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Regularity of weak solution for a coupled system arising from a microwave heating model

HONG-MING YIN[1]† and WEI WEI[2]

[1] *Department of Mathematics, Washington State University, Pullman, WA 99164, USA*
[2] *Department of Mathematics, Guizhou University, Guiyang, Guizhou Province, China*
email: hyin@wsu.edu, wwei@gzu.edu.cn

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In this paper, we study the regularity of a weak solution for a coupled system derived from a microwave-heating model. The main feature of this model is that electric conductivity in the electromagnetic field is assumed to be temperature dependent. It is shown that the weak solution of the coupled system possesses some regularity under certain conditions. In particular, it is shown that the temperature is Hölder continuous, even if electric conductivity has a jump discontinuity with respect to the temperature change. The main idea in the proof is based on an estimate for a linear degenerate system in Campanato space. As an application, the regularity result for the coupled system is used to derive the necessary condition for an optimal control problem arising in microwave heating processes.

Key words: Time-harmonic Maxwell's system; Temperature-dependent electric conductivity; Microwave heating model; Regularity of weak solutions

1 Introduction

Let Ω be a bounded, simply connected domain in R^3 with $C^{2+\alpha}$ boundary $\partial\Omega$ and $Q_T = \Omega \times (0, T]$ for any fixed $T > 0$. In this paper, we study the following coupled system. Find a complex electric field $\mathbf{E}(x, t)$ and a temperature distribution $u(x, t)$ such that

$$\nabla \times [\gamma(x)\nabla \times \mathbf{E}] + \xi(x, u)\mathbf{E} = 0, \qquad (x, t) \in Q_T, \tag{1.1}$$

$$u_t - \nabla[k(x, t, u)\nabla u] = \frac{1}{2} Im\xi(x, u)|\mathbf{E}|^2, \qquad (x, t) \in Q_T, \tag{1.2}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}(x), \qquad (x, t) \in S_T, \tag{1.3}$$

$$u_{\mathbf{n}}(x, t) = 0, \qquad (x, t) \in S_T, \tag{1.4}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.5}$$

where $Im\xi(x, u)$ represents the imaginary part of the complex coefficient $\xi(x, u)$ (see the explicit definition in (1.6)–(1.11) below), $\mathbf{n}$ is the outward unit normal on $S = \partial\Omega$, $u_{\mathbf{n}}(x, t) = \nabla u \cdot \mathbf{n}$ represents the normal derivative on $S_T = S \times [0, T]$ and $\mathbf{G}(x)$ is the applied electric field on S, generated by an external source. The functions $\gamma(x)$, $\xi(x, u)$ and $k(x, t, u)$

† Corresponding author.

are known. Here and thereafter, a bold letter represents a vector or a vector function in R^3.

The coupled system (1.1)–(1.5) is derived from a model which describes a microwave heating process [16,17,21,24]. Indeed, it is well known that the classical Maxwell's system in a conductive medium satisfies the following system [11]:

$$\varepsilon \mathbf{E}_t + \sigma \mathbf{E} = \nabla \times \mathbf{H}, \qquad \mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0,$$

where ε, μ and σ represent, respectively, the electric permittivity, the magnetic permeability and the electric conductivity of the medium.

If we assume that the electric and magnetic fields are time harmonic with frequency ω:

$$\mathbf{E}(x,t) = \hat{\mathbf{E}}(x)e^{i\omega t}, \qquad \mathbf{H}(x,t) = \hat{\mathbf{H}}(x)e^{i\omega t},$$

then it is easy to see that $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{H}}(x)$ satisfy the following system:

$$(i\omega\varepsilon + \sigma)\hat{\mathbf{E}} = \nabla \times \hat{\mathbf{H}}, \qquad i\omega\mu\hat{\mathbf{H}} + \nabla \times \hat{\mathbf{E}} = 0.$$

It follows that the time-harmonic electric field $\hat{\mathbf{E}}(x)$ satisfies

$$\nabla \times [\gamma \nabla \times \hat{\mathbf{E}}] + \xi \hat{\mathbf{E}} = 0, \qquad x \in \Omega,$$

where

$$\gamma = \frac{1}{\mu}, \qquad \xi = i\omega[i\omega\varepsilon + \sigma].$$

In many industrial applications, electric conductivity σ is often used as a switch device via temperature. A classical example is the thermistor model (see [1, 4, 9]), where the electric field is assumed to be a gradient of a potential function. The resulting model consists of an elliptic–parabolic system. This model has been studied extensively by many researchers, for example, see [1,3,4,9]. These applications motivate us to study the system (1.1)–(1.2) along with initial-boundary conditions (1.3)–(1.5). We focus on the case where electric conductivity strongly depends on the temperature:

$$\sigma = \sigma(x, u),$$

while the electric permittivity ε and magnetic permeability μ are assumed only to be functions of x. Those assumptions are typical in electronic device simulations (see [16,17]).

In dealing with microwave and inductive heating, we follow the monograph [17] in using a unified approach and assume that the electric permittivity and magnetic permeability are complex functions:

$$\varepsilon = \varepsilon_0[\varepsilon_1(x) - i\varepsilon_2(x)],$$
$$\mu = \mu_0[\mu_1(x) - i\mu_2(x)].$$

With the above assumption, one can easily see that the coefficients in system (1.1) are defined precisely as follows:

$$\gamma(x) = \gamma_1(x) + i\gamma_2(x), \tag{1.6}$$

$$\xi(x,u) = -\xi_1(x,u) + i\xi_2(x,u), \tag{1.7}$$

$$\gamma_1(x) = \frac{\mu_1(x)}{\mu_1(x)^2 + \mu_2(x)^2} > 0, \tag{1.8}$$

$$\gamma_2(x) = \frac{\mu_2(x)}{\mu_1(x)^2 + \mu_2(x)^2} > 0, \tag{1.9}$$

$$\xi_1(x,u) = \omega^2\varepsilon_0\varepsilon_1(x) > 0, \tag{1.10}$$

$$\xi_2(x,u) = \omega[\sigma(x,u) + \omega\varepsilon_0\varepsilon_2(x)] > 0. \tag{1.11}$$

The reader is referred to the monograph [17] for a detailed explanation of physical interpretation for all parameters.

Note that the time-harmonic system (1.1) is just an approximation of Maxwell's equations since temperature is a function of t. However, this approximation can be justified due to the fact that the time change for temperature is much slower than that for an electromagnetic field. This approximation is frequently used in various industrial applications for governing electric and magnetic fields (see [16,17]). Note that, if we assume that the electric field is just the gradient of a potential function, then the system (1.1)–(1.2) becomes the classical model for a thermistor device.

The system (1.1)–(1.5) is first studied in [25], where the existence of a global weak solution was established under certain conditions. A generalized existence result is established in [21]. However, the regularity of the weak solution is not investigated in [21]. In particular, it is an open question about whether or not the temperature is uniformly bounded. This is extremely important in device designing because of safety reasons [9]. We will answer this question in the present paper. We will show that the temperature is Hölder continuous if $\sigma(x,u)$ is bounded. This regularity result includes an interesting case where $\sigma(x,u)$ acts like a switch [3], namely,

$$\sigma(x,u) = \begin{cases} 1, & \text{if } u \leqslant K, \\ 0, & \text{if } u > K, \end{cases}$$

where K is a constant.

The main idea of our proof follows from a crucial result in [25], which deals with the regularity of the following linear system:

$$\nabla \times [\gamma(x)\nabla \times \mathbf{E}] + \xi(x)\mathbf{E} = 0, \qquad x \in \Omega. \tag{1.12}$$

It is shown in [25] that the weak solution $\mathbf{E}(x)$ is Hölder continuous if $\gamma(x)$ is bounded, its real part has a positive lower bound and $\xi(x)$ is Lipschitz continuous in Ω. This regularity result is optimal (see [8,24] for a special case). However, when σ depends on the temperature, the coefficient $\xi(x,u)$ will also depend on temperature and the assumption for ξ in [25] does not hold. Therefore, the known result cannot be used directly here since $\mathbf{E}(x)$ may not be continuous when $\sigma = \sigma(x,u)$ has a jump discontinuity with respect to u-variable. We overcome this difficulty by using a certain symmetric feature for the system (1.1). With some careful analysis, we are able to show that a magnetic field possesses the desired regularity in Campanato space. With the help of a Campanato-type estimate for a linear parabolic problem [22], we are able to show that temperature is indeed Hölder continuous, even if $\sigma(x,u)$ is discontinuous (see Remark 3.2). As an application, the regularity result obtained in this paper is used to obtain the necessary condition for an optimal control problem considered in [21] (also see [12] for the original model).

The rest of the paper is organized as follows. In Section 2, after reviewing some known results we prove a regularity result for the weak solution to a linear system (1.12) under certain conditions. In Section 3, we prove the main regularity result for the weak solution of (1.1)–(1.5), which includes the Hölder continuity for the temperature. Moreover, it is shown that the weak solution is regular if the known data and coefficients are regular. In Section 4, the necessary condition for an optimal control problem is established. A concluding remark is given in Section 5.

2 Regularity of weak solution for a linear system

In this section, we recall some known results for the reader's convenience since some of those results such as Lemmas 2.3 and 2.4 are not standard in the literature.

We first introduce some basic Banach spaces. For a Banach space B and a positive integer N, B^N is the usual product space equipped with the product norm. For brevity, we use B instead of B^N without causing any confusion:

$$H(curl,\Omega)=\{\mathbf{G}(x)\in L^2(\Omega):\nabla\times\mathbf{G}\in L^2(\Omega)\},$$
$$H_0(curl,\Omega)=\{\mathbf{G}(x)\in L^2(\Omega):\nabla\times\mathbf{G}\in L^2(\Omega),\mathbf{n}\times\mathbf{G}=0\ \text{ on }\partial\Omega,\},$$
$$H(div,\Omega)=\{\mathbf{G}(x)\in L^2(\Omega):\nabla\cdot\mathbf{G}\in L^2(\Omega)\},$$

$H_0(curl,\Omega)$ and $H(curl,\Omega)$ are Hilbert spaces equipped with the inner product

$$<\mathbf{G},\mathbf{F}>=\int_\Omega[(\nabla\times\mathbf{G})\cdot(\nabla\times\mathbf{F}^*)+\mathbf{G}\cdot\mathbf{F}^*]dx,$$

where $\mathbf{F}^*$ represents the complex conjugate of $\mathbf{F}$ (see [7]).

For a complex function $\xi(x)\in L^\infty(\Omega)$ with $Re(\xi)\geqslant\xi_0$ on Ω (or $Re(\xi)<-\xi_0$) for some constant $\xi_0>0$, we define a weighted Sobolev space $H_0(curl,div_\xi,\Omega)$:

$$H_0(curl,div_\xi,\Omega)=\{\mathbf{G}\in H_0(curl,\Omega):div(\xi\mathbf{G})\in L^2(\Omega)\}$$

equipped with the following norm

$$||\mathbf{G}||^2_{H_0(curl,div_\xi,\Omega)}=\int_\Omega[|\nabla\times\mathbf{G}|^2+|\nabla(\xi\mathbf{G})|^2+|\mathbf{G}|^2]dx.$$

It is known that $H_0(curl,div_\xi,\Omega)$ is also a Banach space. Moreover, the embedding operator from $H_0(curl,div_\xi,\Omega)$ into $L^2(\Omega)$ is compact [7].

Other spaces such as $H^1(\Omega), W^{2,p}(\Omega)$ and $W^{2,1}_p(Q_T)$ are the usual Sobolev spaces (see [10]). We also recall the Morrey–John–Nirenberg–Campanato space here. For any $x_0\in\Omega$ and $\rho>0$, let

$$B(x_0,\rho)=\{x\in\Omega:|x-x_0|<\rho\}.$$

For a function in $L^2(\Omega)$ and a nonnegative constant $\mu \geqslant 0$, define

$$[u]_{2,\mu} = \sup_{\rho>0, x_0\in\Omega} \rho^{-\mu} \int_{B(x_0,\rho)} |u - (u)_{x_0}|^2 dx,$$

where

$$(u)_{x_0} = \frac{1}{|B(x_0,\rho)|} \int_{B(x_0,\rho)} u(x)dx.$$

Define

$$L^{2,\mu}(\Omega) = \{f(x) \in L^2(\Omega) : ||f||_{2,\mu} = ||f||_{L^2(\Omega)} + [f]_{2,\mu} < \infty\}.$$

It is well known that $L^{2,\mu}(\Omega)$ is a Banach space for any $\mu \geqslant 0$ (see [19]).

A more interesting result is the following lemma.

Lemma 2.1 (see [6,19]) *For $\mu \in (n, n+2)$, the space $L^{2,\mu}(\Omega)$ is equivalent to $C^\alpha(\bar{\Omega})$ algebraically and topologically, where $\alpha = \frac{\mu-n}{2}$.*

When dealing with parabolic equations, we replace $B(x_0,\rho)$ by

$$Q(x_0,t_0;\rho) = B(x_0,\rho) \times (t_0-\rho^2, t_0], \qquad (x_0,t_0) \in Q_T.$$

One has a similar result to Lemma 2.1 with dimension n replaced by $n+2$.

Before investigating the coupled system (1.1)–(1.5), we consider the following linear system:

$$\nabla \times [\gamma(x)\nabla \times \mathbf{W}] + \xi(x)\mathbf{W} = \mathbf{J}_1(x) + \nabla \times \mathbf{J}_2(x), \qquad x \in \Omega, \tag{2.1}$$

$$\mathbf{n} \times \mathbf{W}(x) = 0, \qquad x \in S = \partial\Omega, \tag{2.2}$$

where $\gamma(x) = \gamma_1(x) + i\gamma_2(x), \xi(x) = -\xi_1(x) + i\xi_2(x)$. The vector fields $\mathbf{J}_1(x)$ and $\mathbf{J}_2(x)$ are given on Ω.

It is clear that the linear system (2.1) is degenerate (sub-elliptic). The classical regularity theory for elliptic systems (see [6]) is not valid here. Our question is to find conditions such that the weak solution $\mathbf{W}(x)$ is Hölder continuous. First of all, the following conditions are always assumed to ensure the existence of a unique weak solution.

H(2.1). (a) Assume that functions $\gamma_1(x), \gamma_2(x), \xi_1(x)$ and $\xi_2(x)$ are real and measurable in Ω. There exist positive constants a_0 and A_0 such that

$$\gamma_1(x), \xi_1(x) \geqslant a_0 > 0, \qquad ||\gamma||_{L^\infty(\Omega)} + ||\xi||_{L^\infty(\Omega)} \leqslant A_0.$$

(b) Assume that $\mathbf{J}_1(x), \mathbf{J}_2(x) \in L^2(\Omega)$. There exists a constant B_0 such that

$$||\mathbf{J}_1||_{L^2(\Omega)} + ||\mathbf{J}_2||_{L^2(\Omega)} \leqslant B_0.$$

The existence of a unique weak solution to (2.1)–(2.1) is well known (see, for example, [5,7]). A different proof is presented in [25] under weaker regularity assumptions for the coefficients in (2.1).

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Lemma 2.2 *Under the assumption H(2.1), the linear system (2.1)–(2.2) has a unique weak solution with* $\mathbf{W}(x) \in H_0(curl, div_\xi, \Omega)$. *Moreover, there exists a constant* C_1 *such that*

$$\int_\Omega |\nabla \times \mathbf{W}|^2 dx + \int_\Omega |\nabla(\xi \mathbf{W})|^2 dx \leqslant C_1[||\mathbf{J}_1||_{L^2(\Omega)} + ||\mathbf{J}_2||_{L^2(\Omega)}], \tag{2.3}$$

where C_1 *depends only on the constants in H(2.1).*

The estimate in Lemma 2.2 is directly derived from the energy estimate.

To improve the regularity of $\mathbf{W}(x)$, *we need additional conditions for the coefficients* $\xi(x)$ *and inhomogeneous terms* $\mathbf{J}_1(x)$ *and* $\mathbf{J}_2(x)$.

H(2.2). (a) Let $\xi(x) \in W^{1,\infty}(\Omega)$ with

$$||\nabla \xi||_{L^\infty(\Omega)} \leqslant A_1.$$

(b) Let $\mathbf{J}_1(x), \mathbf{J}_2(x) \in L^{2,\mu_0}(\Omega)$ for some $\mu_0 \in (1, 2)$ with

$$||\mathbf{J}_1||_{L^{2,\mu_0}(\Omega)} + ||\mathbf{J}_2||_{L^{2,\mu_0}(\Omega)} \leqslant A_2.$$

Lemma 2.3 *Under the assumptions H(2.1)–H(2.2), the weak solution* $\mathbf{W}(x)$ *is Hölder continuous on* $\bar{\Omega}$. *Moreover, there exists a constant C such that*

$$||\nabla \times \mathbf{W}||_{L^{2,\mu_0}(\Omega)} + ||\mathbf{W}||_{C^\alpha(\bar{\Omega})} \leqslant C_2[||\mathbf{J}_1||_{L^{2,\mu_0}(\Omega)} + ||\mathbf{J}_2||_{L^{2,\mu_0}(\Omega)}],$$

where the Hölder exponent $\alpha = \frac{\mu_0 - 1}{2}$ *and the constant* C_2 *depends only on* a_0, A_0, B_0, A_1 *and* A_2.

Proof. The proof follows the same idea as in [24]. Here, we just give an outline of the proof for the reader's convenience. To prove the Hölder continuity, we may assume that $\mathbf{W}(x)$ is a classical solution in the derivation since all estimates are independent of the smoothness of the coefficients. Otherwise, one can simply use the smooth approximation for all coefficients and known functions. All of the estimates depend only on the known constants in H(2.1)–H(2.2) and the domain Ω. Indeed, since Ω is simply-connected, from

$$\nabla(\xi(x)\mathbf{W}) = 0, \qquad x \in \Omega,$$

we see that $\xi(x)\mathbf{W}(x)$ can be expressed by

$$\begin{aligned} &\xi(x)\mathbf{W}(x) = \nabla \times \mathbf{G}(x), \qquad \nabla \cdot \mathbf{G}(x) = 0, \qquad x \in \Omega, \\ &\mathbf{n} \cdot \mathbf{G}(x) = 0, \qquad x \in S. \end{aligned}$$

Since $Re(\xi(x)) \geqslant a_0 > 0$ and $\xi(x) \in W^{1,\infty}(\Omega)$, it follows that the weak solution $\mathbf{W}(x) \in H^1(\Omega)$. Moreover,

$$||\mathbf{W}||_{H^1(\Omega)} \leqslant C[||\mathbf{J}_1||_{L^2(\Omega)} + ||\mathbf{J}_2||_{L^2(\Omega)}].$$

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Furthermore, by using the same argument as for elliptic systems (see [19]), we can show that $\nabla \times \mathbf{W}, \nabla \cdot \mathbf{W} \in L^{2,\mu_0}(\Omega)$ (see [24]). Moreover, there exists a constant C such that

$$||\nabla \times \mathbf{W}||_{L^{2,\mu_0}(\Omega)} + ||\nabla \cdot \mathbf{W}||_{L^{2,\mu_0}(\Omega)} \leqslant C[||\mathbf{J}_1||_{L^2(\Omega)} + ||\mathbf{J}_2||_{L^{2,\mu_0}(\Omega)} + ||\mathbf{W}||_{H^1(\Omega)}],$$

where C depends only on known data.

Now we apply the embedding theorem to find that $\mathbf{W}(x) \in L^{2,\mu_0+2}(\Omega)$ and

$$||\mathbf{W}||_{L^{2,2+\mu_0}(\Omega)} \leqslant C[||\nabla \times \mathbf{W}||_{L^{2,\mu_0}(\Omega)} + ||\nabla \cdot \mathbf{W}||_{L^{2,\mu}(\Omega)} + ||\mathbf{W}||_{L^2(\Omega)}] \leqslant C.$$

Since the space dimension for the system (2.1) $n = 3$ and $\mu = \mu_0 + 2 > 3$ by H(2.2)(b), we see $\mathbf{W}(x) \in C^{\alpha}(\bar{\Omega})$ with $\alpha = \frac{\mu_0 - 1}{2}$. □

Next, we state an estimate in Campanato space for the solution of a parabolic equation. This result is essential in the proof of the main result in Section 3. Consider the following parabolic equation:

$$\begin{aligned} &u_t - L[u] = f_0(x,t) + \sum_{i=1}^{n} (f_i(x,t))_{x_i}, \qquad (x,t) \in Q_T, \\ &u_n(x,t) = 0, \qquad (x,t) \in S_T, \\ &u(x,0) = u_0(x), \qquad x \in \Omega. \end{aligned}$$

where

$$L[u] = (a_{ij}(x,t)u_{x_i})_{x_j} + b_i(x,t)u_{x_i} + c(x,t)u.$$

The following basic assumptions are needed.

H(2.3). (a) Let $a_{ij}(x,t)$ be measurable in Q_T and satisfy the ellipticity condition:

$$a_1|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant a_2|\xi|^2, \qquad \xi \in R^n, a_1, a_2 > 0.$$

(b) Let $b_i(x,t)$ and $c_i(x,t)$ be of class $L^\infty(Q_T)$ with

$$\sum_{i=1}^{n} ||b_i||_{L^\infty(Q_T)} + ||c||_{L^\infty(Q_T)} \leqslant A_3.$$

(c) Let $f_0(x,t) \in L^{2,(\mu-2)^+}(Q_T), f_i(x,t) \in L^{2,\mu}(Q_T)$ for some $\mu > 0$.

Lemma 2.4 ([22]) *Under the assumptions of H(2.3), the weak solution $u(x,t) \in C([0,T];H^1(\Omega))$ satisfies the following estimate:*

$$||\nabla u||_{L^{2,\mu}(Q_T)} \leqslant C[||f_0||_{L^{2,(\mu-2)^+}(Q_T)} + \sum_{i=1}^{n} ||f_i||_{L^{2,\mu}(Q_T)}],$$

where C is a constant that depends only on known data a_1, a_2, A_3 and Q_T. Particularly, when $\mu \in (n, n+2)$, $u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ with $\alpha = \frac{\mu - n}{2}$.

Remark 2.1 The significance of Lemma 2.4 is that when $\mu \in (n+2, n+4)$, the conditions of f_0, f_i for the Hölder continuity of a weak solution are weaker than those for classical results [2, 10, 14].

3 Regularity of weak solution

Now, we consider the system (1.1)–(1.5). To ensure the existence of a unique weak solution, we assume the following conditions to be satisfied throughout this section.

H(3.1). (a) Assume that functions $\gamma_1(x), \gamma_2(x), \sigma(x,u), \xi_1(x)$ and $\xi_2(x)$ are real and measurable. Moreover, there exist positive constants a_0, A_0 and A_1 such that

$$\gamma_1(x), \xi_1(x) \geqslant a_0 > 0, \qquad |\gamma(x)| + |\xi(x,u)| \leqslant A_0,$$
$$0 \leqslant \sigma(x,u) \leqslant A_1.$$

(b) The function $\gamma(x)$ is of class $W^{1,\infty}(\Omega)$ and there exists a constant B_0 such that

$$||\gamma||_{W^{1,\infty}(\Omega)} \leqslant B_0.$$

(c) Let $\mathbf{G}(x), \nabla \times \mathbf{G}(x) \in H(curl, \Omega) \bigcap L^{2,\mu_0}(\Omega)$ for some $\mu_0 \in (1,2)$.

H(3.2). (a) Let $k(x,t,u)$ be a measurable function and there exist two constants k_0, k_1 such that

$$0 < k_0 \leqslant k(x,t,u) \leqslant k_1.$$

(b) $u_0(x) \in C^{\alpha}(\bar{\Omega})$.

Now we are ready to state the main regularity result in this paper.

Theorem 3.1 *Under the assumption of H(3.1)–H(3.2), there exists a unique weak solution* $(\mathbf{E}(x,t), u(x,t))$ *for the problem (1.1)–(1.5):*

$$\mathbf{E}(x,t) - \mathbf{G}(x) \in L^{\infty}(0,T; H_0(curl, div_{\xi}, \Omega)), u(x,t) \in C([0,T]; H^1(\Omega)).$$

Moreover, the weak solution possesses the following regularity:

$$\nabla \times \mathbf{E}(x,t) \in L^{\infty}(0,T; L^{2,\mu_0}(\Omega)), \nabla u(x,t) \in L^{2,\mu_0+2}(Q_T).$$

In particular,

$$u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T),$$

where $\alpha = \frac{\mu_0 - 1}{2}$.

Proof. First of all, the existence of a weak solution for (1.1)–(1.5) is a special case in Theorem 3.1 of a recent paper [25]. Moreover, the weak solution $(\mathbf{E}(x,t), u(x,t))$ has the following regularity:

$$\mathbf{E}(x,t) - \mathbf{G}(x,t) \in L^{\infty}(0,T; H_0(curl, div_{\xi}, \Omega)), u(x,t) \in C([0,T]; H^1(\Omega)).$$

To illustrate the idea, we may assume that $\mathbf{G}(x,t) = 0$ on $\partial\Omega$. Otherwise, we just define

$$\hat{\mathbf{E}}(x,t) = \mathbf{E}(x,t) - \mathbf{G}(x).$$

Then, $\hat{\mathbf{E}}$ satisfies (2.1) with

$$\mathbf{J}_1(x,t) = -\xi(x,u)\mathbf{G}(x), \mathbf{J}_2(x,t) = \gamma(x)\nabla \times \mathbf{G}(x).$$

We note that $\xi(x,u)$ clearly does not satisfy the conditions in Lemma 2.3 since we do not know whether or not that ∇u is bounded. Moreover, when $\sigma(x,u)$ has a jump discontinuity with respect to u-variable, $\xi(x,u)$ may not be continuous.

To overcome this difficulty, we define

$$\mathbf{H}(x,t) = \gamma(x)\nabla \times \mathbf{E}(x,t). \qquad (x,t) \in Q_T.$$

Then we have

$$\nabla \times \mathbf{E}(x,t) = \frac{1}{\gamma(x)}\mathbf{H}(x,t). \tag{3.1}$$

On the other hand, from equation (1.1) we see

$$\nabla \times \mathbf{H} + \xi(x,u)\mathbf{E} = 0, \qquad (x,t) \in Q_T. \tag{3.2}$$

It follows that

$$\mathbf{E}(x,t) = -\frac{1}{\xi(x,u)}\nabla \times \mathbf{H}(x,t).$$

Hence, by applying curl operation we see

$$\nabla \times \left[\frac{1}{\xi(x,u)}\nabla \times \mathbf{H}\right] + \frac{1}{\gamma(x)}\mathbf{H} = 0, \qquad (x,t) \in Q_T. \tag{3.3}$$

Now by the assumption H(3.1), we see that the coefficients $\frac{1}{\xi(x,u)}$ and $\frac{1}{\gamma(x)}$ satisfy the conditions in Lemma 2.3. There exists a number $\mu_0 \in (1,2)$ such that $\mathbf{H}(x,t)$ has the following regularity:

$$\sup_{t\in[0,T]} ||\nabla \times \mathbf{H}||_{L^{2,\mu_0}(\Omega)} + \sup_{t\in[0,T]} ||\nabla \cdot \mathbf{H}||_{L^{2,\mu_0}(\Omega)} \leqslant C,$$

where C depends only on known data.

It follows by the embedding property (see [23]) that $\mathbf{H}(x,t) \in L^{2,\mu_0+2}(\Omega)$ for each $t \in [0,T]$. Consequently, $\mathbf{H}(x,t)$ is Hölder continuous with respect to x for each $t \in [0,T]$.

Clearly, from the definition of Campanato space for a parabolic region Q_T we have

$$||\nabla \times \mathbf{H}||_{L^{2,\mu_0+2}(Q_T)} + ||\nabla \cdot \mathbf{H}||_{L^{2,\mu_0+2}(Q_T)} \leqslant C, \tag{3.4}$$

where C depends only on known data.

It follows that

$$||\mathbf{H}||_{C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)} \leqslant C, \tag{3.5}$$

where C depends only on known data.

From equation (3.2), we see

$$\sup_{t\in[0,T]} ||\mathbf{E}||_{L^{2,\mu_0}(\Omega))} \leqslant C. \tag{3.6}$$

Hence, in the parabolic region Q_T, we find

$$||\mathbf{E}||_{L^{2,\mu_0+2}(Q_T))} \leqslant C. \tag{3.7}$$

To obtain more regularity for $u(x,t)$, we perform the following conversion for the heat source. From

$$\mathbf{E}(x,t) = \frac{1}{\xi(x,u)} \nabla \times \mathbf{H}(x,t),$$

we see that the local heat density can be expressed in terms of $\nabla \times \mathbf{H}$:

$$\xi_2(x,u)|\mathbf{E}|^2 = \frac{\xi_2(x,u)}{|\xi(x,u)|^2} |\nabla \times \mathbf{H}|^2$$
$$= \rho(x,u)|\nabla \times \mathbf{H}|^2,$$

where

$$\rho(x,u) = \frac{\xi_2(x,u)}{|\xi(x,u)|^2}.$$

Note that by (3.2),

$$\nabla \times \mathbf{H} + \xi(x,u)\mathbf{E} = 0,$$

it follows that

$$\rho(x,u)\nabla \times \mathbf{H} = -\rho(x,u)\xi(x,u)\mathbf{E}.$$

Hence,

$$\mathbf{H} \times [\rho(x,u)\nabla \times \mathbf{H}] = \rho(x,u)\xi(x,u)\mathbf{E} \times \mathbf{H}.$$

Now, we use an identity of vector operation:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

to find

$$\rho(x,u)|\nabla \times \mathbf{H}|^2 = \nabla \cdot [\mathbf{H} \times (\rho(x,u)\nabla \times \mathbf{H}] + \nabla \cdot [\rho(x,u)\xi(x,u)\mathbf{E} \times \mathbf{H}].$$

Now, we know that $\mathbf{H}$ is uniformly bounded in $L^\infty(Q_T)$. Moreover, the space $L^\infty(Q_T)$ is a multiplier of $L^{2,\mu}(Q_T)$ if $3 < \mu < 5$ (see [19] for space dimension $N = 3$).

Next, we can apply $L^{2,\mu}$-theory for parabolic equations, Lemma 2.4, to obtain

$$||\nabla u||_{L^{2,\mu_0+2}(Q_T)} \leqslant C[||\nabla \times \mathbf{H}||_{L^{2,\mu_0+2}(Q_T)} + ||\mathbf{E}||_{L^{2,\mu_0+2}(Q_T)}].$$

By standard embedding (see [2]), we see

$$u(x,t) \in L^{2,\mu_0+4}(Q_T).$$

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Since $\mu_0 \in (1, 2)$, we see $\mu_0 + 4 \in (5, 7)$. It follows that

$$u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T),$$

where $\alpha = \frac{\mu_0 - 1}{2}$. □

Corollary 3.2 Under the assumptions of H(3.1)–H(3.2), the weak solution $(\mathbf{E}, u)$ is regular if all coefficients and known data are smooth and the consistency conditions hold on $S_T \bigcap \{t = 0\}$.

Proof. The proof is based on the bootstrap technique. Indeed, since $u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$, from equation (3.3), the coefficient $\frac{1}{\gamma(x,u)}$ is Hölder continuous and the real part has a positive lower bound. Moreover, the coefficient $\frac{1}{\gamma(x)}$ is uniformly bounded in Ω. It follows from [23] that $\mathbf{H}(x,t) \in C^{1+\alpha}(\Omega)$ for each $t \in [0, T]$. Moreover,

$$\sup_{0 \leqslant t \leqslant T} ||u||_{C^{1+\alpha}(\bar{\Omega})} \leqslant C.$$

It follows that the local heat source

$$\xi_2(x,u)|\mathbf{E}|^2 = \rho(x,u)|\nabla \times \mathbf{H}|^2$$

is Hölder continuous in $\bar{Q}_T$.

Now we use the regularity theory for the parabolic equation (1.2) to obtain

$$u \in C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{Q}_T).$$

Since $k(x,t,u)$ is smooth, we can use Schauder's theory to conclude that

$$u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T).$$

Moreover, from equation (1.1) we use the theory of elliptic systems to conclude that $\mathbf{E}(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T)$.

Remark 3.1 From the proof of Theorem 3.1, we see that the magnetic field $\mathbf{H}(x,t)$ is Hölder continuous. However, the electric field $\mathbf{E}(x,t)$ may not be continuous.

Remark 3.2 The regularity result in Theorem 3.1 includes a large number of important cases in applications for the electric conductivity σ.

For example, in the microwave heating model, electric conductivity for certain materials has the following form:

$$\sigma(x,u) = (1+u)^{-p}, (p > 0), \quad \text{or} \quad \sigma(x,u) = 1 + e^{-u}.$$

Then, the problem (1.1)–(1.5) possesses a unique weak solution. In particular, the temperature $u(x,t)$ is uniformly bounded and is Hölder continuous in Q_T.

Another important case in practice is that $\sigma(x,u)$ serves as a switch:

$$\sigma(x,u)=1,\ \text{if } u\leqslant K \text{ and } \sigma(x,u)=0, \text{if } u>K.$$

The regularity result in Theorem 3.1 yields that the temperature is Hölder continuous. For the scalar case, this result is established by Chen and Friedman in [3].

4 A necessary condition for a optimal control problem

In a recent paper [21], the authors considered an optimal control problem associated with the system (1.1)–(1.5). The external electric field $\mathbf{G}(x)$ is chosen to be the control variable such that the terminal temperature reaches the desired distribution. We recall the problem here for the reader's convenience.

Optimal control problem (P). Given $T>0$ and a desired temperature $u_T(x)\in L^2(\Omega)$ at a final time T, find an optimal control $\mathbf{G}^0\in U_{ad}$ such that the cost functional

$$J(\mathbf{G};\mathbf{E},u):=\int_\Omega |u(x,T)-u_T(x)|^2dx+\frac{\lambda}{2}\int_{\partial\Omega}|\mathbf{G}(x)|^2ds \tag{4.1}$$

reaches its minimum at $(u^0,\mathbf{E}^0)$ for all $\mathbf{G}\in U_{ad}$, where $(\mathbf{E},u)$ and $(\mathbf{E}^0,u^0)$ are weak solutions of the coupled system (1.1)–(1.5) corresponding to $\mathbf{G}$ and $\mathbf{G}^0$, respectively. The number $\lambda>0$ is the typical regularization parameter (see [15, 18, 20]).

The admissible control set U_{ad} is defined as follows:

$$U_{ad}=\{\mathbf{G}\in L^2(S):\|\mathbf{G}\|_{L^2(S)}\leqslant A_0<\infty\},$$

where A_0 is a constant.

It is shown in [21] that there exists an optimal control $\mathbf{G}^0\in U_{ad}$. However, due to the lack of the desired regularity of the temperature $u(x,t)$, the necessary condition is derived only for a special case. Now with the result of Theorem 3.1, we can derive the necessary condition.

Theorem 4.1 *In addition to the assumptions H(3.1) and H(3.2), assume that $\xi(x,t)$ is differentiable with respect to u, then the mapping $\mathbf{G}\mapsto(\mathbf{E},u)$ is differentiable in the following sense:*

$$\mathbf{E}_\varepsilon(x,t):=\frac{\mathbf{E}(\mathbf{G}+\varepsilon\mathbf{H})-\mathbf{E}(\mathbf{G})}{\varepsilon}\longrightarrow\mathbf{E}(x,t),\textit{weakly-* in } L^\infty(0,T;H^1(\Omega)),$$
$$U_\varepsilon(x,t):=\frac{u(\mathbf{G}+\varepsilon\mathbf{H})-u(\mathbf{G})}{\varepsilon}\longrightarrow U(x,t),\textit{weakly in } W[0,T],$$

for any $\mathbf{G},\mathbf{H}\in U_{ad}$ such that $\mathbf{G}+\varepsilon\mathbf{H}\in U_{ad}$ for sufficiently small ε. $\mathbf{E}\in W_0[0,T]$ and $U\in H^1(0,T;L^2(\Omega))\bigcap C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ satisfy

$$\begin{aligned}&\nabla\times[\gamma(x)\nabla\times\mathbf{E}]+[-\xi_1(x,u)+i\xi_2(x,u)]\mathbf{E}\\&\quad=[(\xi_1)_u(x,u)-i(\xi_2)_u(x,u)]U\mathbf{E},\qquad \textit{in } Q_T,\end{aligned}\tag{4.2}$$

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$$U_t - \nabla[k(x,t,u)\nabla U] = \nabla[(k_u(x,t,u)\nabla u)U]$$
$$+\frac{1}{2}(\xi_2)_u(x,u)|w|^2 U + \xi_2(x,u)wW, \qquad in\, Q_T, \tag{4.3}$$
$$\mathbf{n} \times \mathbf{E}(x,t) = \mathbf{n} \times \mathbf{H}(x), \qquad on\ S_T, \tag{4.4}$$
$$U_{\mathbf{n}}(x,t) = 0, \qquad on\ \ S_T, \tag{4.5}$$
$$U(x,0) = 0, \qquad in\ \Omega, \tag{4.6}$$

where we denote by $u = u(\mathbf{G})$ and $\mathbf{E} = \mathbf{E}(\mathbf{G})$ the solution of (1.1)–(1.5) corresponding to $\mathbf{G}(x)$.

Proof. The proof follows the same steps as in [21]. Here, we only give an outline.

We write $u^\varepsilon = u(\mathbf{G}^\varepsilon), \mathbf{E}^\varepsilon = \mathbf{E}(\mathbf{G}^\varepsilon)$ the solution of (1.1)–(1.5) corresponding to $\mathbf{G}^\varepsilon := \mathbf{G} + \varepsilon\mathbf{H}$ for any $\mathbf{H} \in U_{ad}$.

Step 1. There exist constants C_1 and C_2 such that

$$\int_\Omega [|\nabla \times \mathbf{E}_\varepsilon|^2 + |\nabla \cdot (\xi \mathbf{W}|^2 + |W_\varepsilon|^2]dx \leqslant C_1,$$
$$\sup_{0\leqslant t\leqslant T} \int_\Omega |U_\varepsilon|^2 dx + \int\int_{Q_T} |\nabla U_\varepsilon|^2 dx\, dt \leqslant C_2.$$

These estimates follow the same steps as in [21]. Since u is uniformly bounded, all energy estimates can be carried out for $\mathbf{E}$.

Step 2. Now we are ready to derive the system (4.2)–(4.6). The estimates in step 1 imply that there exist a subsequence of $\varepsilon \to 0$ and an $U \in W[0,T]$ and $\mathbf{E} \in L^\infty(0,T;H^1(\Omega))$ such that

$$U_\varepsilon = \frac{u^\varepsilon - u}{\varepsilon} \longrightarrow U, \text{ weakly in } W[0,T];$$
$$W_\varepsilon = \frac{w^\varepsilon - w}{\varepsilon} \longrightarrow W, \text{ weakly in } L^\infty(0,T;H^1(\Omega));$$
$$U_\varepsilon \longrightarrow U, \text{ strongly in } L^2(Q_T);$$
$$W_\varepsilon \longrightarrow W, \text{ strongly in } L^\infty(0,T;L^2(\Omega))$$

as $\varepsilon \to 0$. Furthermore, by selecting a subsequence if necessary, for a.e. $t \in [0,T]$,

$$U_\varepsilon \longrightarrow U(x,t), a.e., x \in \Omega, \tag{4.7}$$
$$W_\varepsilon \longrightarrow W(x,t), a.e., x \in \Omega, \tag{4.8}$$

as $\varepsilon \to 0$.

The rest of the calculation is similar to the scalar case (see [21]). □

One can also derive the adjoint system corresponding to the optimal control (see [20,21]).

Theorem 4.2 *Under the conditions of Theorem 4.1, if $\mathbf{G}^0 \in U_{ad}$ is an optimal control and corresponding state $(\mathbf{E}^0, u^0)$ which satisfies (1.1)–(1.5), then there exist $(p, \mathbf{Q}) \in W[0,T] \times$*

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$H(0,T;H^1(\Omega))$, which satisfy the adjoint system:

$$\nabla \times (\gamma(x)\nabla \times \mathbf{Q}) + [(-\xi_1(x,u^0) + i\xi_2(x,u^0))]p\mathbf{Q} = 0, \qquad in\ Q_T, \tag{4.9}$$

$$p_t - \nabla[k(x,t,u)\nabla p] = \frac{1}{2}(\xi_2(x,u))_u|\mathbf{E}^0|^2 p$$
$$+(-(\xi_1(x,u^0))_u + i(\xi_2(x,u^0)_u)\mathbf{E}^0 \cdot \mathbf{Q}, in\ Q_T, \tag{4.10}$$

$$p_{\mathbf{n}}(x,t) = 0, \qquad (x,t) \in S_T, \tag{4.11}$$

$$\mathbf{n} \times \mathbf{Q}(x,t) = 0, \qquad (x,t) \in S_T, \tag{4.12}$$

$$p(x,T) = u^0(x,T) - u_T(x), \qquad x \in \Omega. \tag{4.13}$$

Moreover, the following inequality is satisfied:

$$\int_0^T \int_S [(\gamma(x)\mathbf{G} + \lambda\mathbf{G}^0) \cdot (\mathbf{G} - \mathbf{G}^0)]ds\,dt \geqslant 0, \qquad \forall \mathbf{G} \in U_{ad}. \tag{4.14}$$

The proof is similar to the standard calculation for a single elliptic or parabolic equation (see [13,15,20]). We skip the detail here.

5 Conclusion

In this paper, we studied an elliptic–parabolic system which models microwave heating. The electric and magnetic fields are assumed to be time harmonic and electric conductivity is assumed to be temperature dependent. The Hölder continuity of temperature is established without the smoothness condition on the electric conductivity. In particular, this regularity result includes the important case where the electric conductivity serves like a switch. This result provides the theoretical foundation in designing thermally controlled electric devices. On the other hand, the method developed in this paper provides a new tool to deal with time-harmonic Maxwell's system with rough coefficients.

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References

[1] Allegretto, W. & Xie, H. (1992) Existence of solutions for the time-dependent thermistor equations. *IMA J. Appl. Math.* **48**(3), 271–281.

[2] Chen, Y. Z. (2003) *The Second-Order Parabolic Equations*, Peking University Press, Beijing, China.

[3] Chen, X. & Friedman, A. (1993) The thermistor problem for conductivity which vanishes at large temperature. *Quart. Appl. Math.* **51**(1), 101–115.

[4] Cimatti, G. & Prodi, G. (1988) Existence results for a nonlinear elliptic system modelling a temperature dependent electrical resistor. *Ann. Mat. Pura Appl.* **152**(4), 227–236.

[5] Costabel, M. (1991) A coercive bilinear form for Maxwell's equations. *J. Math. Anal. Appl.* **157**, 527–541.

[6] Giaquinta, M. (1983) *Multiple Integrals in the Calculus of Variation and Nonlinear Elliptic Systems*, Annals of Mathematics Studies, Princeton University Press, NJ.

[7] Hazard, C. & Lenoir, M. (1996) On the solution of time-harmonic scattering problems for Maxwell's equations. *SIAM J. Math. Anal.* **27**, 1597–1630.

[8] Kang, K. & Kim, S. (2004) On the Hölder continuity of solutions of a certain system related to Maxwell's equations. *SIAM J. Math. Anal.* **34**, 87–100.

[9] Lacey, A. (1995) Thermal runaway in a non-local problem modelling Ohmic heating. I. Model derivation and some special cases. *Eur. J. Appl. Math.* **6**(2), 127–144; II. General proof of blow-up and asymptotics of runaway. *Eur. J. Appl. Math.* **6**(3), 201–224.

[10] La dyzenskaja, O. A., Solonnikov, V. A. & Ural'ceva, N. N. (1968) *Linear and Quasi-linear Equations of Parabolic Type*, AMS Transactions, 23, AMS, Providence, RI.

[11] Landau, L. D. & Lifshitz, E. M. (1960) *Electrodynamics of Continuous Media*, Pergamon Press, New York.

[12] Li, B., Tang, J. & Yin, H. M. (2002) Optimal control of microwave sterilization in food processing. *Int. J. Appl. Math.* **10**, 13–31.

[13] Li, X. & Yong, J. (1995) *Optimal Control Theory for Infinite Dimensional Equations*, Birkhäuser, Boston, MA.

[14] Lieberman, G. M. (1996) *Second Order Parabolic Differential Equations*, World Scientific, Singapore.

[15] Lions, J. L. (1971) *Optimal Control of Systems Governed by Partial Differential Equations*, Springer Verlag, Berlin.

[16] Metaxas, A. C. (1996) *Foundations of Electroheat: A Unified Approach*, John Wiley & Sons, New York.

[17] Metaxas, A. C. & Meredith, R. J. (1983) *Industrial Microwave Heating*, IEE Power Engineering Series, Vol. 4, Per Peregrimus Ltd., London.

[18] Russell, D. L. (1986) The Dirichlet–Neumann boundary control problem associated with Maxwell's equations in a cylindrical region, *SIAM J. Control Optim.* **24**, 199–229.

[19] Troianiello, G. (1987) *Elliptic Differential Equations and Obstacle Problems*, Plenum Press, New York.

[20] Troltzsch, F. (2010) *Optimal Control of Partial Differential Equations, Theory, Methods and Applications*, Graduate Studies in Mathematics, Vol. 112, AMS, Providence, RI.

[21] Wei, W., Yin, H. M. & Tang, J. (2012) An optimal control problem for microwave heating. *Nonlinear Anal. Theory Methods Appl.* **75**, 2024–2036.

[22] Yin, H. M. (1997) $L^{2,\mu}$-theory for linear parabolic equations. *J. Partial Differ. Equ.* **10**(1), 31–44.

[23] Yin, H. M. (1997) Regularity of solutions to Maxwell's system in quasi-stationary electromagnetic fields and applications. *Commun. PDE* **22**(7–8), 1029–1053.

[24] Yin, H. M. (2002) Optimal regularity of solution to a degenerate elliptic system arising in electromagnetic fields. *Commun. Pure Appl. Anal.* **1**(1), 127–134.

[25] Yin, H. M. (2004) Regularity of weak solution to Maxwell's equations and applications to microwave heating. *J. Differ. Equ.* **200**, 137–161.

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