$

6.2. Matching of semisimple classes

Recall the following definitions.

Definition 6.2.1. (a) Two semisimple elements $\gamma \in \underline{\mathrm{H}}(\bar{F})$ and $\delta \in \tilde{\underline{\mathsf{G}}}(\bar{F})$, or equivalently their conjugacy classes, are said to match if there exist $g \in \underline{\mathsf{G}}(\bar{F})$ and $h \in \underline{\mathrm{H}}(\bar{F})$ such that $g\delta g^{-1} \in \underline{\mathrm{T}}(\bar{F})\tilde{\theta}$, $h\gamma h^{-1} \in \underline{\mathrm{T}}_{\underline{\mathrm{H}}}(\bar{F})$, and, writing $g\delta g^{-1} = \tau\tilde{\theta}$, $\xi(\tau) = h\gamma h^{-1}$ (this notion depends on the $\nu \in F^\times$ of § 2.1.A).

(b) Two semisimple elements $Y \in \mathfrak{h}_{\mathrm{sc}}(\bar{F}) = \mathfrak{h}(\bar{F})$ and $X \in \mathfrak{g}_{\theta,\mathrm{sc}}(\bar{F}) = \mathfrak{g}_\theta(\bar{F})$ (or equivalently the $\underline{\mathrm{H}}(F)$-conjugacy class of Y and the $\underline{\mathsf{G}}^\theta(F)$-conjugacy class of X) are said to match if there exist $g \in \underline{\mathsf{G}}_\theta(\bar{F}), h \in \underline{\mathrm{H}}(\bar{F})$ such that $\mathrm{Ad}\, g(X) \in \mathfrak{t}_\theta(\bar{F})$, $\mathrm{Ad}\, h(Y) \in \mathfrak{t}_{\underline{\mathrm{H}}}(\bar{F})$ and $j_*(\mathrm{Ad}\, h(Y)) = \mathrm{Ad}\, g(X)$.

Remark 6.2.2. If $\Omega_{\underline{\mathrm{H}}}$ and $\Omega_{\underline{\mathsf{G}}_\theta}$ denote the Weyl groups of $\underline{\mathrm{T}}_{\underline{\mathrm{H}}}$ and $\underline{\mathrm{T}}_{\underline{\mathsf{G}}_\theta}$ in $\underline{\mathrm{H}}$ and $\underline{\mathsf{G}}_\theta$, respectively, then, as in [82, § 1.8], we have an isomorphism $\mathfrak{t}_{\underline{\mathrm{H}}}/\Omega_{\underline{\mathrm{H}}} \cong \mathfrak{t}_{\underline{\mathsf{G}}_\theta}/\Omega_{\underline{\mathsf{G}}_\theta}$ of varieties over F. Recall that these are the varieties of semisimple conjugacy classes in $\underline{\mathrm{H}}$ and $\underline{\mathsf{G}}_\theta$. If $Y \in \mathfrak{h}(F)$ is semisimple, there always exists $X \in \mathfrak{g}_\theta(F)$ that matches Y. Further, for any such Y and X, Y is regular if and only if X is (cf. [82, § 1.7]).

6.3. Matching and Cayley transform

Lemma 6.3.1. *Let $Y \in \mathfrak{h}_{\mathrm{sc}}(F)_{\mathrm{tn}} := \mathfrak{h}(F)_{\mathrm{tn}}$, $X \in \mathfrak{g}_{\theta,\mathrm{sc}}(F)_{\mathrm{tn}} := \mathfrak{g}_\theta(F)_{\mathrm{tn}}$. Then Y and X match if and only if $\mathfrak{c}'(Y)$ and $\mathfrak{c}(X)\tilde{\theta}$ match.*

Proof. Since $\mathfrak{c}'$ and $\mathfrak{c}$ are both conjugation equivariant, we are reduced to showing that, on $\mathfrak{t}_\theta$, $\mathfrak{c}' \circ (j_*)^{-1} = \xi \circ \mathfrak{c}$. To do this, it suffices to show that, after identifying $\underline{\mathrm{T}}_\theta$ and $\underline{\mathrm{T}}_{\underline{\mathrm{H}}}$ with $\mathbb{G}_m^n$ suitably, the map $\underline{\mathrm{T}}_\theta \to \underline{\mathrm{T}}_{\underline{\mathrm{H}}}$ induced by ξ is given by $x \mapsto x^2$, j_*^{-1} by $x \mapsto 2x$, and that modulo these identifications $\mathfrak{c}$ and $\mathfrak{c}'$ are given by

$$(x_1, \ldots, x_n) \mapsto \left(\frac{1 + (x_1/2)}{1 - (x_1/2)}, \ldots, \frac{1 + (x_n/2)}{1 - (x_n/2)}\right)$$

and

$$(y_1, \ldots, y_n) \mapsto \left(\left(\frac{1 + (y_1/4)}{1 - (y_1/4)}\right)^2, \ldots, \left(\frac{1 + (y_n/4)}{1 - (y_n/4)}\right)^2\right),$$

respectively.

For this, let us make the constructions recalled in this subsection a bit more explicit. Choose $(\underline{\mathrm{B}}_{\underline{\mathrm{H}}}, \underline{\mathrm{T}}_{\underline{\mathrm{H}}})$ and $(\underline{\mathrm{B}}, \underline{\mathrm{T}})$ using ordered bases for the vector spaces W and V on which $\underline{\mathrm{H}}$ and $\underline{\mathsf{G}}$ are realized. These ordered bases give 'obvious' identifications $\underline{\mathrm{T}} \cong \mathbb{G}_m^d$ and $\underline{\mathrm{T}}_{\underline{\mathrm{H}}} \cong \mathbb{G}_m^n$, where $d = \dim V$ (so $d = 2n$ or $2n + 1$). The identification $\underline{\mathrm{T}} \cong \mathbb{G}_m^d$ induces an identification $\underline{\mathrm{T}}_\theta \cong \mathbb{G}_m^n$, with respect to which the inclusion $\underline{\mathrm{T}}_\theta \hookrightarrow \underline{\mathrm{T}}$ becomes $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1, x_n^{-1}, \ldots, x_1^{-1})$, where the factor 1 should be ignored if d is even. The map ξ being dual to a similarly formulated inclusion can be written as $(x_1, \ldots, x_d) \mapsto (x_1 x_d^{-1}, x_2 x_{d-1}^{-1}, \ldots, x_n x_{d+1-n}^{-1})$. From this, the assertions at the end of the preceding paragraph are easy to check. □

Remark 6.3.2. The explication in the previous proof shows that semisimple elements $Y \in \mathfrak{h}(F) \subset \operatorname{End}(W)$ and $X \in \mathfrak{g}_\theta(F) \subset \operatorname{End}(V)$ match if and only if the multiset $\operatorname{Eig} X$ of eigenvalues of X is related to the multiset $\operatorname{Eig} Y$ of eigenvalues of Y as $\operatorname{Eig} X = (1/2) \operatorname{Eig} Y \cup \{0\}$ (if $\dim V - \dim W = 1$) or $\operatorname{Eig} X \cup \{0\} = (1/2) \operatorname{Eig} Y$ (if $\dim W - \dim V = 1$). In the context of invoking this remark, for simplicity of notation we will express this condition as $\operatorname{Eig} X \doteq (1/2) \operatorname{Eig} Y$.

Notation 6.3.3. We introduce some notation for later use. If E/F is a finite extension and T is an endomorphism of a finite-dimensional vector space over E, we denote by $\operatorname{Eig}_E T$ the multiset of eigenvalues of T in an algebraic closure of E (a harmless choice). If we are given in addition an F-embedding $\sigma : E \hookrightarrow \bar{F}$, then we can view $\operatorname{Eig}_E T$ as a submultiset of $\bar{F}$ through σ, which we will then denote by $\operatorname{Eig}_{E,\sigma} T$. If $E = F$, we will write $\operatorname{Eig} T$ for $\operatorname{Eig}_F T$ (in this case σ is necessarily the identity).

6.4. Matching and topological nilpotence

Remark 6.4.1. We will continue to identify $\mathfrak{h}_{\mathrm{sc}}$ with $\mathfrak{h}$ and $\mathfrak{g}_{\theta,\mathrm{sc}}$ with $\mathfrak{g}_\theta$. Recall that we are writing $\mathfrak{h}_{\mathrm{sc}}(F)_{\mathrm{tn}}$ for $\mathfrak{h}(F)_{\mathrm{tn}}$, etc.

Lemma 6.4.2. (a) *Suppose that $Y \in \mathfrak{h}(F) \setminus \mathfrak{h}(F)_{\mathrm{tn}} = \mathfrak{h}_{\mathrm{sc}}(F) \setminus \mathfrak{h}_{\mathrm{sc}}(F)_{\mathrm{tn}}$ is semisimple. Then Y does not match any element in $\mathfrak{g}_{\theta,\mathrm{sc}}(F)_{\mathrm{tn}} = \mathfrak{g}_\theta(F)_{\mathrm{tn}}$.*

(b) *Suppose that $\gamma \in \underline{\mathrm{H}}(F) \setminus \underline{\mathrm{H}}(F)_{\mathrm{tu}}$ is semisimple. Then γ does not match any element in $\mathcal{U}$ (cf. Definition 4.0.2).*

Proof. We prove only (b), as (a) is strictly easier. For (b), suppose that γ does match $\delta \in \mathcal{U}$. Choose $g \in \underline{\mathsf{G}}(\bar{F}), h \in \underline{\mathrm{H}}(\bar{F})$ such that $\xi(\tau) = h\gamma h^{-1}$, where $g\delta g^{-1} = \tau\tilde{\theta}$ for some $\tau \in \underline{\mathrm{T}}(\bar{F})$. Since δ is $\underline{\mathsf{G}}(F)$-conjugate to an element of $\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}\tilde{\theta}$, by [46, Lemma 3.2.A] and the sentence in that reference preceding the said lemma, the image of τ in $(\underline{\mathrm{T}}/(1-\theta)\underline{\mathrm{T}})(\bar{F})$ necessarily lies in that of $\underline{\mathrm{T}}_\theta(\bar{F})_{\mathrm{tu}}$. It follows that $h\gamma h^{-1} \in \underline{\mathrm{T}}_{\underline{\mathrm{H}}}(\bar{F})_{\mathrm{tu}}$ (see, for example, the interpretation of the map $\underline{\mathrm{T}}_\theta \to \underline{\mathrm{T}}_{\underline{\mathrm{H}}}$ as $x \mapsto x^2$ as in the proof of Lemma 6.3.1). This forces $\gamma \in \underline{\mathrm{H}}(F)_{\mathrm{tu}}$, a contradiction. □

6.5. Transfer factors and descent

As in [83], the transfer factors Δ we deal with will not involve the Δ_{IV} term, which will be accounted for by the normalization of the orbital integrals (cf. Notation 4.2.1).

Lemma 6.5.1. *The transfer factor Δ can be normalized so that, if strongly regular semisimple elements $\gamma \in \underline{\mathrm{H}}(F)$ and $\delta \in \tilde{\underline{\mathsf{G}}}(F)$ match, then $\Delta(\gamma, \delta) = 1$.*

Proof. This follows from Lemma 1 of Shelstad's appendix to [75], as we are dealing with basic endoscopic data here. Alternatively, one may use [83, Proposition 1.10]; note that, with the notation therein, the character χ is trivial and the set $I^- \supset I^{-*}$ is empty. □

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6.6. Matching and semisimple descent

Fix Haar measures on dg, dm, and dh on $\underline{G}(F)$, $\underline{G}^\theta(F)$, and $\underline{H}(F)$, respectively. Recall the following definitions from [46, § 5.5] and [82, § 1.7], respectively (note that these depend on dg, dm, and dh).

Definition 6.6.1. (i) $f \in C_c^\infty(\tilde{\underline{G}}(F))$ and $f^H \in C_c^\infty(\underline{H}(F))$ have matching orbital integrals if and only if for any $\gamma \in \underline{H}(F)_{\mathrm{srss}}$ that is strongly $\tilde{\underline{G}}$-regular (i.e., matches a strongly regular semisimple element of $\tilde{\underline{G}}(\bar{F})$),

$$\sum_{\{\gamma'\}} I(\gamma', f^H, dh/dt_{\gamma'}) = \sum_{\{\delta\}} \Delta(\gamma, \delta) I(\delta, f, dg/dt_\delta), \tag{6.6.1}$$

where $\{\gamma'\}$ is a set of representatives for the $\underline{H}(F)$-conjugacy classes in the stable conjugacy class of γ, and $\{\delta\}$ is a set of representatives for the $\underline{G}(F)$-conjugacy classes in $\tilde{\underline{G}}(F)$ that match γ.

(ii) $\phi \in C_c^\infty(\mathfrak{g}_{\theta,\mathrm{sc}}(F)) = C_c^\infty(\mathfrak{g}_\theta(F))$ and $\varphi^H \in C_c^\infty(\mathfrak{h}(F)) = C_c^\infty(\mathfrak{h}_{\mathrm{sc}}(F))$ are said to have matching orbital integrals if and only if for all $Y \in \mathfrak{h}(F)_{\mathrm{srss}}$ we have

$$\sum_{\{Y'\}} I(Y', \varphi^H, dh/dt_{Y'}) = \sum_{\{X\}} I(X, \phi, \underline{G}^\theta(F), dm/dt_X), \tag{6.6.2}$$

where $\{Y'\}$ is a set of representatives in $\mathfrak{h}(F) = \mathfrak{h}_{\mathrm{sc}}(F)$ for the set of $\underline{H}(F)$-conjugacy classes in the stable conjugacy class of Y, and $\{X\}$ is a set of representatives in $\mathfrak{g}_\theta(F)$ for the $\underline{G}^\theta(F)$-conjugacy classes in $\mathfrak{g}_\theta(F)$ that match Y.

Remark 6.6.2. (i) To make the above definition meaningful, we need to make stipulations on the measures $dt_{\gamma'}, dt_\delta, dt_{Y'}, dt_X$. Instead of recalling the precise details, we refer the reader to [82, § 3.10]. By the 'Remarque' there, and using that $p \neq 2$, this choice agrees with the one in [46, § 5.5].

(ii) Here, the latter definition above makes sense and agrees with the usual definition (cf. [82, § 1.7]), because working with $\underline{G}_{\theta,\mathrm{sc}}$ instead does not change the stable conjugacy classes, and also thanks to Remark 4.0.3.

(iii) It is easy to see that, in the context of Lemma 6.3.1, the relation imposed by (i) above between each $dt_{Y'}$ and dt_X agrees with that between $dt_{\mathfrak{c}'(Y')}$ and $dt_{\mathfrak{c}(X)\tilde{\theta}}$; i.e., if $a \in \mathbb{C}^\times$ satisfies $dt_{\mathfrak{c}'(Y')} = a \cdot dt_{Y'}$, then $dt_{\mathfrak{c}(X)\tilde{\theta}} = a \cdot dt_X$.

Notation 6.6.3. We denote the left and right sides of equation (6.6.2) by $SI(Y, \varphi^H) = SI(Y, \varphi^H, dh/dt_Y)$ and $I^{\underline{G}^\theta}(Y, \phi) = I^{\underline{G}^\theta}(Y, \phi, dm, dt_Y)$, respectively. We also denote the left and right sides of equation (6.6.1) by $SI(\gamma, f^H) = SI(\gamma, f^H, dh/dt_\gamma)$ and $I^{\tilde{\underline{G}}}(\gamma, f) = I^{\tilde{\underline{G}}}(\gamma, f, dg, dt_\gamma)$, respectively.

Lemma 6.6.4. *Suppose that $f \in C_c^\infty(\mathcal{U}), \phi \in C_c^\infty(\underline{G}_\theta(F)_{\mathrm{tu}})$ and $f^H \in C_c^\infty(\underline{H}(F)_{\mathrm{tu}})$. Suppose that ϕ is obtained from f by semisimple descent with respect to dg and dm. Then f and f^H have matching orbital integrals if and only if $\phi \circ \mathfrak{c} \in C_c^\infty(\mathfrak{g}_\theta(F)_{\mathrm{tn}}) \subset C_c^\infty(\mathfrak{g}_\theta(F))$ and $f^H \circ \mathfrak{c}' \in C_c^\infty(\mathfrak{h}(F)_{\mathrm{tn}}) \subset C_c^\infty(\mathfrak{h}(F))$ have matching orbital integrals.*

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Proof. Recall that $\mathfrak{c}'(\mathfrak{h}(F)_{\mathrm{tn,srss}}) = \underline{\mathrm{H}}(F)_{\mathrm{tu,srss}}$. By Lemma 6.4.2, it suffices to verify that, for $Y \in \mathfrak{h}(F)_{\mathrm{tn}} = \mathfrak{h}_{\mathrm{sc}}(F)_{\mathrm{tn}}$ regular semisimple, $\gamma := \mathfrak{c}'(Y)$ is strongly $\tilde{\underline{\mathrm{G}}}$-regular, and that, setting $\varphi = \phi \circ \mathfrak{c}$ and $\varphi^H = f^H \circ \mathfrak{c}'$, the right sides of equations (6.6.1) and (6.6.2) coincide. Now γ is strongly $\tilde{\underline{\mathrm{G}}}$-regular because, for any $X_1 \in \{X\}$ ($\neq \emptyset$ by Remark 6.2.2!), $\mathfrak{c}(X_1)\tilde{\theta}$ matches γ by Lemma 6.3.1 and is strongly regular by Lemma 4.0.5 and the regularity of X (cf. Remark 6.2.2).

Similar considerations together with Lemma 4.0.1 allow us to assume without loss of generality that $\mathfrak{c}$ induces a bijection $\{X\} \leftrightarrow \{\delta\} \cap \mathcal{U}$. Then the desired equality between the right sides of equations (6.6.1) and (6.6.2) follows from Lemma 6.5.1 and the compatibility of the centralizer measures with semisimple descent (cf. Remark 6.6.2(i)). □

7. Matching and semisimple descent for case (c)

Now we come to case (c). Recall that, in this case, $\dim V = \dim W = 2n$, $\underline{\mathrm{G}} = \mathrm{GL}(V)$ and $\underline{\mathrm{H}} = \mathrm{SO}(W, q_W)$ is quasi-split. If $\underline{\mathrm{H}}$ is split, then it is part of our requirement that $2n > 2$. Note that $\underline{\mathrm{G}}^\theta = \underline{\mathrm{G}}_\theta = \mathrm{Sp}(V, \tilde{\theta})$, the stabilizer of the symplectic form $\tilde{\theta}$ (defined in Remark 2.1.1).

One cannot work with a basic endoscopic datum in this case. Hence we fix the endoscopic datum constructed in 'Cas du groupe linéare tordu avec d pair' on [83, page 51], with $d = d^- = 2n$ and $d^+ = 0$. Most lemmas of the previous section have analogs in this case, but with the difference that the semisimple descent for the transfer involves a standard endoscopic datum rather than a nonstandard endoscopic datum (in fact there is nonstandard endoscopy here too, but it is banal in this case).

7.1. An endoscopic datum for case (c)

We realize $\underline{\mathrm{H}}$ as the endoscopic group underlying an endoscopic datum for $\underline{\mathrm{G}}_\theta$ as in 'Cas symplectique' of [83, page 50], with $d = 2n+1$, $d^- = 2n$, and $d^+ = 1$, using the notation there. Recall that the definition of matching elements has an obvious variant for Lie algebras: given regular semisimple conjugacy classes $Y \in \mathfrak{h}(\bar{F})$ and $X \in \mathfrak{g}_\theta(\bar{F})$, we can talk of what it means for X and Y to match.

7.2. Regular semisimple conjugacy classes

We recall the description of regular semisimple conjugacy classes in the groups/twisted spaces of our interest from [83, § 1.3], but expressed as in [52].

Notation 7.2.1. In this subsection, for an étale F-algebra L, an F-subalgebra $L_\pm$ of L fixed by some involution τ (which is then determined by $L_\pm$) and $c \in L^\times$, q_c will denote the bilinear form on the F-vector space L given by $q_c(a, b) = \mathrm{tr}_{L/F}(\tau(a)bc)$. Further, for $y \in L$, m_y will denote the element of $\mathrm{GL}_F(L)$ or $\mathrm{End}_F(L)$ given as multiplication by y.

(a) Regular semisimple conjugacy classes in $\underline{\mathrm{G}}_\theta(F) = \mathrm{Sp}(V, \tilde{\theta})$ are in bijection with equivalence classes of quadruples $(L, L_\pm, \bar{x}, c)$ (for an appropriate, obvious, notion of equivalence) where L is a $2n$-dimensional étale F-algebra, $L_\pm$ is the subalgebra of L fixed by some involution $\tau \in \mathrm{Aut}_F(L)$, $\bar{x} \in L^\times$ generates L over F and

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satisfies $\bar{x}\tau(\bar{x}) = 1$, and $c \in L^\times/N_{L/L_\pm}(L^\times)$ has a representative $\tilde{c}$ satisfying $\tau(\tilde{c}) = -\tilde{c}$. The conjugacy class associated to such a quadruple $(L, L_\pm, \bar{x}, c)$ is given by

$$\left\{\iota^*(m_{\bar{x}}) \mid \iota : (V, \tilde{\theta}) \xrightarrow{\cong} (L, q_{\tilde{c}}), \tilde{c} \text{ a representative for } c\right\}.$$

(b) Equivalence classes in $\underline{\mathrm{H}}(F) = \mathrm{SO}(W)(F)$ for $\mathrm{O}(W)(\bar{F})$-conjugacy, consisting of regular semisimple elements that are 'very regular' (i.e., without eigenvalue ± 1), are in bijection with equivalence classes of triples $(L, L_\pm, y)$, where $(L, L_\pm, y)$ satisfies the same conditions as the $(L, L_\pm, \bar{x})$ in (a), and, where further, there exists $c \in L_\pm^\times$ such that the quadratic space (L, q_c) is isomorphic to the quadratic space W. The equivalence class corresponding to a given triple is given by

$$\left\{\iota^*(m_y) \mid c \in L_\pm^\times, \iota : (W, q_W) \xrightarrow{\cong} (L, q_c)\right\}.$$

This captures only the equivalence classes of very regular elements, and a regular element like $\mathrm{diag}(1, a, a^{-1}, 1) \in \mathrm{SO}_4(F)$ with $a \neq \pm 1 \in F^\times$ does not arise in the above manner.

(c) The strongly regular conjugacy classes in $\tilde{\underline{\mathsf{G}}}(F)$ are in bijection with equivalence classes of triples $(L, L_\pm, x)$, where $L, L_\pm$ are above, and $x \in L^\times/N_{L/L_\pm}(L^\times)$. The conjugacy class in $\tilde{\underline{\mathsf{G}}}(F)$ associated to $(L, L_\pm, x)$ is given by

$$\left\{\iota^*(q_{\tilde{x}}) \mid \iota : L \cong V, \tilde{x} \text{ a representative for } x\right\}.$$

(d) It follows, for instance using (a) with the Cayley transform and using scaling by $F^\times$ on $\mathfrak{sp}(V, \tilde{\theta})$ (or see [84, § 1.7]), that conjugacy classes in $\mathsf{g}_\theta(F) = \mathfrak{sp}(V, \tilde{\theta})(F)$ are in bijection with equivalence classes of tuples $(L, L_\pm, X, c)$, where $L, L_\pm, c$ are as in (a) and $X \in L$ generates L over F and satisfies $X + \tau(X) = 0$. The conjugacy class corresponding to such a tuple can be given as

$$\left\{\iota^*(m_X) \mid \iota : (V, \tilde{\theta}) \xrightarrow{\cong} (L, q_{\tilde{c}}), \tilde{c} \text{ a representative for } c\right\}.$$

(e) Similarly, it follows that equivalence classes of 'very regular elements' of $\mathfrak{so}(W)(F)$, i.e., regular semisimple elements of $\mathfrak{so}(W)(F)$ without eigenvalue 0, for $\mathrm{O}(W)(\bar{F})$-conjugacy, correspond bijectively to equivalence classes of tuples $(L, L_\pm, Y)$, where $L, L_\pm$ are as in (b), a c has to exist as in (b), and $Y \in L$ generates L over F and satisfies $Y + \tau(Y) = 0$. The equivalence class associated to such a tuple can be given as

$$\left\{\iota^*(m_Y) \mid c \in L_\pm^\times, \iota : (W, q_W) \xrightarrow{\cong} (L, q_c)\right\}.$$

7.3. Parameterization and Cayley transform

A tuple $(L, L_\pm, \bar{x}, c)$ (respectively, $(L, L_\pm, y)$) as in (a) (respectively, (b)) of § 7.2 corresponds to a set of topologically unipotent elements if and only if, for all $\phi \in \mathrm{Hom}_F(L, \bar{F})$ (F-algebra homomorphisms), $|\phi(\bar{x}) - 1| < 1$ (respectively, $|\phi(y) - 1| < 1$). A tuple $(L, L_\pm, X, c)$ (respectively, $(L, L_\pm, Y)$) as in (d) (respectively, (e)) of § 7.2

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corresponds to a set of topologically nilpotent elements if and only if, for all $\phi \in \mathrm{Hom}_F(L, \bar{F})$, $|\phi(X)| < 1$ (respectively, $|\phi(Y)| < 1$). Using this notation, $\mathfrak{c}$ and $\mathfrak{c}'$ take $(L, L_\pm, Y)$ to

$$\left(L, L_\pm, \frac{1+(Y/2)}{1-(Y/2)}\right) \quad \text{and} \quad \left(L, L_\pm, \left(\frac{1+(Y/4)}{1-(Y/4)}\right)^2\right),$$

respectively, and we have analogous formulas for $(L, L_\pm, X, c)$.

7.4. Matching and semisimple descent

Note that the parameterization of regular semisimple conjugacy classes in $\underline{\mathsf{G}}_\theta(F) = \mathrm{Sp}(V, \tilde{\theta})(F)$ discussed in § 7.2(a) depended not just on the isomorphism class of $\mathrm{Sp}(V, \tilde{\theta})$, but rather on that of the symplectic space $(V, \tilde{\theta})$. This is why Lemma 7.4.1 below works. On the other hand, the choice of q_W is of no concern to us, since only the stable conjugacy classes in $\underline{\mathrm{H}}(F)$ matter.

Lemma 7.4.1. *Suppose that the $\underline{\mathsf{G}}_\theta(F)$-conjugacy class of $g \in \underline{\mathsf{G}}_\theta(F)_{\mathrm{srss}}$ corresponds to $(L, L_\pm, \bar{x}, c)$, and that $g\tilde{\theta} \in \tilde{\underline{\mathsf{G}}}(F)$ is regular. Then the $\underline{\mathsf{G}}(F)$-conjugacy class of $g\tilde{\theta} \in \tilde{\underline{\mathsf{G}}}(F)$ is given by $(L, L_\pm, \bar{x}c)$.*

Proof. This is easy. It helps to note that $g\tilde{\theta} = \tilde{\theta}g$. □

Suppose that $\gamma \in \underline{\mathrm{H}}(\bar{F})$ and $\delta \in \tilde{\underline{\mathsf{G}}}(\bar{F})$ (respectively, $\gamma \in \underline{\mathrm{H}}(\bar{F})$ and $\delta \in \underline{\mathsf{G}}_\theta(\bar{F})$) (respectively, $Y \in \mathfrak{h}(\bar{F})$ and $X \in \mathsf{g}_\theta(\bar{F})$) are both semisimple. We have a definition analogous to Definition 6.2.1 for what it means for these two elements to match.

Lemma 7.4.2. *In terms of the parameterizations of § 7.2, the matchings between very regular elements in the contexts of the endoscopic data above are given as follows.*

(a) *The matching between $\underline{\mathrm{H}}$ and $\tilde{\underline{\mathsf{G}}}$ is given by $(L, L_\pm, y)$ matching $(L, L_\pm, x)$, where $x\tau(x)^{-1} = -y$ (by which we mean: given any element γ of the subset of $\underline{\mathrm{H}}(F)$ parameterized by $(L, L_\pm, y)$, γ matches $\delta \in \tilde{\underline{\mathsf{G}}}(F)$ if and only if δ can be parameterized by $(L, L_\pm, x)$ for some x such that $x\tau(x)^{-1} = -y$).*

(b) *The matching between $\underline{\mathrm{H}}$ and $\underline{\mathsf{G}}_\theta$ is given by $(L, L_\pm, y)$ matching $(L, L_\pm, y, c)$ (for any $c \in L^\times$ as in § 7.2(a)).*

(c) *The matching between $\mathfrak{h}$ and g_θ is given by $(L, L_\pm, Y)$ matching $(L, L_\pm, Y, c)$ (again, for any $c \in L^\times$ as in § 7.2(a)).*

Proof. For (a) and (b), cf. [83, § 1.9] (note that the expression $\nu/\tau_i(\nu)$ of [83] is 1 in our case as $\nu \in F^\times$). (c) follows from [84, § X.2]. □

Remark 7.4.3. Since semisimple conjugacy classes in $\mathfrak{sp}_{2n}(\bar{F})$ are completely determined by eigenvalues, it is now easy to see that $Y \in \mathfrak{h}(F) \subset \mathrm{End}(W)$ and $X \in \mathsf{g}_\theta(F) \subset \mathrm{End}(V)$ that are (not necessarily regular) semisimple match if and only if Y and X have the same multiset of eigenvalues, i.e., $\mathrm{Eig}\, Y = \mathrm{Eig}\, X$. For compatibility with the notation in Remark 6.3.2 while treating the cases together later, we will also write this as $\mathrm{Eig}\, Y \doteq \mathrm{Eig}\, X$.

Remark 7.4.4. Lemma 7.4.2 above also gives us an ‘eigenvalue criterion’ for a very regular element $\gamma \in \underline{\mathrm{H}}(F)$ to match a strongly regular element $\delta \in \tilde{\underline{\mathrm{G}}}(F)$. Namely, consider the map $\tilde{\underline{\mathrm{G}}}(F) \to \underline{\mathrm{G}}(F)$ that takes $g_1\tilde{\theta}$ to $g_1\theta(g_1)$, or, more intrinsically, any $\delta \in \tilde{\underline{\mathrm{G}}}(F)$ to $T_\delta \in \underline{\mathrm{G}}(F)$ such that $B(v, w) = -B(T_\delta w, v)$. This map takes θ-conjugacy to conjugacy, and it is easy to see from Lemma 7.4.2(a) and Lemma 3.2.A(1) of [46] that γ matches δ if and only if γ and T_δ have the same multiset of eigenvalues.

Corollary 7.4.5. *Let $Y \in \mathfrak{h}_{\mathrm{sc}}(F)_{\mathrm{tn,srss}} = \mathfrak{h}(F)_{\mathrm{tn,srss}}$ be very regular, and let $X \in \mathfrak{g}_{\theta,\mathrm{sc}}(F)_{\mathrm{tn,srss}} = \mathfrak{g}_\theta(F)_{\mathrm{tn,srss}}$. Note that $\mathfrak{c}'(Y) \in \underline{\mathrm{H}}(F)_{\mathrm{tu,srss}}$, $\mathfrak{c}(X) \in \underline{\mathrm{G}}_\theta(F)_{\mathrm{tu,srss}}$, $\mathfrak{c}(X)\tilde{\theta} \in \mathcal{U}_{\mathrm{srss}}$ by Lemma 4.0.4. Then the following are equivalent (for the appropriate endoscopic data in each case).*

(a) *$Y/2$ and X match,*

(b) *$\mathfrak{c}(Y/2)$ and $\mathfrak{c}(X)$ match,*

(c) *$\mathfrak{c}'(Y)$ and $\mathfrak{c}(X)\tilde{\theta}$ match.*

Proof. Since Y is very regular, so are $\mathfrak{c}(Y/2)$ and $\mathfrak{c}'(Y)$, by § 7.3. Then the equivalence of (a) and (b) follows from (b) and (c) of Lemma 7.4.2 together with the compatibility of the parameterization of § 7.2 with Cayley transform (see § 7.3). The equivalence of (b) with (c), on the other hand, follows from (a) and (b) of Lemma 7.4.2 together with Lemma 7.4.1. □

Corollary 7.4.6. *Both assertions of Lemma 6.4.2 hold in our setting, at least for Y and γ very regular.*

Proof. This follows from Lemma 7.4.2 together with § 7.3. □

7.5. Transfer factors and descent

We normalize transfer factors as in [83]. This normalization involves the choice of an $\eta \in F^\times$, and we require that the η chosen for the twisted endoscopic datum at the beginning of this section is the same as that chosen for the standard endoscopic datum in § 7.1.

7.5.A. Review of Waldspurger’s formulas. Suppose that the equivalence class of a very regular element $Y \in \mathfrak{h}(F)$ (respectively, $h \in \underline{\mathrm{H}}(F)$, respectively, $\tilde{h} \in \underline{\mathrm{H}}(F)$) is parameterized by $(L, L_\pm, \bar{y})$ (respectively, $(L, L_\pm, y)$, respectively, $(L, L_\pm, \tilde{y})$). Suppose that $X \in \mathfrak{g}_\theta(F)$ (respectively, $m \in \underline{\mathrm{G}}_\theta(F)$, respectively, $\delta \in \tilde{\underline{\mathrm{G}}}(F)$) matches Y (respectively, h, respectively, $\tilde{h}$). Suppose that the equivalence class of X (respectively, m, respectively, δ) is parameterized by $(L, L_\pm, \bar{x}, \bar{c})$ (respectively, $(L, L_\pm, x, c)$, respectively, $(L, L_\pm, \tilde{x})$).

Write $L_\pm = \prod_{i\in I} F_{\pm i}$, each $F_{\pm i}$ a field extension of F. We have a corresponding factorization $L = \prod_{i\in I} F_i$, where F_i either a quadratic field extension of $F_{\pm i}$ or $F_{\pm i} \times F_{\pm i}$. Let I^* be the set of $i \in I$ such that F_i is a field. For each $i \in I$, let Φ_i be the set of F-algebra homomorphisms of F_i into $\bar{F}$. Let $\bar{y}_i, y_i, \tilde{y}_i, \bar{c}_i, c_i, \tilde{x}_i$ be the components of $\bar{y}, y, \tilde{y}, \bar{c}, c, \tilde{x}$ along F_i.

Let

$$P_I(T) = \prod_{i\in I}\prod_{\phi\in\Phi_i}(T - \phi(y_i)), \quad \tilde{P}_I(T) = \prod_{i\in I}\prod_{\phi\in\Phi_i}(T - \phi(\tilde{y}_i)), \tag{7.5.1}$$

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$$C_i = -\eta c_i P_I'(y_i) P_I(-1) y_i^{1-n}, \quad \text{and} \quad \tilde{C}_i = \eta \tilde{x}_i^{-1} \tilde{P}_I'(\tilde{y}_i) \tilde{P}_I(-1) \tilde{y}_i^{1-n}(1+\tilde{y}_i). \tag{7.5.2}$$

Then, by [83, Proposition 1.10], we have $C_i, \tilde{C}_i \in F_{\pm i}^\times$, and

$$\Delta(h, m) = \prod_{i \in I^*} \mathrm{sgn}_{F_i/F_{\pm i}}(C_i), \quad \Delta(\tilde{h}, \delta) = \prod_{i \in I^*} \mathrm{sgn}_{F_i/F_{\pm i}}(\tilde{C}_i). \tag{7.5.3}$$

(Note that, in the notation of [83], in our situation we have $I^- = I$.) On the other hand, by [84, Proposition X.8] (which holds without any hypothesis on p, cf. Remark X.1 (2) of [84]), albeit following the conventions of [83] (cf. [83, Remark 1.3]), letting

$$\bar{P}_I(T) = \prod_{i \in I} \prod_{\phi \in \Phi_i} (T - \phi(\bar{y}_i)) = \det(T - Y; W), \quad \text{and} \quad \bar{C}_i = -\eta \bar{c}_i \bar{P}_I'(\bar{y}_i),$$

$\bar{C}_i \in F_{\pm i}^\times$ and

$$\Delta(Y, X) = \prod_{i \in I^*} \mathrm{sgn}_{F_i/F_{\pm i}}(\bar{C}_i). \tag{7.5.4}$$

In fact, one may also deduce equation (7.5.4) from equation (7.5.3) using that $\Delta(Y, X)$ equals $\Delta(\exp t^2 Y, \exp t^2 X)$ for $t \in F^\times$ with $|t|$ sufficiently small.

7.6. Transfer factors and scaling

In this section we study the behavior of the factors for $\mathfrak{h}$ and $\mathfrak{g}_\theta$ with respect to the scaling by $F^\times$ on $\mathfrak{h}$ and $\mathfrak{g}_\theta$. For this the following elementary observation will be needed.

Remark 7.6.1. If (W_1, q_{W_1}) is any even-dimensional quadratic space over F with determinant $\alpha \in F^\times/F^{\times 2}$, and if $X \in \mathfrak{so}(W_1)(F)$ has nonzero determinant, then $\det X$ has image α in $F^\times/F^{\times 2}$. This follows from the fact that, for a (necessarily symmetric) matrix J representing (W_1, q_{W_1}) with respect to some basis of W_1, XJ is skew symmetric with respect to that basis, and hence $\det XJ = \mathrm{pf}(XJ)^2$ is a square in F, pf denoting the Pfaffian.

Lemma 7.6.2. *Let the equivalence class of a very regular element $Y \in \mathfrak{h}$ be parameterized by $(L, L_\pm, y)$. Associate to this triple $I, I^*, F_i, F_{\pm i}$ as before. Then*

$$\prod_{i \in I^*} \mathrm{sgn}_{F_i/F_{\pm i}}|_{F^\times} = \big((-1)^n \det q_W, \cdot\big)_F$$

as characters of $F^\times$, where $(\cdot,\cdot)_F$ stands for the Hilbert symbol on F of order 2 (recall that $\dim W = 2n$*).*

Proof. Note that, for $i \in I^*$, $\mathrm{sgn}_{F_i/F_{\pm i}}(a) = (y_i^2, a)_{F_{\pm i}}$ (here $y_i^2 = -y_i \tau(y_i) \in F_{\pm i}$). This in fact holds good for all $i \in I$, if we interpret $\mathrm{sgn}_{F_i/F_{\pm i}}$ as trivial for $i \in I \setminus I^*$. By the behavior of Hilbert symbols with respect to field extensions, we have, for all $a \in F^\times$,

$$(y_i^2, a)_{F_{\pm i}} = (N_{F_{\pm i}/F}(y_i^2), a)_F = (N_{F_i/F}(y_i), a)_F \cdot (-1, a)_F^{[F_{\pm i}:F]}$$

(again, even when $i \in I \setminus I^*$). Hence, using Remark 7.6.1,

$$\prod_{i \in I^*} \mathrm{sgn}_{F_i/F_{\pm i}}(a) = \prod_{i \in I} \mathrm{sgn}_{F_i/F_{\pm i}}(a) = ((\det Y; W), a)_F \cdot (-1, a)_F^n = \big((-1)^n \det q_W, a\big)_F. \ \square$$

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The following is now easy (cf. equation (7.5.4)).

Corollary 7.6.3. *If Y matches X, then for all $a \in F^\times$ aY matches aX, and*

$$\Delta(aY, aX) = \big((-1)^n \det q_W, a\big)_F \cdot \Delta(Y, X). \qquad \square$$

Remark 7.6.4. Thus, in our situation, the character $\chi_{\underline{\mathsf{G}},\underline{\mathrm{H}}}$ mentioned in [24, pp. 372–373] equals $((-1)^n \det q_W, \cdot)_F$.

Remark 7.6.5. As mentioned earlier, we follow [83] in considering transfer factors without the contribution 'Δ_{IV}' from discriminant factors. But the latter scale well too, since it is easy to see that $|D_{\mathfrak{g}}(tX)| = |t|^{\dim \mathfrak{g} - \mathrm{rk}\, g}|D_{\mathfrak{g}}(X)|$ for regular $X \in \mathfrak{g}$, and similarly for the Lie algebra of any reductive group in place of $\mathfrak{g}$.

7.7. Descent for transfer factors in case (c)

Lemma 7.7.1. *Let $Y \in \mathfrak{h}(F)_{\mathrm{tn,srss}}$ be very regular, and let $X \in \mathfrak{g}_\theta(F)_{\mathrm{tn,srss}}$ be such that $Y/2$ and X match. Then*

$$\Delta(Y/2, X) = \Delta(\mathfrak{c}(Y/2), \mathfrak{c}(X)) = \Delta(\mathfrak{c}'(Y), \mathfrak{c}(X)\tilde{\theta}).$$

Proof. Let the equivalence class of $\mathfrak{c}(Y/2)$ as in § 7.2 be parameterized by $(L, L_\pm, y)$. Then, by § 7.3, the equivalence class of $Y/2$ is parameterized by $(L, L_\pm, \bar{y})$, where $\bar{y} = 2(y-1)/(y+1)$. For proving the first equality, namely $\Delta(Y/2, X) = \Delta(\mathfrak{c}(Y/2), \mathfrak{c}(X))$, it is enough to show, adapting notation from § 7.5.A, that $C_i/\bar{C}_i \in N_{F_i/F_{\pm i}}(F_i^\times)$ for each $i \in I^*$. Since $(-2)^{2n} \in F_{\pm i}^2 \subset N_{F_i/F_{\pm i}}(F_i^\times)$, and since $F_i/F_{\pm i}$ is either unramified or tamely and totally ramified (p being odd), the first equality will follow once we show that

$$\left|\frac{P_I'(y_i)}{\bar{P}_I'(\bar{y}_i)} - 1\right| < 1, \qquad \left|\frac{P_I(-1)}{(-2)^{2n}} - 1\right| < 1, \qquad \left|y_i^{1-n} - 1\right| < 1,$$

which is readily checked using that $Y/2$ is topologically nilpotent (so $|y_i - 1| < 1$). We have used that the set of all elements $a \in \bar{F}$ with $|a-1| < 1$ is closed under multiplication, which is easy to check.

Now we move to the second equality. We know that the conjugacy class of $\mathfrak{c}(X)$ is parameterized by $(L, L_\pm, y, c)$ for some c, that of $\mathfrak{c}'(Y)$ by $(L, L_\pm, \tilde{y})$, where $\tilde{y} := y^2$, and that of $\mathfrak{c}(X)\tilde{\theta}$ by $(L, L_\pm, \tilde{x})$, where $\tilde{x} := yc$. Now we have suitable polynomials P_I, $\tilde{P}_I$ as in equation (7.5.1), and C_i, $\tilde{C}_i$ as in equation (7.5.2).

Fix $i \in I^*$. To finish, it suffices to show that $\tilde{C}_i/C_i \in N_{F_i/F_{\pm i}}(F_i^\times)$. Note (using $\tau(c_i) = -c_i$) that

$$\frac{\tilde{C}_i}{C_i} \cdot \frac{c_i \tau(c_i)}{2^{2n}} = \frac{\tilde{P}_I'(y_i^2)}{2^{2n-1} P_I'(y_i)} \cdot \frac{\tilde{P}_I(-1)}{(-2)^{2n}} \cdot \frac{(-2)^{2n}}{P_I(-1)} \cdot y_i^{-n} \cdot \frac{1+y_i^2}{2}.$$

That $\tilde{C}_i/C_i \in N_{F_i/F_{\pm i}}(F_i^\times)$ will follow if we show that each term a on the right side of the above equation satisfies $|a-1| < 1$. But this is easy to check, completing the proof of the second equality. $\square$

Recall that $\mathcal{U} = \mathrm{tc}(\underline{\mathsf{G}}(F) \times \underline{\mathsf{G}}(F)_{\mathrm{tu}})$ (cf. Notation 4.0.2) and Definition 4.2.2. Fix Haar measures dg, dm, and dh on $\underline{\mathsf{G}}(F)$, $\underline{\mathsf{G}}^\theta(F)$, and $\underline{\mathrm{H}}(F)$ respectively.

Lemma 7.7.2. *Suppose that* $f \in C_c^\infty(\mathcal{U}), \phi \in C_c^\infty(\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}})$ *and* $f^H \in C_c^\infty(\underline{\mathsf{H}}(F)_{\mathrm{tu}})$. *Suppose that* ϕ *is obtained from* f *by semisimple descent with respect to* dg *and* dm. *Then* f *and* f^H *have matching orbital integrals with respect to* dg *and* dh *if and only if* $\phi \circ \mathfrak{c} \in C_c^\infty(\mathfrak{g}_\theta(F))$ *and* $f^H \circ \mathfrak{c}' \circ (Y \mapsto 2Y) \in C_c^\infty(\mathfrak{h}(F))$ *have matching orbital integrals with respect to* dm *and* dh.

Remark 7.7.3. Pull back under $Y \mapsto 2Y$ is what realizes the nonstandard endoscopic transfer implicit in the situation.

Proof of Lemma 7.7.2. Recall that f and f^H have matching orbital integrals if and only if, for all strongly $\tilde{\underline{\mathsf{G}}}$-regular $\gamma \in \underline{\mathsf{H}}(F)$, we have an equality as in equation (6.6.1):

$$\sum_{\{\gamma'\}} I(\gamma', f^H) = \sum_{\{\delta\}} \Delta(\gamma, \delta) I(\delta, f). \tag{7.7.1}$$

$f^H \circ \mathfrak{c}' \circ (Y \mapsto 2Y)$ and ϕ have matching orbital integrals if and only if, for any $Y \in \mathfrak{h}(F)_{\mathrm{srss}}$ that is $\mathfrak{g}_\theta$-regular, we have

$$\sum_{Y'} I(Y', f^H \circ \mathfrak{c}') = \sum_{\{X\}} \Delta(Y/2, X) I(X, \phi \circ \mathfrak{c}, \underline{\mathsf{G}}^\theta(F)), \tag{7.7.2}$$

where $\{Y'\}$ is a set of representatives for the $\underline{\mathsf{H}}(F)$-conjugacy classes in the stable conjugacy class of Y, and $\{X\}$ is a set of representatives in $\mathfrak{g}_\theta(F)$ for the $\underline{\mathsf{G}}^\theta(F) = \underline{\mathsf{G}}_\theta(F)$-conjugacy classes in $\mathfrak{g}_\theta(F)$ that match $Y/2$. In equations (7.7.1) and (7.7.2) again one has to worry about normalizing measures. This is done as in Remark 6.6.2(i).

By continuity, it is enough to test equations (7.7.1) and (7.7.2) on very regular γ and very regular Y, respectively (the existence of endoscopic transfer allows us to see this without appealing to any continuity assertion for transfer factors).

By Corollary 7.4.6, and recalling that $\mathfrak{c}'(\mathfrak{h}(F)_{\mathrm{tn,srss}}) = \underline{\mathsf{H}}(F)_{\mathrm{tu,srss}}$, it suffices to show that, for all very regular $Y \in \mathfrak{h}(F)_{\mathrm{tn,srss}}$, Y is $\mathfrak{g}_\theta$-regular if and only if $\gamma := \mathfrak{c}'(Y)$ is strongly $\tilde{\underline{\mathsf{G}}}$-regular, and that in this case the right sides of equations (7.7.1) and (7.7.2) coincide. The set $\{X\}$ as defined above is nonempty by Lemma 7.4.2 and §7.3. For any $X' \in \{X\}$, $\mathfrak{c}(X')\tilde{\theta}$ matches $\mathfrak{c}'(Y)$ by Lemma 7.4.5, and, by Lemma 4.0.1 and the conjugation equivariance of $\mathfrak{c}$, $X' \in \mathfrak{g}_\theta(F)$ is regular if and only if $\mathfrak{c}(X')\tilde{\theta} \in \tilde{\underline{\mathsf{G}}}(F)$ is strongly regular. Thus, Y is $\mathfrak{g}_\theta$-regular if and only if $\mathfrak{c}'(Y)$ is strongly regular. By Lemma 4.0.4, we may assume without loss of generality that $X' \mapsto \mathfrak{c}(X')\tilde{\theta}$ induces a bijection $\{X\} \leftrightarrow \{\delta\} \cap \mathcal{U}$. Then the desired equality between the right sides of (7.7.1) and (7.7.2) for very regular Y and X follows from Lemma 7.7.1 and the compatibility of centralizer measures with semisimple descent and the analog of Remark 6.6.2(iii). □

Notation 7.7.4. We choose notation so that the left and right sides of equation (7.7.2) are denoted by $SI(Y, f^H \circ \mathfrak{c}') = SI(Y, f^H \circ \mathfrak{c}', dh/dt_Y)$ and $I^{\underline{\mathsf{G}}_\theta}(Y, \phi \circ \mathfrak{c}) = I^{\underline{\mathsf{G}}_\theta}(Y, \phi \circ \mathfrak{c}, \underline{\mathsf{G}}^\theta(F), dm, dt_Y)$, respectively. We also choose notation so that the left and right sides of equation (7.7.1) are denoted by $SI(\gamma, f^H) = SI(\gamma, f^H, dh/dt_\gamma)$ and $I^{\tilde{\underline{\mathsf{G}}}}(\gamma, f) = I^{\tilde{\underline{\mathsf{G}}}}(\gamma, f, dg, dt_\gamma)$, respectively.

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8. Depth comparison

In this section, $\underline{\mathrm{H}}$ is necessarily unramified.

8.1. Congruence filtrations

Fix once and for all an $\mathfrak{O}$-lattice $\Lambda \subset V$. Assume further that $\tilde{\theta}$ is defined on Λ using the recipe of Remark 2.1.1, but using a basis for Λ and taking $\nu \in \mathfrak{O}^\times$ (this is legitimate; cf. Section 2.1).

Λ lets us realize $\underline{\mathsf{G}}$ as a group over $\mathfrak{O}$, whose functor of points is given by $R \rightsquigarrow \mathrm{GL}(\Lambda \otimes_{\mathfrak{O}} R)$. $(\Lambda, \tilde{\theta})$ is unimodular (i.e., for $v \in V$, $\tilde{\theta}(v, \Lambda) \subset \mathfrak{O}$ if and only if $v \in \Lambda$). This gives a smooth $\mathfrak{O}$-model for $\underline{\mathsf{G}}^\theta$ and $\underline{\mathsf{G}}_\theta$ as the stabilizer of $\tilde{\theta}$ in $\underline{\mathsf{G}}$ and its connected component, respectively. We work with these models henceforth.

For all $m \in \mathbb{N}$ ($m \geqslant 1$), $\mathfrak{k}_m = \varpi^m \mathsf{g}(\mathfrak{O}) \subset \mathsf{g}(F)_{\mathrm{tn}}$ satisfies the hypotheses of Lemma 4.2.4, and hence in particular those of Lemma 4.2.3 too. Therefore, using Lemma 4.2.3,

$$K_m = 1 + \mathfrak{k}_m = \mathfrak{c}(\mathfrak{k}_m) = \mathfrak{c}'(\mathfrak{k}_m) = \ker\left(\underline{\mathsf{G}}(\mathfrak{O}) \to \underline{\mathsf{G}}(\mathfrak{O}/\varpi^m\mathfrak{O})\right) \subset \underline{\mathsf{G}}(F)$$

and

$$K_{m,\theta} = \mathfrak{c}(\mathfrak{k}_{m,\theta}) = \mathfrak{c}'(\mathfrak{k}_{m,\theta}) = \ker\left(\underline{\mathsf{G}}_\theta(\mathfrak{O}) \to \underline{\mathsf{G}}_\theta(\mathfrak{O}/\varpi^m\mathfrak{O})\right) \subset \underline{\mathsf{G}}_\theta(F)$$

are compact open subgroups of $\underline{\mathsf{G}}(F)$ and $\underline{\mathsf{G}}_\theta(F)$, respectively.

Further, there exist hyperspecial vertexes $x, \bar{x}$ in the buildings of $\underline{\mathsf{G}}$ and $\underline{\mathsf{G}}_\theta$, respectively, such that $K_m = G_{x,m}$ and $K_m^\theta = G_{\theta,x,m}$.

Since $\underline{\mathrm{H}}$ is unramified, we may similarly assume it to have been defined by a unimodular lattice $(\Lambda_{\underline{\mathrm{H}}}, q_W)$, but we cannot and do not assume q_W to be defined using the prescription of Remark 2.1.1. Then we have a lattice $\mathfrak{k}_{\underline{\mathrm{H}}} = \mathfrak{h}(\mathfrak{O}) \subset \mathfrak{h}(F)$, a hyperspecial subgroup $K_{\underline{\mathrm{H}}} \subset \underline{\mathrm{H}}(F)$, and, for all $m \in \mathbb{N}$, lattices $\mathfrak{k}_{\underline{\mathrm{H}},m} \subset \mathfrak{h}(F)$ and compact open subgroups $K_{H,m} = \mathfrak{c}(\mathfrak{k}_{H,m}) = \mathfrak{c}'(\mathfrak{k}_{H,m}) \subset \underline{\mathrm{H}}(F)$.

8.2. The fundamental lemmas needed

We now state an equivalent version of the fundamental lemmas (the nonstandard fundamental lemma in cases (a) and (b), the fundamental lemma in case (c)) that we need. The following lemma holds for p sufficiently large by [65] together with either of [85] or [15], a fact that has consequences that let one prove this lemma for arbitrary odd p. But we postpone this proof (i.e., that of the lemma for arbitrary odd p) to § 9.

Lemma 8.2.1. (i) *In cases (a) and (b) of § 2.1, (an equivalent version of) the nonstandard fundamental lemma holds for* $(\underline{\mathrm{H}}_{\mathrm{sc}}, \underline{\mathsf{G}}_{\theta,\mathrm{sc}}, j_*)$*; i.e.,* $(\mathrm{meas}\, K_{\underline{\mathrm{H}}})^{-1}\mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}}} \in C_c^\infty(\mathfrak{h}(F))$ *and* $(\mathrm{meas}\, K_\theta)^{-1}\mathbb{1}_{\mathfrak{k}_\theta} \in C_c^\infty(\mathsf{g}_\theta(F)) = C_c^\infty(\mathsf{g}_{\theta,\mathrm{sc}}(F))$ *have matching orbital integrals.*

(ii) *In case (c) of § 2.1, the fundamental lemma for Lie algebras holds for the endoscopic datum of § 7.1, so that* $(\mathrm{meas}\, K_{\underline{\mathrm{H}}})^{-1}\mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}}}$ *and* $(\mathrm{meas}\, K_\theta)^{-1} \cdot \mathbb{1}_{\mathfrak{k}_\theta}$ *have matching orbital integrals.*

Lemma 8.2.2. *For all* $m \in \mathbb{N}$, $a \cdot \mathbb{1}_{K_{H,m}} \in C_c^\infty(\underline{\mathrm{H}}(F))$ *and* $\mathbb{1}_{K_m \tilde{\theta}} \in C_c^\infty(\tilde{\underline{\mathrm{G}}}(F))$ *have matching orbital integrals, where* a *equals* $(\operatorname{meas} K_{\underline{\mathrm{H}},m})^{-1}(\operatorname{meas} K_m)$ *in cases (a) and (b), and*

$$((-1)^n \det q_W, \varpi)_F^m \cdot [K_\theta : K_{m,\theta}] q^{-mn} \cdot \frac{\operatorname{meas} K_m}{\operatorname{meas} K_{\underline{\mathrm{H}}}}$$

in case (c).

Proof. First consider case (c). By Lemma 4.2.4, $(\operatorname{meas} K_{m,\theta})^{-1}\mathbb{1}_{K_{m,\theta}}$ is obtained from $(\operatorname{meas} K_m)^{-1}\mathbb{1}_{K_m\tilde{\theta}}$ by semisimple descent at $\tilde{\theta}$, up to a nonzero scalar. Therefore, by Lemma 7.7.2, it suffices to show that $\mathbb{1}_{K_{m,\theta}} \circ \mathfrak{c} = \mathbb{1}_{\mathfrak{k}_{m,\theta}}$ and

$$\left((-1)^n \det q_W, \varpi\right)_F^m q^{-mn} \cdot \frac{\operatorname{meas} K_\theta}{\operatorname{meas} K_{\underline{\mathrm{H}}}} \cdot \mathbb{1}_{K_{\underline{\mathrm{H}},m}} \circ \mathfrak{c}' \circ (Y \mapsto 2Y) = a \cdot \mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}},m}}$$

have matching orbital integrals. This follows using a standard argument from the fundamental lemma for Lie algebras in this situation, namely Lemma 8.2.1(ii) (see [24], Proposition 3.2.2). One uses Corollary 7.6.3 and Remark 7.6.5 in place of Lemma 3.2.1 of [24].

Cases (a) and (b) are similar. One uses Lemma 4.2.4 as before to reduce to showing that $(\operatorname{meas} K_{m,\theta})^{-1}\mathbb{1}_{K_{m,\theta}} \circ \mathfrak{c} = (\operatorname{meas} K_{m,\theta})^{-1}\mathbb{1}_{\mathfrak{k}_{m,\theta}}$ and $(\operatorname{meas} K_{H,m})^{-1}\mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}},m}}$ have matching orbital integrals. But using that $\#\underline{\mathrm{G}}_\theta(\kappa) = \#\underline{\mathrm{H}}(\kappa)$ and $\dim \underline{\mathrm{G}}_\theta = \dim \underline{\mathrm{H}}$ in these cases, this follows from Lemma 8.2.1(i) together with the nonstandard endoscopic variant of [24, Proposition 3.2.2], which is easier and in fact more or less immediate, using Remark 7.6.5. □

For $\phi \in \tilde{\Phi}_{\mathrm{bdd}}(H)$, let $\tilde{\Pi}_\phi$ denote the tempered packet as in [7, Ch. 2] (slightly coarser than an L-packet in case (c)). Recall that, in case (c), the Langlands parameter ϕ is well defined only 'up to $\mathrm{O}(2n, \mathbb{C})$-conjugacy', while elements $\pi \in \tilde{\Pi}_\phi$ are determined only up to the obvious action of $\mathrm{O}(2n, F)$. However, even in this case, $\operatorname{depth}\pi$ ($\pi \in \tilde{\Pi}_\phi$) has an unambiguous meaning, as also the question of whether or not $\pi^{K_{\underline{\mathrm{H}},m}} \neq 0$ ($m \in \mathbb{N}$). Let $\pi^{\tilde{G}}$ denote the corresponding tempered representation of $\tilde{\underline{\mathrm{G}}}(F)$ (cf. [51] for what a representation of a twisted space means), and π^G the representation of $\underline{\mathrm{G}}(F)$ underlying $\pi^{\tilde{G}}$ with parameter ϕ. It was shown in [7] that, for some complex number $c \neq 0$, whenever $f^H \in C_c^\infty(\underline{\mathrm{H}}(F))$ and $f \in C_c^\infty(\tilde{\underline{\mathrm{G}}}(F))$ have matching orbital integrals, we have

$$\sum_{\pi \in \tilde{\Pi}_\phi} \operatorname{tr}\pi(f^H) = c \cdot \operatorname{tr}\pi^{\tilde{G}}(f). \tag{8.2.1}$$

Lemma 8.2.3. *Let* $\tilde{\Pi}_\phi, \pi^{\tilde{G}}$, *and* π^G *be as above. Then the following hold.*

(a)

$$\min\left\{m \in \mathbb{N} \;\middle|\; \pi^{K_{\underline{\mathrm{H}},m} \neq 0} \textit{ for some } \pi \in \tilde{\Pi}_\phi\right\} \geqslant \min\left\{m \;\middle|\; \left(\pi^G\right)^{K_m} \neq 0\right\}.$$

(b) *For all* $\pi \in \tilde{\Pi}_\phi$, $\operatorname{depth}\phi = \operatorname{depth}\pi^G \leqslant \lceil \operatorname{depth}\pi \rceil + 1$.

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Proof. (a) follows from Lemma 8.2.2 and equation (8.2.1), together with the observation that, since $\theta(K_m) = K_m$ for each m, $\operatorname{tr} \pi^{\tilde{G}}(\mathbb{1}_{K_m\tilde{\theta}}) \neq 0 \Rightarrow (\pi^G)^{K_m} \neq 0$. As for (b), the first equality is [86, Theorem 2.3.6.4]. For the inequality in (b), notice that, if $\pi \in \tilde{\Pi}_\phi$ has depth r, then, by [29, Lemma 8.2], setting $m = \lceil r \rceil + 1$, we have $\pi^{K_{\underline{H},m+1}} \neq 0$, so that $(\pi^G)^{K_{m+1}} \neq 0$ by (a), so that $\operatorname{depth} \pi^G \leqslant m$ by the characterization of depth as in [63, Theorem 3.5] (here, we are using that $K_{m+1} = G_{x,m+}$ for some vertex x in the enlarged Bruhat–Tits building of G). □

9. Proof of Lemma 8.2.1

9.1. Initial observations

We consider cases (a), (b), and (c) together. Recall that it suffices to show, using notation from Notations 6.6.3 and 7.7.4, that, for all $Y \in \mathfrak{h}(F)$ $\mathfrak{g}_\theta$-regular semisimple,

$$\frac{1}{\operatorname{meas} K_{\underline{H}}} SI(Y, \mathbb{1}_{\mathfrak{k}_{\underline{H}}}, dh/dt_Y) = \frac{1}{\operatorname{meas} K_\theta} I^{\underline{G}^\theta}(Y, \mathbb{1}_{\mathfrak{k}_\theta}, dm, dt_Y). \tag{9.1.1}$$

9.2. Matching on the topologically nilpotent elements

Lemma 9.2.1. $\mathfrak{c}'(\mathfrak{k}_{\underline{H},\mathrm{tn}}) = \mathfrak{c}(\mathfrak{k}_{\underline{H},\mathrm{tn}}) = K_{\underline{H},\mathrm{tu}}$, $\mathfrak{c}(\mathfrak{k}_{\theta,\mathrm{tn}}) = K_{\theta,\mathrm{tu}}$.

Proof. It suffices to prove the statement about $\underline{H}$. By Remarks 3.2.6 and 3.2.8, it suffices to prove that, for $h = \mathfrak{c}(Y) = \mathfrak{c}'(Y') \in \underline{H}(F)_{\mathrm{tu}}$, one of h, Y, Y' preserves the lattice $\Lambda_{\underline{H}}$ if and only if the other two do. If Y (respectively, h) preserves $\Lambda_{\underline{H}}$, then so do $1-(Y/2)$ and $1+(Y/2)$ (respectively, $h-1$ and $h+1$), and, since $1+(Y/2)$ (respectively, $h+1$) has unit determinant, so does $(1+(Y/2))^{-1}$ (respectively, $(h+1)^{-1}$). This gives that Y preserves $\Lambda_{\underline{H}}$ if and only if h does. Let $h_1 = \mathfrak{c}(Y'/2)$, so that $h = h_1^2$. Since Y' preserves $\Lambda_{\underline{H}}$ if and only if $\mathfrak{c}(Y')$ does, it now suffices to show that h preserves $\Lambda_{\underline{H}}$ if and only if h_1 does. This is because h_1 lies in the closure of the subgroup generated by h (cf. the proof of 'surjectivity' in Lemma 3.2.7). □

Let $\mathcal{E}$ equal $\{1\}$ if $\dim V$ is even (i.e., $\underline{G}_\theta = \underline{G}^\theta = \mathrm{Sp}(V, \tilde{\theta})$), and let $\mathcal{E}$ be a set of representatives for $\mathfrak{O}^\times/\mathfrak{O}^{\times 2}$ containing 1 if $\dim V$ is odd (i.e., $\underline{G}_\theta = \mathrm{SO}(V, \tilde{\theta})$). For $a \in \mathcal{E}$, we denote by $\sqrt{a}$ a fixed square root of a in $\bar{F}^\times$, viewed as a scalar in $\underline{G}(\bar{F})$.

Lemma 9.2.2. *If $g \in \underline{G}(\bar{F})$ is such that $g^{-1}\tilde{\theta}g \in K\tilde{\theta}$, then $g \in \sqrt{a} \cdot \underline{G}^\theta(\bar{F}) \cdot K$ for a unique $a \in \mathcal{E}$. If, further, $g \in \underline{G}(F)$, then $a = 1$.*

Proof. $g^{-1}\tilde{\theta}g$ represents a quadratic or symplectic form over $\bar{F}$, of the same sign as $\tilde{\theta}$. Since $g^{-1}\tilde{\theta}g \in K\tilde{\theta}$, this form is defined over F, and moreover $(\Lambda, g^{-1}\tilde{\theta}g)$ is a unimodular lattice (i.e., $g^{-1}\tilde{\theta}g$ induces an isomorphism $\Lambda \to \operatorname{Hom}_{\mathfrak{O}}(\Lambda, \mathfrak{O})$ of $\mathfrak{O}$-modules).

Now the result follows from well-known facts about symplectic and quadratic unimodular lattices. Namely, it is well known (see, for example, [28], Proposition 4.2) that there is a unique isomorphism class of symplectic unimodular lattices of a given even rank, and that there exist exactly two isomorphism classes of quadratic unimodular lattices of any given rank (though only the odd case is what concerns us here), classified by the discriminant (valued in $\mathfrak{O}^\times/\mathfrak{O}^{\times 2}$) (see [66, 92:1]). □

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Lemma 9.2.3. *Equation (9.1.1) holds for all* $Y \in \mathfrak{k}_{\underline{H},\mathrm{tn}}$.

Proof. First we consider cases (a) and (b). Suppose that $Y \in \mathfrak{k}_{\underline{H},\mathrm{tn}}$ is strongly $\mathfrak{g}_\theta$-regular. The stable conjugacy class of Y is contained in $\mathfrak{h}(F)_{\mathrm{tn}}$, so that, by Lemma 4.1.3, Lemma 9.2.1 and the conjugation equivariance of $\mathfrak{c}$ and $\mathfrak{c}'$, we conclude that $SI(Y, \mathbb{1}_{\mathfrak{k}_{\underline{H}}}) = SI(\mathfrak{c}'(Y), \mathbb{1}_{K_{\underline{H}}})$. We know from the fundamental lemma for the pair $(\underline{H}, \underline{G})$ (cf. [48, Proposition 4.13]; see also [7, pp. 412 to 413]) that $(\text{meas } K_{\underline{H}})^{-1} SI(\mathfrak{c}'(Y), \mathbb{1}_{K_{\underline{H}}}) = (\text{meas } K)^{-1} I^{\tilde{\underline{G}}}(\mathfrak{c}'(Y), \mathbb{1}_{K\tilde{\theta}})$, which by Lemma 6.3.1 and Lemma 6.5.1 equals $SI(\mathfrak{c}(X_0)\tilde{\theta}, \mathbb{1}_{K\tilde{\theta}})$, for any $X_0 \in \mathfrak{g}_\theta$ that matches Y. Hence, using Lemma 4.1.3, it suffices to show (thanks to compatible centralizer measures) that

$$\frac{1}{\text{meas } \mathfrak{k}_\theta} \sum_{\{X\}} O(X, \mathbb{1}_{\mathfrak{k}_\theta}, \underline{G}^\theta(F)) = \frac{\text{meas } K_{\underline{H}}}{(\text{meas } K)(\text{meas } \mathfrak{k}_{\underline{H}})} \sum_{\{\delta\}} O(\delta, \mathbb{1}_{K\tilde{\theta}}),$$

where $\{X\}$ is a set of representatives in $\mathfrak{g}_\theta(F)_{\mathrm{tn}}$ for the $\underline{G}^\theta(F)$-conjugacy classes stably conjugate to X_0, and $\{\delta\}$ a set of representatives for the $\underline{G}(F)$-conjugacy classes that are stably conjugate to $\mathfrak{c}(X_0)\tilde{\theta}$ *and* intersect $K\tilde{\theta}$.

We view elements of $\mathcal{E}$ as scalar matrices in $\mathrm{GL}(V)$. Using Lemma 4.2.4(ii), it follows that

$$\begin{aligned}\frac{1}{\text{meas } \mathfrak{k}_\theta} O(X, \mathbb{1}_{\mathfrak{k}_\theta}, \underline{G}^\theta(F)) &= \frac{1}{\text{meas } \mathfrak{k}_\theta} O\left(X, \mathbb{1}_{\mathfrak{k}_{\theta,\mathrm{tn}}}, \underline{G}^\theta(F)\right) \\ &= \frac{\text{meas } K^\theta}{(\text{meas } \mathfrak{k}_\theta)(\text{meas } K)} \cdot O\left(\mathfrak{c}(X)\tilde{\theta}, \mathbb{1}_{\mathrm{tc}(K\times\mathfrak{k}_{\theta,\mathrm{tn}})}\right).\end{aligned}$$

Now, since $\#\mathcal{E} = [K^\theta : K_\theta]$, it suffices to prove that

$$O\left(\mathfrak{c}(X)\tilde{\theta}, \mathbb{1}_{\mathrm{tc}(K\times\mathfrak{k}_{\theta,\mathrm{tn}})}\right) = O\left(a\mathfrak{c}(X)\tilde{\theta}, \mathbb{1}_{K\tilde{\theta}}\right) \ \forall a \in \mathcal{E}, \tag{9.2.1}$$

and that

$$\text{One can choose } \{\delta\} = \bigsqcup_{a\in\mathcal{E}} \bigsqcup_{\{X\}} a\mathfrak{c}(X)\tilde{\theta}. \tag{9.2.2}$$

Given Lemma 9.2.2, together with Lemma 4.0.1, both (9.2.1) and (9.2.2) will follow if we show that any $g \in \underline{G}(\bar{F})$ with $g^{-1}\mathfrak{c}(X)\tilde{\theta}g \in K\tilde{\theta}$ automatically satisfies $g^{-1}\tilde{\theta}g \in K\tilde{\theta}$. However, this follows from the topological Jordan decomposition (more explicitly, identifying $\underline{G}\tilde{\theta}$ with $\underline{G} \rtimes \theta \subset \underline{G} \rtimes \langle\theta\rangle$ in the obvious way,

$$g^{-1}\theta(g) \rtimes \theta = \lim_{n\to\infty} g^{-1}(\mathfrak{c}(X) \rtimes \theta)^{p^n} g = \lim_{n\to\infty} (g^{-1}(\mathfrak{c}(X) \rtimes \theta)g)^{p^n} \in K \rtimes \theta$$

).

Now we come to case (c). This is similar but easier, using the fundamental lemma for the standard endoscopic datum for $(\underline{H}, \underline{G}_\theta)$ constructed in § 7.1. Namely, for $Y \in \underline{H}(F)_{\mathrm{tu}}$, we have, using Lemma 9.2.1, that

$$\frac{1}{\text{meas } K_{\underline{H}}} SI(Y, \mathbb{1}_{\mathfrak{k}_{\underline{H}}}) = \frac{1}{\text{meas } K_{\underline{H}}} SI(\mathfrak{c}(Y), \mathbb{1}_{K_{\underline{H}}}) = \frac{1}{\text{meas } K^\theta} I^{\underline{G}_\theta}\left(\mathfrak{c}(Y), \mathbb{1}_{K^\theta}\right),$$

which equals $(\text{meas } K^\theta)^{-1} \cdot I^{\underline{G}_\theta}(Y, \mathbb{1}_{\mathfrak{k}_\theta})$ by Lemmas 9.2.1 and 7.7.1. □

9.3. Eigenvalue criterion for Y having a stable conjugate in $\mathfrak{k}_{\underline{H}}$

Lemma 9.3.1. *A regular semisimple element $Y \in \mathfrak{h}(F) \subset \mathrm{End}(W)(F)$ is stably conjugate to an element of $\mathfrak{k}_{\underline{H}}$ if and only if all of its eigenvalues are integral over $\mathfrak{O}$.*

Proof. The 'only if' part of the assertion is clear, as stable conjugacy with an element of $\mathfrak{k}_{\underline{H}}$ forces the characteristic polynomial of Y to have coefficients in $\mathfrak{O}$. So we need to show that any $Y \in \mathfrak{h}(F)$ with integral eigenvalues has a stable conjugate in $\mathfrak{k}_{\underline{H}}$.

Suppose that $\underline{T}_{\underline{H}}$ is the maximal torus of some Borel subgroup of $\underline{H}$ and that $W_{\underline{H}}$ is the associated Weyl group. Then $\mathfrak{t}_{\underline{H}}/W_{\underline{H}}$ is naturally defined over $\mathfrak{O}$, and we have an adjoint quotient map $\mathfrak{h} \to \mathfrak{t}_{\underline{H}}/W_{\underline{H}}$ defined over $\mathfrak{O}$ that is surjective at the level of $\mathfrak{O}$-points (see [42, Lemma 3.1.2(b)]). Thus, we need to show that the image of Y in $\mathfrak{t}_{\underline{H}}/W_{\underline{H}}(F)$ is defined over $\mathfrak{O}$, or, equivalently, over the ring of integers of an unramified extension of F that splits $\underline{H}$. This follows from the explicit description of the adjoint quotient of $\underline{H}$; cf. [13, Theorems 1.1 and 1.3] (the Pfaffian of Y is integral too as its square is). □

9.4. A crude substitute for the Topological Jordan Decomposition

We now need to define a construction for certain elements of the Lie algebras $\mathfrak{h}$ and g_θ that crudely imitates the topological Jordan decomposition at the group level.

However, while there is a well-known notion of a topologically nilpotent element in a Lie algebra, we are not able to think of a suitable definition for a 'topologically semisimple' element of a Lie algebra (though, if F were a field like $\mathbb{C}((t))$, such a notion is known; see, for example, [43, § 10.1]). On the other hand, Mœglin and Waldspurger had suggested to us that a decomposition like the topological Jordan decomposition could be used for Lie algebras.

In this subsection, we will define such a decomposition for $\mathfrak{h}$. All that we will use about $\underline{H}$ is that it is given as the group scheme of isometries of a unimodular quadratic or symplectic lattice. Hence, all the constructions we define for $\underline{H}$ will be applicable to $\underline{\mathsf{G}}_\theta$ as well. Later we will need to relate the decompositions we define on $\mathfrak{h}$ and g_θ, for reducing the proof of equation (9.1.1) for a general $Y \in \mathfrak{k}_{\underline{H}}$ to that of a similar equation for a topologically nilpotent element in a smaller Lie algebra.

Our decomposition will depend on a choice, namely that of a set-theoretic lift $\bar{\lambda} \mapsto \lambda_s$ from $\bar{\kappa}$ to $\mathfrak{O}_{F^{\mathrm{unr}}}$, which is invariant under the obvious action of $\mathrm{Gal}(\bar{F}/F)$ as well as under multiplication by -1. Fix one such, as we may (for example, the Teichmüller lift).

Notation 9.4.1. Henceforth, if $\lambda \in \mathfrak{O}_{\bar{F}}$, we denote by $\bar{\lambda}$ its image in $\bar{\kappa}$ and by λ_s the lift constructed above.

Suppose that $Y \in \mathfrak{k}_{\underline{H}}$ is semisimple. We wish to write Y as $Y_s + Y_n$, where $Y_s \in \mathfrak{k}_{\underline{H}}$ has all eigenvalues in $\mathfrak{O}^{\times}_{F^{\mathrm{unr}}}$, $Y_n \in \mathfrak{k}_{\underline{H},\mathrm{tn}}$, and $[Y_s, Y_n] = 0$. Moreover, we want any two eigenvalues of Y_s to be either equal or have different images in $\bar{\kappa}$. This condition seems analogous to one in [44, Proposition 7.1]. It will be useful to us through the following elementary linear algebra lemma.

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Lemma 9.4.2. *Let W_1 be a vector space over F and $\Lambda_1 \subset W_1$ a lattice. Suppose that $T \in \mathrm{End}_F(W_1)$ preserves Λ_1. Let $W_1 = W_1' \oplus W_1''$ be a direct sum decomposition of W_1 into T-invariant subspaces, such that, if λ and μ are eigenvalues of T on $W_1' \otimes_F \bar{F}$ and $W_1'' \otimes_F \bar{F}$, respectively, then $\bar{\lambda} \neq \bar{\mu}$ (note that $\lambda, \mu \in \mathfrak{O}_{\bar{F}}$). Then*

$$\Lambda_1 = (\Lambda_1 \cap W_1') \oplus (\Lambda_1 \cap W_1'').$$

Proof. We need to show that the projection from W_1 to each of W_1' and W_2' preserves Λ_1. It is enough to prove this after base changing to a finite extension E of F. Thus, we may assume that all the eigenvalues of T are contained in F. Let f_1 and f_2 be the characteristic polynomials of T on W_1' and W_2'. Since T preserves Λ_1, it is enough to show that f_1 and f_2 generate the unit ideal in $\mathfrak{O}_F[T]$. In other words, we need to show that any prime ideal of $\mathfrak{O}_F[T]$ that contains f_1 and f_2 is the unit ideal. But such an ideal contains $T - \lambda$ and $T - \mu$ for an eigenvalue λ of T on W_1' and an eigenvalue μ of T on W_1'', and hence also $\lambda - \mu \in \mathfrak{O}_F^\times$. □

9.4.A. Construction of Y_s and Y_n. Each eigenvalue λ of Y in $\bar{F}$ belongs to $\mathfrak{O}_{\bar{F}}$ (as $Y \in \mathfrak{k}_{\underline{\mathrm{H}}}$), so we can talk of its reduction $\bar{\lambda} \in \bar{\kappa}$, and the lift $\lambda_s \in \mathfrak{O}_{F^{\mathrm{unr}}}$ fixed in the above paragraph. Let Y_s be the element of $\mathfrak{gl}(W)(\bar{F})$ that, for each eigenvalue λ of Y in $\bar{F}$, acts on the corresponding eigenspace in $W \otimes_F \bar{F}$ by λ_s. Set $Y_n = Y - Y_s$.

Lemma 9.4.3. *$Y_s, Y_n \in \mathfrak{k}_{\underline{\mathrm{H}}}$, $[Y_s, Y_n] = 0$, and Y_n is topologically nilpotent.*

Proof. The $\mathrm{Gal}(\bar{F}/F)$-equivariance of $\lambda \mapsto \bar{\lambda} \mapsto \lambda_s$ implies that Y_s is defined over F. The invariance of $\lambda \mapsto \bar{\lambda} \mapsto \lambda_s$ under $\{\pm 1\}$ implies that $Y_s \in \mathfrak{h}(\bar{F})$. It is clear that the linear transformation $Y - Y_s$ is topologically nilpotent. To finish, it suffices to show that Y_s preserves the lattice $\Lambda_{\underline{\mathrm{H}}}$ that defines $\underline{\mathrm{H}}$. Since Y_s preserves $W \subset W \otimes_F \bar{F}$, this in turn will follow if we show that Y_s preserves $\Lambda_{\underline{\mathrm{H}}} \otimes_{\mathfrak{O}} \mathfrak{O}_E \subset W \otimes_F E$ for some finite extension $E \subset \bar{F}$ of F. Choose E to contain all the eigenvalues of Y in $\bar{F}$.

For $\lambda \in \mathfrak{O}_E$, let $W_\lambda \subset W \otimes_F E$ be the λ-eigenspace of Y. For each $\bar{\lambda} \in \kappa$, let

$$W_{\bar{\lambda}} = \bigoplus_{\substack{\mu \in \mathfrak{O}_E \\ \bar{\mu} = \bar{\lambda}}} W_\mu.$$

Y_s acts on each $W_{\bar{\lambda}}$ by λ_s. Therefore it suffices to show that

$$\Lambda_{\underline{\mathrm{H}}} \otimes_{\mathfrak{O}} \mathfrak{O}_E = \bigoplus_{\bar{\lambda} \in \kappa} \left((\Lambda_{\underline{\mathrm{H}}} \otimes_{\mathfrak{O}} \mathfrak{O}_E) \cap W_{\bar{\lambda}} \right).$$

This follows from Lemma 9.4.2. □

Let $\underline{\mathrm{H}}_{Y_s}$ be the centralizer of Y_s in $\underline{\mathrm{H}}$ over $\mathfrak{O}$; i.e., for any $\mathfrak{O}$-algebra R,

$$\underline{\mathrm{H}}_{Y_s}(R) = \left\{ g \in \underline{\mathrm{H}}(R) \mid \mathrm{Ad}\, g(Y_s) = Y_s \right\}.$$

Write the set of $\mathrm{Gal}(\bar{F}/F)$-orbits of *nonzero* eigenvalues of Y_s as $I_{1,+} \cup I_2 \cup I_{1,-}$, where multiplication by $\{\pm 1\}$ fixes each element of I_2 and induces a bijection $\mathcal{O}_+ \mapsto \mathcal{O}_-$ from $I_{1,+}$ to $I_{1,-}$. The proof of Lemma 9.4.2 says that we may view $\mathfrak{O}_F[Y_s] \subset \mathrm{End}_{\mathfrak{O}}(\Lambda)$ as a product

$$\prod_{\mathcal{O}_+ \in I_{1,+}} \mathfrak{O}_F[T]/(f_{\mathcal{O}_+}) \times \prod_{\mathcal{O} \in I_2} \mathfrak{O}_F[T]/(f_{\mathcal{O}}) \times \prod_{\mathcal{O} \in I_{1,-}} \mathfrak{O}_F[T]/(f_{\mathcal{O}_-}) \times \mathfrak{O}_F[T]/(T) \qquad (9.4.1)$$

(the last term should be ignored if 0 is not an eigenvalue of Y_s), yielding a corresponding decomposition

$$\Lambda_{\underline{H}} = \bigoplus_{\mathcal{O}_+ \in I_{1,+}} \Lambda_{\underline{H},\mathcal{O}_+} \oplus \bigoplus_{\mathcal{O} \in I_2} \Lambda_{\underline{H},\mathcal{O}} \oplus \bigoplus_{\mathcal{O}_- \in I_{1,-}} \Lambda_{\underline{H},\mathcal{O}_-} \oplus \Lambda_{\underline{H},0} \tag{9.4.2}$$

of $\Lambda_{\underline{H}}$ as an $\mathfrak{O}_F[Y_s]$-module.

Let $\mathcal{O}' \in I_{1,+} \cup I_2 \cup I_{1,-}$. In equation (9.4.1), denote by $E_{\mathcal{O}'}$ the field extension $F[T]/(f_{\mathcal{O}'})$. Then, by the condition on the eigenvalues of Y_s, the field $E_{\mathcal{O}'}$ is unramified over F, and its ring $\mathfrak{O}_{E_{\mathcal{O}'}}$ of integers equals $\mathfrak{O}_F[T]/(f_{\mathcal{O}'})$. Each $\Lambda_{\underline{H},\mathcal{O}'}$ has a natural structure of an $\mathfrak{O}_{E_{\mathcal{O}'}}$-module, extending the $\mathfrak{O}_F$-module structure on it. If $\mathcal{O}' = \mathcal{O} \in I_2$, denote by $E_{\mathcal{O},\pm}$ the fixed field of the involution induced by $T \mapsto -T$, and by $\mathfrak{O}_{E_{\mathcal{O},\pm}}$ its ring of integers.

Remark 9.4.4. It now follows using determinant considerations that $\underline{H}_{Y_s}$ is the subgroup scheme of

$$\prod_{\mathcal{O}_+ \in I_{1,+}} \operatorname{Res}_{\mathfrak{O}_{E_{\mathcal{O}_+}}/\mathfrak{O}_F} \mathrm{GL}(\Lambda_{\underline{H},\mathcal{O}_+}) \times \prod_{\mathcal{O} \in I_2} \operatorname{Res}_{\mathfrak{O}_{E_{\mathcal{O}}}/\mathfrak{O}_F} \mathrm{GL}(\Lambda_{\underline{H},\mathcal{O}})$$
$$\times \prod_{\mathcal{O}_- \in I_{1,-}} \operatorname{Res}_{\mathfrak{O}_{E_{\mathcal{O}_-}}/\mathfrak{O}_F} \mathrm{GL}(\Lambda_{\underline{H},\mathcal{O}_-}) \times \mathrm{Isom}(\Lambda_{\underline{H},0}, q_W|_{\Lambda_{H,0}})^0$$

that fixes q_W in the obvious sense. We now make this more explicit to compute $\underline{H}_{Y_s}$.

9.4.B. The case of an $\mathcal{O}_+ \in I_{1,+}$. Let $\mathcal{O}_+ \in I_{1,+}$, and set $\mathcal{O}_- = -\mathcal{O}_+$. We have a natural identification $E_{\mathcal{O}_+} = E_{\mathcal{O}_-}$, induced by $T \mapsto -T$. q_W induces a perfect pairing between $\Lambda_{\underline{H},\mathcal{O}_+}$ and $\Lambda_{\underline{H},\mathcal{O}_-}$ (i.e., q_W is trivial on each of these two sublattices and unimodular on their direct sum). Since $E_{\mathcal{O}_+}/F$ is unramified, we may define

$$\tilde{q}_{\underline{H},\mathcal{O}_+} : \Lambda_{\underline{H},\mathcal{O}_+} \times \Lambda_{\underline{H},\mathcal{O}_-} \to \mathfrak{O}_{E_{\mathcal{O}_+}}$$

requiring that

$$\mathrm{tr}_{E_{\mathcal{O}_+}/F}\left(a \cdot \tilde{q}_{\underline{H},\mathcal{O}_+}(v, w)\right) = q_W(av, w)$$

for all $a \in \mathfrak{O}_{E_{\mathcal{O}_+}}$. Since $q_W(av, w) = q_W(v, aw)$ for $a \in \mathfrak{O}_{E_{\mathcal{O}_+}} = \mathfrak{O}_{E_{\mathcal{O}_-}}$ (via the identification $T \mapsto -T$), $\tilde{q}_{\underline{H},\mathcal{O}_+}$ is perfect and $\mathfrak{O}_{E_{\mathcal{O}_+}}$-linear. For any $\mathfrak{O}_F$-algebra R, any given

$$(g_+, g_-) \in \operatorname{Res}_{\mathfrak{O}_{E_{\mathcal{O}_+}}/\mathfrak{O}_F} \mathrm{GL}(\Lambda_{\underline{H},\mathcal{O}_+})(R) \times \operatorname{Res}_{\mathfrak{O}_{E_{\mathcal{O}_-}}/\mathfrak{O}_F} \mathrm{GL}(\Lambda_{\underline{H},\mathcal{O}_-})(R)$$
$$= \mathrm{GL}_{\mathfrak{O}_{E_{\mathcal{O}_+}} \otimes_{\mathfrak{O}_F} R}\left(\Lambda_{\underline{H},\mathcal{O}_+} \otimes_{\mathfrak{O}_F} R\right) \times \mathrm{GL}_{\mathfrak{O}_{E_{\mathcal{O}_-}} \otimes_{\mathfrak{O}_F} R}\left(\Lambda_{\underline{H},\mathcal{O}_-} \otimes_{\mathfrak{O}_F} R\right)$$

preserves $q_W|_{\Lambda_{\underline{H},\mathcal{O}_+ \otimes_{\mathfrak{O}_F} R} \oplus \Lambda_{\underline{H},\mathcal{O}_-} \otimes_{\mathfrak{O}_F} R}$ if and only if $g_- = {}^t g_+^{-1}$, the transpose being defined with respect to the pairing $\tilde{q}_{\underline{H},\mathcal{O}_+}$.

9.4.C. The case of an $\mathcal{O} \in I_2$. Let $\mathcal{O} \in I_2$. Since the trace form from $\mathfrak{O}_{E_{\mathcal{O}}}$ to $\mathfrak{O}_F$ is nondegenerate, there is a unique pairing

$$\tilde{q}_{\underline{H},\mathcal{O}} : \Lambda_{\underline{H},\mathcal{O}} \times \Lambda_{\underline{H},\mathcal{O}} \to \mathfrak{O}_{E_{\mathcal{O}}}$$

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such that for all $a \in \mathfrak{O}_{E_\mathcal{O}}$ we have

$$q_W(av, w) = \mathrm{tr}_{E_\mathcal{O}/F}\left(a \cdot \tilde{q}_{\underline{\mathrm{H}},\mathcal{O}}(v, w)\right).$$

The restriction of q_W to $\Lambda_{\underline{\mathrm{H}},\mathcal{O}}$ satisfies $q_W(av, w) = q_W(v, \tau(a)w)$ for all $a \in \mathfrak{O}_{E_\mathcal{O}}$, τ being the nontrivial element of $\mathrm{Gal}(E_\mathcal{O}/E_{\mathcal{O},\pm})$. This forces $\tilde{q}_{\underline{\mathrm{H}},\mathcal{O}}$ to be $\mathfrak{O}_{E_\mathcal{O}}/\mathfrak{O}_{E_{\mathcal{O},\pm}}$-$\epsilon_W$-Hermitian (where $\epsilon_W = 1$ or -1 depending on whether q_W is quadratic or symplectic), and realizes $\Lambda_{\underline{\mathrm{H}},\mathcal{O}}$ as a unimodular $\mathfrak{O}_{E_\mathcal{O}}/\mathfrak{O}_{E_{\mathcal{O},\pm}}$-Hermitian lattice. Moreover, for any $\mathfrak{O}_F$-algebra R, any given

$$g \in \mathrm{Res}_{\mathfrak{O}_{E_\mathcal{O}}/\mathfrak{O}_F} \mathrm{GL}(\Lambda_{\underline{\mathrm{H}},\mathcal{O}})(R) = \mathrm{GL}_{\mathfrak{O}_{E_\mathcal{O}} \otimes_{\mathfrak{O}_F} R}(\Lambda_{\underline{\mathrm{H}},\mathcal{O}} \otimes_{\mathfrak{O}_F} R)$$

preserves $q_W|_{\Lambda_{\underline{\mathrm{H}},\mathcal{O}}}$ if and only if it preserves $\tilde{q}_{\underline{\mathrm{H}},\mathcal{O}}$.

Lemma 9.4.5. (a) *There is an obvious isomorphism*

$$\underline{\mathrm{H}}_{Y_s} \cong \prod_{\mathcal{O}_+ \in I_{1,+}} \underline{\mathrm{H}}_{\mathcal{O}_+} \times \prod_{\mathcal{O} \in I_2} \underline{\mathrm{H}}_\mathcal{O} \times \underline{\mathrm{H}}_0,$$

where $\underline{\mathrm{H}}_{\mathcal{O}_+} = \mathrm{Res}_{\mathfrak{O}_{E_{\mathcal{O}_+}}/\mathfrak{O}_F} \mathrm{GL}(\Lambda_{\underline{\mathrm{H}},\mathcal{O}_+})$, $\underline{\mathrm{H}}_\mathcal{O} = \mathrm{Res}_{\mathfrak{O}_{E_{\mathcal{O},\pm}}/\mathfrak{O}_F} \mathrm{U}(\Lambda_{\underline{\mathrm{H}},\mathcal{O}}, \tilde{q}_{\underline{\mathrm{H}},\mathcal{O}})$, *and* $\underline{\mathrm{H}}_0$ *is the special orthogonal/symplectic group scheme associated to the (necessarily unimodular) quadratic/symplectic lattice* $(\Lambda_{\underline{\mathrm{H}},0}, q_W)$.

(b) *If* $g \in \underline{\mathrm{H}}(\bar{F})$ *is such that* $\mathrm{Ad}\, g(Y_s) \in \mathfrak{k}_{\underline{\mathrm{H}}}$, *then* $g \in K_{\underline{\mathrm{H}}} \cdot \underline{\mathrm{H}}_{Y_s}(\bar{F})$.

Remark 9.4.6. While $\underline{\mathrm{H}}_0$ is a group of the same 'absolute' type as $\underline{\mathrm{H}}$ but of smaller rank, and both $\underline{\mathrm{H}}$ and $\underline{\mathrm{H}}_0$ are unramified, it can happen that $\underline{\mathrm{H}}$ is split but $\underline{\mathrm{H}}_0$ is not. Hence, even if we wish to prove only the fundamental lemma involving a split even special orthogonal Lie algebra, it is necessary to accommodate its unramified counterpart through the arguments, for reasons of induction.

Proof of Lemma 9.4.5. (a) is immediate from Remark 9.4.4 and §9.4.B and §9.4.C. So let us focus on (b). We need to show that there exists $k \in K_{\underline{\mathrm{H}}}$ such that $\mathrm{Ad}\, k(Y_s) = \mathrm{Ad}\, g(Y_s)$. Now Y_s and $\mathrm{Ad}\, g(Y_s)$ have the same multisets of eigenvalues, so that $\Lambda_{\underline{\mathrm{H}}}$ has decompositions as

$$\bigoplus_{\mathcal{O}_+ \in I_{1,+}} \Lambda_{\underline{\mathrm{H}},\mathcal{O}_+} \oplus \bigoplus_{\mathcal{O} \in I_2} \Lambda_{\underline{\mathrm{H}},\mathcal{O}} \oplus \bigoplus_{\mathcal{O}_- \in I_{1,-}} \Lambda_{\underline{\mathrm{H}},\mathcal{O}_-} \oplus \Lambda_{\underline{\mathrm{H}},0}$$
$$= \bigoplus_{\mathcal{O}_+ \in I_{1,+}} \Lambda'_{\underline{\mathrm{H}},\mathcal{O}_+} \oplus \bigoplus_{\mathcal{O} \in I_2} \Lambda'_{\underline{\mathrm{H}},\mathcal{O}} \oplus \bigoplus_{\mathcal{O}_- \in I_{1,-}} \Lambda'_{\underline{\mathrm{H}},\mathcal{O}_-} \oplus \Lambda'_{\underline{\mathrm{H}},0}$$

as in equation (9.4.2) corresponding to Y_s and $\mathrm{Ad}\, g(Y_s)$, respectively. It suffices to show that we have isomorphisms of quadratic/symplectic lattices

$$(\Lambda_{\underline{\mathrm{H}},\mathcal{O}_+} \oplus \Lambda_{\underline{\mathrm{H}},\mathcal{O}_-}, q_W) \cong (\Lambda'_{\underline{\mathrm{H}},\mathcal{O}_+} \oplus \Lambda'_{\underline{\mathrm{H}},\mathcal{O}_-}, q_W) \quad (\mathcal{O}_+ \in I_{1,+}),$$
$$(\Lambda_{\underline{\mathrm{H}},\mathcal{O}}, q_W) \cong (\Lambda'_{\underline{\mathrm{H}},\mathcal{O}}, q_W) \quad (\mathcal{O} \in I_2),$$
$$(\Lambda_{H,0}, q_W) \cong (\Lambda'_{H,0}, q_W)$$

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(if q_W is orthogonal, this would only give $g = kh$ with $k \in \mathrm{O}(\Lambda_{\underline{\mathrm{H}}}, q_W)$ and h commuting with Y_s, but this would force $\Lambda_{\underline{\mathrm{H}},0} \neq 0$, allowing us to modify k and h so as to have determinant one). The first of these is obvious, and the third would follow from the second by the cancelation law for quadratic lattices in odd residue characteristic (this goes back to at least [23] and in our case would follow easily, for example, from [66] 92:1). To show the second set of isomorphisms, note that the discussion of § 9.4.C gives two $\mathfrak{O}_{E_\mathcal{O}}/\mathfrak{O}_{E_{\mathcal{O},\pm}}$-$\epsilon_W$-Hermitian unimodular lattices $(\Lambda_{\underline{\mathrm{H}},\mathcal{O}}, \tilde{q}_{\underline{\mathrm{H}},\mathcal{O}}), (\Lambda'_{\underline{\mathrm{H}},\mathcal{O}}, \tilde{q}'_{\underline{\mathrm{H}},\mathcal{O}})$ satisfying $q_W = \mathrm{tr}_{E_\mathcal{O}/F} \circ \tilde{q}_{\underline{\mathrm{H}},\mathcal{O}}$ and $q'_W = \mathrm{tr}_{E_\mathcal{O}/F} \circ \tilde{q}'_{\underline{\mathrm{H}},\mathcal{O}}$. It suffices to show that these lattices are isomorphic. This is the case, since they have the same rank and are unimodular, and since $E_\mathcal{O}/E_{\mathcal{O},\pm}$ is unramified (see, for example, [28, Proposition 4.2]). □

Corollary 9.4.7. (a) *Let $\{Y'_n\}$ be a set of representatives for the set of those $\underline{\mathrm{H}}_{Y_s}(F)$-conjugacy classes in the $\underline{\mathrm{H}}_{Y_s}$-stable conjugacy class of Y_n that intersect $\mathfrak{k}_{\underline{\mathrm{H}},\mathrm{tn}}$. Then $\{Y_s + Y'_n\}$ is a set of representatives for those $\underline{\mathrm{H}}(F)$-conjugacy classes in the $\underline{\mathrm{H}}$-stable conjugacy class of Y that intersect $\mathfrak{k}_{\underline{\mathrm{H}}}$.*

(b) *For each Y'_n as in (a), the centralizer $\underline{\mathrm{H}}_{Y_s+Y'_n}$ of $Y_s + Y'_n$ in $\underline{\mathrm{H}}$ coincides with the centralizer of Y'_n in $\underline{\mathrm{H}}_{Y_s}$, and, choosing any common measure dY'_n on these, and letting $\mathfrak{k}_{\underline{\mathrm{H}}_{Y_s}} = \mathfrak{k} \cap \mathfrak{h}_{Y_s}(F)$, $K_{\underline{\mathrm{H}}_{Y_s}} = K \cap \underline{\mathrm{H}}_{Y_s}(F)$,*

$$\frac{1}{\text{meas } K_{\underline{\mathrm{H}}_{Y_s}}} O(Y'_n, \mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}_{Y_s}}}, \underline{\mathrm{H}}_{Y_s}(F), dh_{Y_s}/dY'_n) = \frac{1}{\text{meas } K_{\underline{\mathrm{H}}}} O(Y_s + Y'_n, \mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}}}, \underline{\mathrm{H}}(F), dh/dY'_n).$$

Further, for any such Y'_n, $D_{\mathfrak{h}_{Y_s}}(Y'_n) = D_{\mathfrak{h}}(Y_s + Y'_n)$, so that we may replace the unnormalized orbital integrals O by the normalized orbital integrals I in the above equality.

(c) *The normalized $\underline{\mathrm{H}}$-stable orbital integral of $(\text{meas } K_{\underline{\mathrm{H}}})^{-1}\mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}}}$ at Y coincides with the normalized $\underline{\mathrm{H}}_{Y_s}$-stable orbital integral of $(\text{meas } K_{\underline{\mathrm{H}}_{Y_s}})^{-1}\mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}_{Y_s}}}$ at Y_n.*

Proof. (a) Follows from Lemma 9.4.5(b) together with the fact that the process of the assignment $Y \mapsto Y_s$ commuted with $\underline{\mathrm{H}}$-conjugation. (c) follows once (a) and (b) are proved. The equality of the unnormalized orbital integrals in (b) follows from Lemma 4.2.5 applied with $\underline{\mathrm{H}}(F), \underline{\mathrm{H}}_{Y_s}(F), \underline{\mathrm{H}}_{Y_s+Y'_n}(F) = \underline{\mathrm{H}}_{Y_s}(F), K_{\underline{\mathrm{H}}}$ and $h \mapsto \mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}}}(h^{-1}(Y_s + Y'_n)h)$ in place of G_1, H_1, I_1, K_1, and f, respectively, using Lemma 9.4.5(b) to justify the hypotheses (note that the process of obtaining Y_s from Y gives Y_s from $Y_s + Y'_n$ as well).

It remains to show that $D_{\mathfrak{h}_{Y_s}}(Y'_n) = D_{\mathfrak{h}}(Y_s + Y'_n)$, or, equivalently, that $D_{\mathfrak{h}}(Y) = D_{\mathfrak{h}_{Y_s}}(Y_n)$. For this it is enough to show that, for every root α of the maximal torus $\underline{\mathrm{T}}$ in $\underline{\mathrm{H}}$ such that $Y \in \mathfrak{t}(F)$ (and hence clearly also $Y_s \in \mathfrak{t}(F)$), $|\alpha(Y)| = 1$ unless α is a root of $\underline{\mathrm{T}}$ in $\underline{\mathrm{H}}_{Y_s}$. This follows from the fact that such an $\alpha(Y)$ is either the difference or sum of two eigenvalues λ_1, λ_2 of Y, or an eigenvalue λ of Y, or twice an eigenvalue of Y, while $|\lambda_1 \pm \lambda_2| = 1$ (respectively, $|\lambda| = 1$, respectively, $|2\lambda| = 1$) unless $\lambda_{1,s} \pm \lambda_{2,s} = 0$ (respectively, $\lambda_s = 0$, respectively, $\lambda_s = 0$). □

9.5. Stable conjugacy classes in unitary groups and their Lie algebras

Remark 9.5.1. Consider a unitary group $\mathrm{U}(W_1)$, where W_1 is a Hermitian space over some field extension $E/E_\pm/F$ contained in $\bar{F}$. Then two semisimple elements

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$g, h \in \mathrm{U}(W_1)(E_\pm)$ belong to the same stable conjugacy class if and only if $g, h \in \mathrm{GL}(W_1)(E)$ have the same multisets of eigenvalues. Indeed, this follows from the fact that an identification $\mathrm{U}(W_1) \otimes_{E_\pm} E \cong \mathrm{GL}(W_1)(E)$ may be made so that obvious map $\mathrm{U}(W_1)(E_\pm) \hookrightarrow \mathrm{U}(W_1)(E) \cong \mathrm{GL}(W_1)(E)$ becomes the obvious inclusion. The same clearly holds for the Lie algebra $\mathfrak{u}(W_1)(E_\pm)$ too.

9.6. Back to $\mathbf{g}_\theta$

Henceforth, write $n_0 = 1$ if $\underline{\mathrm{H}} = \mathrm{SO}(W)$ with $\dim W$ even, and $n_0 = 2$ otherwise.

Now suppose we have semisimple elements $Y \in \mathfrak{h}(F)$, $X \in \mathsf{g}_\theta(F)$ with Y matching X. Write $Y = Y_s + Y_n$ as in §9.4.A. Moreover, write $X = X_s + X_n$ applying the construction of §9.4.A but using the section

$$\bar{\lambda} \mapsto (1/n_0)(n_0\bar{\lambda})_s$$

in place of $\bar{\lambda} \mapsto \lambda_s$ (see Notation 9.4.1), which as well is $\mathrm{Gal}(\bar{F}/F) \times \{\pm 1\}$-equivariant. Then it follows from Remarks 6.3.2 and 7.4.3 that Y_s and X_s match as well.

Let $\mathcal{O}'$ be a $\mathrm{Gal}(\bar{\kappa}/\kappa)$-orbit of the form $\mathcal{O}_+$, $\mathcal{O}$ or $\mathcal{O}_-$, associated to the multiset $\mathrm{Eig}\, Y$ of nonzero eigenvalues of Y. Corresponding to the decomposition (9.4.1) of $\mathfrak{O}_F[Y_s]$, we have the following decomposition for $\mathfrak{O}_F[X_s]$:

$$\prod_{\mathcal{O}_+ \in I_{1,+}} \mathfrak{O}[T]/(f_{\mathcal{O}_+, n_0}) \times \prod_{\mathcal{O} \in I_2} \mathfrak{O}[T]/(f_{\mathcal{O}, n_0}) \times \prod_{\mathcal{O} \in I_{1,-}} \mathfrak{O}[T]/(f_{\mathcal{O}_-, n_0}) \times \mathfrak{O}[T]/(T), \quad (9.6.1)$$

where the last term should be ignored if 0 is not an eigenvalue of X_s, and where we have written $f_{\mathcal{O}', n_0}$ for the polynomial $T \mapsto f_{\mathcal{O}'}(n_0 T)$. This gives a decomposition

$$\Lambda = \bigoplus_{\mathcal{O}_+ \in I_{1,+}} \Lambda_{\mathcal{O}_+} \oplus \bigoplus_{\mathcal{O} \in I_2} \Lambda_{\mathcal{O}} \oplus \bigoplus_{\mathcal{O}_- \in I_{1,-}} \Lambda_{\mathcal{O}_-} \oplus \Lambda_0, \quad (9.6.2)$$

where, this time, for $\mathcal{O}' = \mathcal{O}_+$, $\mathcal{O}$ or $\mathcal{O}_-$, the eigenvalues of X_s on $\Lambda_{\mathcal{O}'}$ are $(1/n_0)$ of those of Y_s on $\Lambda_{\underline{\mathrm{H}}, \mathcal{O}'}$. For $\mathcal{O}' \in I_{1,+} \cup I_2 \cup I_{1,-}$, $\mathfrak{O}_{E_{\mathcal{O}'}}$ acts on $\Lambda_{\mathcal{O}'}$ via the map

$$\mathfrak{O}_{E_{\mathcal{O}'}} = \mathfrak{O}[T]/(f_{\mathcal{O}'}) \to \mathfrak{O}[T]/(f_{\mathcal{O}', n_0}),$$

where the last map is induced by $T \mapsto n_0^{-1} T$. By Lemma 9.4.5(a), applied to $(\Lambda, \tilde{\theta}, X_s)$ in place of $(\Lambda_{\underline{\mathrm{H}}}, q_W, Y_s)$, the centralizer $\underline{\mathsf{G}}_{\theta, X_s}$ of X_s in $\underline{\mathsf{G}}_\theta$ is the source of an obvious isomorphism

$$\underline{\mathsf{G}}_{\theta, X_s} \cong \prod_{\mathcal{O}_+ \in I_{1,+}} \underline{\mathsf{G}}_{\mathcal{O}_+} \times \prod_{\mathcal{O} \in I_2} \underline{\mathsf{G}}_{\mathcal{O}} \times \underline{\mathsf{G}}_{\theta, 0}, \quad (9.6.3)$$

where $\underline{\mathsf{G}}_{\mathcal{O}_+} = \mathrm{Res}_{\mathfrak{O}_{E_{\mathcal{O}_+}}/\mathfrak{O}_F} \mathrm{GL}(\Lambda_{\mathcal{O}_+})$, $\underline{\mathsf{G}}_{\mathcal{O}} = \mathrm{Res}_{\mathfrak{O}_{E_{\mathcal{O},\pm}}/\mathfrak{O}_F} \mathrm{U}(\Lambda_{\mathcal{O}}, \tilde{q}_{\mathcal{O}})$ for a suitable unimodular $E_{\mathcal{O}}/E_{\mathcal{O},\pm}$-Hermitian/skew-Hermitian form $\tilde{q}_{\mathcal{O}}$ constructed from $\tilde{\theta}$ just as $\tilde{q}_{\underline{\mathrm{H}}, \mathcal{O}}$ was constructed from q_W in §9.4.C, and $\underline{\mathsf{G}}_{\theta, 0}$ equals the special orthogonal/symplectic group scheme associated to $(\Lambda_0, \tilde{\theta})$.

9.7. Semisimple descent and transfer for $(\mathfrak{h}, \mathbf{g}_\theta)$

Set, for all $\mathcal{O}' \in I_{1,+} \cup I_2$,

$$W_{\mathcal{O}'} = \Lambda_{\underline{\mathrm{H}}, \mathcal{O}'} \otimes_{\mathfrak{O}_{E_{\mathcal{O}'}}} E_{\mathcal{O}'}, \quad V_{\mathcal{O}'} = \Lambda_{\mathcal{O}'} \otimes_{\mathfrak{O}_{E_{\mathcal{O}'}}} E_{\mathcal{O}'}.$$

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Lemma 9.7.1. *Let regular semisimple $Y' \in \mathfrak{h}(F)$, $X' \in \mathfrak{g}_\theta(F)$ be such that $Y'_s = Y_s$, $X'_s = X_s$. Write, using the isomorphisms of Lemma 9.4.5(a) and equation (9.6.3),*

$$Y' = \left((Y'_{\mathcal{O}'})_{\mathcal{O}'}, Y'_0\right), X' = \left((X'_{\mathcal{O}'})_{\mathcal{O}'}, X'_0\right).$$

Similarly, define $Y'_{n,\mathcal{O}'}$, $Y'_{n,0}$ $(=Y'_0)$, $X'_{n,\mathcal{O}'}$, $X'_{n,0}$ $(=X'_0)$. Then X' matches Y' if and only the following two conditions hold.

(i) *For each $\mathcal{O}'$, $(1/n_0)Y'_{n,\mathcal{O}'} \in \mathfrak{h}_{\mathcal{O}'}(F) \subset \operatorname{End}_{E_{\mathcal{O}'}}(W_{\mathcal{O}'})$ and $X'_{n,\mathcal{O}'} \in \mathfrak{g}_{\mathcal{O}'}(F) \subset \operatorname{End}_{E_{\mathcal{O}'}}(V_{\mathcal{O}'})$ have the same multisets of eigenvalues.*

(ii) *Y'_0 and X'_0 match.*

Proof. Recall that, from Remarks 6.3.2 and 7.4.3, we can express the condition of X' matching Y' as

$$\operatorname{Eig}_F X' \doteq (1/n_0) \operatorname{Eig}_F Y'. \tag{9.7.1}$$

On the other hand, (i) and (ii) hold if and only if

$$\operatorname{Eig}_{E_{\mathcal{O}'}} X_{n,\mathcal{O}'} = (1/n_0) \operatorname{Eig}_{E_{\mathcal{O}'}} Y_{n,\mathcal{O}'} \,\forall\, \mathcal{O}', \quad \text{and} \quad \operatorname{Eig}_F X'_0 \doteq (1/n_0) \operatorname{Eig}_F Y'_0. \tag{9.7.2}$$

Denote by $\lambda_{\mathcal{O}'}$ the image of T in $E_{\mathcal{O}'} \cong \mathfrak{O}[T]/(f_{\mathcal{O}'})$. Then X_s, Y_s have images $1/n_0\lambda_{\mathcal{O}'}$ and $\lambda_{\mathcal{O}'}$, respectively, in $E_{\mathcal{O}'}$. Then we have, with notation as in Remark 6.3.2 and Notation 6.3.3,

$$\operatorname{Eig}_F X' = \bigsqcup_{\mathcal{O}'} \operatorname{Eig}_F X'_{\mathcal{O}'} \cup \operatorname{Eig}_F X'_0 = \bigsqcup_{\mathcal{O}'} \bigsqcup_{\sigma \in \operatorname{Hom}_{F\text{-alg}}(E_{\mathcal{O}'}, \bar{F})} \sigma(\operatorname{Eig}_{E_{\mathcal{O}'}} X_{\mathcal{O}'}) \cup \operatorname{Eig}_F X'_0,$$

or, equivalently,

$$\operatorname{Eig}_F X' = \bigsqcup_{\mathcal{O}'} \bigsqcup_{\sigma \in \operatorname{Hom}_{F\text{-alg}}(E_{\mathcal{O}'}, \bar{F})} \left(\frac{1}{n_0}\sigma(\lambda_{\mathcal{O}'}) + \sigma\left(\operatorname{Eig}_{E_{\mathcal{O}'}} X_{n,\mathcal{O}'}\right)\right) \sqcup \operatorname{Eig}_F X'_0. \tag{9.7.3}$$

Similarly,

$$\operatorname{Eig}_F Y' = \bigsqcup_{\mathcal{O}'} \bigsqcup_{\sigma \in \operatorname{Hom}_{F\text{-alg}}(E_{\mathcal{O}'}, \bar{F})} \left(\sigma(\lambda_{\mathcal{O}'}) + \sigma\left(\operatorname{Eig}_{E_{\mathcal{O}'}} Y_{n,\mathcal{O}'}\right)\right) \sqcup \operatorname{Eig}_F Y'_0. \tag{9.7.4}$$

By equations (9.7.3) and (9.7.4), we have (9.7.2) $\Rightarrow$ (9.7.1), showing one of our implications. Moreover, the converse implication also follows since, on the one hand, $\operatorname{Eig}_F X'_0$ (respectively, $\operatorname{Eig}_F Y'_0$) can be recovered from $\operatorname{Eig}_F X'$ and $\operatorname{Eig}_F Y'$ as the elements with image 0 in $\bar{\kappa}$, while, on the other, for each $\mathcal{O}'$, because $\lambda_{\mathcal{O}'}$ generates $E_{\mathcal{O}'}$, we can choose any $\sigma \in \operatorname{Hom}_{F\text{-alg}}(E_{\mathcal{O}'}, \bar{F})$ and recover

$$\operatorname{Eig}_{E_{\mathcal{O}'},\sigma} X_{n,\mathcal{O}'} = \left\{\mu - \sigma(\lambda_{\mathcal{O}'}) \mid \mu \in \operatorname{Eig}_F X',\ \mu \text{ and } \sigma(\lambda_{\mathcal{O}'}) \text{ have the same image in } \bar{\kappa}\right\},$$

and similarly for $Y_{n,\mathcal{O}'}$. □

Remark 9.7.2. The proof above also shows the following. Let $Y = Y_s + Y_n$ and $Y' = Y_s + Y'_n$, with $Y'_s = Y_s$, and write $Y_n = ((Y_{n,\mathcal{O}'}), Y_0)$, $Y'_n = (Y'_{n,\mathcal{O}'}, Y'_0)$. Then, by Remark 9.5.1, Y and Y' are $\underline{\mathrm{H}}$-stably conjugate if and only if the following hold.

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(i) For each $\mathcal{O}'$, $Y_{n,\mathcal{O}'}, Y'_{n,\mathcal{O}'}$ are stably conjugate as elements of $\mathfrak{gl}(W_{\mathcal{O}'})$ or $\mathfrak{u}(W_{\mathcal{O}'})$ (depending on whether $\mathcal{O}' \in I_{1,\pm}$ or I_2).

(ii) Y_0, Y'_0 are stably conjugate in $\underline{\mathrm{H}}_0$.

This assertion is in fact contained in Corollary 9.4.7(a), since restriction of scalars respects stable conjugacy.

Lemma 9.7.3. *In the situation of Lemma 9.7.1, if Y' and X' match and we are in case (c), $\Delta(Y', X') = \Delta(Y'_0, X'_0)$.*

Proof. Let the conjugacy class of X' be parameterized by $(L, L_\pm, y, c)$ as in § 7.2, so that the equivalence class of Y' is parameterized by $(L, L_\pm, y)$. Write $L = \prod_{i\in I} F_i$, $L_\pm = \prod_{i\in I} F_{\pm i}$, where each $F_{\pm i}$ is a field, and F_i is a degree-2 étale algebra over $F_{\pm i}$. Let τ be the involution of L that restricts to each F_i as the unique nontrivial involution of F_i fixing $F_{\pm i}$ (we will abuse notation to denote the latter involution by τ as well). Let I^* be the set of $i \in I$ such that F_i is a field.

For $\mathcal{O} \in I_2$, let $I_{\mathcal{O}}, I^*_{\mathcal{O}}$ denote the subsets of I, I^* consisting of i such that $\bar{y}_i \in \mathcal{O}$. It is easy to see that F_i is never a field if $\bar{y}_i \in \mathcal{O}_\pm \in I_{1,\pm}$.

For $\mathcal{O}_+ \in I_{1,+}$ (respectively, $\mathcal{O} \in I_2$), denote by $P_{\mathcal{O}_+}$ (respectively, $P_{\mathcal{O}}$) the characteristic polynomial of $Y'_{\mathcal{O}_+} \oplus Y'_{\mathcal{O}_-}$ (respectively, $Y'_{\mathcal{O}}$) acting on $W_{\mathcal{O}_+} \oplus W_{\mathcal{O}_-}$ (respectively, $W_{\mathcal{O}}$). Let P, P_0 be the characteristic polynomials of Y', Y'_0 on W, W_0, respectively. Thus,

$$P = \prod_{\mathcal{O}_+ \in I_{1,+}} P_{\mathcal{O}_+} \cdot \prod_{\mathcal{O}\in I_2} P_{\mathcal{O}} \cdot P_0.$$

Then we get (see § 7.5.A)

$$\Delta(Y', X') = \prod_{\mathcal{O}\in I_2} \prod_{i \in I^*_{\mathcal{O}}} \operatorname{sgn}_{F_i/F_{\pm i}} \left(-\eta c_i \cdot P'_{\mathcal{O}}(y_i) \cdot \prod_{\substack{\mathcal{O}' \in I_{1,+}\cup I_2 \\ \mathcal{O}' \neq \mathcal{O}}} P_{\mathcal{O}'}(y_i) \cdot P_0(y_i) \right)$$
$$\cdot \prod_{\substack{i\in I^* \\ \bar{y}_i = 0}} \operatorname{sgn}_{F_i/F_{\pm i}} \left(-\eta c_i P'_0(y_i) \cdot \prod_{\mathcal{O}' \in I_{1,+}\cup I_2} P_{\mathcal{O}'}(y_i) \right).$$

On the other hand,

$$\Delta(Y'_0, X'_0) = \prod_{\substack{i\in I^* \\ \bar{y}_i = 0}} \operatorname{sgn}_{F_i/F_{\pm i}} \left(-\eta c_i P'_0(y_i) \right).$$

Thus it is enough to show that

$$\prod_{\substack{i\in I^* \\ \bar{y}_i = 0}} \operatorname{sgn}_{F_i/F_{\pm i}} \left(\prod_{\mathcal{O}' \in I_{1,+}\cup I_2} P_{\mathcal{O}'}(y_i) \right) = 1 \tag{9.7.5}$$

(the term in the parentheses clearly belongs to $F_{\pm i}$) and

$$\prod_{\mathcal{O}\in I_2}\prod_{i\in I^*_{\mathcal{O}}}\operatorname{sgn}_{F_i/F_{\pm i}}\left(-\eta c_i\cdot P'_{\mathcal{O}}(y_i)\cdot\prod_{\substack{\mathcal{O}'\in I_{1,+}\cup I_2\\ \mathcal{O}'\neq\mathcal{O}}}P_{\mathcal{O}'}(y_i)\cdot P_0(y_i)\right)=1. \tag{9.7.6}$$

Let us prove equation (9.7.5) first. For each y_i occurring in that equation, $P_{\mathcal{O}'}(y_i)$ and $P_{\mathcal{O}'}(0)$ are units having the same image in $\bar{\kappa}$ (as $\bar{y}_i=0$). Therefore, $F_i/F_{\pm i}$ being tamely ramified, equation (9.7.5) will follow once we show that

$$\prod_{\substack{i\in I^*\\ \bar{y}_i=0}}\operatorname{sgn}_{F_i/F_{\pm i}}$$

is trivial on $\mathfrak{O}^\times$. Since $\underline{\mathrm{H}}_0$ is unramified, this follows from Lemma 7.6.2.

Now we move to equation (9.7.6). Note that, whenever $i\in I^*_{\mathcal{O}}$, $\tau(y_{i,s})=\tau(y_i)_s=-y_{i,s}$ (cf. Notation 9.4.1), forcing $E_{\mathcal{O}}\not\subset F_{\pm i}$, so that $F_i=F_{\pm i}\otimes_{E_{\mathcal{O},\pm}}E_{\mathcal{O}}$. In particular, $F_i/F_{\pm i}$ is unramified, so that $\operatorname{sgn}_{F_i/F_{\pm i}}$ is given by the parity of the valuation $\operatorname{val}_{F_{\pm i}}$. This immediately gives that, for each $\mathcal{O}\in I_2$,

$$\prod_{i\in I^*_{\mathcal{O}}}\operatorname{sgn}_{F_i/F_{\pm i}}\left(\prod_{\substack{\mathcal{O}'\in I_{1,+}\cup I_2\\ \mathcal{O}'\neq\mathcal{O}}}P_{\mathcal{O}'}(y_i)\cdot P_0(y_i)\right)=1,$$

as the parenthetical term here is a unit in $\mathfrak{O}_{F_{\pm i}}$. Note also that $P_{\mathcal{O}}$ can be written over $E_{\mathcal{O}}$ as $P_{\mathcal{O},E_{\mathcal{O}}}\cdot\tilde{P}_{\mathcal{O}}$, where $P_{\mathcal{O},E_{\mathcal{O}}}$ is the characteristic polynomial of multiplication by $Y'|_{W_{\mathcal{O}}}$ viewed in $\mathrm{End}_{E_{\mathcal{O}}}(W_{\mathcal{O}})$. Since $\lambda_{\mathcal{O}}$ generates $E_{\mathcal{O}}$, it follows that $\tilde{P}_{\mathcal{O}}(y_i)\in\mathfrak{O}^\times_{F_{\pm i}}$. Thus, we are reduced to showing that, for each $\mathcal{O}\in I_2$,

$$\prod_{i\in I^*_{\mathcal{O}}}\operatorname{sgn}_{F_i/F_{\pm i}}\left(-\eta c_i P'_{\mathcal{O},E_{\mathcal{O}}}(y_i)\right)=1. \tag{9.7.7}$$

We consider two ways of viewing $\mathrm{U}(W_{\mathcal{O}})$ as its own endoscopic group. The first, which can perhaps be called the 'tautological' way, is as in [83, pp. 51–52], under the heading 'Cas unitaire', with $d^-=0$ and $d^+=\dim_{E_{\mathcal{O}}}W_{\mathcal{O}}$. The choice of μ^+,μ^-,z^+ and z^- as in [83] may be made arbitrarily. The second is similar, except that d^- and d^+ are interchanged (and accordingly μ^+ and μ^-, and z^+ and z^-). In either of these cases, the correspondence between stable conjugacy classes may be taken to be the identity (cf. § 1.9 of [83]). Moreover, these two endoscopic data are in fact equivalent, as mentioned in the penultimate line of page 52 in [83]. Let us denote the transfer factors for these two realizations by $\Delta_{[1]}$ and $\Delta_{[2]}$, respectively.

Then Waldspurger's formula for transfer factors for Lie algebras of unitary groups [84, Proposition X.8] tells us that the left and right sides of equation (9.7.7) can be viewed as $\Delta_{[2]}(Y'_{\mathcal{O}},Y'_{\mathcal{O}})$ and $\Delta_{[1]}(Y'_{\mathcal{O}},Y'_{\mathcal{O}})$, respectively. But, these realizations being equivalent to each other, and $\mathrm{U}(W_{\mathcal{O}})$ being quasi-split, these two transfer factors are equal (cf. [83, § 1.11(1)]; the cocycle u of [83] is trivial for us, as we have implicitly taken the inner twist ψ of [83] to be trivial, as we may). This proves equation (9.7.6). □

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Remark 9.7.4. The proof of equation (9.7.7) above may perhaps seem a bit indirect. However, given how semisimple descent works, it was only natural that transfer factors between $\mathfrak{h}_{\mathcal{O}} = \mathrm{Res}_{E_{\mathcal{O},\pm}/F}\,\mathfrak{u}(W_{\mathcal{O}})$ and $\mathfrak{g}_{\theta,\mathcal{O}} = \mathrm{Res}_{E_{\mathcal{O},\pm}/F}\,\mathfrak{u}(V_{\mathcal{O}}) \cong \mathfrak{h}_{\mathcal{O}}$ appear. Furthermore, transfer factors are compatible with restriction of scalars, as mentioned in [32, page 981]. In any case, one can give a more elementary proof for equation (9.7.7), using among other things a familiar argument used to study the different ideal in terms of the derivative of the minimal polynomial of a generator of a number field (see, for example, [19]).

End of Proof of Lemma 8.2.1. Let $Y \in \mathfrak{h}(F)$ be $\mathfrak{g}_\theta$-regular semisimple. We need to prove equation (9.1.1). If $Y \in \mathfrak{h}(F)_{\mathrm{tn}}$, then by Lemma 9.3.1 we may assume that $Y \in \mathfrak{k}_{\underline{\mathrm{H}},\mathrm{tn}}$, and the result follows from Lemma 9.2.3. If Y is not stably conjugate to an element of $\mathfrak{k}_{\underline{\mathrm{H}}}$, then, by Lemma 9.3.1, Y has an eigenvalue in $\bar{F}$ which lies outside $\bar{\mathfrak{O}}_F$. But then so does any X that matches Y, from the explicit descriptions of matching elements, showing that the right side of equation (9.1.1) vanishes too. Therefore, we may and do assume that $Y \in \mathfrak{k}_{\underline{\mathrm{H}}}$. Write $Y = Y_s + Y_n$ as in § 9.4.A. Fix X that matches Y. Correspondingly we have a decomposition $X = X_s + X_n$ (cf. the beginning of § 9.6). Let $\{Y_n'\}$ be as in Corollary 9.4.7. Write $Y = ((Y_{\mathcal{O}'})_{\mathcal{O}'}, Y_0)$, and $Y_s = ((Y_{s,\mathcal{O}'})_{\mathcal{O}'}, Y_{s,0})$, so $Y_{s,0} = 0$.

Then, by Corollary 9.4.7 and Remark 9.7.2, we can write, using an obvious notation, the left side of equation (9.1.1) as

$$\prod_{\mathcal{O}' \in I_{1,+} \cup I_2} \frac{1}{\mathrm{meas}\, K_{\underline{\mathrm{H}}_{\mathcal{O}'}}} SI\left(Y_{n,\mathcal{O}'}, \mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}_{\mathcal{O}'}}}, dh_{\mathcal{O}'}/dt_{Y_{n,\mathcal{O}'}}\right) \cdot \frac{1}{\mathrm{meas}\, K_{\underline{\mathrm{H}}_0}} SI(Y_0, \mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}_0}}, dh_0/dt_{Y_0}). \tag{9.7.8}$$

In the above equation, the discriminant factor in the term corresponding to each $\mathcal{O}'$ is to be, a priori, the discriminant factor $D_{\mathfrak{h}_{\mathcal{O}_+}}(Y_{\mathcal{O}_+})$ or $D_{\mathfrak{h}_{\mathcal{O}}}(Y_{\mathcal{O}})$ (which is what Corollary 9.4.7 gives), but it also coincides with $D_{\mathfrak{gl}_{E_{\mathcal{O}_+}}(W_{\mathcal{O}_+})}(Y_{\mathcal{O}_+})$ or $D_{\mathfrak{u}(W_{\mathcal{O}})}(Y_{\mathcal{O}})$, provided these two latter factors are defined using the normalized absolute value on $E_{\mathcal{O}_+}$ or $E_{\mathcal{O}_\pm}$, respectively.

Similarly, but using in addition Lemma 9.7.3 if we are in case (c), the right side of equation (9.1.1) equals (using obvious notation again)

$$\prod_{\mathcal{O}' \in I_{1,+} \cup I_2} \frac{1}{\mathrm{meas}\, K_{\underline{\mathsf{G}}_{\mathcal{O}'}}} SI\left(X_{n,\mathcal{O}'}, \mathbb{1}_{\mathfrak{k}_{\underline{\mathsf{G}}_{\mathcal{O}'}}}, dg_{\mathcal{O}'}/dt_{X_{n,\mathcal{O}'}}\right) \cdot \frac{1}{\mathrm{meas}\, K_{\underline{\mathsf{G}}_{\theta,0}}} I^{\mathsf{G}_\theta}(Y_0, \mathbb{1}_{\mathfrak{k}_{\mathfrak{g}_{\theta,0}}}, dg_{\theta,0}, dt_{Y_0}).$$

The centralizer measures $dt_{X_{n,\mathcal{O}'}}$ here are to be taken compatibly with the $dt_{Y_{n,\mathcal{O}'}}$ of equation (9.7.8), following the isomorphism $\underline{\mathrm{H}}_{Y_s,Y_n} = \underline{\mathrm{H}}_Y \cong \underline{\mathsf{G}}_{\theta,X} \cong \underline{\mathsf{G}}_{\theta,X_s,X_n}$. Thus, we need to prove that

$$\begin{aligned} &\frac{1}{\mathrm{meas}\, K_{\underline{\mathrm{H}}_{\mathcal{O}'}}} SI\left(Y_{n,\mathcal{O}'}, \mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}_{\mathcal{O}'}}}, dh_{\mathcal{O}'}/dt_{Y_{n,\mathcal{O}'}}\right) \\ &\quad = \frac{1}{\mathrm{meas}\, K_{\underline{\mathsf{G}}_{\mathcal{O}'}}} SI\left(X_{n,\mathcal{O}'}, \mathbb{1}_{\mathfrak{k}_{\underline{\mathsf{G}}_{\mathcal{O}'}}}, dg_{\mathcal{O}'}/dt_{X_{n,\mathcal{O}'}}\right) \forall\, \mathcal{O}', \end{aligned}$$

and

$$\frac{1}{\mathrm{meas}\, K_{\underline{\mathrm{H}}_0}} SI(Y_0, \mathbb{1}_{\mathfrak{k}_{\underline{\mathrm{H}}_0}}, dh_0/dt_{Y_0}) = \frac{1}{\mathrm{meas}\, K_{\underline{\mathsf{G}}_{\theta,0}}} I^{\mathsf{G}_\theta}(Y_0, \mathbb{1}_{\mathfrak{k}_{\mathfrak{g}_{\theta,0}}}, dg_{\theta,0}, dt_{Y_0}).$$

The first of these follows from the condition $n_0^{-1}\operatorname{Eig}_{E_{\mathcal{O}'}} Y_{n,\mathcal{O}'} = \operatorname{Eig}_{E_{\mathcal{O}'}} X_{n,\mathcal{O}'}$ from Lemma 9.7.1 together with the fact that hyperspecial subgroups of any unramified group are all adjoint conjugate (in our case we can see this directly from [28], Proposition 4.2), and the second follows since Y_0 and X_0 match and are topologically nilpotent, a case handled at the beginning of this proof.

10. The other depth bound

We want to prove that, if an irreducible admissible tempered representation π of $\underline{H}(F)$ has a Langlands parameter that maps to a Langlands parameter ϕ of $\underline{G}(F)$, then $\operatorname{depth}\pi \leqslant \operatorname{depth}\phi$. Our idea to prove this involves an explicit region of validity for the Harish-Chandra–Howe character expansion that is due to DeBacker (in the nontwisted case), cf. [20], and Adler and Korman (in the twisted case), cf. [4]. Because we will need to work with analogous questions for endoscopic groups of $\underline{H}$, we cannot restrict it to be unramified anymore. In particular, $\underline{H}$ is as in cases (a), (b), or (c), but possibly ramified quasi-split. Whatever results we have will be accordingly more general. To study the interaction between character expansion and endoscopy, we find some of the formalism used by Arthur in [6] particularly convenient. Let us recall some notation from [2, 6].

10.1. Some notation

In this section, let $\underline{G}$ be any connected reductive group over F. Recall that, if x lies in the (enlarged) Bruhat–Tits building $\mathcal{B}(\underline{G}, F)$ of $\underline{G}(F)$, then we have Moy–Prasad filtration subgroups $G_{x,r} \subset \underline{G}(F)$, for $r \in \mathbb{R}_{\geqslant 0}$, and Moy–Prasad filtration sublattices $\mathfrak{g}_{x,r} \subset \mathfrak{g}(F)$ and $\mathfrak{g}^*_{x,r} \subset \mathfrak{g}^*(F)$, for $r \in \mathbb{R}$. These subgroups and sublattices were defined in [62, 63], but we normalize them as in [2]. In particular, we have that $\mathfrak{g}_{x,r+1} = \varpi_F \mathfrak{g}_{x,r}$. For $r \geqslant 0$ we may set (see, for example, [2])

$$G_{x,r+} = \bigcup_{s>r} G_{x,r} = G_{x,r+\varepsilon} \quad \text{for all sufficiently small } \varepsilon > 0,$$

and analogous prescriptions define lattices $\mathfrak{g}_{x,r+}$ and $\mathfrak{g}^*_{x,r+}$, for all $r \in \mathbb{R}$.

Notation 10.1.1. (a) For $r \in \mathbb{R}$, $r \geqslant 0$, set

$$G_r = \bigcup_{x\in\mathcal{B}(\underline{G},F)} G_{x,r}, \quad \text{and} \quad G_{r+} = \bigcup_{x\in\mathcal{B}(\underline{G},F)} G_{x,r+}\ \forall r \geqslant 0.$$

(b) For $r \in \mathbb{R}$, set

$$\mathfrak{g}_r = \bigcup_{x\in\mathcal{B}(\underline{G},F)} \mathfrak{g}_{x,r} \quad \text{and} \quad \mathfrak{g}_{r+} = \bigcup_{x\in\mathcal{B}(\underline{G},F)} \mathfrak{g}_{x,r+}.$$

(c) For any $\operatorname{Int}\underline{G}(F)$-invariant (respectively, $\operatorname{Ad}\underline{G}(F)$-invariant) open subset $\mathcal{V}$ of $\underline{G}(F)$ (respectively, of $\mathfrak{g}(F)$), let $\Gamma(\mathcal{V})$ denote the set of $\underline{G}(F)$-conjugacy classes in the set of strongly regular semisimple elements in $\mathcal{V}$, given the quotient topology from that obtained by restricting the topology on $\underline{G}(F)$ (respectively, $\mathfrak{g}(F)$).

(d) Given $\mathcal{V}$ as above, let $\Delta(\mathcal{V})$ denote the analogous topological space with 'conjugacy classes' replaced by 'equivalence classes under $\underline{G}(\bar{F})$-conjugacy'.

(e) Let $\mathcal{V}$ be as above. Fix a Haar measure dg on $\underline{G}(F)$. For each maximal torus $\underline{T}$ of $\underline{G}$, choose a Haar measure $dt_{\underline{T}}$ on $\underline{T}(F)$, such that isomorphic tori get compatible measures. Then we get a map from $C_c^\infty(\mathcal{V})$ to the space of continuous functions on $\Gamma(\mathcal{V})$, given by taking normalized orbital integrals

$$f \mapsto I(f) := \left(\gamma \mapsto I\left(g_\gamma, f, dg/dt_{\underline{T}_{g_\gamma}}\right)\right),$$

where g_γ is a representative for γ and $\underline{T}_{g_\gamma}$ is the centralizer of g_γ. Denote by $\mathcal{I}(\mathcal{V})$ the image of $C_c^\infty(\mathcal{V})$ under this map.

(f) Assume that $\mathcal{V}$ above is closed under stable conjugacy. Replacing orbital integrals by stable orbital integrals, we get a map $f \mapsto SI(f)$ from $C_c^\infty(\mathcal{V})$ to continuous functions on $\Delta(\mathcal{V})$. Denote the image by $\mathcal{SI}(\mathcal{V})$.

(h) The definitions of $\Gamma, \Delta, \mathcal{I}$ and $\mathcal{SI}$ above will also apply in the context of any open subsets of $\underline{\tilde{G}}(F)$ having appropriate invariance properties under $\underline{G}$, where $\underline{\tilde{G}}$ is any twisted space under $\underline{G}$ (except that the analogs of the $dt_{\underline{T}}$ will be measures on abelian, not necessarily connected, diagonalizable groups).

(i) Let $\hat{J}(\mathcal{N}_{\mathfrak{g}})$ denote the space of distributions on $\mathfrak{g}(F)$ obtained by taking Fourier transforms of distributions supported on the nilpotent cone $\mathcal{N}_{\mathfrak{g}}$ of $\mathfrak{g}(F)$ (the Fourier transform may be taken with respect to any nondegenerate bicharacter without affecting the definition).

The following theorem is due to Harish-Chandra (cf. [33, Theorem 3.1], the same proof works in the cases of groups and twisted spaces).

Theorem 10.1.2. *For any $\mathcal{V}$ as above, any $\underline{G}(F)$-invariant distribution $C_c^\infty(\mathcal{V}) \to \mathbb{C}$ factors through the map $C_c^\infty(\mathcal{V}) \to \mathcal{I}(\mathcal{V})$ discussed in (e) above.*

Notation 10.1.3. Henceforth, we will identify invariant distributions on $\mathcal{V}$ with the complex vector space dual $\mathcal{I}(\mathcal{V})^*$ of $\mathcal{I}(\mathcal{V})$.

Remark 10.1.4. $G_r \subset \underline{G}(F)$ and $\mathfrak{g}_r \subset \mathfrak{g}(F)$ are open and closed subsets of $\underline{G}(F)$ (cf. [2, Corollary 3.4.3 and Corollary 3.7.21]).

Remark 10.1.5. If every maximal F-torus of $\underline{G}$ splits over a tamely ramified extension, then these regions have interpretations in terms of 'eigenvalues'. More precisely, given a regular semisimple element $X \in \mathfrak{g}(F)$ (respectively, $g \in \underline{G}(F)$), let $\underline{T}$ be the maximal torus of $\underline{G}$ centralizing it. Then X belongs to $\mathfrak{g}_r$ (respectively, $g \in G_r, r > 0$) if and only if, for all $\chi \in X^*(\underline{T})$, $|d\chi(X)| < \#\kappa^{-r}$ (respectively, g belongs to the parahoric subgroup of $\underline{T}$ and $|\chi(g) - 1| < \#\kappa^{-r}$) (cf. [2, § 3.6], [3, Corollary 2.2.7 and Lemma 2.2.9], keeping in mind that we are following the normalization of the Moy–Prasad filtrations as in [2, 3]). Note that, for a quasi-split classical group $\underline{G}$ of rank m, the condition that every maximal F-torus of $\underline{G}$ splits over a tamely ramified extension is automatic if $p > 2m$.

Let $r > 0$. Let $\mathfrak{c}_1$ denote an $\mathrm{Ad}(\underline{G}(F))$-invariant homeomorphism from $\mathfrak{g}_r$ to G_r (if it exists).

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Definition 10.1.6. For a virtual admissible representation π of $\underline{G}(F)$, denote by Θ_π its character, a distribution on $\underline{G}(F)$. A character expansion for Θ_π on G_r with respect to $\mathfrak{c}_1$ is a distribution $\theta_\pi \in \hat{J}(\mathcal{N}_{\mathfrak{g}})$ such that we have an equality of distributions on $\mathfrak{g}_r$,

$$\left(\Theta_\pi|_{\mathcal{I}(G_r)}\right) \circ (\mathfrak{c}_1^{-1})^* = \theta_\pi|_{\mathcal{I}(\mathfrak{g}_r)}.$$

Our notation θ_π should not cause any confusion with the automorphism θ of $\underline{\mathsf{G}}$ or the element $\tilde{\theta} \in \underline{\tilde{\mathsf{G}}}$; we now go back to working with our objects $\underline{\mathsf{G}}, \tilde{\theta}, \underline{\tilde{\mathsf{G}}}$ and $\underline{\mathrm{H}}$ etc.

10.2. Review of constructions associated to the Bruhat–Tits building

We will need some well-known properties of the Bruhat–Tits building of $\underline{\mathsf{G}}(F)$ as well as the groups $\underline{\mathsf{G}}_\theta(F)$. These will of course follow from analogous results for $\underline{\mathrm{H}}(F)$, which we no longer assume to be unramified. Thus, in this section (§ 10.2), we let $\underline{\mathrm{H}}$ be any quasi-split classical group defined by a quadratic or symplectic space (W, q_W).

Lemma 10.2.1. *For each x in the Bruhat–Tits building $\mathcal{B}(\underline{\mathrm{H}}, F)$ of $\underline{\mathrm{H}}(F)$, and for all $r > 0$, we have*

$$\mathfrak{c}(\mathfrak{h}_{x,r}) = \mathfrak{c}'(\mathfrak{h}_{x,r}) = H_{x,r}.$$

The analogous result holds for $\underline{\mathsf{G}}_W := \mathrm{GL}(W)$ as well.

Proof. Since this is well known (for example, it is mentioned in [14, page 535] in the split case), we will only sketch a proof for $\underline{\mathrm{H}}$; the analogous result for $\underline{\mathsf{G}}_W$ can be proved in an easier but similar manner. By [1, Proposition 1.4.1], one reduces to the case where $\underline{\mathrm{H}}$ is split and hence defined over $\mathfrak{O}$. Fix such an $\mathfrak{O}$-structure on $\underline{\mathrm{H}}$, and a maximal $\mathfrak{O}$-split torus in it. Let $\Phi(\underline{\mathrm{H}}, \underline{\mathrm{T}}_{\underline{\mathrm{H}}})$ denote the set of roots of $\underline{\mathrm{T}}_{\underline{\mathrm{H}}}$ in $\underline{\mathrm{H}}$. Accordingly, for each root $\alpha \in \Phi(\underline{\mathrm{H}}, \underline{\mathrm{T}}_{\underline{\mathrm{H}}})$, we have a root subgroup $\underline{\mathrm{U}}_\alpha$ defined over $\mathfrak{O}$, and an $\mathfrak{O}$-isomorphism $u_\alpha : \mathbb{G}_a \to \underline{\mathrm{U}}_\alpha$. Without loss of generality, the hyperspecial point given by the chosen $\mathfrak{O}$-structure belongs to the apartment of $\underline{\mathrm{T}}_{\underline{\mathrm{H}}}$, which may, using this point, be identified with $X_*(\underline{\mathrm{T}}_{\underline{\mathrm{H}}}) \otimes \mathbb{R}$. Assume without loss of generality that x belongs to the apartment of $\underline{\mathrm{T}}_{\underline{\mathrm{H}}}$, and hence corresponds to, say, $\lambda \in X_*(\underline{\mathrm{T}}_{\underline{\mathrm{H}}}) \otimes \mathbb{R}$. We have formulas

$$\begin{aligned} \mathfrak{h}_{x,r} &= \mathfrak{t}_{\underline{\mathrm{H}},r} \oplus \bigoplus_{\alpha \in \Phi(\underline{\mathrm{H}}, \underline{\mathrm{T}}_{\underline{\mathrm{H}}})} du_\alpha\left(\varpi^{\lceil r - \langle \alpha, \lambda \rangle \rceil} \mathfrak{O}\right), \quad \text{and} \\ H_{x,r} &= \left\langle T_{H,r}, u_\alpha\left(\varpi^{\lceil r - \langle \alpha, \lambda \rangle \rceil} \mathfrak{O}\right) \mid \alpha \in \Phi(\underline{\mathrm{T}}_{\underline{\mathrm{H}}}, \underline{\mathrm{H}})\right\rangle. \end{aligned} \tag{10.2.1}$$

One readily verifies that

$$\mathfrak{c}(\mathfrak{t}_{\underline{\mathrm{H}},r}) = T_{H,r}, \quad \text{and} \quad \mathfrak{c}(du_\alpha(\varpi^m \mathfrak{O})) = u_\alpha(\varpi^m \mathfrak{O}) \quad \text{for each } m \in \mathbb{Z}. \tag{10.2.2}$$

Together with Lemma 4.2.3 (since $r > 0$, $\mathfrak{h}_{x,r}$ consists entirely of topologically nilpotent elements, for example, this follows from equation (10.2.3) below), it follows that $\mathfrak{c}(\mathfrak{h}_{x,r})$ is a group containing $H_{x,r}$. We need to show that $\mathfrak{c}(\mathfrak{h}_{x,r}) = H_{x,r}$. Since we know that $H_{x,r}/H_{x,2r}$ is abelian, it is easy to check from (10.2.2) that $\mathfrak{c}^{-1}$ induces a surjective homomorphism of abelian groups $H_{x,r}/H_{x,2r} \to \mathfrak{h}_{x,r}/\mathfrak{h}_{x,2r}$, so that

$$\mathfrak{c}(\mathfrak{h}_{x,r}) \subset H_{x,r}\mathfrak{c}(\mathfrak{h}_{x,2r}) \subset H_{x,r}H_{x,2r}\mathfrak{c}(\mathfrak{h}_{x,4r}) \ldots,$$

giving $\mathfrak{c}(\mathfrak{h}_{x,r}) = H_{x,r}$. □

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Remark 10.2.2. One can identify the Bruhat–Tits building of $\underline{G}_W = \mathrm{GL}(W)$ in a $\underline{G}_W(F)$-equivariant fashion with the set of $\mathfrak{O}$-lattice functions on W, which are 'decreasing left-continuous' functions $r \mapsto \Lambda(r)$ from $\mathbb{R}$ to the set of $\mathfrak{O}$-lattices in W, such that $\Lambda(r+1) = \varpi\Lambda(r)$ (cf. [11], Section 2, [50]). This identification made, one can use q_W to define an involution on $\mathcal{B}(\underline{G}_W, F)$, the fixed points of which, namely the 'selfdual lattice functions', can be identified $\underline{H}(F)$-equivariantly with $\mathcal{B}(\underline{H}, F)$. For $x \in \mathcal{B}(\underline{H}, F) \subset \mathcal{B}(\underline{G}_W, F)$, write Λ_x for the associated lattice function. Then [11, appendix A] and [50, Theorem 1.8] give

$$\mathfrak{g}_{W,x,r} = \left\{X \in \mathfrak{g}_W(F) \mid X(\Lambda_x(s)) \subset \Lambda_x(s+r)\,\forall\, s \in \mathbb{R}\right\}, \quad \text{and} \quad \mathfrak{h}_{x,r} = \mathfrak{g}_{W,x,r} \cap \mathfrak{h}(F). \tag{10.2.3}$$

Thus, we conclude, using the notations above, with the following lemma.

Lemma 10.2.3. *For $x \in \mathcal{B}(\underline{H}, F) \subset \mathcal{B}(\underline{G}_W, F)$ and $r, s > 0$, we have the following.*

(a) $\mathfrak{c}(\mathfrak{h}_{x,r}) = \mathfrak{c}'(\mathfrak{h}_{x,r}) = H_{x,r}$, $\mathfrak{c}(\mathfrak{g}_{W,x,r}) = \mathfrak{c}'(\mathfrak{g}_{W,x,r}) = G_{W,x,r}$, $\mathfrak{c}(\mathfrak{h}_r) = \mathfrak{c}'(\mathfrak{h}_r) = H_r$, $\mathfrak{c}(\mathfrak{g}_{W,r}) = \mathfrak{c}'(\mathfrak{g}_{W,r}) = G_{W,r}$.

(b) *If $r \leqslant s \leqslant 2r$, $\mathfrak{c}$ induces an isomorphism of abelian groups between $\mathfrak{g}_{W,x,r}/\mathfrak{g}_{W,x,s}$ and $G_{W,x,r}/G_{W,x,s}$, which restricts to an isomorphism between the abelian groups $\mathfrak{h}_{x,r}/\mathfrak{h}_{x,s}$ and $H_{x,r}/H_{x,s}$.*

(c) $\mathfrak{h}_{x,r} = \mathfrak{g}_{W,x,r} \cap \mathfrak{h}(F)$, $H_{x,r} = G_{W,x,r} \cap \underline{H}(F)$, $\mathfrak{h}_r = \mathfrak{g}_{W,r} \cap \mathfrak{h}(F)$, $H_r = G_{W,r} \cap \underline{H}(F)$.

(d) $\mathfrak{g}_{W,x,r} \cdot \mathfrak{g}_{W,x,s} \subset \mathfrak{g}_{W,x,r+s}$. □

Here we remark that, in (c) above, the last two assertions follow from the first two, together with a usual identification of $\mathcal{B}(\underline{H}, F)$ as the set of fixed points on an involution on $\mathcal{B}(G_W, F)$, and the fact that for two points $x, y \in \mathcal{B}(G_W, F)$, and any point z in the geodesic connecting x and y,

$$\mathfrak{g}_{W,x,r} \cap \mathfrak{g}_{W,y,r} \subset \mathfrak{g}_{W,z,r}, \quad \text{and} \quad G_{W,x,r} \cap G_{W,y,r} \subset G_{W,z,r}$$

(this follows, for example, from equation (10.2.1)).

10.3. Hypotheses from [4, 20, 21]

We now use Lemma 10.2.3 to find conditions under which various hypotheses from [4, 20, 21] hold.

We wish to show that the hypotheses under consideration apply to the group $\underline{G}_W \rtimes \langle\theta_W\rangle$, at the point $1 \rtimes \theta_W$ (hence we will be able to use $\tilde{\theta}$-twisted character expansions for representations of $\tilde{\underline{G}}(F)$ in the next section). The centralizer of this point can be naturally identified with $\underline{H}$.

The following lemma consists of well-known assertions.

Lemma 10.3.1. *Hypotheses 8.1 to 8.6 of [4] are satisfied, with either of $\mathfrak{c}$ or $\mathfrak{c}'$ as the exponential map 'e' of Hypothesis 8.5 of [4] provided that p is odd. The hypotheses of [20, § 3] (namely Hypotheses 3.2.1, 3.4.1 and 3.4.3 therein) are satisfied by $\mathfrak{h}$ too.*

Proof. Hypotheses 8.1 and 8.2 of [4] are satisfied as the eigenvalues of $d\theta_W$ on $\mathfrak{g}_W$ are ± 1 and because $p \neq 2$, respectively. Hypothesis 8.4 of [4] needs us to give a

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$\underline{\mathsf{G}}_W \rtimes \langle\theta_W\rangle$-invariant symmetric nondegenerate bilinear form on $\mathfrak{g}_W(F)$ that identifies each $\mathfrak{g}_{W,x,r}$ with $\mathfrak{g}^*_{W,x,r}$. Proposition 4.1 of [5] gives us such a form, because $\underline{\mathsf{G}}_W$ is a general linear group, and because the θ_W invariance is automatic (the bilinear form being unique up to scaling separately on the one-dimensional center and the derived subalgebra of $\mathfrak{g}_W$). By the θ-invariance of this form, using that $p \neq 2$, it is easy to see that this form identifies each $\mathfrak{h}_{x,r}$ with $\mathfrak{h}^*_{x,r}$ as well. Using this, Hypothesis 8.3 of [4] follows from Lemma 10.2.3(c). Hypothesis 8.6 of [4] follows from [68], since F has characteristic 0. We claim that Hypothesis 8.5 of [4] is satisfied (with $\mathfrak{c}$ or $\mathfrak{c}'$ in place of e, and with 0 in place of e). Then the unnumbered conditions of the hypothesis follow from Lemma 10.2.3(a) and (b), while (2) of the hypothesis follows from the conjugation equivariance of $\mathfrak{c}$ and $\mathfrak{c}'$.

By Lemma 10.2.3(c), we only need to verify Hypothesis 8.5(1) and (3) by replacing $\underline{\mathrm{H}}$ with $\underline{\mathsf{G}}_W$ in the former. For any power series $p \in t + t^2\mathfrak{O}[[t]] \subset \mathfrak{O}[[t]]$, using Lemma 10.2.3(d) and the binomial theorem, one can make sense of $p(X)$ for $X \in \mathfrak{g}_{W,x,r}$, and also show that for $X \in \mathfrak{g}_{W,x,r}$ and $Y \in \mathfrak{g}_{W,x,s}$ we have $p(X) \in \mathfrak{g}_{W,x,r}$, $p(Y) \in \mathfrak{g}_{W,x,s}$, $p(X+Y) \in p(X) + p(Y) + \mathfrak{g}_{W,x,r+s}$ $(r, s > 0)$. Since both $\mathfrak{c} - 1$ and $\mathfrak{c}' - 1$ are given by such power series, the required assertions are now easy to check.

Finally, Hypotheses 3.2.1 and 3.4.3 of [20] for $\mathfrak{h}$ are already included in the hypotheses above (the preservation of Haar measures follows from the paragraph following Hypothesis 8.5 of [4]), while Hypothesis 3.4.1 has already been seen above. □

Remark 10.3.2. Finally, we need [4, Hypothesis 8.7], and for this it remains to consider the hypotheses of [20, § 2], or equivalently those of [21, § 4], for $\mathfrak{h}$. As mentioned in [20, § 2], all these hypotheses hold if p is larger than some constant that depends only on the absolute root datum of $\underline{\mathrm{H}}$. This should be explicitly computable, but we do not know its value yet. It suffices for us that it does not depend on the ramification degree of F over $\mathbb{Q}_p$. If for instance $\underline{\mathrm{H}}$ is symplectic or odd orthogonal, it should be at least $\dim W + 2$ (cf. [20, Hypothesis 2.2.4]).

Hypothesis 10.3.3. For the rest of § 10, we assume that p is large enough for the hypotheses of [20], Section 4, to hold, for $\underline{\mathsf{G}}_\theta$, $\underline{\mathrm{H}}$, and all the endoscopic groups of $\underline{\mathrm{H}}$.

In particular, by Remark 10.3.2, Remark 10.1.5 applies to $\underline{\mathrm{H}}$.

10.4. On the twisted character expansion

We will be concerned with the twisted character expansion only on twisted general linear groups, and hence will focus on $\underline{\mathsf{G}}$ here. In this context we will use $\mathfrak{c}$ as the mock exponential map.

Notation 10.4.1. Henceforth, for $r > 0$, set $\mathcal{U}_r = \mathrm{tc}(\underline{\mathsf{G}}(F), G_{\theta,r})$, an open subset of $\underline{\mathsf{G}}(F)$ by Remark 10.1.4 and Lemma 4.0.6.

The twisted character expansion involves a marginally different version of semisimple descent (as defined earlier), which we now recall and relate to the semisimple descent defined earlier.

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10.4.A. Another kind of semisimple descent. Recall that the map tc is submersive on $\underline{G}(F) \times \underline{G}_\theta(F)_{\rm tu}$. The theory of integration on fibers says that there exists a unique surjective map $C_c^\infty(\underline{G}(F) \times \underline{G}_\theta(F)_{\rm tu}) \to C_c^\infty(\mathcal{U})$, such that under this map $\alpha \mapsto f_\alpha$ if and only if supp $f_\alpha \subset$ tc(supp α) and

$$\int_{\mathcal{U}} f(x) f_\alpha(x)\, dx = \int_{\underline{G}(F)\times\underline{G}_\theta(F)_{\rm tu}} (f \circ {\rm tc})(g,m)\alpha(g,m)\, dg\, dm, \tag{10.4.1}$$

for all $f \in C_c^\infty(\mathcal{U})$ (for example, this is stated as [4, Theorem 7.1]). Clearly this map takes $C_c^\infty(\underline{G}(F) \times G_{\theta,r})$ to $C_c^\infty(\mathcal{U}_r)$, and this restriction, thanks to Lemma 4.0.1 and the $\underline{G}^\theta(F)$-conjugation invariance of $G_{\theta,r}$, is given by integration along the fibers of tc too.

Of course the map $\alpha \mapsto f_\alpha$ depends on the choice of the measures dg and dm, which we fix now.

Given $\alpha \in C_c^\infty(\underline{G}(F) \times \underline{G}_\theta(F)_{\rm tu})$, define $\beta_\alpha \in C_c^\infty(\underline{G}_\theta(F)_{\rm tu})$ by

$$\beta_\alpha(m) = \int_{\underline{G}(F)} \alpha(g,m)\, dg.$$

Lemma 10.4.2. (a) *The inclusion $\underline{G}_\theta(F)_{\rm tu} \hookrightarrow \mathcal{U}$ induces an identification $\Gamma(\underline{G}_\theta(F)_{\rm tu}) = \Gamma(\mathcal{U})$ as topological spaces, where $\underline{G}_\theta(F)_{\rm tu}$ and $\mathcal{U}$ are viewed as open subsets of $\underline{G}_\theta(F)$ and $\tilde{\underline{G}}(F)$, respectively.*

(b) *Assume that, in the definitions of $\mathcal{I}(\underline{G}_\theta(F)_{\rm tu})$ and $\mathcal{I}(\mathcal{U})$, the centralizer measures are chosen compatibly (this makes sense by Lemma 4.0.1). Then the identification $\Gamma(\underline{G}_\theta(F)_{\rm tu}) = \Gamma(\mathcal{U})$ induces an identification $\mathcal{I}(\underline{G}_\theta(F)_{\rm tu}) = \mathcal{I}(\mathcal{U})$.*

(c) *For $\alpha \in C_c^\infty(\underline{G}(F) \times \underline{G}_\theta(F)_{\rm tu})$, $f_\alpha \in C_c^\infty(\mathcal{U})$ is given by*

$$f_\alpha(g^{-1}x\tilde{\theta}g) = \int_{\underline{G}^\theta(F)} \alpha(mg, mxm^{-1})\, dm \quad \forall g \in \underline{G}(F), x \in \underline{G}_\theta(F)_{\rm tu}. \tag{10.4.2}$$

(d) *For $\alpha \in C_c^\infty(\underline{G}(F) \times \underline{G}_\theta(F)_{\rm tu})$, $I(f_\alpha) = I(\beta_\alpha)$ as elements of $\mathcal{I}(\underline{G}_\theta(F)_{\rm tu}) = \mathcal{I}(\mathcal{U})$.*

(e) *For any $r > 0$, the above results remain valid on replacing $\underline{G}_\theta(F)_{\rm tu}$ and $\mathcal{U}$ by $G_{\theta,r}$ and $\mathcal{U}_r$, respectively.*

Proof. The proof of (e) is similar to that of (a)–(d), so we will focus on the first four.

Let us prove (a) first. By Lemma 4.0.1 and Remark 4.0.3, we have a natural identification of sets $\Gamma(\mathcal{U}) = \Gamma(\underline{G}_\theta(F)_{\rm tu})$ (where $\mathcal{U}$ and $\underline{G}_\theta(F)_{\rm tu}$ are viewed as open subsets of $\tilde{\underline{G}}(F)$ and $\underline{G}_\theta(F)$, respectively). We claim that this is also an identification as topological spaces. This is because on the one hand the topology this set has as $\Gamma(\underline{G}_\theta(F)_{\rm tu})$ (respectively, as $\Gamma(\mathcal{U})$) is defined by the condition that, for each maximal torus $\underline{T}'_\theta$ of $\underline{G}_\theta$, with centralizer, say $\underline{T}'$ in $\underline{G}$ (a torus by [46, Theorem 1.1]), the obvious map from the regular semisimple set $\underline{T}'_\theta(F)_{\rm tu,srss}$ (respectively, the image of $\underline{T}'_\theta(F)_{\rm tu,srss}$ in $\underline{T}'(F)/(1-\theta)\underline{T}'(F)$) to $\Gamma(\underline{G}_\theta(F)_{\rm tu}) = \Gamma(\mathcal{U})$ is a local homeomorphism. On the other hand, the map from $\underline{T}'_\theta(F)_{\rm tu}$ to $\underline{T}'(F)/(1-\theta)\underline{T}'(F)$ is a local homeomorphism onto its image by Lemma 4.0.1, since this map is submersive and hence open.

The assertion of (b) is standard semisimple descent; for example, it can be proved exactly as in [82, § 2.4]. In any case, we will see in the proof of (c) that f_α and β_α

have the same orbital integrals at regular semisimple elements of $\underline{G}_\theta(F)_{\mathrm{tu}}$, and then the surjectivity of $\alpha \mapsto f_\alpha$ and $\alpha \mapsto \beta_\alpha$ will yield (b). Denote $\Gamma(\underline{G}_\theta(F)_{\mathrm{tu}}) = \Gamma(\mathcal{U})$ and $\mathcal{I}(\Gamma(\underline{G}_\theta(F)_{\mathrm{tu}})) = \mathcal{I}(\Gamma(\mathcal{U}))$ by Γ and $\mathcal{I}(\Gamma)$, respectively, for the purposes of this lemma.

Now we move to (c). It is easy to see that the integral on the right side of equation (10.4.2) is indeed convergent, and that it defines some *locally constant* function, say f'_α, on $C_c^\infty(\mathcal{U})$ (the local constancy using that tc is open; see Lemma 4.0.6). Further, $\operatorname{supp} f'_\alpha \subset \mathrm{tc}(\operatorname{supp}\alpha)$ is compact. Hence $f'_\alpha \in C_c^\infty(\mathcal{U})$. Thus, letting $f \in C_c^\infty(\mathcal{U})$, we need to show that equation (10.4.1) holds good with f'_α in place of f_α. Note that equation (10.4.2) is valid with $f'_\alpha f$ (as opposed to f'_α) in place of f_α and $\alpha\cdot(f\circ\mathrm{tc})$ in place of α (as opposed to α). Thus, we may work with $\alpha\cdot(f\circ\mathrm{tc})$ and $f'_\alpha\cdot f$ in place of α and f'_α, respectively, and are reduced to showing that

$$\int_{\mathcal{U}} f'_\alpha(x)dx = \int_{\underline{G}(F)\times\underline{G}_\theta(F)_{\mathrm{tu}}} \alpha(g,m)\,dg\,dm \left(= \int_{\underline{G}_\theta(F)_{\mathrm{tu}}} \beta_\alpha(m)\,dm\right).$$

Now the idea is to show that orbital integrals of f'_α and β_α give the same function on Γ, and that Γ inherits the same measure from $\mathcal{U}$ as it does from $\underline{G}_\theta(F)_{\mathrm{tu}}$.

First, let $\gamma \in \underline{G}_\theta(F)_{\mathrm{tu}}$, and let us show that $I(f'_\alpha) = I(\beta_\alpha)$ as functions on Γ; i.e.,

$$I(\gamma\tilde\theta, f'_\alpha, dg/dt_\gamma) = I(\gamma, \beta_\alpha, dm/dt_\gamma) \tag{10.4.3}$$

for a choice of dt_γ consistent with ones already made. Indeed, the unnormalized version $O(\gamma\tilde\theta, f'_\alpha, dg/dt_\gamma)$ of the left side equals (using Lemma 4.0.1)

$$\begin{aligned}
&\int_{\underline{G}^{\gamma\tilde\theta}(F)\backslash\underline{G}(F)}\int_{\underline{G}^{\theta}(F)} \alpha(mg, m\gamma m^{-1})\,dm\,d\dot g\\
&\quad= \int_{\underline{G}^{\gamma\tilde\theta}(F)\backslash\underline{G}(F)}\int_{\underline{G}^{\theta,\gamma}(F)\backslash\underline{G}^{\theta}(F)}\int_{\underline{G}^{\theta,\gamma}(F)} \alpha(mig, m\gamma m^{-1})di\,d\dot m\,d\dot g\\
&\quad= \int_{\underline{G}^{\theta,\gamma}(F)\backslash\underline{G}^{\theta}(F)}\int_{\underline{G}(F)} \alpha(mg, m\gamma m^{-1})\,dg\,d\dot m\\
&\quad\overset{g\mapsto m^{-1}g}{=} \int_{\underline{G}^{\theta,\gamma}(F)\backslash\underline{G}^{\theta}(F)}\int_{\underline{G}(F)} \alpha(g, m\gamma m^{-1})\,dg\,d\dot m,
\end{aligned}$$

which equals $O(\gamma, \beta_\alpha, dm/dt_\gamma)$. Here, we have used that everything in sight is convergent.

By Lemma 4.1.3, equation (10.4.3) follows.

We have measures $d\mu, d\mu'$ on Γ such that for all $f' \in C_c^\infty(\mathcal{U})$ and for all $\beta' \in C_c^\infty(\underline{G}_\theta(F)_{\mathrm{tu}})$ we have (letting $d\tilde g$ denote the transfer of dg to a measure on $\tilde{\underline{G}}(F)$)

$$\int_{\mathcal{U}} f'(\tilde g)\,d\tilde g = \int_\Gamma I(f')\,d\mu, \quad \text{and} \quad \int_{\underline{G}_\theta(F)_{\mathrm{tu}}} \beta'(m)\,dm = \int_\Gamma I(\beta')\,d\mu'.$$

Since $I(f'_\alpha) = I(\beta_\alpha)$ by equation (10.4.3), it is now enough to show that $\mu = \mu'$. We do this torus by torus, using the Weyl integration formulas for $\underline{G}_\theta(F)$ and for $\tilde{\underline{G}}(F)$.

Let $\underline{T}'_\theta$ be a maximal torus of $\underline{G}_\theta$ with centralizer $\underline{T}'$ in $\underline{G}$. Suppose that the chosen measure on $\underline{T}'_\theta(F)$ restricts to a measure dt' on $\underline{T}'_\theta(F)_{\mathrm{tu,srss}}$ and its image $[\underline{T}'_\theta(F)_{\mathrm{tu,srss}}]$

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in $\underline{\mathrm{T}}'(F)/(1-\theta)\underline{\mathrm{T}}'(F)$. We can talk of its image $\overline{dt'}$ on (the relevant open subset of) Γ. We also have well-defined measures $D_{\tilde{\underline{\mathsf{G}}}}(\cdot)^{1/2}\overline{dt'}$ and $D_{\underline{\mathsf{G}}_\theta}^{1/2}(\cdot)\overline{dt'}$ on Γ. It suffices to show that $d\mu = D_{\underline{\mathsf{G}}_\theta}(\cdot)^{1/2}\overline{dt'}$ and $d\mu' = D_{\tilde{\underline{\mathsf{G}}}}(\cdot)^{1/2}\overline{dt'}$, because $D_{\tilde{\underline{\mathsf{G}}}}$ and $D_{\underline{\mathsf{G}}_\theta}$ coincide on $\underline{\mathrm{T}}_\theta(F)_{\mathrm{tu}}$ by Lemma 4.1.3. These statements follow by the usual Weyl integration formula arguments (cf. [51, Lemma 5.3.5 and Proposition 5.3.6] for the case of $\tilde{\underline{\mathsf{G}}}(F)$, whose choice of measure relations, made shortly after Lemma 5.3.5 therein, is compatible with ours, as in [82, Remark 3.10], using that $p \neq 2$).

Finally, (d) follows from (c) and equation (10.4.3). □

Definition 10.4.3. Let $\tilde{\pi}$ be a representation of the twisted space $\tilde{\underline{\mathsf{G}}}(F)$. Let $r > 0$, so that $\mathfrak{c}$ induces a homeomorphism $\mathsf{g}_{\theta,r} \to G_{\theta,r}$ (see Lemma 10.2.3(a)). Then a character expansion for the Harish-Chandra character $\Theta_{\tilde{\pi}}$ of $\tilde{\pi}$ at the element $\tilde{\theta} \in \tilde{\underline{\mathsf{G}}}(F)$ on $\mathcal{U}_r$ with respect to $\mathfrak{c}$ is a distribution $\theta_{\tilde{\pi}} \in \hat{J}_{\mathcal{N}_{\mathsf{g}_\theta}}$ such that we have an equality of distributions on $\mathsf{g}_{\theta,r}$,

$$\Theta_{\tilde{\pi}}|_{\mathcal{I}(G_{\theta,r})} \circ (\mathfrak{c}^{-1})^* = \theta_{\tilde{\pi}}|_{\mathcal{I}(\mathsf{g}_{\theta,r})},$$

where the left hand side is made sense of using the identification $\mathcal{I}(G_{\theta,r}) = \mathcal{I}(\mathcal{U}_r)$ (cf. Lemma 10.4.2) and Notation 10.1.3.

We can analogously talk of $\tilde{\pi}_{\underline{\mathsf{G}}}$ having a character expansion on $\mathrm{tc}(\underline{\mathsf{G}}(F), G_{\theta,r+})$. The main result of [4], or more conveniently Corollary 12.9 there, in our context (where what is denoted $s(\gamma)$ there is 0; cf. Definition 4.1 of [4]), specialized to our situation, can now be stated as follows.

Theorem 10.4.4 (Adler, Korman). *Suppose that the hypotheses of [4], Section 8, are satisfied (see § 10.3). Let $\tilde{\pi}_{\underline{\mathsf{G}}}$ be an irreducible admissible representation of $\tilde{\underline{\mathsf{G}}}(F)$. Suppose that the underlying representation $\pi_{\underline{\mathsf{G}}}$ of $\underline{\mathsf{G}}(F)$ has depth less than r. Then $\Theta_{\tilde{\pi}}$ has a character expansion on $\mathcal{U}_r$.* □

Note that the above description of the character expansion is slightly different from the one in [4], which involves $\beta_\alpha \leftarrow \alpha \to f_\alpha$. But this turns out to be equivalent, thanks to Lemma 10.4.2 and Theorem 10.1.2.

10.5. Matching under Arthur's formalism

Let us recall, following [6], how endoscopic transfer may be described using the language from there recalled in § 10.1. First suppose that we are in case (a) or case (b). Set $\bar{\underline{\mathsf{G}}} = \underline{\mathsf{G}}_\theta$, and let $\overline{\mathrm{end}} : \mathcal{I}(\mathsf{g}_\theta(F)) \to \mathcal{SI}(\bar{\mathsf{g}}(F))$ be the obvious map, i.e., summing along the fibers of $\Gamma(\mathsf{g}_\theta(F)) \to \Delta(\bar{\mathsf{g}}(F)) = \Delta(\mathsf{g}_\theta(F))$. Further, by Remark 6.2.2 we have a natural homeomorphism $\Delta(\mathfrak{h}(F)) \cong \Delta(\bar{\mathsf{g}}(F))$, restricting to homeomorphisms $\Delta(\mathfrak{h}(F)_{\mathrm{tn}}) \cong \Delta(\bar{\mathsf{g}}(F)_{\mathrm{tn}})$ and $\Delta(\mathfrak{h}_r) \cong \Delta(\bar{\mathsf{g}}_r)$ (see Remark 10.1.5 and the end of § 10.3). Keeping in mind Remark 4.0.3, and for suitable normalizations of measures, Definition 6.6.1(ii) can then be interpreted as simply saying that $\varphi \in C_c^\infty(\bar{\mathsf{g}}(F))$ and $\varphi^H \in C_c^\infty(\mathfrak{h}(F))$ have matching orbital integrals if and only if $SI(\varphi) \in \mathcal{SI}(\bar{\mathsf{g}}(F))$ and $SI(\varphi^H) \in \mathcal{SI}(\mathfrak{h}(F))$

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are obtained from each other by pulling back under the homeomorphism $\Delta(\mathfrak{h}_r) \cong \Delta(\bar{\mathsf{g}}_r)$. The nonstandard transfer conjecture, proved in [82, § 1.8] using the nonstandard fundamental lemma of [65], then says that pull back under $\Delta(\mathfrak{h}(F)) \cong \Delta(\bar{\mathsf{g}}(F))$ induces a well-defined isomorphism $\mathrm{nst} : \mathcal{SI}(\bar{\mathsf{g}}(F)) \cong \mathcal{SI}(\mathfrak{h}(F))$. Strictly speaking, this is stated in [82] for simply connected groups, but it is easy to see that our statement is implied by the obvious analogue for $\mathfrak{h}_{\mathrm{sc}}$ and $\bar{\mathsf{g}}_{\mathrm{sc}}$, since the isomorphisms $\mathfrak{h}_{\mathrm{sc}}(F) \cong \mathfrak{h}(F)$ and $\bar{\mathsf{g}}_{\mathrm{sc}}(F) \cong \bar{\mathsf{g}}(F)$ respect stable conjugacy and the notion of invariance of measures on stable orbits. Since $\mathfrak{h}(F)_{\mathrm{tn}}$ and the $\mathfrak{h}_r$ are open and closed subsets of $\underline{\mathrm{H}}(F)$, and similarly with $\bar{\mathsf{g}}(F)_{\mathrm{tn}}$ and the $\bar{\mathsf{g}}_r$, we have well-defined isomorphisms $\overline{\mathrm{nst}}$,

$$\overline{\mathrm{nst}} : \mathcal{SI}(\bar{\mathsf{g}}_r) \xrightarrow{\cong} \mathcal{SI}(\mathfrak{h}_r), \tag{10.5.1}$$

induced by pulling back under $\Delta(\mathfrak{h}_r) \cong \Delta(\bar{\mathsf{g}}_r)$.

Now suppose that we are in case (c). In this case, set $\bar{\underline{\mathsf{G}}} = \underline{\mathrm{H}}$. This time what we have is a map

$$\mathcal{I}(\mathsf{g}_\theta(F)) \to \text{functions on the `strongly } \mathsf{g}_\theta\text{-regular' subset of } \Delta(\bar{\mathsf{g}}(F)),$$

given by

$$I(\phi) \mapsto \left(Y \mapsto \sum_{X \in \Gamma(\mathsf{g}_\theta(F))} \Delta(Y, X) I(X, \phi) \right),$$

again keeping in mind Remark 4.0.3, and for suitable normalizations of measures. The usual transfer conjecture for Lie algebras, proved as [82, Theorem 1.5] using [65], says that the above map has image in $\mathcal{SI}(\bar{\mathsf{g}}(F))$. Thus, we now have a map $\overline{\mathrm{end}} : \mathcal{I}(\mathsf{g}_\theta(F)) \to \mathcal{SI}(\bar{\mathsf{g}}(F))$, the endoscopic transfer map. Take $\overline{\mathrm{nst}} : \mathcal{SI}(\bar{\mathsf{g}}(F)) \to \mathcal{SI}(\mathfrak{h}(F))$ to be the pull back under $Y \mapsto 2Y$. Using Remark 10.1.5 (see also the end of Section 10.3) and Remark 10.1.4, we see that $\overline{\mathrm{end}}$ and $\overline{\mathrm{nst}}$ restrict to maps

$$\overline{\mathrm{end}} : \mathcal{I}(\mathsf{g}_{\theta,r}) \to \mathcal{SI}(\bar{\mathsf{g}}_r) \quad \text{and} \quad \overline{\mathrm{nst}} : \mathcal{SI}(\bar{\mathsf{g}}_r) \to \mathcal{SI}(\mathfrak{h}_r),$$

for all $r > 0$, and we have a similar analogue involving the topologically nilpotent sets.

In all the three cases, write end for the endoscopic transfer map from $\mathcal{I}(\tilde{\underline{\mathsf{G}}}(F))$ to $\mathcal{SI}(\underline{\mathrm{H}}(F))$. By Lemma 6.4.2/Lemma 7.4.6, $\mathrm{end}(\mathcal{I}(\mathcal{U})) \subset \mathcal{SI}(\underline{\mathrm{H}}(F)_{\mathrm{tu}})$. In the same way, it follows that $\mathrm{end}(\mathcal{I}(\mathcal{U}_r)) \subset \mathcal{SI}(H_r)$.

Now Lemmas 6.6.4 and 7.7.2 can be rephrased as saying that the following diagram commutes (cf. Lemma 10.4.2(b) for the equality in the top left entry):

$$\begin{array}{ccccc}
\mathcal{I}(\mathcal{U}) = \mathcal{I}(\underline{\mathsf{G}}_\theta(F)_{\mathrm{tu}}) & & \xrightarrow{\quad\mathrm{end}\quad} & & \mathcal{SI}(\underline{\mathrm{H}}(F)_{\mathrm{tu}}) \\
\mathfrak{c}^{-1*} \uparrow \cong & & & & \cong \uparrow \mathfrak{c}'^{-1*} \\
\mathcal{I}(\mathsf{g}_\theta(F)_{\mathrm{tn}}) & \xrightarrow{\overline{\mathrm{end}}} & \mathcal{SI}(\bar{\mathsf{g}}(F)_{\mathrm{tn}}) & \xrightarrow[\cong]{\overline{\mathrm{nst}}} & \mathcal{SI}(\mathfrak{h}(F)_{\mathrm{tn}}).
\end{array} \tag{10.5.2}$$

Remark 10.5.1. It is also easy to see, by an obvious modification of Lemmas 6.4.2 and 7.4.6, that we may restrict the above diagram to get a variant where the 'tu' and the 'tn' are replaced by r for some $r > 0$.

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Recall that $\widetilde{\mathrm{Out}}(\underline{H})$ denoted the group of outer automorphisms of $\underline{H}$ (see § 2.2). Identify this group with the subgroup of the group of F-automorphisms that preserve a fixed splitting. The objects $\Gamma(\mathcal{U})$ and $\Delta(\mathcal{U})$ defined earlier have analogues that now include orbits under the subgroup $\mathrm{Int}(\underline{H}(F)) \cdot \widetilde{\mathrm{Out}}(\underline{H})$ of the group of F-automorphisms of $\underline{H}$. Taking orbital integrals of functions invariant under $\widetilde{\mathrm{Out}}(\underline{H})$ therefore yields subspaces $\tilde{\mathcal{I}}(\underline{H}(F)) \subset \mathcal{I}(\underline{H}(F))$ and $\widetilde{\mathcal{SI}}(\underline{H}(F)) \subset \mathcal{SI}(\underline{H}(F))$, respectively. Similarly we have $\tilde{\mathcal{I}}(\underline{H}(F)_{\mathrm{tu}}), \tilde{\mathcal{I}}(\mathfrak{h}(F)_{\mathrm{tn}})$, etc.

It is easy to see that, as mentioned in [7, page 56], $\mathrm{end}(\mathcal{I}(\underline{\mathsf{G}}(F))) \subset \widetilde{\mathcal{SI}}(\underline{H}(F))$. It also follows from the definition of matching that $\overline{\mathrm{end}}(\mathcal{I}(\mathsf{g}_\theta(F))) \subset \widetilde{\mathcal{SI}}(\bar{\mathsf{g}}(F))$ and $\overline{\mathrm{nst}}(\mathcal{SI}(\bar{\mathsf{g}}(F))) = \widetilde{\mathcal{SI}}(\mathfrak{h}(F))$ (in cases (a) and (b) this is tautological, and in case (c) the nontautological part is a consequence of the invariance properties of the transfer factors involved in $\overline{\mathrm{end}}$, namely $\Delta(Y, X) = \Delta(Y', X)$ if Y, Y' are in the same $\widetilde{\mathrm{Out}}(\underline{H})$-orbit; see § 7.5.A).

Taking this and Remark 10.5.1 into account, and dualizing to get maps at the level of distributions, we obtain a commutative diagram:

$$\begin{array}{ccccc}
\widetilde{\mathcal{SI}}(H_r)^* & & \xrightarrow{\qquad\mathrm{end}^*\qquad} & & \mathcal{I}(\mathcal{U}_r)^* = \mathcal{I}(\mathsf{G}_{\theta,r})^* \\
{\scriptstyle\cong}\big\downarrow{\scriptstyle _\circ\mathfrak{c}'^{-1*}} & & & & {\scriptstyle _\circ\mathfrak{c}^{-1*}}\big\downarrow{\scriptstyle\cong} \\
\widetilde{\mathcal{SI}}(\mathfrak{h}_r)^* & \underset{\cong}{\xrightarrow{\overline{\mathrm{nst}}^*}} & \widetilde{\mathcal{SI}}(\bar{\mathsf{g}}_r)^* & \xrightarrow{\quad\overline{\mathrm{end}}^*\quad} & \mathcal{I}(\mathsf{g}_{\theta,r})^*
\end{array} \tag{10.5.3}$$

Now let ϕ be a Langlands parameter for $\underline{H}(F)$. As recalled in § 2.2, we have a finite packet $\tilde{\Pi}_\phi$ of elements of $\tilde{\Pi}_{\mathrm{temp}}(H)$, indexed by the characters of the finite group $\mathcal{S}_\phi$ discussed earlier.

$\tilde{\Pi}_\phi$ then determines a distribution in $\widetilde{\mathcal{SI}}(\underline{H}(F))^*$, namely the stable Harish-Chandra character $S\Theta_\phi$. By [20] (cf. § 10.3 for a check of the relevant hypotheses), this distribution has a character expansion $\theta_\phi \in \hat{J}(\mathcal{N}_{\mathfrak{h}})$ with respect to $\mathfrak{c}'$ (note that we are using $\mathfrak{c}'$ instead of $\mathfrak{c}$) on H_s, for s large enough (this character expansion is uniquely defined and independent of s by [74, Lemma 9.9]).

On the other hand, ϕ determines a Langlands parameter for $\underline{\mathsf{G}}$, and hence a representation $\pi_{\underline{\mathsf{G}}}$ of $\underline{\mathsf{G}}(F)$ invariant under θ. Extend $\pi_{\underline{\mathsf{G}}}$ (noncanonically) to a representation $\pi_{\tilde{\underline{\mathsf{G}}}}$ of $\tilde{\underline{\mathsf{G}}}(F)$. Write $\Theta_{\phi,\tilde{\underline{\mathsf{G}}}}$ for its Harish-Chandra character. By the theorem of Adler and Korman, namely Theorem 10.4.4 above, $\Theta_{\phi,\tilde{\underline{\mathsf{G}}}}$ has a character expansion $\theta_{\phi,\tilde{\underline{\mathsf{G}}}} \in \hat{J}(\mathcal{N}_{\mathsf{g}_\theta})$ with respect to $\mathfrak{c}$ on $\mathcal{U}_s$ for large s. Again, by [74, Lemma 9.9], $\theta_{\phi,\tilde{\underline{\mathsf{G}}}}$ is well defined and independent of s.

Remark 10.5.2. While $\widetilde{\mathcal{SI}}(\underline{H}(F))^*$ has been defined as a quotient of $\mathcal{SI}(\underline{H}(F))^*$, by 'averaging under $\widetilde{\mathrm{Out}}(\underline{H})\,\mathrm{Int}(\underline{H}(F))/\,\mathrm{Int}(\underline{H}(F))$', we may and do also view $\widetilde{\mathcal{SI}}(\underline{H}(F))^*$ as the subspace of distributions in $\mathcal{SI}(\underline{H}(F))^*$ invariant under $\widetilde{\mathrm{Out}}(\underline{H})$. Since $\tilde{\Pi}_\phi$ consists of $\widetilde{\mathrm{Out}}(\underline{H})$-orbits of representations, there is an obvious way to view $S\Theta_\phi$ as an element of $\widetilde{\mathcal{SI}}(\underline{H}(F))^*$. Similarly, we may and do view θ_ϕ as an element of $\widetilde{\mathcal{SI}}(\mathfrak{h}(F))^* \subset \mathcal{SI}(\mathfrak{h}(F))^*$.

Theorem 2.2.1 of [7] gives us (for a suitable normalization of $\pi_{\tilde{\underline{G}}}$)

$$\mathrm{end}^*(S\Theta_\phi) = \Theta_{\phi,\tilde{\underline{G}}}. \tag{10.5.4}$$

Lemma 10.5.3. *We have*

$$\overline{\mathrm{end}}^* \circ \overline{\mathrm{nst}}^* (\theta_\phi) = \theta_{\phi,\tilde{\underline{G}}}.$$

Proof. Let $f_{\mathfrak{g}_\theta} \in C_c^\infty(\mathfrak{g}_\theta(F))$, and let $f_{\mathfrak{h}} = \overline{\mathrm{nst}} \circ \overline{\mathrm{end}}(f_{\mathfrak{g}_\theta}) \in C_c^\infty(\mathfrak{h}(F))$. We need to show that $\theta_\phi(f_{\mathfrak{h}}) = \theta_{\phi,\tilde{\underline{G}}}(f_{\mathfrak{g}_\theta})$.

From diagram (10.5.3), we know this to hold if $f_{\mathfrak{h}} \in C_c^\infty(\mathfrak{h}_r)$ and $f_{\mathfrak{g}_\theta} \in C_c^\infty(\mathfrak{g}_{\theta,r})$, where r is large enough so that the character expansions for $S\Theta_\phi$ and $\Theta_{\phi,\tilde{\underline{G}}}$ hold with respect to $\mathfrak{c}'$ and $\mathfrak{c}$ on H_r and $\mathcal{U}_r$, respectively. Fix such an r.

For $t \in F^\times$, let $f_{\mathfrak{h},t}$ and $f_{\mathfrak{g}_{\theta,t}}$ denote the functions $Y \mapsto f_{\mathfrak{h}}(t^{-2}Y)$ and $X \mapsto f_{\mathfrak{g}_\theta}(t^{-2}X)$, respectively. By Corollary 7.6.3 and Remark 7.6.5 in case (c), and readily in cases (a) and (b), we get, as in Lemma 8.2.2 (cf. [74, Lemma 9.7], [24, Lemma 3.2.1]), that

$$\overline{\mathrm{nst}} \circ \overline{\mathrm{end}}(f_{\mathfrak{g}_{\theta,t}}) = |t|^{n_0} f_{\mathfrak{h},t}, \tag{10.5.5}$$

where $n_0 = \dim \mathfrak{g}_\theta - \dim \bar{\mathfrak{g}}$. Moreover, by the homogeneity properties of Fourier transforms of nilpotent orbital integrals, $P_{\tilde{\underline{G}}}(t) := \theta_{\phi,\tilde{\underline{G}}}(f_{\mathfrak{g}_\theta,t})$ and $P(t) := \theta_\phi(|t|^{n_0} f_{\mathfrak{h},t})$ are polynomials in $|t|$ (for example, see the last equation in [74, page 325]). We need to show that $P_{\tilde{\underline{G}}}(1) = P(1)$, which will follow if we show that $P_{\tilde{\underline{G}}}(t) = P(t)$ for infinitely many $|t|$. But for t such that $|t|$ is small enough, $f_{\mathfrak{g}_{\theta,t}} \in C_c^\infty(\mathfrak{g}_{\theta,r})$ and $f_{\mathfrak{h},t} \in C_c^\infty(\mathfrak{h}_r)$, so that, by (10.5.5) and the choice of r, $P_{\tilde{\underline{G}}}(t) = P(t)$, as we wanted. □

Let $d_\phi = \operatorname{depth} \phi$. Then by Theorem 10.4.4 and the fact that the LLC for general linear groups preserves depth (cf. [86, Theorem 2.3.6.4], a result of J.-K. Yu), the character expansion for $\Theta_{\phi,\tilde{\underline{G}}}$ at $\tilde{\theta}$ with respect to $\mathfrak{c}$ is valid on $\mathrm{tc}(\underline{G}(F), G_{\theta,d_\phi+})$.

Lemma 10.5.4. *The character expansion for $S\Theta_\phi$ with respect to $\mathfrak{c}'$ is valid on $H_{d_\phi+}$.*

Proof. By (10.5.4), Lemma 10.5.3, the commutativity of (10.5.3), Remark 10.5.2, and the definition of character expansion, it suffices to show that the map $\overline{\mathrm{end}}^* \circ \overline{\mathrm{nst}}^*$ of (10.5.3), or equivalently the map end^*, with $d_\phi+$ in place of r, is injective (recall that the maps denoted $\overline{\mathrm{nst}}$ are isomorphisms, cf. (10.5.1) in cases (a) and (b), and tautologically in case (c)).

In cases (a) and (b) this is tautological since $\overline{\mathrm{end}}^*$ is the identity map. So, let us focus on case (c), where we need to show that the map $\mathrm{end} : \mathcal{I}(\mathcal{U}_r) \to \widetilde{\mathcal{SI}}(H_r)$ is surjective. Now we crucially appeal to [7, Corollary 2.1.2], according to which $\mathrm{end} : \mathcal{I}(\tilde{\underline{G}}(F)) \to \widetilde{\mathcal{SI}}(\underline{H}(F))$ is surjective. Therefore, it is enough to prove the following two assertions.

(i) If $\gamma \in \underline{H}(F)$ matches a strongly regular semisimple element $\delta \in \tilde{\underline{G}}(F)$, then $\gamma \in H_r$ if and only if $\delta \in \mathcal{U}_r$.

(ii) $\mathcal{U}_r$ is open and closed in $\tilde{\underline{G}}(F)$.

Recall from Remark 7.4.4 the map $\tilde{\underline{G}}(F) \to \underline{G}(F)$ given by $\gamma \mapsto T_\gamma$, where $\gamma(v, w) = \gamma(-T_\gamma w, v)$. If we show that $\mathcal{U}_r$ is the inverse image of G_r under $\gamma \mapsto T_\gamma$, then (ii) will

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follow (as G_r is open and closed in $\underline{G}(F)$), and so will (i) (by Remark 7.4.4 and Remark 10.1.5). One implication here is clear: if $\gamma = g^{-1}m\tilde{\theta}g \in \mathcal{U}_r$ with $g \in \underline{G}(F)$ and $m \in G_{\theta,r}$, then $T_\gamma = g^{-1}m^2g \in G_r$.

Conversely, suppose that $\gamma = g_1\tilde{\theta} \in \tilde{\underline{G}}(F)$ is such that $T_\gamma \in G_r$. By Lemma 3.2.7, we have a unique $m \in G_r$ such that $m^2 = T_\gamma$. Therefore, $\operatorname{Int}\gamma = \operatorname{Int} g_1 \circ \theta$ fixes $T_\gamma = m^2$ and hence m too (by uniqueness). In other words, $\gamma(mv, mw) = \gamma(v, w)$ for all $v, w \in V$. Then

$$\begin{aligned}(m^{-1}\gamma)(v, w) &= \gamma(mv, w) = -\gamma(T_\gamma w, mv) = -\gamma(m^2w, mv)\\ &= -\gamma(mw, v) = -(m^{-1}\gamma)(w, v).\end{aligned}$$

Therefore $m^{-1}\gamma$ is a symplectic form, so that $m^{-1}\gamma = g^{-1}\tilde{\theta}g$ for some $g \in \underline{G}(F)$. Further, gmg^{-1} commutes with $g(m^{-1}\gamma)g^{-1} = \tilde{\theta}$, so that $gmg^{-1} \in G_{\theta,r}$. Therefore,

$$\gamma = m \cdot m^{-1}\gamma = \operatorname{tc}(g, gmg^{-1}) \in \operatorname{tc}(\underline{G}(F), G_{\theta,r}) = \mathcal{U}_r,$$

as needed. This shows (ii). □

From the above lemma we prefer to get a character expansion for $S\Theta_\phi$ with respect to $\mathfrak{c}$ rather than $\mathfrak{c}'$, and to this end we have the following.

Lemma 10.5.5. *Let $r > 0$, and let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two $\operatorname{Ad}\underline{H}(F)$-invariant homeomorphisms from $\mathfrak{h}_r$ to H_r satisfying the hypotheses of [20] and such that, for $s \geqslant r$, the self-bijection of $\mathfrak{h}_{x,s}/\mathfrak{h}_{x,s+}$ induced by $\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1$ using [20, Hypothesis 3.2.1] is the identity. Then, if a virtual character of $\underline{H}(F)$ has a character expansion with respect to $\mathfrak{c}_1$ on H_r, then it has one with respect to $\mathfrak{c}_2$ too. In particular, by Lemma 10.5.4, $S\Theta_\phi$ has a character expansion on $H_{d_\phi+}$ with respect to $\mathfrak{c}$.*

Proof. We will use a homogeneity result of DeBacker, namely [20, Theorem 2.1.5], to which end we recall a few definitions from [20]. For $r \in \mathbb{R}$, let

$$\tilde{J}_{(-r)+} = \bigcap_{x \in \mathcal{B}(\underline{H}, F)} \bigcap_{s \leqslant -r} \tilde{J}_{x,s,(-r)+},$$

where

$$\begin{aligned}\tilde{J}_{x,s,(-r)+} = \{T \in J(\mathfrak{h}) \mid \forall f \in C(\mathfrak{h}_{x,s}/\mathfrak{h}_{x,(-r)+}),\ &\text{if } \operatorname{supp} f \cap (\mathcal{N}_{\mathfrak{h}} + \mathfrak{h}_{x,s^+}) = \emptyset,\\ &\text{then } T(f) = 0\}.\end{aligned}$$

In the above, $J(\mathfrak{h})$ denotes the set of invariant distributions on $\mathfrak{h}(F)$, which is identified with $\mathcal{I}(\mathfrak{h}(F))^*$ by Notation 10.1.3, and $\mathcal{N}_{\mathfrak{h}}$ denotes the nilpotent cone of $\mathfrak{h}(F)$. Recall also the definition

$$\mathcal{D}_{(-r)^+} = \sum_{x \in \mathcal{B}(\underline{H}, F)} C_c(\mathfrak{h}(F)/\mathfrak{h}_{x,(-r)+}) = \mathcal{F}\left(C_c^\infty(\mathfrak{h}_r)\right),$$

where the Fourier transform $\mathcal{F}$, as in [20], is taken with respect to a fixed additive character $\Lambda : F \to \mathbb{C}^\times$ trivial on $\varpi\mathfrak{O}$ but not on $\mathfrak{O}$, and a symmetric bilinear form $\mathfrak{tr}$ that identifies each $\mathfrak{h}_{x,r}$ with $\mathfrak{h}^*_{x,r}$ (cf. Hypothesis 3.4.1 of [20], which is satisfied by $\mathfrak{h}$, see Lemma 10.3.1).

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Theorem 2.1.5 of [20] says that, if the hypotheses of Section 2.2 there hold, and F has characteristic 0 (which is so in our case),

$$\operatorname{res}_{\mathcal{D}_{(-r)+}} \tilde{J}_{(-r)+} = \operatorname{res}_{\mathcal{D}_{(-r)+}} J(\mathcal{N}_{\mathfrak{h}}).$$

From [20, Remark 2.1.7], we have $J(\mathcal{N}_{\mathfrak{h}}) \subset \tilde{J}_{(-r)+}$. Now, let Θ be a virtual character of $\underline{\mathrm{H}}(F)$ that has a character expansion with respect to $\mathfrak{c}_1$ on $\underline{\mathrm{H}}_r$. This implies that there exists $D_1 \in \tilde{J}_{(-r)+}$ such that, for each $f \in C_c^\infty(\mathfrak{h}_r)$, we have $D_1(\mathcal{F} \circ f) = \Theta(f \circ \mathfrak{c}_1^{-1})$. Let $D_2 \in J(\mathfrak{h})$ be such that $D_2(\mathcal{F} \circ f) = \Theta(f \circ \mathfrak{c}_2^{-1})$ for $f \in C_c^\infty(\mathfrak{h}_r)$. Using the above homogeneity result, to prove that Θ has a character expansion with respect to $\mathfrak{c}_2$ on H_r, it suffices to show that $D_2 \in \tilde{J}_{(-r)+}$. We will show that $D_2 \in \tilde{J}_{x,s,(-r)+}$ for each $x \in \mathcal{B}(\underline{\mathrm{H}}, F)$ and each $s \leqslant -r$. Note that $D_2 = D_1 \circ \mathcal{F} \circ (\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1)^* \circ \mathcal{F}^{-1}$ on $C_c^\infty(\mathfrak{h}_r)$.

Therefore, to show that $D_2 \in \tilde{J}_{x,s,(-r)+}$, it is enough to show that that, for $X \in \mathfrak{h}_{x,s}$,

$$\mathcal{F} \circ (\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1)^* \circ \mathcal{F}^{-1} \left(\mathbb{1}_{X+\mathfrak{h}_{x,(-r)+}}\right) \in \sum_{X' \in X+\mathfrak{h}_{x,s+}} \mathbb{C} \cdot \mathbb{1}_{X'+\mathfrak{h}_{x,(-r)+}}.$$

For $X' \in \mathfrak{h}_{x,s}$, $\mathbb{1}_{X'+\mathfrak{h}_{x,(-r)+}}$ is the Fourier transform of the function $(\operatorname{meas} \mathfrak{h}_{x,r})^{-1} \cdot \chi_{-X'} \in C_c(\mathfrak{h}_{x,r}/\mathfrak{h}_{x,(-s)+}) \subset C_c^\infty(\mathfrak{h})$, whose value on $Y \in \mathfrak{h}_{x,r}$ equals $(\operatorname{meas} \mathfrak{h}_{x,(-r)+}) \cdot \Lambda \circ \operatorname{tr}(-X', Y)$. Thus, we need to show that, for each $X \in \mathfrak{h}_{x,s}$,

$$\chi_X \circ (\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1) \in \sum_{X' \in X+\mathfrak{h}_{x,s+}} \mathbb{C} \cdot \chi_{X'}. \tag{10.5.6}$$

This will follow once we show that $\chi_X \circ (\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1)$ satisfies

$$\chi_X \circ (\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1)(Y+Y') = \chi_X \circ (\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1)(Y) \cdot \chi_X(Y'), \quad \forall\, Y \in \mathfrak{h}_{x,r},\ Y' \in \mathfrak{h}_{x,-s}.$$

This in turn reduces to showing that

$$\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1(Y+Y') \in \mathfrak{c}_2^{-1} \circ \mathfrak{c}_1(Y) + Y' + \mathfrak{h}_{x,(-s)+} \quad \forall\, Y \in \mathfrak{h}_{x,r},\ Y' \in \mathfrak{h}_{x,-s}.$$

This follows from [20, Hypothesis 3.2.1(b)] together with our assumption that the automorphism of $\mathfrak{h}_{x,-s}/\mathfrak{h}_{x,(-s)+}$ induced by $\mathfrak{c}_2^{-1} \circ \mathfrak{c}_1$ is the identity.

For the last assertion of the lemma, we should verify that $\mathfrak{c} \circ \mathfrak{c}'^{-1}$, or equivalently $\mathfrak{c}' \circ \mathfrak{c}^{-1}$, induces the identity on each $\mathfrak{h}_{x,s}/\mathfrak{h}_{x,s+}$ $(s \geqslant r)$. But this follows from the fact that $\mathfrak{c}'(X) = \mathfrak{c}(X/2)^2$, and the fact that $\mathfrak{c}$ itself induces an isomorphism $\mathfrak{h}_{x,s}/\mathfrak{h}_{x,s+} \to H_{x,s}/H_{x,s+}$ of groups. □

Recall that for each $\pi \in \tilde{\Pi}_\phi$ we have a character ϵ_π of $\mathcal{S}_\phi$. Write $\langle s, \pi \rangle$ for $\epsilon_\pi(s)$. Then $\langle \cdot, \cdot \rangle$ is a perfect pairing between $\mathcal{S}_\phi$ and π (see [7, Theorem 2.2.1]).

10.6. Endoscopic transfer and depth bound

Now, suppose that $s \neq 1$. Then s determines an endoscopic group $\underline{\mathrm{H}}_s$ of $\underline{\mathrm{H}}$, which may be written as a product $\underline{\mathrm{H}}_{s,1} \times \underline{\mathrm{H}}_{s,2}$ of quasi-split classical groups (one of which may be trivial; for example, it can happen that $\underline{\mathrm{H}} = \mathrm{Sp}_{2n}$ and $\underline{\mathrm{H}}_s$ is a form of SO_{2n}). Accordingly, ϕ is the image of a product $\phi_{s,1} \times \phi_{s,2}$, each $\phi_{s,i}$ being a Langlands parameter for $\underline{\mathrm{H}}_{s,i}$. Denote

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both the associated endoscopic transfer maps $\mathcal{I}(\underline{\mathrm{H}}(F)) \to \mathcal{SI}(\underline{\mathrm{H}}_s(F))$ and $\mathcal{I}(\mathfrak{h}(F)) \to \mathcal{SI}(\mathfrak{h}_s(F))$ by the same letter end_s. We have

$$\mathcal{SI}(\underline{\mathrm{H}}_s(F)) \cong \mathcal{SI}(\underline{\mathrm{H}}_{s,1}(F)) \otimes_{\mathbb{C}} \mathcal{SI}(\underline{\mathrm{H}}_{s,2}(F)),$$

$$\mathcal{SI}(\underline{\mathrm{H}}_{s,1}(F))^* \otimes_{\mathbb{C}} \mathcal{SI}(\underline{\mathrm{H}}_{s,2}(F))^* \subset \mathcal{SI}(\underline{\mathrm{H}}_s(F))^*,$$

and similarly at the Lie algebra level. By [7, Theorem 2.2.1], up to a normalizing scalar,

$$\Theta^s_\phi = \mathrm{end}_s^*(S\Theta_{\phi_{s,1}} \otimes S\Theta_{\phi_{s,2}}).$$

Lemma 10.6.1. *For every $r > 0$, we have a commutative diagram analogous to (10.5.3), namely*

$$\begin{array}{ccccc}
\widetilde{\mathcal{SI}}(H_{s,1,r})^* \otimes \widetilde{\mathcal{SI}}(H_{s,2,r})^* & \longrightarrow & \widetilde{\mathcal{SI}}(H_{s,r})^* & \xrightarrow{\mathrm{end}_s^*} & \tilde{\mathcal{I}}(H_r)^* \\
{\scriptstyle _\circ\mathfrak{c}^{-1*}\otimes_\circ\mathfrak{c}^{-1*}} \downarrow \cong & & {\scriptstyle _\circ(\mathfrak{c}^{-1}\times\mathfrak{c}^{-1})^*} \downarrow \cong & & \cong \downarrow {\scriptstyle _\circ\mathfrak{c}^{-1*}} \\
\widetilde{\mathcal{SI}}(\mathfrak{h}_{s,1,r})^* \otimes \widetilde{\mathcal{SI}}(\mathfrak{h}_{s,2,r})^* & \longrightarrow & \widetilde{\mathcal{SI}}(\mathfrak{h}_{s,r})^* & \xrightarrow{\mathrm{end}_s^*} & \tilde{\mathcal{I}}(\mathfrak{h}_r)^*
\end{array}$$

Proof. By Lemma 4.1.3 and the conjugation equivariance of $\mathfrak{c}$ it is enough to show that the following hold.

(i) No $\mathfrak{h}$-regular (respectively, $\underline{\mathrm{H}}$-regular) semisimple element of $\mathfrak{h}_s(F) \setminus \mathfrak{h}_{s,r}$ (respectively, $\underline{\mathrm{H}}_s(F) \setminus H_{s,r}$) matches an element of $\mathfrak{h}_r$ (respectively, H_r).

(ii) For $Y_i \in \mathfrak{h}_{s,i,r}$ ($i = 1, 2$) and $X \in \mathfrak{h}_r$, (Y_1, Y_2) matches X if and only if $(\mathfrak{c}(Y_1), \mathfrak{c}(Y_2)) \in \underline{\mathrm{H}}_s(F)$ matches $\mathfrak{c}(X) \in H_r$, and moreover in this case we have

$$\Delta((Y_1, Y_2), X) = \Delta((\mathfrak{c}(Y_1), \mathfrak{c}(Y_2)), \mathfrak{c}(X)).$$

(i) follows from the fact that, if the equivalence class of $Y_i \in \mathfrak{h}_{s,i}(F)$ (respectively, $\gamma_i \in \underline{\mathrm{H}}_{s,i}(F)$) is parameterized by $(L^{(i)}, L^{(i)}_\pm, y^{(i)})$, for $i = 1, 2$, and if (Y_1, Y_2) (respectively (γ_1, γ_2)) matches $X \in \mathfrak{h}(F)$ (respectively, $\gamma \in \underline{\mathrm{H}}(F)$), then the equivalence class of X (respectively, γ) may be parameterized by $(L^{(1)} \times L^{(2)}, L^{(1)}_\pm \times L^{(2)}_\pm, (y^{(1)}, y^{(2)}))$; cf. [84, § X.2] (respectively, [83, § 1.9]) (in cases (a) and (b) this also proves the first assertion of (ii), but not in case (c)). For the first assertion of (ii), what we need to show is that, if $\iota : \underline{\mathrm{T}}_1 \times \underline{\mathrm{T}}_2 \to \underline{\mathrm{T}}$ is an admissible embedding of tori, then $\iota \circ \mathfrak{c} = \mathfrak{c} \circ \iota_*$. This follows from the above (partial) description of matching, which shows that every such admissible embedding arises in an obvious manner from an isomorphism $(L^{(1)} \times L^{(2)}, L^{(1)}_\pm \times L^{(2)}_\pm) \cong (L, L_\pm)$ of pairs consisting of an étale F-algebra and the subalgebra fixed by an F-involution.

Now let us prove the second assertion of (ii). Let Y_1, Y_2, X be as in that assertion, and let the conjugacy class of $\mathfrak{c}(X)$ be parameterized by $(L, L_\pm, x, c)$. Then the conjugacy class of X is parameterized by $(L, L_\pm, \bar{x}, c)$, with $\bar{x} = 2(x-1)(x+1)^{-1}$ (cf. § 7.3). Let m_0 be 0 when $\underline{\mathrm{H}}$ is a symplectic group, and 1 otherwise. Let n_0 be 1 when $\underline{\mathrm{H}}$ is an odd special orthogonal group, and 0 otherwise. Write $L = \prod_{i \in I} L_i$, $x = (x_i)_{i \in I}$ and $\bar{x} = (\bar{x}_i)$ as before. We have sets I^* and I^{-*} and I^{+*}, and an $\eta \in F^\times$ to normalize the transfer factors, as in § 7.5.A. Let d be the dimension of the standard representation of $\underline{\mathrm{H}}$. Set

$$P_I(T) = \prod_{i \in I} \prod_{\phi \in \mathrm{Hom}_{F\text{-alg}}(F_i, \bar{F})} (T - \phi(x_i)).$$

By [83, Proposition 1.10], we have that $\Delta(\mathfrak{c}(Y), \mathfrak{c}(X))$ is the product over $i \in I^{-*}$ of $\operatorname{sgn}_{F_i/F_{\pm i}}(C_i)$, where

$$C_i = (-1)^{m_0-n_0+1}\eta c_i P_I'(x_i) P_I(-1) x_i^{(2+n_0-d)/2} \left(\frac{2(1+x_i)}{x_i-1}\right)^{m_0}.$$

On the other hand, set

$$\bar{P}_I(T) = T^{n_0} \cdot \prod_{i\in I} \prod_{\phi\in \operatorname{Hom}_{F\text{-alg}}(F_i,\bar{F})} (T-\phi(\bar{x}_i)).$$

Then, by [84, Proposition X.8], $\Delta(Y, X)$ is the product over $i \in I^{-*}$ of $\operatorname{sgn}_{F_i/F_{\pm i}}(\bar{C}_i)$, where (as defined in Section X.7 of [84], but keeping in mind [83, Remark 1.3])

$$\bar{C}_i = \eta c_i^{-1} \bar{x}_i^{n_0-m_0} \bar{P}_I'(\bar{x}_i).$$

It is enough to show that $\operatorname{sgn}_{F_i/F_{\pm i}}(C_i) = \operatorname{sgn}_{F_i/F_{\pm i}}(\bar{C}_i)$ for each $i \in I^{-*}$. It is readily computed, for instance first by showing that

$$\bar{P}_I'(\bar{x}_i) = \left(\frac{2(x_i-1)}{1+x_i}\right)^{n_0} \cdot 4^{d-n_0-1} P_I'(x_i) \cdot P_I(-1)^{-1} (x_i+1)^{-(d-n_0-2)},$$

that

$$C_i/\bar{C}_i = (-1)^{m_0-n_0+1} c_i^2 \frac{P_I(-1)^2}{4^{d-n_0-1}} \left(\frac{x_i}{(1+x_i)^2}\right)^{(2+n_0-d)/2} \cdot 2^{2(m_0-n_0)} \cdot \left(\frac{1+x_i}{x_i-1}\right)^{2n_0}.$$

Since each $x_i/(1+x_i)^2$ is a norm, and since so are 4, $2^{2(m_0-n_0)}$ and $P_I(-1)^2$, we are reduced to checking that

$$(-1)^{m_0-n_0+1} \cdot c_i^2 \cdot \left(\frac{1+x_i}{x_i-1}\right)^{2n_0} = N_{F_i/F_{\pm i}}\left(c_i \left(\frac{1+x_i}{x_i-1}\right)^{n_0}\right),$$

as can be seen in each case. □

Remark 10.6.2. Of course, the above proof shows that the Cayley transform, where it is defined, behaves well with respect to the product of the transfer factors excluding the discriminant factor (not just on the topologically nilpotent set). However, this does not mean the same about endoscopic transfer, for topological nilpotence is really needed for Lemma 4.1.3.

Lemma 10.6.3. *For every $s \in \mathcal{S}_\phi$,*

$$\Theta_\phi^s := \sum_{\tilde{\pi}\in\tilde{\Pi}_\phi} \langle s, \tilde{\pi}\rangle \Theta_{\tilde{\pi}}$$

has a character expansion on $H_{d_\phi+}$ with respect to $\mathfrak{c}$.

Proof. By Lemmas 10.5.4 and 10.5.5 applied to $\underline{\mathrm{H}}_{s,1}$ and $\underline{\mathrm{H}}_{s,2}$, the character expansions $\theta_{\phi,1}$ and $\theta_{\phi,2}$ of $S\Theta_{\phi_{s,1}}$ and $S\Theta_{\phi_{s,2}}$ with respect to $\mathfrak{c}$ (and any sufficiently large $r' > 0$) are valid on $H_{s,1,d_\phi+}$ and $H_{s,2,d_\phi+}$, respectively. Exactly as in Lemma 10.5.3, the character expansion θ_ϕ^s for Θ_ϕ^s with respect to $\mathfrak{c}$ and any large r' can be given as $\operatorname{end}^*(\theta_{\phi_1} \otimes \theta_{\phi_2})$. By Lemma 10.6.1, it follows that the character expansion for Θ_ϕ^s with respect to $\mathfrak{c}$ is valid on $H_{d_\phi+}$ as well. □

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Corollary 10.6.4. *Let ϕ be a Langlands parameter for $\underline{\mathrm{H}}(F)$, and let π be an irreducible admissible tempered representation of $\underline{\mathrm{H}}(F)$ whose image in $\tilde{\Pi}_{\mathrm{temp}}(H)$ belongs to $\tilde{\Pi}_{\phi}$. Then*

$$\operatorname{depth}\pi \leqslant \operatorname{depth}\phi.$$

Proof. Let Θ_π denote the character of π in cases where the $\widetilde{\mathrm{Out}}(\underline{\mathrm{H}})$-orbit of π is singleton. If the $\widetilde{\mathrm{Out}}(\underline{\mathrm{H}})$-orbit of π has two elements, let this orbit be $\{\pi, \pi'\}$. Set $\Theta_\pi = (\operatorname{tr}\pi + \operatorname{tr}\pi')/2$. Since π and π' clearly have the same depth whenever π' is defined, in all of the cases it is enough to show that $\Theta_\pi(\mathbb{1}_{H_{x,r+}}) \neq 0$ for some $x \in \mathcal{B}(\underline{\mathrm{H}}, F)$ and every $r > \operatorname{depth}\phi$; this would imply that $\operatorname{depth}\pi \leqslant r$ for all $r > \operatorname{depth}\phi$, and hence that $\operatorname{depth}\pi \leqslant \operatorname{depth}\phi$. Let $r > \operatorname{depth}\phi$. By Lemma 10.6.3 and the fact that the pairing $\langle\cdot,\cdot\rangle$ is perfect on $\mathcal{S}_\phi \times \tilde{\Pi}_\phi$ [7, Theorem 2.2.1], Θ_π has a character expansion with respect to $\mathfrak{c}$ that is valid on H_r.

Write

$$\theta_\pi = \sum c_{\mathcal{O}'}\hat{\nu}_{\mathcal{O}'} \in \hat{J}(\mathcal{N}_{\mathfrak{h}})$$

for this character expansion of Θ_π, as $\mathcal{O}'$ ranges over the nilpotent $\underline{\mathrm{H}}(F)$-orbits in $\mathfrak{h}(F)$, $\nu_{\mathcal{O}'}$ being a choice of an $\underline{\mathrm{H}}(F)$-invariant measure on $\mathcal{O}'$ whose Fourier transform (using $\Lambda\circ\operatorname{tr}$ as in the proof of Lemma 10.5.5) we denote by $\hat{\nu}_{\mathcal{O}'}$. Let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{h}(F)$ such that $c_{\mathcal{O}} \neq 0$ and is maximal with respect to this property. Note that $\mathcal{O} \neq \{0\}$ (if $\mathcal{O} = \{0\}$, then it follows from the character expansion that, for sufficiently large l, the dimension of π^{K_l} is a constant independent of l, forcing π to be finite dimensional). The classification of nilpotent orbits from [21] gives us (since the hypotheses of [21] are valid), cf. [20, Remark 2.5.4], a generalized $-r$-facet F^* in $\mathcal{B}(\underline{\mathrm{H}}, F)$ and an element $X \in \mathfrak{h}_{F^*} = \mathfrak{h}_{x,-r}$ for $x \in F^*$ (see [20, Definition 1.5.5]) such that $X \in \mathcal{O}$ and such that, if a nilpotent orbit $\mathcal{O}'$ meets $X + \mathfrak{h}_{F^*}^+$ (where $\mathfrak{h}_{F^*}^+ = \mathfrak{h}_{x,(-r)+}$), then $\mathcal{O} \subset \overline{\mathcal{O}'}$. Furthermore, since $\mathcal{O} \neq \{0\}$, we know that $\mathfrak{h}_{x,-r} \neq \mathfrak{h}_{x,(-r)+}$ and $\mathfrak{h}_{x,r} \neq \mathfrak{h}_{x,r+}$.

Consider the function χ_X in $C_c^\infty(\mathfrak{h}_{x,r}) \subset C_c^\infty(\mathfrak{h}(F))$ that takes $Y \in \mathfrak{h}_{x,r}$ to $\Lambda\circ\operatorname{tr}(-XY)$. Since $\mathfrak{c}$ induces an isomorphism of groups $\mathfrak{h}_{x,r}/\mathfrak{h}_{x,r+} \cong H_{x,r}/H_{x,r+}$ (as [20, Hypothesis 3.2.1] is satisfied), $\chi_X \circ \mathfrak{c}$ is a character of $H_{x,r}$ trivial on $H_{x,r+}$. It is enough to show that $\Theta_\pi(\chi_X \circ \mathfrak{c}) \neq 0$. The Fourier transform of χ_X equals $(\operatorname{meas}\mathfrak{h}_{x,r})\mathbb{1}_{X+\mathfrak{h}_{x,(-r)+}}$. By the previous paragraph, we get

$$\begin{aligned}\Theta_\pi(\chi_X) = \sum_{\mathcal{O}'} c_{\mathcal{O}'}\hat{\nu}_{\mathcal{O}'}(\chi_X) &= (\operatorname{meas}\mathfrak{h}_{x,r}) \cdot \sum_{\mathcal{O}'} c_{\mathcal{O}'}\nu_{\mathcal{O}'}(\mathbb{1}_{X+\mathfrak{h}_{x,(-r)+}}) \\ &= (\operatorname{meas}\mathfrak{h}_{x,r})c_{\mathcal{O}} \cdot \nu_{\mathcal{O}}(X + \mathfrak{h}_{x,(-r)+}) \neq 0\end{aligned}$$

(as $X \in \mathcal{O}$ and as $\mathcal{O}$ is maximal for the property $c_{\mathcal{O}} \neq 0$), as needed. □

11. The other depth bound for generic representations of SO_{2n+1}

In Corollary 10.6.4, we showed a depth bound for all members in an L-packet in terms of the depth of the Langlands parameter. This was under Hypothesis 10.3.3. However, for SO_{2n+1}, we can give a crude estimate of the depth of a generic tempered representation in terms of that of its Langlands parameter using the work of [79] without any assumptions on the residue characteristic. More precisely, we have the following lemma.

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Lemma 11.0.1. *Let F be any non-Archimedean local field of characteristic 0. Let π be an irreducible, admissible, generic, tempered representation of* $\mathrm{SO}_{2n+1}(F)$ *that lies in the L-packet Π_ϕ. Then*

$$\mathrm{depth}(\pi) \leqslant 2n\,\mathrm{depth}(\phi) + 2n.$$

Proof. We reduce ourselves to the case when π is generic supercuspidal using [63, Theorem 4.5] and the Mœglin–Tadic classification [60] (see [16, § 7]). Let us explain why the depth bound for generic supercuspidal π is a consequence of [79]. Let $K(\mathfrak{p}^m)$ be the paramodular subgroup of level m defined in [79, Definition 1.2.1]. Note that $K_m \subset K(\mathfrak{p}^m)$. Let a_π be the conductor of π. Theorem 1.2.5 of [79] says that π has a nonzero fixed vector under $K(\mathfrak{p}^{a_\pi})$, and hence under K_{a_π}. Consequently, $\mathrm{depth}(\pi) \leqslant a_\pi$. By Theorem 12.8.1(b), we have that $a_\pi = \mathrm{cond}(\phi)$, and furthermore, using the relationship between depth and conductor for representations of GL_{2n} (see [86, Theorem 2.3.6.4]), we have that $\mathrm{cond}(\phi) \leqslant 2n\,\mathrm{depth}(\phi) + 2n$. □

12. Recharacterization of the LLC in characteristic 0

In this section, we show how the works of [7, 16, 60] let us give a characterization of the LLC for split classical groups using L-functions, ϵ-factors, and Plancherel measures. To do this, we will first recall some preliminaries about these local factors and Plancherel measures.

12.1. Some notation

From now on, $\underline{G}$ will denote a split connected reductive group defined over $\mathbb{Z}$. Let $\underline{B} = \underline{T}\underline{U}$ be a Borel subgroup of $\underline{G}$ with maximal torus $\underline{T}$ and unipotent radical $\underline{U}$, all defined over $\mathbb{Z}$. Let $X^*(\underline{T})$ (respectively, $X_*(\underline{T})$) be the character lattice (respectively, cocharacter lattice), $\Phi \subset X^*(\underline{T})$ the set of roots of $\underline{T}$ in $\underline{G}$, Φ^+ the set of positive roots of $\underline{T}$ (i.e., in $\underline{B}$) and Δ the set of simple roots. Let $Z(\underline{G})$ denote the center of $\underline{G}$ and $\underline{N}_{\underline{G}}(\underline{T})$ the normalizer of $\underline{T}$ in $\underline{G}$. For $\Omega \subset \Delta$, let $\underline{P}_\Omega = \underline{M}_\Omega \underline{N}_\Omega$ be the Levi decomposition of the corresponding standard parabolic subgroup of $\underline{G}$, where $\underline{M}_\Omega$ is chosen to contain $\underline{T}$.

Let $\underline{A}_\Omega$ be the connected component of $\cap_{\alpha\in\Omega} \mathrm{Ker}\,\alpha$. One has $\underline{M}_\Omega = \mathrm{Cent}_{\underline{G}}(\underline{A}_\Omega)$, and $\underline{A}_\Omega$ is the maximal split torus in the center of $\underline{M}_\Omega$. $\underline{T} = \underline{A}_\emptyset$ is of course a maximal torus in $\underline{M}_\Omega$ as well. For $\Omega \subset \Delta$, let Φ_Ω be the set of roots in the linear span of Ω, $\Phi_\Omega^+ = \Phi^+ \cap \Phi_\Omega$, and let W_Ω be the Weyl group of $\underline{M}_\Omega$ with respect to $\underline{T}$. We write W for W_Δ. There is a natural inclusion $W_\Omega \hookrightarrow W$. For $\alpha \in \Phi^+$, let $s_\alpha \in W$ denote the reflection with respect to the root α. Then $W = \langle s_\alpha | \alpha \in \Delta\rangle$ and $W_\Omega = \langle s_\alpha | \alpha \in \Omega\rangle$. Let $\mathfrak{a}_\Omega^* = X^*(\underline{A}_\Omega) \otimes \mathbb{R}$, and let $\mathfrak{a}_{\Omega,\mathbb{C}}^*$ denote its complexification.

We fix a Chevalley basis $\{\mathbf{u}_\alpha \mid \alpha \in \Phi\}$, where $\mathbf{u}_\alpha : \mathbb{G}_a \to \underline{U}_\alpha$ (here $\underline{U}_\alpha$ denotes the root subgroup) is an isomorphism. For each $\alpha \in \Phi$ there is a $\mathbb{Z}$-homomorphism $\phi_\alpha : \mathrm{SL}_2 \to \underline{G}$ such that $\phi_\alpha\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) = \mathbf{u}_\alpha(t)$ and $\phi_\alpha\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) = \mathbf{u}_{-\alpha}(t)$, and $\phi_\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = \alpha^\vee(t)$. Let $\mathbf{w}_\alpha(t) = \phi_\alpha\left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0 \end{smallmatrix}\right)$ for each $\alpha \in \Phi^+$. Then

$$\mathbf{w}_\alpha(t) = \mathbf{u}_\alpha(t)\mathbf{u}_{-\alpha}(-t^{-1})\mathbf{u}_\alpha(t),$$

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$\mathbf{w}_\alpha(1)$ is a representative of the reflection s_α in $N_G(T)$, and there exist universal signs $\epsilon^{\mathbb{Z}}_{\alpha,\beta} \in \{\pm 1\}$ depending only α and β such that, for $t \in F$,

$$\mathbf{w}_\alpha(1)\mathbf{u}_\beta(t)\mathbf{w}_\alpha(1)^{-1} = \mathbf{u}_{s_\alpha(\beta)}(\epsilon_{\alpha,\beta}t),$$

where $\epsilon_{\alpha,\beta}$ is the corresponding element of $\mathfrak{O}_F$. Now let $w \in W$. Write a minimal decomposition of w as $w = s_{\alpha_1} \cdots s_{\alpha_r}$. We write $\tilde{w} = \tilde{s}_{\alpha_1} \cdots \tilde{s}_{\alpha_r}$ to denote the representative of w in $N_G(T)$. By [77, § 9.3.3], we know that this representative is independent of the choice of the minimal decomposition of w.

Let $G = \underline{G}(F)$. We similarly have B, T, U, and so on. Let $U_{\alpha,\mathfrak{O}_F} := \underline{U}_\alpha(\mathfrak{O}_F)$ and $U_{\alpha,\mathfrak{p}_F^m} = \mathrm{Ker}(\underline{U}_\alpha(\mathfrak{O}_F) \to \underline{U}_\alpha(\mathfrak{O}_F/\mathfrak{p}_F^m))$ for $m \in \mathbb{N}$. Similarly, let $T_{\mathfrak{p}_F^m} = \mathrm{Ker}(\underline{T}(\mathfrak{O}_F) \to \underline{T}(\mathfrak{O}_F/\mathfrak{p}_F^m))$ for $m \in \mathbb{N}$.

12.2. The theory of intertwining operators

12.2.A. Induced representations. For the rest of this section, we will assume $\underline{M}_\Omega$ to be maximal, and let α be the simple root of Δ such that $\Omega = \Delta\backslash\{\alpha\} \subset \Delta$. Let $w_0 = w_{l,\Delta} w_{l,\Omega}$, where $w_{l,\Delta}$ and $w_{l,\Omega}$ are the longest elements of W and W_Ω, respectively, and let $\Psi = w_0(\Omega) \subset \Delta$. Let $\underline{P}_\Psi$ denote the corresponding standard parabolic subgroup with Levi subgroup $\underline{M}_\Psi \supset \underline{T}$ and unipotent radical $\underline{N}_\Psi$. Let (π, V) be an irreducible admissible representation of M_Ω. Let $\mathrm{Unr}(M_\Omega)$ denote the set of unramified characters of M_Ω. For $\eta \in \mathrm{Unr}(M_\Omega)$, let

$$I(\eta, \pi) = \mathrm{ind}_{P_\Omega}^G \pi\eta$$

denote the normalized parabolically induced representation.

Let $w_0(\pi)$ be an irreducible admissible representation of M_Ψ defined by

$$w_0(\pi)(m)(v) = \pi(\tilde{w}_0^{-1} m \tilde{w}_0)(v) \quad \text{for all } v \in V,$$

and let

$$I(w_0(\eta), w_0(\pi)) = \mathrm{ind}_{P_\Psi}^G w_0(\pi\eta).$$

Let $\mathcal{U}_\pi = \{\eta \in \mathrm{Unr}(M_\Omega) | I(\eta, \pi) \text{ is irreducible}\}$. Then $\mathcal{U}_\pi$ is a nonempty Zariski open subset of $\mathrm{Unr}(M_\Omega)$ (cf. [71, Theorem 3.2] and [69, Remark 1.8.6.2]). Define

$$\mathcal{U}_1 = \mathcal{U}_\pi \cap w_0^{-1} \mathcal{U}_{w_0(\pi)}.$$

Then $\mathcal{U}_1$ is a nonempty Zariski open subset of $\mathrm{Unr}(M_\Omega)$ and, for $\eta \in \mathcal{U}_1$,

$$I(\eta, \pi) \quad \text{and} \quad I(w_0(\eta), w_0(\pi)) \text{ are both irreducible.}$$

Let $H_\Omega : M_\Omega \to \mathfrak{a}_\Omega$ be defined by requiring that

$$q^{\langle \lambda, H_\Omega(m)\rangle} = |\lambda(m)|, \quad \forall m \in M_\Omega, \lambda \in X^*(\underline{M}_\Omega).$$

Using the surjection $\mathfrak{a}^*_{\Omega,\mathbb{C}} \to \mathrm{Unr}(M_\Omega)$, $\nu \to \eta_\nu$, where

$$\eta_\nu(m) = q^{\langle \nu, H_\Omega(m)\rangle}, \quad m \in M_\Omega,$$

we write $I(\nu, \pi)$ instead of $I(\eta_\nu, \pi)$. By the above, there exists an open dense subset $\mathcal{V}_1$ of $\mathfrak{a}^*_{\Omega,\mathbb{C}}$ such that, for $\nu \in \mathcal{V}_1$,

$$I(\nu, \pi) \quad \text{and} \quad I(w_0(\nu), w_0(\pi)) \text{ are both irreducible.}$$

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12.2.B. Intertwining operators. We retain the assumptions of § 12.2.A. We shall recall the theory of intertwining integrals. Let $\underline{\bar{N}}_\Omega$ be the unipotent radical of the parabolic subgroup $\underline{\bar{P}}_\Omega$ that is opposite to $\underline{P}_\Omega$ with respect to $\underline{M}_\Omega$ (so $\underline{\bar{P}}_\Omega = \underline{M}_\Omega\underline{\bar{N}}_\Omega$). Since $\underline{P}_\Omega$ is maximal and $w_0 = w_{l,\Delta}w_{l,\Omega}$, a simple calculation shows that

$$\tilde{w}_0\bar{N}_\Omega\tilde{w}_0^{-1} = N_\Psi \subset U.$$

Let ψ be a nontrivial additive character of F. We fix measures du, dn_Ω, and dn_Ψ on U, N_Ω, and N_Ψ, respectively, using the self-dual Haar measure on F induced by ψ and our choice of splittings. We fix measures $d\bar{n}_\Omega$ and $d\bar{n}_\Psi$ on $\bar{N}_\Omega$ and $\bar{N}_\Psi$ by transport of structure. Given $f \in I(\nu, \pi)$, define

$$A(\nu, \pi, \tilde{w}_0)f(g) = A(\eta_\nu, \pi, \tilde{w}_0)f(g) = \int_{N_\Psi} f(\tilde{w}_0^{-1}ng)\, dn_\Psi. \tag{12.2.1}$$

This integral converges absolutely whenever the following condition holds:

$$\langle Re(\nu), \beta^\vee\rangle \gg 0 \quad \text{for each } \beta \in \Phi^+\backslash\Phi^+_\Omega. \tag{12.2.2}$$

For such ν, $A(\nu, \pi, \tilde{w}_0)f \in I(w_0(\nu), w_0(\pi))$. Moreover, this is a meromorphic function of ν, and in fact a rational function of η_ν ([73, § 2] and [81, Theorem IV.I.I]). In fact, Muić [64] explicitly constructed a Zariski open dense subset $\mathcal{U}(\pi, w_0)$ of $\mathrm{Unr}(M_\Omega)$ where the intertwining operator is defined (cf. [64, Lemma 4.6, Remark 4.16 and Theorem 5.6(ii)]); that is,

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(I(\nu, \pi), I(w_0(\nu), w_0(\pi)) = 1. \tag{12.2.3}$$

Let $\mathcal{V}(\pi, w_0)$ denote the corresponding open dense subset of $\mathfrak{a}^*_{\Omega,\mathbb{C}}$ so that $A(\nu, \pi, \tilde{w}_0)$ is defined for all $\nu \in \mathcal{V}(\pi, w_0)$. Let $[W_\Omega\backslash W/W_\Psi] = \{w \in W | w^{-1}\Omega > 0, w\Psi > 0\}$. Then

$$G = \coprod_{w\in[W_\Omega\backslash W/W_\Psi]} P_\Omega w P_\Psi,$$

and there is a total order $\leqslant$ on $[W_\Omega\backslash W/W_\Psi]$ such that, for each $w \in [W_\Omega\backslash W/W_\Psi]$, the set

$$G^{\leqslant w} := \bigcup_{w_1\leqslant w} P_\Omega w_1 P_\Psi$$

is open (cf. [64, § 3]). With $I(\nu, \pi)^{\leqslant w_0^{-1}} = \{f \in I(\nu, \pi) | \operatorname{supp}(f) \subset G^{\leqslant w_0^{-1}}\}$, he showed that for each $\eta_\nu \in \mathcal{U}(\pi, w_0)$ the intertwining operator is determined by (G-equivariance and) the following requirement:

$$A(\nu, \pi, \tilde{w}_0)(f)(1) = \int_{N_\Psi} f(\tilde{w}_0^{-1}n)dn_\Psi, \quad f \in I(\nu, \pi)^{\leqslant w_0^{-1}}. \tag{12.2.4}$$

(see [64, Equation (4.20) and Lemma 4.21]).

12.3. Local coefficients

Given a generic representation (π, V) of M_Ω, there is a Whittaker functional associated to the representation $I(\nu, \pi)$ making the induced representation generic.

More precisely, let $\chi : U \to \mathbb{C}^\times$ be a generic character of U, and let χ_Ω denote its restriction to $U_{M_\Omega} := U \cap M_\Omega$. Assume that χ and $\tilde{w}_0$ are compatible; that is,

$$\chi(\tilde{w}_0 u \tilde{w}_0^{-1}) = \chi(u) \quad \forall u \in U_{M_\Omega},$$

where $\tilde{w}_0$ is the lifting of w_0 as described in § 12.1. Assume that (π, V) is χ_Ω-generic, and let $\lambda : V \to \mathbb{C}$ be a nonzero Whittaker functional on V satisfying

$$\lambda(\pi(u)v) = \chi_\Omega(u)\lambda(v) \quad \forall u \in U_{M_\Omega}, v \in V.$$

With χ as above, the induced representation $I(\nu, \pi)$ is χ-generic (see [73, Proposition 3.1]). Let $\lambda_\chi(\nu, \pi)$ denote the Whittaker functional on the induced space constructed in that Proposition. This is an entire function of ν, and there exists a function $f \in I(\nu, \pi)$ such that $\lambda_\chi(\nu, \pi) f$ is nonzero. In fact, $\lambda_\chi(\nu, \pi)$ is a polynomial in η_ν (see [54, § 1.2]).

The local coefficient arises by studying the intertwining operators between certain parabolically induced generic representations and the uniqueness of Whittaker models of these representations. More precisely, we have the following theorem.

Theorem 12.3.1 [73, Theorem 3.1]. *There exists a complex number $C_\chi(\nu, \pi, \tilde{w}_0)$ such that*

$$\lambda_\chi(\nu, \pi) = C_\chi(\nu, \pi, \tilde{w}_0)\lambda_\chi(w_0(\nu), w_0(\pi)) \circ A(\nu, \pi, \tilde{w}_0). \tag{12.3.1}$$

Furthermore, as a function of ν, it is meromorphic in $\mathfrak{a}^*_{\Omega,\mathbb{C}}$, and its value depends only on the class of π.

The scalar $C_\chi(\nu, \pi, \tilde{w}_0)$ is called the *local coefficient* associated to ν and π. In fact, it can be shown that $C_\chi(\nu, \pi, \tilde{w}_0)$ is a rational function of η_ν (cf. [54, Theorem 2.1]).

12.4. Plancherel measures

For an irreducible admissible representation (π, V) of M_Ω (maximal) and $\nu \in \mathfrak{a}^*_{\Omega,\mathbb{C}}$, we consider the induced representations

$$I(\nu, \pi) \quad \text{and} \quad I(w_0(\nu), w_0(\pi)).$$

There exists a constant $\mu(\nu, \pi, \psi)$ such that

$$A(w_0(\nu), w_0(\pi), \tilde{w}_0^{-1}) \circ A(\nu, \pi, \tilde{w}_0) = \mu(\nu, \pi, \psi)^{-1}. \tag{12.4.1}$$

The scalar $\mu(\nu, \pi, \psi)$ is a meromorphic function of ν, and it is called the Plancherel measure associated to π. Note that this depends on the choice of measures used to define the intertwining operator.

Notation 12.4.1. From now on, the group $\underline{H}$ will be as in cases (a)–(c) of § 2.1, but additionally *split* if in case (c). To ease the presentation of the results in the remaining sections, we write $\underline{H}_n$ instead of $\underline{H}$ in cases (a)–(c) when $\dim(W) = 2n$ or $2n+1$. We accordingly let N denote the dimension of V.

12.5. The split classical groups and their structure theory

We fix a Chevalley model for $\underline{\mathrm{H}}_n$ over $\mathbb{Z}$ as follows. Let

$$J(n) = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$$

and let

$$J'(2n) = \begin{pmatrix} 0 & -J(n) \\ J(n) & 0 \end{pmatrix}.$$

The group Sp_{2n} is the functor on the category of commutative rings

$$R \rightsquigarrow \{g \in \mathrm{GL}_{2n}(R) \mid {}^t g \cdot J'(2n) \cdot g = J'(2n)\}.$$

To define special orthogonal groups in a characteristic free way, let R denote any commutative ring, and let q_n denote the quadratic form on R^n given by

$$q_n = \sum_{i=1}^{n} x_i x_{n+1-i}.$$

The group $\mathrm{O}(q_n)$ is the functor

$$R \rightsquigarrow \{g \in \mathrm{GL}_n(R) \mid q_n(gx) = q_n(x)\}.$$

Then, by definition, $\mathrm{SO}_{2n+1} := \mathrm{Ker}(\det|_{\mathrm{O}(q_{2n+1})})$ and $\mathrm{SO}_{2n} := \mathrm{Ker}(D(q_{2n}))$, where $D(q_{2n})$ denotes the Dickson invariant (see [18, Appendix C]). Furthermore, when R is a $\mathbb{Z}[1/2]$-algebra, we have that

$$\mathrm{SO}_{2n}(R) = \{g \in \mathrm{SL}_n(R) \mid q_{2n}(gx) = q_{2n}(x)\}.$$

With these choices, we have a standard choice of a Borel subgroup and a maximal torus for each $\underline{\mathrm{H}}_n$, specified by the standard ordered basis for R^{2n} or R^{2n+1} depending on the case. Next, we describe the root datum of $\underline{\mathrm{H}}_n$ using standard notation.

(I) For $\underline{\mathrm{H}}_n = \mathrm{Sp}_{2n}$, $\Phi^+ = \{\pm e_i \pm e_j\ (i \neq j),\ \pm 2e_i\}$ and

$$\Delta = \{\alpha_1 := e_1 - e_2,\ \alpha_2 := e_2 - e_3, \ldots,\ \alpha_{n-1} := e_{n-1} - e_n, \alpha_n :=\ 2e_n\}.$$

(II) For $\underline{\mathrm{H}}_n = \mathrm{SO}_{2n+1}$, $\Phi^+ = \{\pm e_i \pm e_j\ (i \neq j), \pm e_i\}$ and

$$\Delta = \{\alpha_1 := e_1 - e_2,\ \alpha_2 := e_2 - e_3, \ldots,\ \alpha_{n-1} := e_{n-1} - e_n,\ \alpha_n := e_n\}.$$

(III) For $\underline{\mathrm{H}}_n = \mathrm{SO}_{2n}$, $\Phi^+ = \{\pm e_i \pm e_j, i \neq j\}$ and

$$\Delta = \{\alpha_1 := e_1 - e_2,\ \alpha_2 := e_2 - e_3, \ldots,\ \alpha_{n-1} := e_{n-1} - e_n,\ \alpha_n := e_{n-1} + e_n\}$$

12.6. The local factors for split classical groups

Let $\underline{\mathrm{P}}_\Omega$ be the 'standard' maximal parabolic subgroup of $\underline{\mathrm{H}}_n$ for a suitable $\Omega \subset \Delta$, with Levi decomposition $\underline{\mathrm{P}}_\Omega = \underline{\mathrm{M}}_\Omega \underline{\mathrm{N}}_\Omega$, where $\underline{\mathrm{N}}_\Omega \subset \underline{\mathrm{U}}$. Note that $\underline{\mathrm{M}}_\Omega \cong \mathrm{GL}_k \times \underline{\mathrm{H}}_t$, where $\underline{\mathrm{H}}_t$ is a smaller rank classical group of the same type and $t + k = n$. Now, consider the adjoint action r of ${}^L\underline{\mathrm{M}}_\Omega$ on ${}^L\mathfrak{n}_\Omega$; cf. [74]. Let ρ_k denote the standard representation of $\mathrm{GL}_k(\mathbb{C})$, and let τ_n denote the natural embedding of $\hat{\underline{\mathrm{H}}}_n$ in $\mathrm{GL}_N(\mathbb{C})$. Then

$$r = r_1 \oplus r_2,$$

where $r_1 = \rho_k \otimes \tau_t$ and

$$r_2 = \begin{cases} \wedge^2 \rho_k & \text{if } \underline{\mathrm{H}}_n(F) = \mathrm{Sp}_{2n}(F),\ \mathrm{SO}_{2n}(F) \\ \mathrm{Sym}^2 \rho_k & \text{if } \underline{\mathrm{H}}_n(F) = \mathrm{SO}_{2n+1}(F). \end{cases}$$

Let σ and π be irreducible admissible generic representations of $\mathrm{GL}_k(F)$ and $\underline{\mathrm{H}}_t(F)$, respectively. The Langlands–Shahidi method defines L- and γ-factors

$$\begin{cases} L(s, \pi \boxtimes \sigma, r_i) \\ \gamma(s, \pi \boxtimes \sigma, r_i, \psi), \quad i = 1, 2 \end{cases}$$

via the theory of local coefficients that satisfies a number of properties (see [74] when $\mathrm{char}\,(F) = 0$ and [36, 54] when $\mathrm{char}\,(F) > 0$). The first L- and γ-factors will be used in the recharacterization in Theorem 12.8.1, and we write

$$L(s, \pi \times \sigma) = L(s, \pi \boxtimes \sigma, r_1), \quad \gamma(s, \pi \times \sigma, \psi) = \gamma(s, \pi \boxtimes \sigma, r_1, \psi).$$

We also note that the second L- and γ-factors are the symmetric and exterior square local factors in the Langlands–Shahidi method for classical groups. The symmetric square factor arises as the local coefficient by viewing GL_k as the standard Seigel Levi subgroup of SO_{2k+1}, and the exterior square factor arises as the local coefficient by considering GL_k as the standard Seigel Levi subgroup of SO_{2k}. We simply write $L(s, \sigma, r_2)$ and $\gamma(s, \sigma, r_2, \psi)$ to denote these factors.

12.7. On discrete series representations of classical groups

Let F be a non-Archimedean local field of characteristic 0. In this section, we recall the main results of [60] and [59]. For $\phi \in \tilde{\Phi}_2(\underline{\mathrm{H}}_n(F))$, we write

$$\mathrm{Jord}(\phi) := \{(\rho, a) | \phi_\rho \otimes S_a \text{ appears in } \phi\}.$$

In the above, ρ denotes an irreducible self-dual supercuspidal representation of $\mathrm{GL}_{d_\rho}(F)$ (for a suitable $d_\rho \in \mathbb{N}$), and S_a is the irreducible a-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.

Now, let π be a discrete series representation of $\underline{\mathrm{H}}_n(F)$. For a supercuspidal representation ρ of $\mathrm{GL}_{d_\rho}(F)$ and $a \in \mathbb{Z}$, let $\delta(\rho, a)$ denote the irreducible essentially square integrable representation of $\mathrm{GL}_{ad_\rho}(F)$ that is the unique irreducible quotient of the induced representation $\rho|-|^{\frac{a-1}{2}} \boxtimes \rho|-|^{\frac{a-3}{2}} \cdots \boxtimes \rho|-|^{\frac{-(a-1)}{2}}$, where $\rho|-|$ stands for $\rho|\det(\cdot)|$. Note that, if ρ is self-dual, then so is $\delta(\rho, a)$. We recall the definition of $\mathrm{Jord}(\pi)$ from [56].

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Definition 12.7.1. Define $\mathrm{Jord}(\pi)$ to be the set of pairs (ρ, a) satisfying the following.

(a) ρ is a unitary, supercuspidal, self-dual representation of $\mathrm{GL}_{d_\rho}(F)$.

(b) a is an integer that is even if $L(s, \rho, r_2)$ has a pole at $s = 0$ and is odd otherwise.

(c) $\delta(\rho, a) \rtimes \pi^+$ is irreducible, where $\pi^+ = \pi$ if $\underline{\mathrm{H}}_n$ is in case (a) or case (b), and π^+ is any irreducible representation of $\mathrm{O}_{2n}(F)$ whose restriction to $\mathrm{SO}_{2n}(F)$ contains π if $\underline{\mathrm{H}}_n$ is in case (c).

Now, it follows from [59, §§ 7.1 and 7.2] (which also refers to [7] for certain groups) that

$$\mathrm{Jord}(\pi) = \mathrm{Jord}(\phi), \quad \text{if } \pi \in \Pi_\phi. \tag{12.7.1}$$

In [56], Mœglin has defined a partially defined function $\epsilon_\pi : \mathrm{Jord}(\pi) \to \{\pm 1\}$, and to each irreducible discrete series representation π of $\underline{\mathrm{H}}_n(F)$, has associated a triple $(\mathrm{Jord}(\pi), \pi_{\mathrm{cusp}}, \Delta_\pi)$. Here π_{cusp} is a supercuspidal representation of a smaller rank classical group of the same type as $\underline{\mathrm{H}}_n(F)$ that is in the cuspidal support of π, and Δ_π is defined via ϵ_π on a certain subset of $\mathrm{Jord}(\pi) \sqcup \mathrm{Jord}(\pi) \times \mathrm{Jord}(\pi)$ (see [60, page 729]). With the notion of admissible triple as in [60], the main results of [56] and [60] prove that the map

$$\pi \to (\mathrm{Jord}(\pi), \pi_{\mathrm{cusp}}, \Delta_\pi)$$

from the set of discrete series representations to the set of admissible triples is bijective.

12.8. The Langlands parameter for tempered representations

Let F be a non-Archimedean local field of characteristic 0. For an irreducible admissible tempered representation (π, V) of H_n, we will often write π^{GL} for its local functorial lift to $\mathrm{GL}(N)$ and ϕ_π for its Langlands parameter, as in § 2.2. We prove the following recharacterization of the LLC in Theorem 2.2.1. We note that, if we did not have any restriction on $\mathsf{depth}(\sigma)$ below, this would essentially follow from the results of Arthur [7] (combined with some lemmas in [16]). However, this restriction on the depth of σ is essential for our main theorem in § 13.6.

Theorem 12.8.1. *For $m \geqslant 1$, let $l(m, N) := Nm + 2N$. Let (π, V) be an irreducible admissible tempered representation of H_n with $\mathsf{depth}(\pi) \leqslant m$. The parameter $\phi_\pi \in \tilde{\Phi}_{\mathrm{bdd}}(H_n)$ satisfies the following properties.*

(a) *If π is a discrete series representation, then ϕ_π does not factor through any proper Levi subgroup of $\hat{\underline{\mathrm{H}}}_n$.*

(b) *If π is generic, then, for each irreducible admissible supercuspidal representation σ of $\mathrm{GL}_r(F)$, where $r \leqslant N - 1$ and $\mathsf{depth}(\sigma) \leqslant 2l(m, N)$, we have*

$$L(s, \pi \times \sigma) = L(s, \phi_\pi \otimes \phi_\sigma)$$
$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi).$$

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(c) *If π is nongeneric, then, for each irreducible admissible discrete series representation σ of $\mathrm{GL}_r(F)$, where $r \leqslant N-1$, and* $\mathrm{depth}(\sigma) \leqslant m+1$,

$$\mu(s,\pi\times\sigma,\psi) \\ = \gamma(s,\phi_\pi\otimes\phi_\sigma,\psi)\gamma(-s,\phi_\pi\otimes\phi_\sigma^\vee,\bar{\psi})\gamma(2s,r_2\circ\phi_\sigma,\psi)\gamma(-2s,r_2\circ\phi_\sigma^\vee,\bar{\psi}),$$

where $\mu(s,\pi\times\sigma,\psi)$ is as in equation (12.4.1).

Now suppose that π is a discrete series representation. When $\underline{\mathrm{H}}_n = \mathrm{Sp}_{2n}$ and SO_{2n+1}, the parameter ϕ_π is uniquely determined by these properties, and when $\underline{\mathrm{H}}_n = \mathrm{SO}_{2n}$, the parameter ϕ_π is determined up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy.

Proof. We begin by showing that the parameter ϕ_π satisfies Properties (a)–(c). (a) follows from the construction of [7] (see Proposition 2.4.3 of [7]). We next explain why Properties (b) and (c) hold without any restriction on the depth of σ. Property (b) is basically a consequence of [16, Lemma 7.1, Propositions 7.2 and 7.3]. Let us elaborate on this. First, suppose that π is generic supercuspidal. Then we know that π occurs as a local component of a globally generic cuspidal automorphic representation by [74, Lemma 5.1]. More precisely, suppose that χ is a generic character of U such that π is a χ-generic supercuspidal representation of H_n; then there exist a number field K, a generic character $\dot{\chi}$ of $\underline{\mathrm{U}}(K)\backslash\underline{\mathrm{U}}(\mathbb{A}_K)$, and a globally $\dot{\chi}$-generic cuspidal representation $\dot{\pi} = \otimes'\dot{\pi}_v$ of $\underline{\mathrm{H}}_n(\mathbb{A}_K)$ such that the following hold.

- $K_{v_0} = F$ for some place v_0 of K.
- $\dot{\chi}_{v_0} = \chi$.
- $\dot{\pi}_{v_0} = \pi$.
- For every other finite place $v \neq v_0$, $\dot{\pi}_v$ is unramified.

For such a $\dot{\pi}$, the authors of [16] construct a global functorial lift $\dot{\Pi}$ to $\mathrm{GL}_N(\mathbb{A}_K)$ that agrees with the Satake classification [70] at all the unramified places and the arithmetic Langlands classification [10, 47] at the Archimedean places. The works of [31, 76] characterize the global image of globally generic cuspidal automorphic representations of classical groups (also see [16, § 7.1]). Combining this with the strong multiplicity one theorem for isobaric representations of GL_N [39, 40], we know that this has to agree with the global functorial lift of $\dot{\pi}$ constructed by Arthur. In particular, we have that $\dot{\Pi}_{v_0} \cong \pi^{\mathrm{GL}}$. Now [16, Proposition 7.2] gives that

$$\gamma(s,\pi\times\sigma,\psi) = \gamma(s,\pi^{\mathrm{GL}}\times\sigma,\psi)$$

and the proof of Lemma 7.1 of [16] gives that

$$L(s,\pi\times\sigma) = L(s,\pi^{\mathrm{GL}}\times\sigma).$$

This finishes the proof of (b) for generic supercuspidal π. Next, the local functorial lift of a generic discrete series representation π is obtained in [16] using the Mœglin–Tadic classification of discrete series representations (see [16, Equations (7.2) and (7.3)]). It now follows that (b) holds for any generic discrete series representation π by the results of [59, § 7] and [16, Proposition 7.3]. If π is tempered, then, for a Levi subgroup

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$\underline{\mathrm{M}} = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k} \times \underline{\mathrm{H}}_t$ of $\underline{\mathrm{H}}_n$ and a discrete series representation $\sigma = \delta(\rho_1, a_1) \boxtimes \delta(\rho_2, a_2) \boxtimes \cdots \boxtimes \delta(\rho_k, a_k) \boxtimes \pi_-$ of M, π occurs as an irreducible summand of $\mathrm{Ind}_M^{H_n} \sigma$. Then $\phi_\pi = \bigoplus_{i=1}^k (\phi_{\rho_i} \otimes S_{a_i}) \oplus \phi_{\pi_-} \oplus \bigoplus_{i=1}^k (\phi_{\rho_i} \otimes S_{a_i})^\vee$, where ϕ_{π_-} is the discrete series parameter of π_- as above. Now the fact that (b) holds for generic tempered π is a consequence of the multiplicativity property of the local factors (see [16, Page 209]).

Let π be nongeneric. The properties of Arthur's normalization of intertwining operators (see [7, Ch. 2.3]) imply that the Plancherel measure $\mu(s, \pi \times \sigma, \psi)$ is constant on the L-packet. Let us explain this briefly. With $A(s, \pi \times \sigma, \tilde{w}_0)$ as in § 12.2.B, let

$$R(s, \pi \times \sigma, w_0) = r(s, \pi^{\mathrm{GL}}, \sigma, w_0)^{-1} A(s, \pi \times \sigma, \tilde{w}_0)$$

be the normalized intertwining operator in [7, Equation (2.3.25) and (2.3.26)], with the normalizing factor $r(s, \pi^{\mathrm{GL}}, \sigma, w_0)$ defined in [7, Equation (2.3.27)]. Then, by the cocycle relation in [7, Equation (2.3.28)], we see that

$$R(w_0(s), w_0(\pi \times \sigma), w_0^{-1}) \circ R(s, \pi \times \sigma, w_0) = 1.$$

Since the normalizing factor $r(s, \pi^{\mathrm{GL}}, \sigma, w_0)$ depends on π^{GL} and not on the specific member π of the L-packet, we see that the Plancherel measure $\mu(s, \pi \times \sigma, \psi)$ in the right side of equation (12.4.1) is constant on the L-packet. Writing this more explicitly in terms of the normalizing factors,

$$\begin{aligned}\mu(s, \pi \times \sigma, \psi) &= \gamma(s, \pi^{\mathrm{GL}} \times \sigma, r_1, \psi)\gamma(-s, \pi^{\mathrm{GL}} \times \sigma, r_1^\vee, \bar{\psi})\gamma(2s, \sigma, r_2, \psi)\gamma(-2s, \sigma, r_2^\vee, \bar{\psi}) \\ &= \gamma(s, \pi^{\mathrm{GL}} \times \sigma, r_1, \psi)\gamma(-s, \pi^{\mathrm{GL}} \times \sigma, r_1^\vee, \bar{\psi}) \frac{\gamma(2s, \sigma, r_2, \psi)}{\gamma(1-2s, \sigma, r_2, \psi)} \\ &= \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi)\gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \bar{\psi}) \frac{\alpha.\gamma(2s, r_2 \circ \phi_\sigma, \psi)}{\alpha.\gamma(1-2s, r_2 \circ \phi_\sigma, \psi)} \\ &= \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi)\gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \bar{\psi})\gamma(2s, r_2 \circ \phi_\sigma, \psi)\gamma(-2s, r_2 \circ \phi_\sigma^\vee, \bar{\psi}).\end{aligned}$$

In the above, the second equality follows from [74, Equation (3.10)], the third holds as a consequence of [35] for the factors involving r_2 and because the LLC for GL_N preserves Rankin–Selberg factors, and the fourth is again a consequence of [74, Equation (3.10)] (In fact, by recent work of [17], we can obtain the fourth equality directly from the first.) This completes the proof of (c).

Finally, we let π be a discrete series representation, and proceed to show that ϕ_π is uniquely characterized by Properties (a)–(c). Let $\phi : \mathrm{WD}_F \to \hat{\underline{\mathrm{H}}}_n$ be another parameter attached to π that satisfies Properties (a)–(c) of the theorem. We want to show that $\phi \cong \phi_\pi$. Composing with the standard embedding $\tau_n : \hat{\underline{\mathrm{H}}}_n \to \mathrm{GL}_N(\mathbb{C})$, we will first show that ϕ and ϕ_π are isomorphic as N-dimensional representations.

Suppose that π is generic with $\mathsf{depth}(\pi) \leqslant m$. Property (b) implies that, for each irreducible, admissible, supercuspidal representation σ of $\mathrm{GL}_r(F)$, where $r \leqslant N-1$ and $\mathsf{depth}(\sigma) \leqslant 2l(m, N)$,

$$\begin{aligned} L(s, \phi_\pi \otimes \phi_\sigma) &= L(s, \phi \otimes \phi_\sigma) \\ \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi) &= \gamma(s, \phi \otimes \phi_\sigma, \psi). \end{aligned} \tag{12.8.1}$$

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Note that, had there not been any restriction on the depth of σ, then the above would imply that $\phi \cong \phi_\pi$ by [34, Theorems 1.4 and 1.7]. However, this restriction on the depth of σ is essential for our main theorem. For $\Phi = \phi_\pi$ or ϕ, let $\mathrm{Jord}(\Phi)$ be as in § 12.7. Let us now make some simple observations.

(1) Note that (12.8.1) also implies the equality of the corresponding ϵ-factors, and, in particular, we have

$$\epsilon(s, \phi_\pi, \psi) = \epsilon(s, \phi, \psi). \tag{12.8.2}$$

This implies that $\mathrm{cond}(\phi) = \mathrm{cond}(\phi_\pi)$.

(2) If ϕ and ϕ_π are both irreducible representations of W_F, then we have

$$\mathrm{depth}(\phi) = \frac{\mathrm{cond}(\phi) - N}{N} = \frac{\mathrm{cond}(\phi_\pi) - N}{N} = \mathrm{depth}(\phi_\pi).$$

by [86, Theorem 2.3.6.4]. Even without the irreducibility assumptions, since $\mathrm{depth}(\phi_\pi) \leqslant m + 1$ by Lemma 8.2.3, it follows that

$$\mathrm{depth}(\phi) \leqslant \mathrm{cond}(\phi) = \mathrm{cond}(\phi_\pi) \leqslant N\,\mathrm{depth}(\phi_\pi) + N \leqslant Nm + 2N.$$

(3) For $(\rho, a) \in \mathrm{Jord}(\Phi)$, we have that $\det(\rho)$ takes values in $\{\pm 1\}$ by Property (a) of the theorem.

To prove that $\phi = \phi_\pi$, first suppose that ϕ and ϕ_π are irreducible representations of W_F. Since ϕ and ϕ_π factor through $\hat{\underline{\mathrm{H}}}_n$, we see that $\det(\phi) = \det(\phi_\pi) = 1$. Moreover, we have by [34, § 3.3] that

$$L(s, \phi_\pi \otimes \phi_\sigma) = 1 = L(s, \phi \otimes \phi_\sigma)$$

and

$$\epsilon(s, \phi_\pi \otimes \phi_\sigma, \psi) = \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi) = \gamma(s, \phi \otimes \phi_\sigma, \psi) = \epsilon(s, \phi \otimes \phi_\sigma, \psi)$$

for each irreducible admissible supercuspidal representation σ of $\mathrm{GL}_r(F)$, where $r \leqslant N - 1$ and $\mathrm{depth}(\sigma) \leqslant 2l(m, N)$. Now, using [29, Theorem 7.5], we obtain that $\phi \cong \phi_\pi$, as desired.

Now, we drop the assumption that ϕ and ϕ_π are irreducible representations of W_F. Without loss of generality, let us assume that ϕ is not an irreducible representation of W_F. Then, for each $(\rho, a) \in \mathrm{Jord}(\phi)$, we have $\dim(\phi_\rho) < N$. Since $\phi_\rho \otimes S_a$ appears in ϕ, we see that $\mathrm{depth}(\phi_\rho) \leqslant Nm + 2N$. Furthermore, $L(s, \phi \otimes \phi_\rho^\vee)$ has a pole at $s = -\frac{a-1}{2}$. By (b), we see that $L(s, \phi_\pi \otimes \phi_\rho^\vee)$ also has a pole at $s = -\frac{a-1}{2}$. Then, for some irreducible summand $\phi_{\rho_1} \otimes S_{a_1}$ of ϕ_π, we get that $L(s + \frac{a_1-1}{2}, \phi_{\rho_1} \otimes \phi_\rho^\vee)$ has a unique simple pole at $s = -\frac{a-1}{2}$, or, in other words, $L(s, \phi_{\rho_1} \otimes \phi_\rho^\vee)$ has a pole at $s = \frac{a_1-1}{2} - \frac{a-1}{2}$. Since ϕ_{ρ_1} and ϕ_ρ are both irreducible representations of W_F, it follows by [34, § 3.3] that

$$\phi_{\rho_1} \cong \phi_\rho |-|^{-\left(\frac{a_1-1}{2} - \frac{a-1}{2}\right)}.$$

Now, by (3), we obtain that $a_1 = a$ and $\phi_{\rho_1} \cong \phi_\rho$. Hence $(\rho, a) \in \mathrm{Jord}(\phi_\pi)$. We have shown that $\mathrm{Jord}(\phi) \subset \mathrm{Jord}(\phi_\pi)$, and we obtain equality by dimension considerations.

Finally, suppose that π is nongeneric. We will show that $\mathrm{Jord}(\phi_\pi) \subset \mathrm{Jord}(\phi)$. We first note that, in this case, ϕ_π necessarily reduces. To see this, note that, if ϕ_π is irreducible, then the component group $\mathcal{S}_{\phi_\pi}$ is trivial by Schur's lemma. This implies that $\tilde{\Pi}_{\phi_\pi}$ is a singleton, and [7, Proposition 8.3.2] forces π to be generic. Let $(\rho, a) \in \mathrm{Jord}(\phi_\pi)$. Then $\mathrm{depth}(\rho) \leqslant m+1$, and $\delta(\rho, a)$ is a representation of $\mathrm{GL}_r(F)$ with $r \leqslant N-1$. By Property (c), we have that

$$\begin{aligned}\gamma(s, \phi_\pi \otimes (\phi_\rho \otimes S_a)^\vee, \psi)\gamma(-s, \phi_\pi \otimes (\phi_\rho \otimes S_a), \bar{\psi})\\ = \gamma(s, \phi \otimes (\phi_\rho \otimes S_a)^\vee, \psi)\gamma(-s, \phi \otimes (\phi_\rho \otimes S_a), \bar{\psi}).\end{aligned}$$

Note that the left side in terms of L-functions and ϵ-factors becomes

$$(\epsilon\text{-factors}) \cdot \frac{L(1-s, \phi_\pi \otimes (\phi_\rho \otimes S_a))L(1+s, \phi_\pi \otimes (\phi_\rho \otimes S_a)^\vee)}{L(s, \phi_\pi \otimes (\phi_\rho \otimes S_a)^\vee)L(-s, \phi_\pi \otimes (\phi_\rho \otimes S_a))}.$$

Since $(\rho, a) \in \mathrm{Jord}(\phi_\pi)$, we have that $L(s, \phi_\pi \otimes (\phi_\rho \otimes S_a)^\vee)$ has a pole at $s=0$. This implies that $\gamma(s, \phi_\pi \otimes (\phi_\rho \otimes S_a)^\vee, \psi)\gamma(-s, \phi_\pi \otimes (\phi_\rho \otimes S_a), \bar{\psi})$ has a zero at $s=0$, and, consequently, we obtain that $L(s, \phi \otimes (\phi_\rho \otimes S_a)^\vee)L(-s, \phi \otimes (\phi_\rho \otimes S_a))$ has a pole at $s=0$. Suppose that $L(s, \phi \otimes (\phi_\rho \otimes S_a)^\vee)$ has a pole at $s=0$. Then, for some irreducible summand $\phi_{\rho_1} \otimes S_{a_1}$ of ϕ, we have that

$$L(s, (\phi_{\rho_1} \otimes S_{a_1}) \otimes (\phi_\rho \otimes S_a)^\vee) = \prod_{j=0}^{\min(a_1,a)-1} L\left(s + \frac{a_1+a-2j-2}{2}, \phi_{\rho_1} \otimes \phi_\rho^\vee\right)$$

has a pole at $s=0$. Let $j_0, 0 \leqslant j_0 \leqslant \min(a_1, a)-1$ be such that $L(s+\frac{a_1+a-2j_0-2}{2}, \phi_{\rho_1} \otimes \phi_\rho^\vee)$ has a pole at $s=0$. By [34, § 3.3], we have

$$\phi_{\rho_1} \cong \phi_\rho|-|^{-(\frac{a_1+a-2j_0-2}{2})}.$$

Using (3), we have

$$\frac{a_1+a-2j_0-2}{2} = 0 \quad \text{and} \quad \phi_{\rho_1} \cong \phi_\rho.$$

But $\frac{a_1+a}{2} = j_0+1$ for some j_0 with $0 \leqslant j_0 \leqslant \min(a_1, a)-1$ implies that $a_1 = a$. Hence $\phi_{\rho_1} \otimes S_{a_1} \cong \phi_\rho \otimes S_a$. The case where $L(-s, \phi \otimes (\phi_\rho \otimes S_a))$ has a pole at $s=0$ is similar, and we omit the details. We have proved that $\mathrm{Jord}(\phi_\pi) \subset \mathrm{Jord}(\phi)$. Now, since both ϕ and ϕ_π are discrete parameters, we have that

$$\sum_{(\rho,a)\in \mathrm{Jord}(\phi)} a d_\rho = N = \sum_{(\rho,a)\in \mathrm{Jord}(\phi_\pi)} a d_\rho.$$

Consequently, we obtain that $\phi \cong \phi_\pi$ as N-dimensional representations. It is well known that this gives the equivalence of ϕ and ϕ_π in $\tilde{\Phi}_2(H_n)$ (see, for example, [25, Theorem 8.1]). □

13. The main theorem

In this section, we use the Deligne–Kazhdan theory to attach a Langlands parameter $\phi_{\pi'}$ to π', where π' is a discrete series representation of $\underline{\mathrm{H}}_n(F')$, F' being a local field of odd positive characteristic. We then prove that it is uniquely characterized by a

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list of properties, analogous to Theorem 12.8.1. Let us begin by briefly reviewing the Deligne–Kazhdan theory.

Recall that two non-Archimedean local fields F and F' are *m-close* if $\mathfrak{O}_F/\mathfrak{p}_F^m \cong \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ as rings. For example, the fields $\mathbb{F}_p((t))$ and $\mathbb{Q}_p(p^{1/m})$ are m-close.

A non-Archimedean local field of characteristic p can be viewed as a limit of non-Archimedean local fields of characteristic 0. More precisely, given a local field F' of characteristic p and an integer $m \geqslant 1$, we can always find a local field F of characteristic 0 such that F' is m-close to F. We just have to choose the field F to be ramified enough.

Notation 13.0.1. From now on, for an object X associated to the field F, we will write X' to denote the analogous object over F'.

13.1. Deligne's theory

Let $m \geqslant 1$. Let $\bar{F}$ be a separable closure of F. Let I_F be the inertia group of F and I_F^m its mth higher ramification subgroup with upper numbering (cf. [72, Ch. IV]). Let us summarize the results of Deligne [22] that will be used later in this work. Deligne considered the triplet $\mathrm{tr}_m(F) = (\mathfrak{O}_F/\mathfrak{p}_F^m, \mathfrak{p}_F/\mathfrak{p}_F^{m+1}, \epsilon)$, where ϵ is the natural projection of $\mathfrak{p}_F/\mathfrak{p}_F^{m+1}$ on $\mathfrak{p}_F/\mathfrak{p}_F^m$, and proved that

$$\mathrm{Gal}(\bar{F}/F)/I_F^m,$$

together with its upper numbering filtration, is canonically determined by $\mathrm{tr}_m(F)$. If the fields F and F' are m-close, an isomorphism of triplets $\mathrm{tr}_m(F) \xrightarrow{\mathrm{cl}_m} \mathrm{tr}_m(F')$ gives an isomorphism

$$\mathrm{Gal}(\bar{F}/F)/I_F^m \xrightarrow{\mathrm{Del}_m} \mathrm{Gal}(\bar{F}'/F')/I_{F'}^m, \tag{13.1.1}$$

which is unique up to inner automorphisms [22, Equation (3.5.1)]. Here is a partial description of the map [22, § 1.3]. Let L be a finite totally ramified Galois extension of F satisfying $I(L/F)^m = 1$ (here $I(L/F)$ is the inertia group of L/F). Then $L = F(\alpha)$, where α is a root of an Eisenstein polynomial

$$P(x) = x^n + \pi \sum a_i x^i$$

for $a_i \in \mathfrak{O}_F$. Let $a_i' \in \mathfrak{O}_{F'}$ be such that $a_i \mod \mathfrak{p}_F^m \mapsto a_i' \mod \mathfrak{p}_{F'}^m$. So a_i' is well defined $\mod \mathfrak{p}_{F'}^m$. Then the corresponding extension L'/F' can be obtained as $L' = F'(\alpha')$, where α' is a root of the polynomial

$$P'(x) = x^n + \pi' \sum a_i' x^i,$$

where $\pi \mod \mathfrak{p}_F^m \to \pi' \mod \mathfrak{p}_{F'}^m$. The assumption that $I(L/F)^m = 1$ ensures that the extension L' does not depend on the choice of a_i' [22, Remark A.6.3 and A.6.4].

Henceforth, whenever we talk of F and F' as m-close, we will implicitly assume that an isomorphism $\mathrm{cl}_m : \mathrm{tr}_m(F) \to \mathrm{tr}_m(F')$ has been fixed. Our final results will be independent of our choices of close local fields, and in particular independent of this choice of the isomorphisms cl_m.

Deligne proved some very interesting properties of the map Del_m, which we list below.

(i) Note that, when the fields F and F' are m-close, cl_m determines an isomorphism $F^\times/(1+\mathfrak{p}_F^m) \xrightarrow{\mathrm{cl}_m} F'^\times/(1+\mathfrak{p}_{F'}^m)$. Also, the map Del_m naturally induces an isomorphism between the abelianizations of the corresponding Galois groups. These isomorphisms commute with local class field theory (LCFT), that is, the diagram

$$\begin{array}{ccc} (\mathrm{Gal}(\bar{F}/F)/I_F^m)^{ab} & \xrightarrow{\mathrm{Del}_m} & (\mathrm{Gal}(\bar{F}'/F')/I_{F'}^m)^{ab} \\ \mathrm{LCFT}\downarrow & & \downarrow\mathrm{LCFT} \\ (F^\times/(1+\mathfrak{p}_F^m))\widehat{\ } & \xrightarrow[\mathrm{cl}_m]{} & (F'^\times/(1+\mathfrak{p}_{F'}^m))\widehat{\ } \end{array} \tag{13.1.2}$$

is commutative, where $\widehat{\ }$ denotes profinite completion [22, Proposition 3.6.1].

(ii) The obvious variants of the above properties hold when $\mathrm{Gal}(\bar{F}/F)$ is replaced by W_F, the Weil group of F, or more generally the Weil–Deligne group of F (see [22, § 3.7]).

(iii) Note that the isomorphism Del_m induces a bijection

$$\begin{aligned} &\{\text{Isomorphism classes of representations of } \mathrm{Gal}(\bar{F}/F) \text{ trivial on } I_F^m\} \\ &\longleftrightarrow \{\text{Isomorphism classes of representations of } \mathrm{Gal}(\bar{F}'/F') \text{ trivial on } I_{F'}^m\}. \end{aligned} \tag{13.1.3}$$

Recall that, for a homomorphism $\phi : \mathrm{WD}_F \to {}^LG$,

$$\mathrm{depth}(\phi) = \inf_r\{r \mid \phi|_{I_F^{r+}} = 1\},$$

where the filtration is the upper numbering filtration of the inertia group I_F. So if $\mathrm{depth}(\phi) < m$ then $\phi|_{I_F^m} = 1$. Hence, when the fields F and F' are m-close, the Deligne isomorphism also induces a bijection

$$\begin{aligned} &\{\text{Homomorphisms } \phi : \mathrm{WD}_F \to {}^LG \text{ with } \mathrm{depth}(\phi) < m\} \\ &\longleftrightarrow \{\text{Homomorphisms } \phi' : \mathrm{WD}_{F'} \to {}^LG \text{ with } \mathrm{depth}(\phi') < m\}, \end{aligned} \tag{13.1.4}$$

and we write $\phi \sim_m \phi'$ for homomorphisms that correspond as above.

(iv) Let ψ be a nontrivial additive character of F and $k = \mathrm{cond}(\psi)$. Let ψ' be a character of F' that satisfies the following conditions:

- $\mathrm{cond}(\psi') = k$,
- $\psi'|_{\mathfrak{p}_{F'}^{k-m}/\mathfrak{p}_{F'}^k} = \psi|_{\mathfrak{p}_F^{k-m}/\mathfrak{p}_F^k}$ as in [22, § 3.7].

When ψ and ψ' are related in this manner, we write $\psi' \sim_m \psi$. Let $\phi \sim_m \phi'$ as in (13.1.4). Then their Artin L- and ϵ-factors remain the same; that is,

$$\begin{aligned} L(s,\phi) &= L(s,\phi'), \\ \epsilon(s,\phi,\psi) &= \epsilon(s,\phi',\psi'). \end{aligned} \tag{13.1.5}$$

This is [22, Proposition 3.7.1].

13.2. Kazhdan's theory

Let us recall the results of [41]. Let $\underline{G}$ be as in § 12.1. Let $K_m = \mathrm{Ker}(\underline{G}(\mathfrak{O}_F) \to \underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m))$ be the *mth* usual congruence subgroup of G. Fix a Haar measure dg on G. Let

$$t_x = \mathrm{vol}(K_m; dg)^{-1}\, \mathrm{char}(K_m x K_m),$$

where $\mathrm{char}(K_m x K_m)$ denotes the characteristic function of the coset $K_m x K_m$. The set $\{t_x | x \in G\}$ forms a $\mathbb{C}$-basis of the Hecke algebra $\mathcal{H}(G, K_m)$ (of compactly supported K_m-biinvariant complex-valued functions on G). Let

$$X_*(\underline{T})_- = \{\lambda \in X_*(\underline{T}) \mid \langle \alpha, \lambda \rangle \leqslant 0\ \forall\, \alpha \in \Phi^+\}.$$

Let $\varpi_\lambda = \lambda(\varpi)$ for $\lambda \in X_*(\underline{T})_-$. Consider the Cartan decomposition of G:

$$G = \coprod_{\lambda \in X_*(\underline{T})_-} \underline{G}(\mathfrak{O}_F)\varpi_\lambda \underline{G}(\mathfrak{O}_F).$$

The set $\underline{G}(\mathfrak{O}_F)\varpi_\lambda\underline{G}(\mathfrak{O}_F)$ is a homogeneous space of the group $\underline{G}(\mathfrak{O}_F) \times \underline{G}(\mathfrak{O}_F)$ under the action $(a, b).g = agb^{-1}$. The set $\{K_m x K_m | x \in \underline{G}(\mathfrak{O}_F)\varpi_\lambda\underline{G}(\mathfrak{O}_F)\}$ is then a homogeneous space of the finite group $\underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m) \times \underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m)$. Let $\Gamma_\lambda \subset \underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m) \times \underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m)$ be the stabilizer of the double coset $K_m\varpi_\lambda K_m$. Kazhdan observed that the obvious isomorphism $\underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m) \times \underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m) \to \underline{G}(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m) \times \underline{G}(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m)$ (such an isomorphism would exist if the fields are m-close) maps $\Gamma_\lambda \to \Gamma'_\lambda$, where Γ'_λ is the corresponding object for F'. Let $T_\lambda \subset \underline{G}(\mathfrak{O}_F) \times \underline{G}(\mathfrak{O}_F)$ be a set of representatives of $(\underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m) \times \underline{G}(\mathfrak{O}_F/\mathfrak{p}_F^m))/\Gamma_\lambda$. Similarly define T'_λ. Then we have a bijection $T_\lambda \to T'_\lambda$. Kazhdan constructed an isomorphism of $\mathbb{C}$-vector spaces

$$\mathcal{H}(G, K_m) \xrightarrow{\mathrm{Kaz}_m} \mathcal{H}(G', K'_m)$$

by requiring that

$$t_{a_i \pi_\lambda a_j^{-1}} \mapsto t_{a'_i \pi'_\lambda a'^{-1}_j}$$

for all $\lambda \in X_*(\underline{T})_-$ and $(a_i, a_j) \in T_\lambda$, where (a'_i, a'_j) is the image of (a_i, a_j) under the bijection $T_\lambda \to T'_\lambda$. He then proved the following theorem.

Theorem 13.2.1 [41, Theorem A]. *Given $m \geqslant 1$, there exists $l \geqslant m$ such that, if F and F' are l-close, the map* Kaz_m *constructed above is an algebra isomorphism.*

An irreducible admissible representation (π, V) of G such that $\pi^{K_m} \neq 0$ naturally becomes an $\mathcal{H}(G, K_m)$-module. Hence, if the fields F and F' are sufficiently close, Kaz_m gives a bijection

$$\begin{aligned} &\{\text{Iso. classes of irr. ad. representations } (\pi, V) \text{ of } G \text{ with } \pi^{K_m} \neq 0\} \\ &\longleftrightarrow \{\text{Iso. classes of irr. ad. representations } (\pi', V') \text{ of } G' \text{ with } \pi'^{K'_m} \neq 0\}. \qquad (13.2.1)\end{aligned}$$

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13.3. A variant of the Kazhdan isomorphism

A useful variant of the Kazhdan isomorphism is now available for split reductive groups. Let I be the standard Iwahori subgroup of G, defined as the inverse image under $\underline{G}(\mathfrak{O}_F) \to \underline{G}(\mathfrak{O}_F/\mathfrak{p}_F)$ of $\underline{B}(\mathfrak{O}_F/\mathfrak{p}_F)$. By [78, Ch. 3], there is a smooth affine group scheme $\underline{I}$ defined over $\mathfrak{O}_F$ with generic fiber $\underline{G} \times_{\mathbb{Z}} F$ such that $\underline{I}(\mathfrak{O}_F) = I$. Define $I_m := \mathrm{Ker}(\underline{I}(\mathfrak{O}_F) \to \underline{I}(\mathfrak{O}_F/\mathfrak{p}_F^m))$. Explicitly, $I = \langle U_{\alpha,\mathfrak{O}_F}, \underline{T}(\mathfrak{O}_F), U_{-\alpha,\mathfrak{p}_F} \mid \alpha \in \Phi^+\rangle$ and $I_m = \langle U_{\alpha,\mathfrak{p}_F^m}, T_{\mathfrak{p}_F^m}, U_{-\alpha,\mathfrak{p}_F^{m+1}} \mid \alpha \in \Phi^+\rangle$. Let $W_a = N_G(T)/\underline{T}(\mathfrak{O}_F)$ denote the extended affine Weyl group of $\underline{G}$. Via the isomorphisms $W \cong N_{\underline{G}(\mathfrak{O}_F)}(T)/\underline{T}(\mathfrak{O}_F)$ and $X_*(\underline{T}) \cong T/\underline{T}(\mathfrak{O}_F)$, we can realize these groups inside W_a, and in fact $W_a \cong X_*(\underline{T}) \rtimes W$, where W acts on $X_*(\underline{T})$ in the obvious way.

Let A be a set of representatives for W_a in $N_G(T)$. Then, by [38, Theorem 2.16], we have that $G = IAI$. Hence $G = \bigcup_{w\in A, x,y\in I} I_m xwy I_m$. Fix a Haar measure dg on G such that $\mathrm{vol}(I_m; dg) = 1$. For $g \in G$, let f_g denote the characteristic function of the double coset $I_m g I_m$. Then, using the above decomposition, we see that the set $\{f_{xwy} \mid w \in A \text{ and } x, y \in I\}$ is a $\mathbb{C}$-basis for the Hecke algebra $\mathcal{H}(G, I_m)$. In [29, § 3], a presentation has been written down for this Hecke algebra $\mathcal{H}(G, I_m)$ (extending [37, Theorem 2.1] for GL_n). Furthermore, if the fields F and F' are m-close, then [29, § 3.4.A] gives an isomorphism

$$\beta : I/I_m \to I'/I'_m.$$

If under this isomorphism

$$b \bmod I_m \mapsto b' \bmod I'_m, \tag{13.3.1}$$

we will also write $b \sim_\beta b'$. Using the presentation and this isomorphism, one gets an obvious map

$$\zeta_m : \mathcal{H}(G, I_m) \to \mathcal{H}(G', I'_m),$$

when the fields F and F' are m-close (also see [49] for GL_n), which was shown in [29] to be an isomorphism of rings. Hence we obtain a bijection

$$\begin{aligned} &\{\text{Iso. classes of irr. ad. representations } (\pi, V) \text{ of } G \text{ with } \pi^{I_m} \neq 0\} \\ &\longleftrightarrow \{\text{Iso. classes of irr. ad. representations } (\pi', V') \text{ of } G' \text{ with } \pi'^{I'_m} \neq 0\}. \end{aligned} \tag{13.3.2}$$

In fact, more is true. Let $\mathfrak{R}(G)$ be the category of smooth complex representations of G. Let $\mathfrak{R}^m(G)$ be the subcategory of $\mathfrak{R}(G)$ of representations (π, V) of G generated by their I_m-fixed vectors, that is, such that the $\mathbb{C}$-linear span of $\pi(G)(V^{I_m})$ equals V; i.e., $\pi(\mathcal{H}(G))(V^{I_m}) = V$, where $\mathcal{H}(G)$ denotes the Hecke algebra of compactly supported locally constant complex-valued functions on G. Let $\mathcal{H}(G, I_m)$-mod be the category of $\mathcal{H}(G, I_m)$-modules. Then the category $\mathfrak{R}^m(G)$ is closed under subquotients, and the functor

$$\begin{aligned} J_m : \mathfrak{R}^m(G) &\longrightarrow \mathcal{H}(G, I_m)\text{-mod}, \\ (\pi, V) &\longrightarrow V^{I_m}, \end{aligned}$$

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is an equivalence of categories with left adjoint

$$j_m : \mathcal{H}(G, I_m)\text{-mod} \longrightarrow \mathfrak{R}^m(G), \qquad V^{I_m} \longrightarrow \mathcal{H}(G) \otimes_{\mathcal{H}(G,I_m)} V^{I_m}. \tag{13.3.3}$$

(See [29, Proposition 3.16].) Now, for $\pi \in \mathfrak{R}^m(G)$ and π' the corresponding representation in $\mathfrak{R}^m(G')$ obtained via ζ_m, we write $\pi' \sim_m \pi$.

Note that $I_m \subset K_m$, so that $\mathcal{H}(G, K_m)$ is a subalgebra of $\mathcal{H}(G, I_m)$. It was shown in [29, Corollary 3.15] that Kaz_m is an algebra isomorphism when the fields F and F' are m-close, and furthermore that ζ_m is compatible with Kaz_m; that is,

$$\zeta_m|_{\mathcal{H}(G,K_m)} = \mathrm{Kaz}_m .$$

In particular, we can take $l = m$ in Theorem 13.2.1. Now, let (π, V) be an irreducible admissible representation of G with $\mathrm{depth}(\pi) = r$, where depth is as defined in [62, 63]. With $m = \lceil r \rceil + 1$, it follows that $\pi^{I_m} \neq 0$ and $\pi^{K_{m+1}} \neq 0$ (see [29, Lemma 7.2]). Assume that F and F' are m-close, and let π' be the representation of G' with $\pi \sim_m \pi'$. Then it is easy to see that $\mathrm{depth}(\pi') = r$.

13.4. Properties

We now recall some representation theoretic properties that are preserved by the Kazhdan isomorphism (and its variant). These properties are needed for our main theorem.

(I) (*Supercuspidal, square integrable, and tempered representations*) Let (π, V) be an irreducible, admissible supercuspidal (respectively, discrete series, respectively, tempered) representation of G with $\mathrm{depth}(\pi) < m$. Assume that F and F' are $(m+1)$-close, and let (π', V') be such that $\pi \sim_{m+1} \pi'$. Then π' is supercuspidal (respectively, discrete series, respectively, tempered). See [29, Theorem 4.6].

(II) (*Parabolic induction*) Let $\Omega \subset \Delta$, and let $\underline{\mathrm{P}} = \underline{\mathrm{P}}_\Omega$ be the standard parabolic subgroup with Levi subgroup $\underline{\mathrm{M}} = \underline{\mathrm{M}}_\Omega$ and unipotent radical $\underline{\mathrm{N}} = \underline{\mathrm{N}}_\Omega \subset \underline{\mathrm{U}}$. Let π be an irreducible admissible representation of M_Ω such that $\pi^{I_m} \neq 0$. Assume that F' is a field that is $(m+3)$-close to F. Let π' be a representation of M'_Ω such that $\pi \sim_{m+3} \pi'$. Then, by [29, § 4.3], we have

$$\mathrm{Ind}_{P_\Omega}^{G} \pi \sim_m \mathrm{Ind}_{P'_\Omega}^{G'} \pi'.$$

To describe this isomorphism, first note that $G = P_\Omega \underline{\mathrm{G}}(\mathfrak{O}_F) = \coprod_{w \in [W_\Omega \backslash W]} P_\Omega \tilde{w} I$. Here, $\tilde{w}$ denotes the representative of w using a minimal decomposition, as in § 12.1. Let $R(\tilde{w})$ be a system of representatives of $(I \cap \tilde{w}^{-1} P_\Omega \tilde{w}) \backslash I / I^m$ in I. Then, $G = \coprod_{w \in [W_\Omega \backslash W]} \coprod_{b \in R(\tilde{w})} P_\Omega \tilde{w} b I_m$. Then it is easy to see that

$$(\mathrm{Ind}_{P_\Omega}^{G} \pi)^{I_m} \longrightarrow \prod_{w \in [W_\Omega \backslash W]} \prod_{b \in R(\tilde{w})} \pi^{M \cap \tilde{w} I_m \tilde{w}^{-1}}, \qquad h \longrightarrow h(\tilde{w} b),$$

is an isomorphism of $\mathbb{C}$-vector spaces. Hence an element $h \in (\mathrm{Ind}_{P_\Omega}^{G} \pi)^{I_m}$ is completely determined by its values $h(\tilde{w} b)$, $w \in [W_\Omega \backslash W]$, $b \in R(\tilde{w})$. Using the observation that

$M_\Omega \cap \tilde{w} I_m \tilde{w}^{-1} = M_\Omega \cap I_m$ for all $w \in W_\Omega$, and with $\kappa_{m,M_\Omega} : \pi^{M_\Omega \cap I_m} \to \pi'^{M'_\Omega \cap I'_m}$, we obtain a map

$$(\mathrm{Ind}_{P_\Omega}^G \pi)^{I_m} \xrightarrow{\kappa_m} (\mathrm{Ind}_{P'_\Omega}^{G'} \pi')^{I'_m},$$
$$h \to h',$$

where $h'(\tilde{w}'b') = \kappa_{m,M_\Omega}(h(\tilde{w}b))$ for $w \in [W_\Omega \backslash W]$, $b' \in R(\tilde{w}')$ and $b \sim_\beta b'$ as in equation (13.3.1). It was shown in [29, § 4.3] that κ_m is an isomorphism of $\mathbb{C}$-vector spaces that is compatible with the Hecke algebra isomorphism $\mathcal{H}(G, I_m) \cong \mathcal{H}(G', I'_m)$.

(III) (*Intertwining operators*) Let $A(\nu, \pi, \tilde{w}_0)$ be as in § 12.2.B. In [29, Theorem 5.5], it was proved that this intertwining operator is compatible with the theory of close fields. This played a crucial role in proving the compatibility of the Langlands–Shahidi local coefficient and Plancherel measures with the Deligne–Kazhdan theory (see below), and will also be used in § 14. Let us recall this proof from STEP 2, [29, Theorem 5.5]. Let π be an irreducible admissible representation of M_Ω such that $\pi^{I_m} \neq 0$. Let F and F' be $(m+4)$-close, and let $\pi' \sim_{m+4} \pi$. Let ψ be the nontrivial additive character of F that is used in the definition of $A(\nu, \pi, \tilde{w}_0)$ in § 12.2.B. Let ψ' be a character of F' with $\psi' \sim_{m+4} \psi$. Using the self-dual measure on F' induced by ψ' and the same Chevalley basis, we accordingly obtain measures $du, dn'_\Omega, dn'_\Psi, d\bar{n}'_\Omega$, and $d\bar{n}'_\Psi$ on $U, N'_\Omega, N'_\Psi, \bar{N}'_\Omega$, and $\bar{N}'_\Psi$, respectively, as in § 12.2.B. Let $A(\nu, \pi', \tilde{w}'_0)$ be the intertwining operator in § 12.2.B. Let $\mathcal{V}$ be the open dense subset of $\mathfrak{a}^*_{\Omega,\mathbb{C}}$ obtained by taking the intersection of $\mathcal{V}_1$ (defined as in § 12.2.A), $\mathcal{V}(\pi, w_0)$ and $\mathcal{V}(\pi', w_0)$ (these being defined as in § 12.2.B). For $\nu \in \mathcal{V}$, the following hold.

- $I(\nu, \pi)$ and $I(w_0(\nu), w_0(\pi))$ are irreducible.
- $A(\nu, \pi, \tilde{w}_0)$ and $A(\nu, \pi', \tilde{w}'_0)$ are defined and

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(I(\nu, \pi), I(w_0(\nu), w_0(\pi))) = 1 = \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(I(\nu, \pi'), I(w_0(\nu), w_0(\pi'))).$$

Let $\nu \in \mathcal{V}$. Consider the following diagram with $\kappa_{m+1}(\nu)$ and $\kappa_{m+1}(w_0(\nu))$ as in (II) above.

$$\begin{array}{ccc} I(\nu, \pi)^{I_{m+1}} & \xrightarrow{A(\nu,\pi,\tilde{w}_0)} & I(w_0(\nu), w_0(\pi))^{I_{m+1}} \\ \Big\downarrow{\scriptstyle \kappa_{m+1}(\nu)} & & \Big\downarrow{\scriptstyle \kappa_{m+1}(w_0(\nu))} \\ I(\nu, \pi')^{I'_{m+1}} & \xrightarrow[A(\nu,\pi',\tilde{w}'_0)]{} & I(w_0(\nu), w_0(\pi'))^{I'_{m+1}} \end{array} \qquad (13.4.1)$$

Define $A_1 = \kappa_{m+1}(w_0(\nu))^{-1} \circ A(\nu, \pi', \tilde{w}'_0) \circ \kappa_{m+1}(\nu)$ on $I(\nu, \pi)^{I_{m+1}}$, and extend it to $I(\nu, \pi)$ using equation (13.3.3). Then

$$A_1 \in \mathrm{Hom}_G(I(\nu, \pi), I(w_0(\nu), w_0(\pi))).$$

Since $\nu \in \mathcal{V}$, these induced representations are irreducible, the intertwining operator is defined, and

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(I(\nu, \pi), I(w_0(\nu), w_0(\pi))) = 1.$$

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Hence, there exists a scalar $b(\nu,\pi,\tilde{w}_0)\in\mathbb{C}$ such that $A_1=b(\nu,\pi,\tilde{w}_0)\cdot A(\nu,\pi,\tilde{w}_0)$. We proceed to show that $b(\nu,\pi,\tilde{w}_0)=1$. For $0\neq v_0\in\pi^{I_m\cap M_\Omega}$, consider the function f defined as follows.

- $\mathrm{supp}(f)=P_\Omega I_m=P_\Omega(\bar{N}_\Omega\cap I_m)$.
- $f(m_\Omega n i)=\pi(m_\Omega)\eta_\nu(m_\Omega)\delta_{P_\Omega}^{1/2}(m_\Omega)\,\mathrm{vol}(\bar{N}_\Omega\cap I_m,d\bar{n}_\Omega)^{-1}.v_0$ for $m_\Omega\in M_\Omega, n\in N_\Omega, i\in I_m$, where $d\bar{n}_\Omega$ is the Haar measure fixed in this section.

Clearly, $f\in I(\nu,\pi)^{I_m}\subset I(\nu,\pi)^{I_{m+1}}$. Define f' analogously, so that $f'=\kappa_{m+1}(\nu)(f)$. Since $\mathrm{supp}(f)=P_\Omega(\bar{N}_\Omega\cap I_m)$, we see that $\mathrm{supp}(R_{\tilde{w}_0}f)\subset P_\Omega\tilde{w}_0^{-1}N_\Psi$, and hence $R_{\tilde{w}_0}(f)\in I(\nu,\pi)^{\leqslant w_0^{-1}}$ (with notation explained in § 12.2.B). We will show that $A_1(R_{\tilde{w}_0}f)(1)=A(\nu,\pi,\tilde{w}_0)(R_{\tilde{w}_0}f)(1)\neq 0$, which would imply that $b(\nu,\pi,\tilde{w}_0)=1$. By equation (12.2.4), we know that

$$A(\nu,\pi,\tilde{w}_0)(R_{\tilde{w}_0}f)(1)=A(\nu,\pi,\tilde{w}_0)f(\tilde{w}_0)=\int_{N_\Psi}f(\tilde{w}_0^{-1}n\tilde{w}_0)\,dn_\Psi=\int_{\bar{N}_\Omega}f(\bar{n})\,d\bar{n}_\Omega=v_0.$$

We will now compute $A_1(f)(\tilde{w}_0)$. First note that, for $g\in I(w_0(\nu),w_0(\pi))^{I_{m+1}}$,

$$g(\tilde{w}_0)\in(w_0(\pi\eta_\nu))^{M_\Psi\cap\tilde{w}_0I_{m+1}\tilde{w}_0^{-1}}=(w_0(\pi\eta_\nu))^{M_\Psi\cap I_{m+1}}.$$

It is not hard to show that $w_0(\pi\eta_\nu)\sim_l w_0(\pi'\eta'_\nu)$ (see Observation (e) of § 5.6, [29]). Using this and the construction of the isomorphism between the induced representations in (II), we see that the following diagram is commutative.

$$\begin{array}{ccc} I(w_0(\nu),w_0(\pi))^{I_{m+1}} & \xrightarrow{g\to g(\tilde{w}_0)} & (w_0(\pi\eta_\nu))^{M_\Psi\cap I_{m+1}} \\ \uparrow{\scriptstyle \kappa_{m+1}(w_0(\nu))^{-1}} & & \downarrow{\scriptstyle \kappa_{m+1,M_\Psi}} \\ I(w_0(\nu),w_0(\pi'))^{I'_{m+1}} & \xrightarrow[g'\to g'(\tilde{w}'_0)]{} & w_0(\pi'\eta'_\nu)^{M'_\Psi\cap I'_{m+1}} \end{array} \tag{13.4.2}$$

Now,

$$\begin{aligned} A_1(f)(\tilde{w}_0)&=\kappa_{m+1}(w_0(\nu))^{-1}\circ A(\nu,\pi',\tilde{w}'_0)\circ\kappa_{m+1}(\nu)(f)(\tilde{w}_0)\\ &=(\kappa_{m+1}(w_0(\nu))^{-1}\circ A(\nu,\pi',\tilde{w}'_0)(f'))(\tilde{w}_0)\\ &=\kappa_{m+1,M_\Psi}^{-1}\circ(A(\nu,\pi',\tilde{w}'_0)(f')(\tilde{w}'_0))\\ &=\kappa_{m+1,M_\Psi}^{-1}(v'_0)=v_0. \end{aligned}$$

In particular, we have proved that

$$A(\nu,\pi,\tilde{w}_0)(f)=\kappa_{m+1}(w_0(\nu))^{-1}\circ A(\nu,\pi',\tilde{w}'_0)\circ\kappa_{m+1}(\nu)(f)\quad\forall\ f\in I(\nu,\pi)^{I_{m+1}},\nu\in\mathcal{V}. \tag{13.4.3}$$

To finish, we have to observe that $A(\nu,\pi,\tilde{w}_0)$ and $A(\nu,\pi',\tilde{w}'_0)$ also have the same set of zeroes and poles. Let us first briefly recall what it means for $A(\nu,\pi,\tilde{w}_0)$ to be meromorphic in ν: let $I(\pi)_0=\{f|_{\underline{G}(\mathfrak{O}_F)}\mid f\in I(\nu,\pi)\}$, which is $\underline{G}(\mathfrak{O}_F)$-isomorphic to $I(\nu,\pi)$. This space is independent of ν, and the intertwining operator, when defined, is a $\underline{G}(\mathfrak{O}_F)$-equivariant map from $I(\pi)_0$ to $I(w_0(\pi))_0$. For each compact open subgroup K of $\underline{G}(\mathfrak{O}_F)$, the space $I(\pi)_0^K = \{f|_{\underline{G}(\mathfrak{O}_F)} \mid f \in I(\pi)^K\}$ is finite dimensional, and the assertion that the operator $A(\nu, \pi, \tilde{w}_0)$ is meromorphic in ν simply means that, for each compact open subgroup K of $\underline{G}(\mathfrak{O}_F)$, the map $\nu \to A^K(\nu, \pi, \tilde{w}_0) \in \mathrm{Hom}_{\mathbb{C}}(I(\pi)_0^K, I(w_0(\pi))_0^K)$ is meromorphic in the usual sense. Now, since $I(\nu, \pi)$ is generated by its I_m-fixed vectors, we know that $A(\nu, \pi, \tilde{w}_0)$ (respectively, its zeroes and poles) is (are) determined by $A^{I_m}(\nu, \pi, \tilde{w}_0)$ (respectively, its zeroes and poles). Since $A^{I_{m+1}}(\nu, \pi, \tilde{w}_0)$ and $A^{I_{m+1}}(\nu, \pi', \tilde{w}_0')$ are meromorphic functions of ν, and $\mathcal{V}$ is open dense, the desired result follows from (13.4.3).

(IV) (*Generic representations*) Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character of F. Since

$$\underline{U}/[\underline{U}, \underline{U}] = \prod_{\alpha \in \Delta} \underline{U}_\alpha,$$

a character of χ of U can be written as

$$\chi = \prod_{\alpha \in \Delta} \chi_\alpha \circ \mathbf{u}_\alpha^{-1},$$

where χ_α is an additive character of F. Note that there exists $a_\alpha \in F$ such that $\chi_\alpha(x) = \psi(a_\alpha x)\ \forall\ x \in F$. Let $m_\alpha = \mathrm{cond}(\chi_\alpha)$. We will assume that χ is generic; that is, $a_\alpha \in F^\times$ for all $\alpha \in \Delta$. Let (π, V) be a χ-generic representation of G with $\pi^{I_m} \neq 0$. Assume that F and F' are $(m+1)$-close, and let $\pi' \sim_{m+1} \pi$. Furthermore, let $\chi' = \prod_{\alpha \in \Delta} \chi'_\alpha \circ \mathbf{u}_\alpha^{-1}$ be a generic character of U', where $\chi'_\alpha \sim_{m+1} \chi_\alpha$ for each $\alpha \in \Delta$. Then π' is χ'-generic (see [29, § 4.1]).

(V) (*Langlands–Shahidi local coefficients*) Let $\underline{M}_\Omega$ be maximal, and let $C_\chi(\nu, \pi, \tilde{w}_0)$ denote the Langlands–Shahidi local coefficient as in § 12.3 (recall that χ is required to be compatible with $\tilde{w}_0$). Let $m \geqslant 1$ be large enough such that the following hold.

(i) $\pi^{I_m \cap M_\Omega} \neq 0$.

(ii) There exists $v_0 \in V^{I_{m}, M_\Omega}$ such that $W_{v_0}(e) \neq 0$ where $W_{v_0} \in \mathcal{W}(\pi, \chi|_{U \cap M_\Omega})$.

(iii) $\mathrm{cond}(\chi_\alpha) \leqslant m\ \forall\ \alpha \in \Delta$.

Set $l = m+4$, and let F' be l-close to F. Let (π', V') be such that $\pi' \sim_l \pi$. Let $\chi' = \prod_{\alpha \in \Delta} \chi'_\alpha \circ \mathbf{u}_\alpha^{-1}$ with $\chi'_\alpha \sim_l \chi_\alpha$. Then, with Haar measures chosen compatibly as before, we have by [29, Theorem 5.5] that

$$C_\chi(\nu, \pi, \tilde{w}_0) = C_{\chi'}(\nu, \pi', \tilde{w}_0'). \tag{13.4.4}$$

Note that this result can be seen as a crude analog of equation (13.1.5) for the analytic local factors.

(VI) (*Rankin–Selberg factors for pairs*) It can be shown using the above that the Rankin–Selberg L- and ϵ-factors are the same for sufficiently close local fields. More precisely, let $n \geqslant 2$ and $1 \leqslant t \leqslant n$. Fix $m \geqslant 1$. Let σ and τ be two irreducible admissible generic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_t(F)$, respectively. Assume that $\mathrm{depth}(\sigma), \mathrm{depth}(\tau) \leqslant m$. There exists $l = l(m,n) \geqslant m+1$ such that, for any field F' that is l-close to F, and σ', τ' representations of $\mathrm{GL}_n(F')$ and $\mathrm{GL}_t(F')$, respectively,

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with $\sigma \sim_l \sigma'$ and $\tau \sim_l \tau'$, we have

$$L(s, \sigma \times \tau) = L(s, \sigma' \times \tau')$$
$$\gamma(s, \sigma \times \tau, \psi) = \gamma(s, \sigma' \times \tau', \psi'),$$

where $\psi \sim_l \psi'$ (see [29, Theorem 7.5]).

(VII) (*The local Langlands correspondence for* GL_n) Using the above, it can be shown that the LLC for GL_n is compatible with the Deligne–Kazhdan theory. More precisely, for each $m \geqslant 1$, there exists an $l = l(m, n) \geqslant m + 1$ such that, for two fields F and F' that are at least l-close, the following diagram is commutative:

$$\begin{array}{ccc} \{\text{Irr. smooth reps } \pi \text{ of } G \mid \mathrm{depth}(\pi) < m\} & \xrightarrow{\text{LLC}} & \{\text{Hom. } \phi : \mathrm{WD}_F \to \mathrm{GL}_n(\mathbb{C}) \mid \mathrm{depth}(\phi) < m\} \\ \sim_l \downarrow & & \downarrow \sim_l \\ \{\text{Irr. smooth reps } \pi' \text{ of } G' \mid \mathrm{depth}(\pi') < m\} & \xrightarrow{\text{LLC}} & \{\text{Hom. } \phi' : \mathrm{WD}_{F'} \to \mathrm{GL}_n(\mathbb{C}) \mid \mathrm{depth}(\phi') < m\} \end{array}$$

(see [29, § 7]). Note that the LLC for GL_n preserves the depth [86, Theorem 2.3.6.4], making this diagram well defined.

(VIII) (*Symmetric and exterior square local factors*) Using (V), it can be shown that the symmetric and exterior square L- and γ-factors are the same for sufficiently close local fields. More precisely, let (π, V) be a representation of $\mathrm{GL}_n(F)$ with $\mathrm{depth}(\pi) < m$, and let ψ be a nontrivial additive character of F. It was shown in [30] that there exists $l = l(m, n) \geqslant m + 1$ such that, for any field F' that is l-close to F, the representation π' of $\mathrm{GL}_n(F')$ with $\pi' \sim_l \pi$, and character ψ' of F' with $\psi' \sim_l \psi$, we have

$$L(s, \pi, r_2) = L(s, \pi', r_2)$$

$$\gamma(s, \pi, r_2, \psi) = \gamma(s, \pi', r_2, \psi').$$

(IX) (*Plancherel measures*) Let (σ, V) be an irreducible admissible representation of M_Ω, and let $m \geqslant 1$ such that $\sigma^{I_m \cap M_\Omega} \neq 0$. Set $l = m + 4$, and let F' be another non-Archimedean local field that is l-close to F. Let (σ', V') be the irreducible admissible representation of $M_\Omega{}'$ such that $\sigma \sim_l \sigma'$. Then

$$\mu(\nu, \sigma, \psi) = \mu(\nu, \sigma', \psi').$$

by [29, § 6].

13.5. The local L- and γ-factors for classical groups over close local fields

Let χ be a nondegenerate character of U. For split classical groups, adopting the conventions and notation of § 12.5 we observe the following.

Remark 13.5.1. Let ψ be a nontrivial additive character of F. Let $u \in U$ have image $\prod_{\alpha \in \Delta} \mathbf{u}_\alpha(x_\alpha) \in U/[U, U]$, and let $a \in F^\times$. Define

$$\chi_a(u) = \psi(x_{\alpha_1} + \cdots + x_{\alpha_{n-1}} + a x_{\alpha_n}). \tag{13.5.1}$$

Then each nondegenerate character of U is T-conjugate to χ_a for some $a \in F^\times$. For $\underline{\mathrm{H}}_n = \mathrm{SO}_{2n+1}$, we in fact have that each nondegenerate character is conjugate to χ_1.

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For a nondegenerate character χ of U, we write χ_Ω to denote its restriction to $U_{M_\Omega} = U \cap M_\Omega$. For an irreducible admissible χ_Ω-generic representation $\pi \boxtimes \sigma$ of M_Ω, we write $\mathcal{W}(\pi \boxtimes \sigma, \chi_\Omega)$ to denote its Whittaker model.

Lemma 13.5.2. *Let π be an irreducible admissible representation of H_t, and let σ be an irreducible admissible representation of $\mathrm{GL}_k(F)$. Let ψ be a nontrivial additive character of F of conductor 0, and let χ_a be as in Remark 13.5.1 for some $a \in F^\times$. Assume that $\pi \boxtimes \sigma$ is a $\chi_{a,\Omega}$-generic representation of $M_\Omega = H_t \times \mathrm{GL}_k(F)$. Let $m \geqslant 1$ be a natural number such that* $\mathrm{depth}(\pi), \mathrm{depth}(\sigma) < m$.

There exist an integer $l = l(m, n) \geqslant m + 1$, depending only on m and n, and a character χ^* of U that is T-conjugate to χ_a, such that the following conditions are satisfied.

(a) *$\pi \boxtimes \sigma$ is generic with respect to χ^*_Ω, and there exists $v \in (\pi \boxtimes \sigma)^{K_l}$ such that $W^*_v(e) \neq 0$, where $W^*_v \in \mathcal{W}(\pi \boxtimes \sigma, \chi^*_\Omega)$.*

(b) *χ^* is compatible with $\tilde{w}_0$.*

(c) *Writing $\chi^* = \prod_{\alpha \in \Delta} \chi^*_\alpha \circ \mathbf{u}_\alpha^{-1}$, we have* $\mathrm{cond}(\chi^*_\alpha) \leqslant l$ *for all $\alpha \in \Delta$.*

Proof. Write $\chi_a = \chi_{\Omega_1} \times \chi_{\alpha_k} \times \chi_{\Omega_2}$, where $\chi_{\alpha_k} = \psi$, $\chi_{\Omega_1} = \prod_{\alpha \in \Omega_1} \psi \circ \mathbf{u}_\alpha^{-1}$ and $\chi_{\Omega_2} = \prod_{\alpha \in \Omega_2} \chi_\alpha \circ \mathbf{u}_\alpha^{-1}$ (here $\chi_\alpha = \psi$ for $\alpha \neq \alpha_n$ and $\chi_{\alpha_n} = a\psi$). Then σ is χ_{Ω_1}-generic, and π is χ_{Ω_2}-generic. Let K_m be the mth usual congruence subgroup of M_Ω. Then there exist $l = l(m, n) \geqslant m + 2$ and a $v_1 \in \sigma^{K_l \cap \mathrm{GL}_k(F)}$ such that $W_{v_1}(e) \neq 0$ for some $W_{v_1} \in \mathcal{W}(\sigma, \chi_{\Omega_1})$. For example, we can choose any $l \geqslant km + k \geqslant \mathrm{cond}(\sigma)$, and let v_1 be the essential vector (see [12, Theorem 2]). This vector has the property that $W_{v_1}(e) \neq 0$.

Using the Iwasawa decomposition, it is easy to see that there exist $w_1 \in \pi^{K_{m+2} \cap H_t}$ and $\lambda \in X_*(\underline{\mathrm{T}} \cap \underline{\mathrm{H}}_t) \subset X_*(\underline{\mathrm{T}})$ such that $W_{w_1}(\lambda(\varpi)) \neq 0$. Now, for each $\alpha \in \Omega_2$, define $\chi^*_\alpha(x) = \chi_\alpha(\varpi^{\langle \alpha, \lambda \rangle} x)$, and let $\chi^*_{\Omega_2} = \prod_{\alpha \in \Omega_2} \chi^*_\alpha \circ \mathbf{u}_\alpha^{-1}$. Note that $\chi^*_{\Omega_2}$ is $\lambda(\varpi)$-conjugate to χ_{Ω_2}. Hence π is $\chi^*_{\Omega_2}$-generic, and, furthermore, for each $W_w \in \mathcal{W}(\pi, \chi_{\Omega_2})$, we get a corresponding Whittaker function $W^*_w \in \mathcal{W}(\pi, \chi^*_{\Omega_2})$ given by $W^*_w(g) = W_w(\lambda(\varpi) g)$. Note that $W^*_{w_1}(e) \neq 0$.

Now let $\chi^* = \chi_{\Omega_1} \times \chi_{\alpha_k} \times \chi^*_{\Omega_2}$. It is easy to see that χ^* is T-conjugate to χ_a. More precisely, the element $\lambda_0(\varpi)$, where, with standard coordinate identifications,

$$\lambda_0 = (\underbrace{\langle e_{k+1}, \lambda \rangle, \langle e_{k+1}, \lambda \rangle, \ldots, \langle e_{k+1}, \lambda \rangle}_{k \text{ times}}, \langle e_{k+1}, \lambda \rangle, \langle e_{k+2}, \lambda \rangle \cdots \langle e_n, \lambda \rangle) \in X_*(\underline{\mathrm{T}}),$$

conjugates χ to χ^*. Therefore the representation $\pi \boxtimes \sigma$ is χ^*_Ω-generic. Let $v = v_1 \boxtimes w_1$. Then $v \in (\pi \boxtimes \sigma)^{K_l}$, and the function $W^*_v = W_{v_1} \cdot W^*_{w_1} \in \mathcal{W}(\pi \boxtimes \sigma, \chi^*_\Omega)$ has the property that $W^*_v(e) \neq 0$. The element $w_0 = w_{l,\Delta} w_{l,\Omega}$ acts on Ω as follows:

$$w_0 : \alpha_1 \to \alpha_{k-1},\ \alpha_2 \to \alpha_{k-2}, \ldots, \alpha_{k-1} \to \alpha_1,\ \alpha_{k+1} \to \alpha_{k+1}, \ldots, \alpha_n \to \alpha_n.$$

Next, note that, since $\tilde{w}_0$ is a representative of w_0 chosen using the same fixed Chevalley basis, we obtain using [77, Proposition 9.3.5] that

$$\tilde{w}_0 \mathbf{u}_\alpha(x) \tilde{w}_0^{-1} = \mathbf{u}_{w_0 \cdot \alpha}(x) \quad \forall\, \alpha \in \Omega, x \in F.$$

Now, note that w_0 permutes the elements of Ω_1. However, we have chosen $\chi_{w_0 \cdot \alpha} = \psi = \chi_\alpha\ \forall\, \alpha \in \Omega_1$. On the other hand, $w_0 \cdot \alpha = \alpha\ \forall\, \alpha \in \Omega_2$, and hence $\chi_{w_0 \cdot \alpha} = \chi_\alpha\ \forall\, \alpha \in \Omega_2$.

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Therefore, for $u = \prod_{\alpha\in\Omega} \mathbf{u}_\alpha(x_\alpha) \in U_{M_\Omega}/[U_{M_\Omega}, U_{M_\Omega}]$, we have

$$\begin{aligned}\chi^*(\tilde{w}_0 u \tilde{w}_0^{-1}) &= \prod_{\alpha\in\Omega} \chi^*_{w_0\cdot\alpha}(x_\alpha)\\ &= \prod_{\alpha\in\Omega_1} \chi^*_{w_0\cdot\alpha}(x_\alpha) \prod_{\alpha\in\Omega_2} \chi^*_{w_0\cdot\alpha}(x_\alpha)\\ &= \prod_{\alpha\in\Omega_1} \psi(x_\alpha) \prod_{\alpha\in\Omega_2} \chi^*_\alpha(x_\alpha)\\ &= \chi^*(u).\end{aligned}$$

Finally, by the definition of χ^*, we have that $\mathrm{cond}(\chi^*_\alpha) = 0$ for all $\alpha \in \Omega_1 \cup \{\alpha_k\}$. For each $\alpha \in \Omega_2$ and each $x_\alpha \in \mathfrak{p}^l$, we have $\mathbf{u}_\alpha(x_\alpha) \in U \cap K_l$. Hence $\mathbf{u}_\alpha(x_\alpha)w_1 = w_1$. Therefore $\chi^*_\alpha(x_\alpha)W^*_{w_1}(e) = W^*_{w_1}(\mathbf{u}_\alpha(x_\alpha)) = W^*_{w_1}(e)$. Hence, for each $\alpha \in \Omega_2$, we have that $\mathrm{cond}(\chi^*_\alpha) \leqslant l$. Therefore, we have verified (c) as well. □

Proposition 13.5.3. *Let $\pi \boxtimes \sigma$ be an irreducible admissible generic representation of M_Ω. Let $m \geqslant 1$ be such that $\mathrm{depth}(\pi), \mathrm{depth}(\sigma) < m$. There exists an integer $l = l(m, n)$ such that, for each F' that is l-close to F, the following holds:*

$$\begin{aligned}L(s, \pi \times \sigma, r_i) &= L(s, \pi' \times \sigma', r_i),\\ \gamma(s, \pi \times \sigma, r_i, \psi) &= \gamma(s, \pi' \times \sigma', r_i, \psi'),\end{aligned}$$

if $\pi \boxtimes \sigma \sim_l \pi' \boxtimes \sigma'$ and $\psi \sim_l \psi'$.

Proof. Let ψ be a nontrivial additive character of F of conductor 0. Then there exists $a \in F^\times$ such that $\pi \boxtimes \sigma$ is χ_a-generic, where χ_a is the generic character defined using ψ and a as in Remark 13.5.1. Note that, when $\underline{\mathrm{H}}_n = \mathrm{SO}_{2n+1}$, we can take $a = 1$. When $\underline{\mathrm{H}}_n = \mathrm{Sp}_{2n}$ or SO_{2n}, let

$$t_0 = \mathrm{diag}(\underbrace{\sqrt{a}^{-1}, \ldots, \sqrt{a}^{-1}, \sqrt{a}^{-1}}_{n \text{ times}}, \sqrt{a}, \sqrt{a}, \ldots, \sqrt{a}).$$

Then we have that

$$\chi_a(u) = \chi_1 \circ \mathrm{Ad}_{\underline{\mathrm{U}}}(t_0)(u) = \chi_1(t_0^{-1} u t_0) \quad \forall u \in U.$$

Furthermore,

$$\tilde{w}_0(t_0)t_0^{-1} = \mathrm{diag}(\underbrace{a, a, \ldots, a}_{k \text{ times}}, \underbrace{1, 1, \ldots, 1, 1}_{t \text{ times}}, a^{-1}, a^{-1}, \ldots, a^{-1})$$

lies in Z_{M_Ω}. Then the γ-factors satisfy the equation

$$\omega_s(\tilde{w}_0(t_0)t_0^{-1})\gamma(s, \pi \times \sigma, r_1, \psi)\gamma(s, \sigma, r_2, \psi) = C_{\chi_a}(s, \pi \times \sigma, \tilde{w}_0), \tag{13.5.2}$$

where ω_s is the central character of $\sigma\eta_\nu\delta_P^{1/2}$ (see [74, Equation (3.11)] and [54, § 6]). Let l_0 be the integer in Lemma 13.5.2, l_1 the integer in Property (VIII) of § 13.4, and let $l = \max(l_0, l_1) + 4$. Assume that F and F' are l-close, and let $\pi \boxtimes \sigma \sim_l \pi' \boxtimes \sigma'$ and

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$\psi \sim_l \psi'$. Since the γ-factors associated to r_2 agree over close local fields by Property (VIII) of § 13.4, in order to verify the equality of the γ-factors with respect to r_1, we simply need to verify that

$$C_{\chi_a}(s, \pi \boxtimes \sigma, \tilde{w}_0) = C_{\chi'_{a'}}(s, \pi' \boxtimes \sigma', \tilde{w}'_0).$$

(Note that, if $\pi \boxtimes \sigma \sim_l \pi' \boxtimes \sigma'$, then $\omega_s \sim_l \omega'_s$.) Let χ^* be the character in Lemma 13.5.2 obtained from χ_a. Let χ'^* be the corresponding character of U' such that $\chi^* \sim_l \chi'^*$. Then, by Property (V) of § 13.4, we have that

$$C_{\chi^*}(s, \pi \boxtimes \sigma, \tilde{w}_0) = C_{\chi'^*}(s, \pi' \boxtimes \sigma', \tilde{w}'_0). \tag{13.5.3}$$

Since χ^* is T-conjugate to χ_a, that is, $\chi^* = \chi_a \circ \mathrm{Ad}_{\underline{\mathrm{U}}}(t)$ for some $t \in T$, we have

$$C_{\chi^*}(s, \pi \boxtimes \sigma, \tilde{w}_0) = \omega_s(\tilde{w}_0(t)t^{-1}) C_{\chi_a}(s, \pi \boxtimes \sigma, \tilde{w}_0). \tag{13.5.4}$$

With $t \mapsto t'$ under the isomorphism $T/T_{\mathfrak{p}^l_F} \cong T'/T'_{\mathfrak{p}^l_{F'}}$, and $\chi_a \sim_l \chi'_{a'}$, where $a \mapsto a'$ under the isomorphism $F^\times/1+\mathfrak{p}^l_F \cong F'^\times/1+\mathfrak{p}^l_{F'}$, we see that

$$\chi^* = \chi_a \circ \mathrm{Ad}_{\underline{\mathrm{U}}}(t) \sim_l \chi'_{a'} \circ \mathrm{Ad}_{\underline{\mathrm{U}}}(t') \sim_l \chi'^*.$$

Combining the observation that $\omega_s \sim_l \omega'_s$ with equations (13.5.4) and (13.5.3), we obtain that

$$C_{\chi_a}(s, \pi \boxtimes \sigma, \tilde{w}_0) = C_{\chi'_{a'}}(s, \pi' \boxtimes \sigma', \tilde{w}'_0).$$

This proves the equality of first γ-factors when $\mathrm{cond}(\psi) = 0$. Now the equality of the γ-factors with respect to all nontrivial additive characters is obtained by arguing as above (see [53, Property (v), § 1.4], where this dependence on ψ has been explicated).

The equality of the second L-functions over close local fields is Property (VIII) of § 13.4. The first L-function is defined for generic tempered representations using the γ-factor and then extended to the general case via the Langlands classification. The tempered case is automatic from the equality of the first γ-factors. Since temperedness, twists by unramified characters, and normalized parabolic induction are all compatible with the Deligne–Kazhdan correspondence (see [29, § 4]), we obtain the desired result for the first L-function. □

13.6. The main theorem

Let F' be a non-Archimedean local field of odd positive characteristic.

Theorem 13.6.1. *Let $m \geqslant 1$, and let $l(m, N) := Nm + 2N$. Let (π', V') be an irreducible admissible discrete series representation of H'_n with* $\mathrm{depth}(\pi') < m$. *There exists a $\phi_{\pi'}$:* $\mathrm{WD}_{F'} \to \hat{\underline{\mathrm{H}}}_n$ *satisfying the following properties.*

(a) *If π' is in the discrete series, then $\phi_{\pi'}$ does not factor through any proper Levi subgroup of $\hat{\underline{\mathrm{H}}}_n$.*

(b) *If π' is generic, then, for each irreducible admissible supercuspidal representation σ' of $\mathrm{GL}_r(F')$, $r \leqslant N-1$, and* $\mathrm{depth}(\sigma') \leqslant 2l(m, N)$, *we have*

$$L(s, \pi' \times \sigma') = L(s, \phi_{\pi'} \otimes \phi_{\sigma'})$$
$$\gamma(s, \pi' \times \sigma', \psi') = \gamma(s, \phi_{\pi'} \otimes \phi_{\sigma'}, \psi').$$

(c) *If π' is nongeneric, then, for each irreducible admissible discrete series representation σ' of* $\mathrm{GL}_r(F')$, $r \leqslant N-1$, *and* $\mathrm{depth}(\sigma') \leqslant m+1$,

$$\mu(s, \pi' \times \sigma', \psi') = \gamma(s, \phi_{\pi'} \otimes \phi_{\sigma'}^\vee, \psi')\gamma(-s, \phi_{\pi'}^\vee \otimes \phi_{\sigma'}, \bar{\psi}')\gamma(2s, r_2 \circ \phi_{\sigma'}, \psi')\gamma(-2s, r_2 \circ \phi_{\sigma'}^\vee, \bar{\psi}').$$

Furthermore, the parameter $\phi_{\pi'}$ is uniquely determined up to $\hat{\underline{\mathrm{H}}}_n$-conjugacy if $\underline{\mathrm{H}}_n = \mathrm{Sp}_{2n}$ or SO_{2n+1}, and up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy if $\underline{\mathrm{H}}_n = \mathrm{SO}_{2n}$.

Proof. Let $l \geqslant 2(Nm + 2N)$ be an integer large enough so that Properties (I)–(IX) of § 13.4 and Proposition 13.5.3 hold. Note that such an l is completely determined by m and N. Let F be a non-Archimedean local field of characteristic 0 that is l-close to F', and let π be the discrete series representation of $H_n = \underline{\mathrm{H}}_n(F)$ such that $\pi \sim_l \pi'$. Let ϕ_π be the parameter of π as in Theorem 12.8.1. By Lemma 8.2.3, we know that $\mathrm{depth}(\phi_\pi) < m+1$. Let $\phi_{\pi'} : \mathrm{WD}_{F'} \to \hat{\underline{\mathrm{H}}}_n$ be such that $\phi_{\pi'} \sim_l \phi_\pi$ via the Deligne isomorphism. To summarize, $\phi_{\pi'}$ is obtained using the diagram

$$\begin{array}{ccccc} F' & \quad & \pi' & \dashrightarrow & \phi_{\pi'} \\ \vdots\ l\text{-close} & & \downarrow \sim_l & & \uparrow \sim_l \\ F & & \pi & \xrightarrow{\text{Arthur}} & \phi_\pi \end{array} \tag{13.6.1}$$

We claim that $\phi_{\pi'}$ satisfies Properties (a)–(c). Since $\mathrm{Im}(\phi_\pi) = \mathrm{Im}(\phi_{\pi'})$, (a) is clear. For Property (b), note that

$$\phi_\sigma \sim_l \phi_{\sigma'} \tag{13.6.2}$$

by Property (VII) of § 13.4. Now, by Theorem 12.8.1, Property (iv) of § 13.1, and Proposition 13.5.3, it follows that (b) holds. By Property (IX) of § 13.4, Property (iv) of § 13.1, and Theorem 12.8.1, we see that (c) holds.

Next, we explain why the definition of $\phi_{\pi'}$ does not depend on this field F of characteristic 0. To see this, suppose that F_1 is another local field of characteristic 0 that is l-close to F'. We have to observe that the bottom square of the diagram below is commutative,

$$\begin{array}{ccccc} F' & \quad & \pi' & \dashrightarrow & \phi_{\pi'} \\ \vdots\ l\text{-close} & & \downarrow \sim_l & & \uparrow \sim_l \\ F & & \pi & \xrightarrow{\text{Arthur}} & \phi_\pi \\ \vdots\ l\text{-close} & & \downarrow \sim_l & & \vdots\!\uparrow \sim_l ? \\ F_1 & & \pi_1 & \xrightarrow{\text{Arthur}} & \phi_{\pi_1} \end{array}$$

that is, that $\phi_\pi \sim_l \phi_{\pi_1}$. Here both F and F_1 are local fields of characteristic 0, and the parameters ϕ_π and ϕ_{π_1} are as constructed by Arthur in Theorem 2.2.1. Since both ϕ_π and ϕ_{π_1} are uniquely characterized as in Theorem 12.8.1, we see that $\phi_\pi \sim_l \phi_{\pi_1}$ using the same argument as above that all the characterizing properties of Theorem 12.8.1 are compatible with the Deligne–Kazhdan theory.

Now it follows that $\phi_{\pi'}$ is uniquely characterized by Properties (a)–(c) if π' is a discrete series representation, because the corresponding ϕ_π is uniquely characterized by those properties in characteristic 0. $\square$

Remark 13.6.2. The above theorem gives a partition of the discrete series representations of $\underline{\mathrm{H}}_n(F')$ into L-packets (slightly coarsened in the even orthogonal case) indexed by $\phi' \in \tilde{\Phi}_2(H_n')$; that is,

$$\tilde{\Pi}_2(H_n') = \bigsqcup_{\phi' \in \tilde{\Phi}_2(H_n')} \tilde{\Pi}_{\phi'}.$$

Let π' be a tempered representation of H_n' of depth $\leqslant m$. Then we know that π' occurs as an irreducible summand of a representation induced from a discrete series representation; that is, for a proper Levi subgroup $\underline{\mathrm{M}} = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \cdots \mathrm{GL}_{n_k} \times \underline{\mathrm{H}}_t$ of $\underline{\mathrm{H}}_n$ and an essentially discrete series representation $\sigma' = \delta(\rho_1', a_1) \boxtimes \delta(\rho_2', a_2) \boxtimes \cdots \delta(\rho_k', a_k) \boxtimes \pi_-'$ of M', π' occurs in $\mathrm{Ind}_{M'}^{H_n'} \sigma'$. Then we define $\phi_{\pi'} = \bigoplus_{i=1}^k (\phi_{\rho_i'} \otimes S_{a_i}) \oplus \phi_{\pi_-'} \oplus \bigoplus_{i=1}^k (\phi_{\rho_i'} \otimes S_{a_i})^\vee$, where $\phi_{\pi_-'}$ is as in the previous theorem. This gives a partition of the tempered spectrum into L-packets indexed by $\phi' \in \tilde{\Phi}_{\mathrm{bdd}}(H_n')$. Furthermore, it is clear from the properties in § 13.4 and the above theorem that there exists an integer l depending only on m and N such that, if F is a local field of characteristic 0 that is l-close to F' and π is a tempered representation of H_n with $\pi \sim_l \pi'$, then $\phi_\pi \sim_m \phi_{\pi'}$.

14. The enhanced Langlands parameters

In characteristic 0, Theorem 2.2.1 also provides concrete information about the internal structure of each tempered L-packet. In this section, we partially address the analogous question in positive characteristic. Let F' be a non-Archimedean local field of positive characteristic, and let $\phi' : \mathrm{WD}_{F'} \to \hat{\underline{\mathrm{H}}}_n$. One wishes to construct a map between $\tilde{\Pi}_{\phi'}$ and $\hat{\mathcal{S}}_{\phi'}$ that satisfies some important properties, and show that it is compatible with the corresponding map in characteristic 0. We do this partially in this section, namely we address this question for discrete series parameters in the odd orthogonal and symplectic cases. Additionally, our approach requires following conditions, both of which are satisfied at least when p is large enough (see Remark 14.0.1 below). Let p be an odd prime. Consider the following conditions on p.

C1(p): Let F be any non-Archimedean local field of characteristic 0 with residue characteristic p. For every integer $m \geqslant 1$, there exists an integer m_0 depending only on m and N satisfying the following: if $\phi \in \tilde{\Phi}_{\mathrm{bdd}}(H_n)$ and $\mathrm{depth}(\phi) \leqslant m$, then $\mathrm{depth}(\pi) \leqslant m_0$ for every tempered representation π of H_n whose image in $\tilde{\Pi}_{\mathrm{temp}}(H_n)$ lies in $\tilde{\Pi}_\phi$.

C2(p): Let F be any non-Archimedean local field of characteristic 0 with residue characteristic p. For every integer $m \geqslant 1$, there exists an integer m_0 depending only on m and N satisfying the following: if $\phi \in \tilde{\Phi}_{\mathrm{bdd}}(H_n)$ and $\mathrm{depth}(\phi) \leqslant m$, then $\mathrm{depth}(\pi) \leqslant m_0$ for every generic tempered representation π of H_n whose image in $\tilde{\Pi}_{\mathrm{temp}}(H_n)$ lies in $\tilde{\Pi}_\phi$.

Remark 14.0.1. By Corollary 10.6.4, we know that both $\mathbf{C}1(p)$ and $\mathbf{C}2(p)$ can be satisfied with $m_0 = m$, when p is a large enough integer determined by the absolute root datum of $\underline{\mathrm{H}}_n$. When $\underline{\mathrm{H}}_n = \mathrm{SO}_{2n+1}$, $\mathbf{C}2(p)$ can be satisfied with $m_0 = Nm + N$ for all p by Lemma 11.0.1.

Remark 14.0.2. Let $\phi \in \Phi_{\mathrm{bdd}}(H_n)$. Then the strong form of the tempered L-packet conjecture has been proved in [61, § 4]. More precisely, it is shown there that, for each T-orbit of χ_a, with χ_a as in Remark 13.5.1, there exists a unique $\mathrm{Out}(\underline{\mathrm{H}}_n)$-orbit in $\tilde{\Pi}_\phi$ that contains a χ_a-generic representation.

Lemma 14.0.3. *Let p be a prime number. Assume that Condition $\mathbf{C}2(p)$ holds for all local fields F of characteristic 0 with residue characteristic p. Let F' be a non-Archimedean local field of characteristic p. Let $\phi' : \mathrm{WD}_{F'} \to \hat{\underline{\mathrm{H}}}_n$ be a bounded Langlands parameter for $\underline{\mathrm{H}}_n$ over F', and let $\tilde{\Pi}_{\phi'}$ be the L-packet attached to ϕ'. Then the following hold.*

- (a) *For each T'-orbit of $\chi'_{a'}$, with $\chi'_{a'}$ a generic character of U' as in Remark 13.5.1, there is a unique $\mathrm{Out}(\underline{\mathrm{H}}_n)$-orbit in $\tilde{\Pi}_{\phi'}$ that contains a $\chi'_{a'}$-generic representation.*
- (b) *The cardinality of $\tilde{\Pi}_{\phi'}$ is at most the cardinality of $\hat{\mathcal{S}}_{\phi'}$.*

Proof. Let $m \geqslant 1$ be such that $\mathrm{depth}(\phi') \leqslant m$. Let $l \geqslant 2(Nm_0 + 2N)$ be large enough so that the Properties (I)–(IX) of § 13.4 and Proposition 13.5.3 hold for representations of depth $\leqslant m_0$. Note that such an integer depends only on m and N. Let F be a non-Archimedean local field of characteristic 0 such that F' is l-close to F. Let $\phi : \mathrm{WD}_F \to \hat{\underline{\mathrm{H}}}_n$ be such that $\phi \sim_l \phi'$. Then $\mathrm{depth}(\phi) \leqslant m$. Let $\chi'_{a'}$ be a generic character of U' as in Remark 13.5.1, and let χ_a be a generic character of U with $\chi_a \sim_l \chi_{a'}$. Let π be a χ_a-generic tempered representation whose image in $\tilde{\Pi}_{\mathrm{temp}}(H_n)$ lies in $\tilde{\Pi}_\phi$; such a π exists, and it represents the unique orbit in $\tilde{\Pi}_\phi$ containing a χ_a-generic member by [61, § 4]. Then, by Condition $\mathbf{C}2(p)$, $\mathrm{depth}(\pi) \leqslant m_0$. Let π' be the representation of H'_n such that $\pi' \sim_l \pi$. Then π' is $\chi'_{a'}$-generic by Property (IV) of § 13.4, and, furthermore, $\pi' \in \tilde{\Pi}_{\phi'}$ by the proof of Theorem 13.6.1 (see Diagram 13.6.1). The uniqueness of the orbit in $\tilde{\Pi}_{\phi'}$ containing a $\chi'_{a'}$-generic representation follows from the corresponding uniqueness statement in characteristic 0.

Next, let $k = \max\{\mathrm{depth}(\pi') \mid \pi' \in \tilde{\Pi}_{\phi'}\}$. Choose a local field F of characteristic 0 that is l-close to F', where l is large enough so that Properties (I)–(IX) of § 13.4 and Proposition 13.5.3 hold for representations of depth at most k. For each $\pi' \in \tilde{\Pi}_{\phi'}$, we obtain a unique $\pi \in \tilde{\Pi}_\phi$ such that $\pi \sim_l \pi'$. Also, $\phi \sim_l \phi'$ by Theorem 13.6.1. Hence $\#\tilde{\Pi}_{\phi'} \leqslant \#\tilde{\Pi}_\phi$. Now $\#\tilde{\Pi}_\phi = \#\hat{\mathcal{S}}_\phi$ by Theorem 2.2.1, and it is easy to check that the cardinality of $\#\mathcal{S}_\phi = \#\mathcal{S}_{\phi'}$ since $\phi \sim_l \phi'$. Now (b) follows. □

Lemma 14.0.4. *Assume that Condition $\mathbf{C}1(p)$ holds for all local fields F of characteristic 0 with residue characteristic p. Let F' be a non-Archimedean local field of characteristic p. Let $\phi' : \mathrm{WD}_{F'} \to \hat{\underline{\mathrm{H}}}_n$ be a bounded Langlands parameter, and let $\tilde{\Pi}_{\phi'}$ be the L-packet attached to ϕ'. Then the cardinality of $\tilde{\Pi}_{\phi'}$ equals the cardinality of $\hat{\mathcal{S}}_{\phi'}$.*

Proof. Let $m \geqslant 1$ be such that $\mathrm{depth}(\phi') \leqslant m$. Let $l \geqslant Nm_0 + 2N$ be large enough so that Properties (I)–(IX) of § 13.4 and Proposition 13.5.3 hold for representations of depth at most m_0. Note that such an integer depends only on m and N. Let F be a non-Archimedean local field of characteristic 0 such that F' is l-close to F. Let $\phi : WD_F \to \hat{\underline{\mathrm{H}}}_n$ such that $\phi \sim_l \phi'$. Then $\mathrm{depth}(\phi) \leqslant m$. By Hypothesis **C1**$(p)$, we have that, for each representation π whose image in $\tilde{\Pi}_{\mathrm{temp}}(H_n)$ lies in $\tilde{\Pi}_\phi$, $\mathrm{depth}(\pi) \leqslant m_0$. Since F and F' are l-close, for each such π let π' be the representation of H_n' with $\pi' \sim_l \pi$. Then the orbit of π' lies in $\tilde{\Pi}_{\phi'}$ by the proof of Theorem 13.6.1. Hence we obtain a bijection between $\tilde{\Pi}_\phi$ and $\tilde{\Pi}_{\phi'}$. Since $\#\tilde{\Pi}_\phi = \#\hat{\mathcal{S}}_\phi$ by Theorem 2.2.1 and $\#\mathcal{S}_\phi = \#\mathcal{S}_{\phi'}$, the lemma follows. □

14.1. The ϵ_π-conjecture of Mœglin for Sp_{2n} and SO_{2n+1}

For the remainder of this article, we will assume that $\underline{\mathrm{H}}_n = \mathrm{Sp}_{2n}$ or SO_{2n+1}. Let F be a non-Archimedean local field, and let $\phi \in \Phi_2(H_n)$. Now we explain the identification of $\hat{\mathcal{S}}_\phi$ with certain $\mathbb{Z}/2\mathbb{Z}$-valued functions on $\mathrm{Jord}(\phi)$. Since ϕ is a discrete series parameter, we can write $\phi = \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_r$, where ϕ_i is an irreducible self-dual N_i-dimensional representation of WD_F of the same type as $\hat{\underline{\mathrm{H}}}_n$, and the ϕ_i are all distinct. Then it is easy to check that

$$S_\phi = \mathrm{Cent}_{\hat{\underline{\mathrm{H}}}_n}(\mathrm{Im}(\phi)) = \left\{ (\alpha_1, \alpha_2, \ldots \alpha_r) \in (\mathbb{Z}/2\mathbb{Z})^r \;\middle|\; \prod_{i=1}^{r} (\alpha_i)^{N_i} = 1 \right\}.$$

When $\underline{\mathrm{H}}_n = \mathrm{Sp}_{2n}$, using the fact that $S_\phi \cong \mathrm{Cent}_{\mathrm{O}_{2n+1}(\mathbb{C})}(\mathrm{Im}(\phi))/Z(\mathrm{O}_{2n+1}(\mathbb{C}))$, we see that $\mathcal{S}_\phi = S_\phi \cong (\mathbb{Z}/2\mathbb{Z})^r/\langle(-1, -1 \cdots -1)\rangle$. When $\underline{\mathrm{H}}_n = \mathrm{SO}_{2n+1}$, $S_\phi = (\mathbb{Z}/2\mathbb{Z})^r$ (since each N_i is even), and hence $\mathcal{S}_\phi$ is again isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r/\langle(-1, -1 \cdots -1)\rangle$. Hence this allows us to identify $\hat{\mathcal{S}}_\phi$ with the subset of $\mathbb{Z}/2\mathbb{Z}$-valued functions on $(\mathbb{Z}/2\mathbb{Z})^r$ that are trivial on $\langle(-1, -1 \cdots -1)\rangle$. Viewing functions on $(\mathbb{Z}/2\mathbb{Z})^r$ as functions on $\mathrm{Jord}(\phi)$, we obtain the desired identification.

Let F be a non-Archimedean local field of characteristic 0. In this section, we recall a result that Mœglin communicated to us and will appear in [58]. This result uses the work of Arthur to describe the character attached to π in terms of certain normalized intertwining operators. Let $\pi \in \Pi_\phi$ for some $\phi \in \Phi_2(H_n)$. We denote the character attached to π in Theorem 2.2.1 as ϵ_π and view it as a function on $\mathrm{Jord}(\phi)$ as explained above (as Mœglin notes in [58], this agrees with the partially defined function described in § 12.7 on $\mathcal{S}_\phi$ under suitable identifications).

Let $(\rho, a) \in \mathrm{Jord}(\phi)$, and write $k = ad_\rho$, where $d_\rho \in \mathbb{N}$ is such that ρ is a representation of $\mathrm{GL}_{d_\rho}(F)$. Consider the induced representation

$$I(s, \pi \boxtimes \delta(\rho, a)) := \mathrm{Ind}_{H_n \times \mathrm{GL}_k(F)}^{H_{n+2k}} \pi \boxtimes \delta(\rho, a)| - |^s.$$

Let us fix a Whittaker datum (U, χ_b) of H_{n+2k}, where χ_b is as in Remark 13.5.1, whose 'a' we now denote as 'b' to avoid confusion with the 'a' of the above equation. We will often abuse notation to denote by χ_b the analogously defined character of the unipotent radical of the standard Borel subgroup of $\underline{\mathrm{H}}_n$. In particular, χ_b is determined using the fixed $\mathbb{Z}$-splitting $\{\mathbf{u}_\alpha \mid \alpha \in \Delta\}$, a nontrivial character ψ of F, and $a \in F^\times$.

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Let $A(s, \pi \boxtimes \delta(\rho, a), \tilde{w}_0)$ be the standard intertwining operator as in § 12.2.B. This operator has a pole at $s = 0$. We also let $r(s, \pi^{\mathrm{GL}}, \delta(\rho, a), w_0)$ denote the normalizing factor

$$r(s, \pi^{\mathrm{GL}}, \delta(\rho, a), w_0) := \frac{L(s, \pi^{\mathrm{GL}} \times \delta(\rho, a))}{\epsilon(s, \pi^{\mathrm{GL}} \times \delta(\rho, a), \psi) L(1+s, \pi^{\mathrm{GL}} \times \delta(\rho, a))} \cdot \frac{L(2s, \delta(\rho, a), r_2)}{\epsilon(2s, \delta(\rho, a), r_2, \psi) L(1+2s, \delta(\rho, a), r_2)},$$

and let

$$R(s, \pi \boxtimes \delta(\rho, a), w_0) := r(s, \pi^{\mathrm{GL}}, \delta(\rho, a), w_0)^{-1} A(s, \pi \boxtimes \delta(\rho, a), \tilde{w}_0).$$

This operator is holomorphic at $s = 0$, and it satisfies

$$R(-s, w_0(\pi \boxtimes \delta(\rho, a)), w_0^{-1}) R(s, \pi \boxtimes \delta(\rho, a), w_0) = 1$$

by [7, § 2.3]. Since $(\rho, a) \in \mathrm{Jord}(\phi)$, we have $\rho \cong \rho^\vee$, so that $\tilde{w}_0(\pi \boxtimes \delta(\rho, a)|-|^s) = \pi \boxtimes \delta(\rho, a)|-|^{-s}$ for all $s \in \mathbb{C}$. Choose the isomorphism $R_{\tilde{w}_0} : \tilde{w}_0(\pi \boxtimes \delta(\rho, a)) \to \pi \boxtimes \delta(\rho, a)$ such that

- $R_{\tilde{w}_0}(w_0(\pi \boxtimes \delta(\rho, a)))(m) = (\pi \boxtimes \delta(\rho, a))(m) \circ R_{\tilde{w}_0}$ for $m \in H_n \times \mathrm{GL}_k(F)$,
- $R_{\tilde{w}_0} = \mathrm{Id}_\pi \otimes R'_{\tilde{w}_0}$, where $R'_{\tilde{w}_0}$ satisfies $\lambda_{\chi_k} \circ R'_{\tilde{w}_0} = \lambda_{\chi_k}$. Here $\lambda_{\chi_k} : \delta(\rho, a) \to \mathbb{C}$ is the (unique up to scalar) Whittaker functional obtained from the fixed Whittaker datum (U_k, χ_k), where U_k denotes the unipotent radical of the standard Borel subgroup of GL_k, χ_k is the generic character on U_k obtained from the character ψ and the fixed $\mathbb{Z}$-splitting $\{\mathbf{u}_\alpha \mid \alpha \in \Delta_k\}$, with Δ_k denoting the set of simple roots of GL_k (this is just the restriction of the Whittaker datum (U, χ_b) on H_{n+2k} that we fixed shortly after the beginning of this section, to GL_k).

Then

$$\mathcal{R}(\pi \boxtimes \delta(\rho, a), w_0) := R_{\tilde{w}_0} \circ R(0, \pi \boxtimes \delta(\rho, a), w_0)$$

becomes a self-intertwining operator on $I(0, \pi \otimes \delta(\rho, a))$.

Proposition 14.1.1 [58]. *With notation as above, we have the following.*

(a) $\mathcal{R}(\pi \boxtimes \delta(\rho, a), w_0) \in \mathbb{C}^\times$.

(b) *For* $\pi_1, \pi_2 \in \Pi_\phi$,

$$\frac{\epsilon_{\pi_1}(\rho, a)}{\epsilon_{\pi_2}(\rho, a)} = \frac{\mathcal{R}(\pi_1 \boxtimes \delta(\rho, a), w_0)}{\mathcal{R}(\pi_2 \boxtimes \delta(\rho, a), w_0)}.$$

In particular, with π^{gen} *the* χ_b*-generic member in* Π_ϕ *(see Remark 14.0.2), we have*

$$\epsilon_\pi(\rho, a) = \frac{\mathcal{R}(\pi \boxtimes \delta(\rho, a), w_0)}{\mathcal{R}(\pi^{\mathrm{gen}} \boxtimes \delta(\rho, a), w_0)}.$$

14.2. Internal structure of the L-packet $\Pi_{\phi'}$ for $\phi' \in \Phi_2(H'_n)$

Let $\phi' \in \Phi_2(H'_n)$. We discuss the construction of the map $\Pi_{\phi'} \to \hat{\mathcal{S}}_{\phi'}$. Assuming Proposition 14.1.1 allows us to define the map $\Pi_{\phi'} \to \hat{\mathcal{S}}_{\phi'}$ under Condition $\mathbf{C}2(p)$ and prove that it is injective. We will then show surjectivity under Condition $\mathbf{C}1(p)$.

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Remark 14.2.1. Suppose that $\phi' : \mathrm{WD}_{F'} \to \hat{\underline{\mathrm{H}}}_n$, and let $(\rho', a) \in \mathrm{Jord}(\phi')$. If $\mathsf{depth}(\phi') \leqslant m$, then $\mathsf{depth}(\rho') \leqslant m$. Let $l = l(m, N)$ be an integer large enough such that Property (VII) of § 13.4 holds for each $(\rho', a) \in \mathrm{Jord}(\phi')$. Let F be a local field of characteristic 0 that is at least l-close to F', and let $\phi : \mathrm{WD}_F \to \hat{\underline{\mathrm{H}}}_n$ be such that $\phi \sim_l \phi'$. Then we obtain a bijection between $\mathrm{Jord}(\phi)$ and $\mathrm{Jord}(\phi')$, where $(\rho, a) \in \mathrm{Jord}(\phi)$ corresponds to $(\rho', a) \in \mathrm{Jord}(\phi')$ via Del_l. So when $\phi \sim_l \phi'$ we write $\mathrm{Jord}(\phi) \sim_l \mathrm{Jord}(\phi')$ to denote this bijection.

We wish to construct a map from $\Pi_{\phi'}$ to $\hat{\mathcal{S}}_{\phi'}$.

Definition 14.2.2. Let p be a prime such that Condition $\mathbf{C}2(p)$ holds. Let F' be a non-Archimedean local field of characteristic p. Let $\phi' \in \Phi_2(H'_n)$. We fix a Whittaker datum $(U', \chi'_{b'})$, where $\chi'_{b'}$ is as in Remark 13.5.1. We keep in mind that the *same* $\mathbb{Z}$-splitting $\{\mathbf{u}_\alpha \mid \alpha \in \Delta\}$ fixed before, a character ψ' of F', and $b' \in F'^{\times}$ determine $\chi'_{b'}$. In view of Proposition 14.1.1, we define the following. For $\pi' \in \Pi_{\phi'}$,

- define the normalizing factor

$$r(s, \pi'^{\mathrm{GL}}, \delta(\rho', a), w_0) := \frac{L(s, \pi'^{\mathrm{GL}} \times \delta(\rho', a))}{\epsilon(s, \pi'^{\mathrm{GL}} \times \delta(\rho', a), \psi')L(1+s, \pi'^{\mathrm{GL}} \times \delta(\rho', a))} \frac{L(2s, \delta(\rho', a), r_2)}{\epsilon(2s, \delta(\rho', a), r_2, \psi')L(1+2s, \delta(\rho', a), r_2)}$$

- define $R(s, \pi' \boxtimes \delta(\rho', a), w_0) := r(s, \pi'^{\mathrm{GL}}, \delta(\rho', a), w_0)^{-1} A(s, \pi' \boxtimes \delta(\rho, a), \tilde{w}'_0)$.
- define $R_{\tilde{w}'_0}$ completely analogously to $R_{\tilde{w}_0}$ above; in particular, $R_{\tilde{w}'_0} = \mathrm{Id}_{\pi'} \otimes R'_{\tilde{w}'_0}$ using a map $R'_{\tilde{w}'_0}$ satisfying $R'_{\tilde{w}'_0} \circ \lambda_{\chi'_k} = \lambda_{\chi'_k}$. Here $\lambda_{\chi'_k} : \delta(\rho', a) \to \mathbb{C}$ is the (unique up to scalar) Whittaker functional obtained from the fixed Whittaker datum (U'_k, χ'_k), where U'_k denotes the group of F'-points of the unipotent radical of the standard Borel subgroup of $\mathrm{GL}_k(F')$, χ'_k is the generic character on U'_k obtained from the character ψ' and the fixed $\mathbb{Z}$-splitting $\{\mathbf{u}_\alpha \mid \alpha \in \Delta_k\}$, with Δ_k denoting the set of simple roots of GL_k (this is just the restriction of a Whittaker datum $(U', \chi'_{b'})$ on H'_{n+2k} like the one that we fixed in the beginning of this section to $\mathrm{GL}_k(F')$).
- define $\mathcal{R}(\pi' \boxtimes \delta(\rho', a), w_0) = R_{\tilde{w}'_0} \circ R(s, \pi' \boxtimes \delta(\rho', a), w_0)$.

Define $\epsilon_{\pi'} : \mathrm{Jord}(\phi') \to \{\pm 1\}$ as follows. For $(\rho', a) \in \mathrm{Jord}(\phi')$,

$$\epsilon_{\pi'}(\rho', a) := \frac{\mathcal{R}(\pi' \boxtimes \delta(\rho', a), w_0)}{\mathcal{R}(\pi'^{\mathrm{gen}} \boxtimes \delta(\rho', a), w_0)},$$

where π'^{gen} is the unique $\chi'_{b'}$-generic member in $\Pi_{\phi'}$ (see Lemma 14.0.3).

Proposition 14.2.3. *The map $\Pi_{\phi'} \to \hat{\mathcal{S}}_{\phi'}$, $\pi' \mapsto \epsilon_{\pi'}$, has the following properties.*

(a) *Suppose that $\mathsf{depth}(\pi') \leqslant m$. There exists $l = l(m, n)$ large enough such that, for any field F that is l-close to F' and the representation π of H_n with $\pi \sim_l \pi'$ and parameter ϕ, and the character χ_b of F with $\chi_b \sim_l \chi'_{b'}$ (making the Whittaker data*

(U, χ_b) and (U', χ_b') compatible), we have

$$\epsilon_{\pi}(\rho, a) = \epsilon_{\pi'}(\rho', a),$$

where $(\rho, a) \in \mathrm{Jord}(\phi)$, $(\rho, a) \sim_l (\rho', a)$.

(b) *The map is injective.*

(c) *The map is surjective if Condition* $\mathbf{C}1(p)$ *holds.*

Proof. Property (a) follows by Properties (I)–(IX) of § 13.4, Theorem 13.6.1 and Proposition 14.1.1. For (b), let π_1' and π_2' be nonisomorphic representations in $\Pi_{\phi'}$, and let $m = \max\{\mathrm{depth}(\pi_1'), \mathrm{depth}(\pi_2')\}$. Choose l large enough so that (a) holds for π_1' and π_2'. Then, for any local field F of characteristic 0 that is l-close to F', we obtain nonisomorphic π_1 and π_2 in Π_ϕ, with $\pi_i \sim_l \pi_i'$, $i = 1, 2$, $\phi \sim_l \phi'$, and, furthermore,

$$\epsilon_{\pi_i}(\rho, a) = \epsilon_{\pi_i'}(\rho', a), \quad i = 1, 2$$

for all $(\rho, a) \in \mathrm{Jord}(\phi)$ and $(\rho, a) \sim_l (\rho', a)$. Now, since $\pi_1 \not\cong \pi_2$, we know that there exists $(\rho, a) \in \mathrm{Jord}(\phi)$ such that

$$\epsilon_{\pi_1}(\rho, a) \neq \epsilon_{\pi_2}(\rho, a),$$

and (b) follows. (c) follows by Lemma 14.0.4. □

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