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ITERATING BILINEAR HARDY INEQUALITIES

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Abstract An iteration technique for characterizing boundedness of certain types of multilinear operators is presented, reducing the problem to a corresponding linear-operator case. The method gives a simple proof of a characterization of validity of the weighted bilinear Hardy inequality

$$\bigg(\int_a^b\bigg(\int_a^t f\int_a^t g\bigg)^q w(t)\,\mathrm{d}t\bigg)^{1/q}\leqslant C\bigg(\int_a^b f^{p_1}v_1\bigg)^{1/p_1}\bigg(\int_a^b f^{p_2}v_2\bigg)^{1/p_2}$$

for all non-negative f, g on (a,b), for $1<p_1,p_2,q<\infty$. More equivalent characterizing conditions are presented.

The same technique is applied to various further problems, in particular those involving multilinear integral operators of Hardy type.

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1. Introduction

Let $-\infty\leqslant a<b\leqslant\infty$. Let $\mathscr{M}_+$ denote the cone of non-negative Lebesgue-measurable functions on (a,b). The *Hardy operator* H_1 and the *'dual Hardy' operator* H_1' are operators acting on $\mathscr{M}_+$, defined by

$$H_1f(t):=\int_a^t f(s)\,\mathrm{d}s,\quad H_1'f(t):=\int_t^b f(s)\,\mathrm{d}s,\quad t\in(a,b).$$

Recall that the *weighted Lebesgue space* $L^\alpha(u)$ consists of all real-valued Lebesgue-measurable functions f on (a,b) such that

$$\|f\|_{L^\alpha(u)}:=\bigg(\int_a^b|f(t)|^\alpha u(t)\,\mathrm{d}t\bigg)^{1/\alpha}<\infty.$$

Here $1\leqslant\alpha<\infty$ and u is a *weight*, i.e. simply a fixed function $u\in\mathscr{M}_+$.

It is well known under which conditions the operator H_1 is bounded from $L^\alpha(u)$ to $L^\beta(z)$, or, in other words, when the *weighted Hardy inequality*

$$\biggl(\int_a^b\biggl(\int_a^t f\biggr)^\beta z(t)\,\mathrm{d}t\biggr)^{1/\beta}\leqslant C\biggl(\int_a^b f^\alpha u\biggr)^{1/\alpha} \tag{1.1}$$

holds for all $f\in\mathscr{M}_+$. Namely, the following theorems hold (see [**3**, **13**–**15**]).

Theorem 1.1. *Let u, z be weights. For $\alpha,\beta\in(1,\infty)$ set*

$$C_{(1.2)}:=\sup_{f\in\mathscr{M}_+}\biggl(\int_a^b\biggl(\int_a^t f\biggr)^\beta z(t)\,\mathrm{d}t\biggr)^{1/\beta}\biggl(\int_a^b f^\alpha u\biggr)^{-1/\alpha}. \tag{1.2}$$

Then the following hold.

(i) *If $1<\alpha\leqslant\beta<\infty$, then*

$$C_{(1.2)}\simeq\sup_{a<x<b}\biggl(\int_x^b z\biggr)^{1/\beta}\biggl(\int_a^x u^{1-\alpha'}\biggr)^{1/\alpha'}.$$

(ii) *If $1<\beta<\alpha<\infty$ and $\gamma:=\alpha\beta/(\alpha-\beta)$, then*

$$\begin{aligned}C_{(1.2)}&\simeq\biggl(\int_a^b\biggl(\int_x^b z\biggr)^{\gamma/\beta}\biggl(\int_a^x u^{1-\alpha'}\biggr)^{\gamma/\beta'}u^{1-\alpha'}(x)\,\mathrm{d}x\biggr)^{1/\gamma}\\&\simeq\biggl(\int_a^b\biggl(\int_x^b z\biggr)^{\gamma/\alpha}\biggl(\int_a^x u^{1-\alpha'}\biggr)^{\gamma/\alpha'}z(x)\,\mathrm{d}x\biggr)^{1/\gamma}.\end{aligned}$$

Theorem 1.2. *Let u, z be weights. For $\alpha,\beta\in(1,\infty)$ set*

$$C_{(1.3)}:=\sup_{f\in\mathscr{M}_+}\biggl(\int_a^b\biggl(\int_t^b f\biggr)^\beta z(t)\,\mathrm{d}t\biggr)^{1/\beta}\biggl(\int_a^b f^\alpha u\biggr)^{-1/\alpha}. \tag{1.3}$$

Then the following hold.

(i) *If $1<\alpha\leqslant\beta<\infty$, then*

$$C_{(1.3)}\simeq\sup_{a<x<b}\biggl(\int_a^x z\biggr)^{1/\beta}\biggl(\int_x^b u^{1-\alpha'}\biggr)^{1/\alpha'}.$$

(ii) *If $1<\beta<\alpha<\infty$ and $\gamma:=\alpha\beta/(\alpha-\beta)$, then*

$$\begin{aligned}C_{(1.3)}&\simeq\biggl(\int_a^b\biggl(\int_a^x z\biggr)^{\gamma/\beta}\biggl(\int_x^b u^{1-\alpha'}\biggr)^{\gamma/\beta'}u^{1-\alpha'}(x)\,\mathrm{d}x\biggr)^{1/\gamma}\\&\simeq\biggl(\int_a^b\biggl(\int_a^x z\biggr)^{\gamma/\alpha}\biggl(\int_x^b u^{1-\alpha'}\biggr)^{\gamma/\alpha'}z(x)\,\mathrm{d}x\biggr)^{1/\gamma}.\end{aligned}$$

In both these cases, as well as later on, we will use the conventions $1/0 := \infty$, $1/\infty := 0$, $0\times\infty := 0$. Observe that then the two preceding theorems are indeed true even for weights with zero value on a set of non-zero measure. In particular, we may use them for a weight w such that $w = w\chi_{(c,b)}$ for some $c \in (a, b)$. This formal detail will be used at a certain point.

Notice also the two equivalent conditions in each of the cases of part (ii). Existence of such alternative conditions is a common feature in weighted Hardy-type inequalities. Often it proves to be useful to find such equivalent expressions since each of them may be applicable in different particular situations.

Let us now consider the *bilinear Hardy operator* H_2, acting on $\mathscr{M}_+ \times \mathscr{M}_+$ and defined by

$$H_2(f,g)(t) := \int_a^t f(s)\,\mathrm{d}s \int_a^t g(s)\,\mathrm{d}s, \quad t \in (a,b).$$

Recently, Aguilar *et al.* [**1**] characterized the boundedness $H_2\colon L^{p_1}(v_1) \times L^{p_2}(v_2) \to L^q(w)$, or, equivalently, the validity of the *bilinear weighted Hardy inequality*

$$\left(\int_a^b \left(\int_0^t f\right)^q \left(\int_0^t g\right)^q w(t)\,\mathrm{d}t\right)^{1/q} \leqslant C\left(\int_a^b f^{p_1}(t)v_1(t)\,\mathrm{d}t\right)^{1/p_1} \left(\int_a^b f^{p_2}(t)v_2(t)\,\mathrm{d}t\right)^{1/p_2} \qquad (1.4)$$

for all $f, g \in \mathscr{M}_+$. The range of exponents was $1 < p, q < \infty$. To prove these results, the authors used the discretization technique, a standard yet technical method that proves to be rather unnecessarily complicated in this case.

In this paper, we first present a much easier proof of the characterization of (1.4). In most cases we also manage to reduce the number of conditions, compared with those of [**1**]. Our proof technique will be referred to as the 'iteration method'. The idea is to proceed simply in two steps, each time treating the problem as the ordinary Hardy inequality (1.1). Especially in the 'easy case' $p_1, p_2 \leqslant q$, the proof becomes extremely simple. Let us note that the same idea was also used in [**12**] to characterize the bilinear Hardy inequality for decreasing functions.

Having proved the aforementioned characterizations of (1.4) in § 2, we then continue by providing more alternative conditions. Existence of equivalent conditions is a common feature of weighted inequalities, although it was not observed in [**1**].

Fairly obviously, the iteration method is not limited just to the bilinear case and the Hardy operator case. Hence, in the final part we present more applications of this technique to a variety of problems involving other operators.

As a final remark in this introduction, let us recall the following duality property of the $L^p(v)$-spaces. Namely, if $p \in (1, \infty)$ and v is a weight, then for any $f \in \mathscr{M}_+$ it holds that

$$\left(\int_0^\infty f^p(x)v(x)\,\mathrm{d}x\right)^{1/p} = \sup_{h\in\mathscr{M}_+} \frac{\int_0^\infty f(x)h(x)\,\mathrm{d}x}{(\int_0^\infty h^{p'}(x)v^{1-p'}(x)\,\mathrm{d}x)^{1/p'}}. \qquad (1.5)$$

2. Bilinear weighted Hardy inequality

Using the iteration method, in this part we characterize the quantity

$$C_{(2.1)} := \sup_{f,g\in\mathscr{M}_+} \left(\int_a^b \left(\int_a^t f\right)^q \left(\int_a^t g\right)^q w(t)\,\mathrm{d}t\right)^{1/q} \left(\int_a^b f^{p_1} v_1\right)^{-1/p_1} \left(\int_a^b g^{p_2} v_2\right)^{-1/p_2}, \tag{2.1}$$

which is the optimal constant C in the inequality (1.4). The following notation will be used from now on: $F \lesssim G$ means that there exists a constant $C \in (0,\infty)$ such that $F \leqslant CG$ and C is 'independent of relevant quantities in F and G'. More precisely, in this paper this constant C always depends only on the exponents p, p_1, p_2, q. If $F \lesssim G$ and $G \lesssim F$, we write $F \simeq G$.

We will provide such conditions A that $C_{(2.1)} \simeq A$, without explicit estimates on the constants D_1, D_2 such that $D_1 A \leqslant C_{(2.1)} \leqslant D_2 A$. An exact calculation of these constants is left to the interested reader.

Theorem 2.1. *Let v_1, v_2, w be weights, and let $1 < p_1, p_2, q < \infty$, $p_1 \leqslant q$, $p_2 \leqslant q$. Then $C_{(2.1)} \simeq A_{(2.2)}$, where*

$$A_{(2.2)} := \sup_{a<x<b} \left(\int_x^b w\right)^{1/q} \left(\int_a^x v_1^{1-p_1'}\right)^{1/p_1'} \left(\int_a^x v_2^{1-p_2'}\right)^{1/p_2'}. \tag{2.2}$$

Proof. It holds that

$$\begin{aligned}
C_{(2.1)} &= \sup_{g\in\mathscr{M}_+} \sup_{f\in\mathscr{M}_+} \left(\int_a^b \left(\int_a^t f\right)^q \left(\int_a^t g\right)^q w(t)\,\mathrm{d}t\right)^{1/q} \left(\int_a^b f^{p_1} v_1\right)^{-1/p_1} \left(\int_a^b g^{p_2} v_2\right)^{-1/p_2} \\
&\simeq \sup_{g\in\mathscr{M}_+} \sup_{a<x<b} \left(\int_x^b \left(\int_a^y g\right)^q w(y)\,\mathrm{d}y\right)^{1/q} \left(\int_a^x v_1^{1-p_1'}\right)^{1/p_1'} \left(\int_a^b g^{p_2} v_2\right)^{-1/p_2} \\
&= \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'}\right)^{1/p_1'} \sup_{g\in\mathscr{M}_+} \left(\int_x^b \left(\int_a^y g\right)^q w(y)\,\mathrm{d}y\right)^{1/q} \left(\int_a^b g^{p_2} v_2\right)^{-1/p_2} \\
&\simeq \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'}\right)^{1/p_1'} \sup_{x<y<b} \left(\int_y^b w\right)^{1/q} \left(\int_a^y v_2^{1-p_2'}\right)^{1/p_2'} \\
&= A_{(2.2)}.
\end{aligned}$$

The second line follows from Theorem 1.1 (i) with $\alpha := p_1$, $\beta := q$, $u := v_1$, $z(t) := (\int_a^t g)^q w(t)$. The fourth line follows from the same theorem with $\alpha := p_2$, $\beta := q$, $u := v_2$, $z := \chi_{(x,b)} w$. □

Theorem 2.2. *Let v_1, v_2, w be weights, and let $1 < p_1 \leqslant q < p_2 < \infty$ and $r_2 := p_2 q/(p_2 - q)$. Then $C_{(2.1)} \simeq A_{(2.3)}$, where*

$$A_{(2.3)} := \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'}\right)^{1/p_1'} \left(\int_x^b \left(\int_y^b w\right)^{r_2/p_2} \left(\int_a^y v_2^{1-p_2'}\right)^{r_2/p_2'} w(y)\,\mathrm{d}y\right)^{1/r_2}. \tag{2.3}$$

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Proof. In the same way as in Theorem 2.1, using Theorem 1.1 (i) (with $\alpha := p_1$, $\beta := q$, $u := v_1$, $z(t) := (\int_a^t g)^q w(t)$) in the first step and Theorem 1.1 (ii) (with $\alpha := p_2$, $\beta := q$, $u := v_2$, $z := \chi_{(x,b)} w$) in the second one, we get

$$C_{(2.1)} \simeq \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'} \right)^{1/p_1'} \sup_{g \in \mathscr{M}_+} \left(\int_x^b \left(\int_a^y g \right)^q w(y)\,\mathrm{d}y \right)^{1/q} \left(\int_a^b g^{p_2} v_2 \right)^{-1/p_2}$$
$$\simeq A_{(2.3)}. \qquad \square$$

Theorem 2.3. *Let v_1, v_2, w be weights, and let $1 < q < p_i < \infty$, $r_i := p_i q/(p_i - q)$ for $i \in \{1, 2\}$, and $1/q \leqslant 1/p_1 + 1/p_2$. Then $C_{(2.1)} \simeq A_{(2.4)} + A_{(2.5)}$, where*

$$A_{(2.4)} := \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'} \right)^{1/p_1'} \left(\int_x^b \left(\int_y^b w \right)^{r_2/q} \left(\int_a^y v_2^{1-p_2'} \right)^{r_2/q'} v_2^{1-p_2'}(y)\,\mathrm{d}y \right)^{1/r_2}, \tag{2.4}$$

$$A_{(2.5)} := \sup_{a<x<b} \left(\int_a^x v_2^{1-p_2'} \right)^{1/p_2'} \left(\int_x^b \left(\int_y^b w \right)^{r_1/q} \left(\int_a^y v_1^{1-p_1'} \right)^{r_1/q'} v_1^{1-p_1'}(y)\,\mathrm{d}y \right)^{1/r_1}. \tag{2.5}$$

Proof. We have

$$\begin{aligned}
C_{(2.1)} &\simeq \sup_{g \in \mathscr{M}_+} \left(\int_a^b \left(\int_x^b \left(\int_a^y g \right)^q w(y)\,\mathrm{d}y \right)^{r_1/q} \left(\int_a^x v_1^{1-p_1'} \right)^{r_1/q'} v_1^{1-p_1'}(x)\,\mathrm{d}x \right)^{1/r_1} \\
&\qquad \times \left(\int_a^b g^{p_2} v_2 \right)^{-1/p_2} \\
&= \sup_{h \in \mathscr{M}_+} \sup_{g \in \mathscr{M}_+} \frac{(\int_a^b (\int_a^y g)^q w(y) \int_a^y h(t)\,\mathrm{d}t\,\mathrm{d}y)^{1/q} (\int_a^b g^{p_2} v_2)^{-1/p_2}}{(\int_a^b h^{p_1/q}(y) (\int_a^y v_1^{1-p_1'})^{-p_1/q'} v_1^{p_1'/r_1}(y)\,\mathrm{d}y)^{1/p_1}} \\
&\simeq \sup_{h \in \mathscr{M}_+} \frac{(\int_a^b (\int_x^b w(y) \int_a^y h(t)\,\mathrm{d}t\,\mathrm{d}y)^{r_2/q} (\int_a^x v_2^{1-p_2'})^{r_2/q'} v_2^{1-p_2'}(x)\,\mathrm{d}x)^{1/r_2}}{(\int_a^b h^{p_1/q}(y) (\int_a^y v_1^{1-p_1'})^{-p_1/q'} v_1^{p_1'/r_1}(y)\,\mathrm{d}y)^{1/p_1}} \\
&= \sup_{h \in \mathscr{M}_+} \frac{\begin{gathered}(\int_a^b (\int_a^x h(t)\,\mathrm{d}t \int_x^b w(y)\,\mathrm{d}y + \int_x^b h(t) \int_t^b w(y)\,\mathrm{d}y\,\mathrm{d}t)^{r_2/q} \\ \times (\int_a^y v_2^{1-p_2'})^{r_2/q'} v_2^{1-p_2'}(x)\,\mathrm{d}x)^{1/r_2}\end{gathered}}{(\int_a^b h^{p_1/q}(y) (\int_a^x v_1^{1-p_1'})^{-p_1/q'} v_1^{p_1'/r_1}(y)\,\mathrm{d}y)^{1/p_1}} \\
&\simeq \sup_{h \in \mathscr{M}_+} \left[\frac{(\int_a^b (\int_a^x h)^{r_2/q} (\int_x^b w)^{r_2/q} (\int_a^x v_2^{1-p_2'})^{r_2/q'} v_2^{1-p_2'}(x)\,\mathrm{d}x)^{q/r_2}}{(\int_a^b h^{p_1/q}(y) (\int_a^y v_1^{1-p_1'})^{-p_1/q'} v_1^{p_1'/r_1}(y)\,\mathrm{d}y)^{q/p_1}} \right]^{1/q} \\
&\quad + \sup_{h \in \mathscr{M}_+} \left[\frac{(\int_a^b (\int_x^b h)^{r_2/q} (\int_a^x v_2^{1-p_2'})^{r_2/q'} v_2^{1-p_2'}(x)\,\mathrm{d}x)^{q/r_2}}{(\int_a^b h^{p_1/q}(y) (\int_y^b w)^{-p_1/q} (\int_a^y v_1^{1-p_1'})^{-p_1/q'} v_1^{p_1'/r_1}(y)\,\mathrm{d}y)^{q/p_1}} \right]^{1/q} \\
&=: B_1 + B_2.
\end{aligned}$$

Here, the first relation follows by Theorem 1.1 (i), setting $\alpha := p_1$, $\beta := q$, $u := v_1$, $z(t) := (\int_a^t g)^q w(t)$. The second relation is due to duality (see (1.5)). For the third

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relation we use Theorem 1.1 (i) with $\alpha := p_2$, $\beta := q$, $u := v_2$, $z(y) := w(y)\int_a^y h$. The fourth relation holds by the Fubini theorem. Finally, by Theorem 1.1 (i), setting

$$\alpha := p_1/q, \qquad \beta := r_2/q,$$
$$u(y) := \left(\int_a^y v_1^{1-p_1'}\right)^{-p_1/q'} v_1^{p_1'/r_1}(y),$$
$$z(x) := \left(\int_x^b w\right)^{r_2/q} \left(\int_a^x v_2^{1-p_2'}\right)^{r_2/q'} v_2^{1-p_2'}(x),$$

we get $B_1 \simeq A_{(2.4)}$. Similarly, Theorem 1.2 (i) with

$$\alpha := p_1/q, \qquad \beta := r_2/q,$$
$$u(y) := \left(\int_x^b w\right)^{-p_1/q} \left(\int_a^y v_1^{1-p_1'}\right)^{-p_1/q'} v_1^{p_1'/r_1}(y),$$
$$z(x) := \left(\int_a^x v_2^{1-p_2'}\right)^{r_2/q'} v_2^{1-p_2'}(x)$$

yields $B_2 \simeq A_{(2.5)}$. □

Theorem 2.4. *Let v_1, v_2, w be weights, let $1 < q < p_i < \infty$, $r_i := p_i q/(p_i - q)$ for $i \in \{1, 2\}$, and let $1/q \leqslant 1/p_1 + 1/p_2$. Let $1/s = 1/q - 1/p_1 - 1/p_2$. Then $C_{(2.1)} \simeq A_{(2.6)} + A_{(2.7)}$, where*

$$A_{(2.6)} := \left(\int_a^b \left(\int_x^b \left(\int_y^b w\right)^{r_2/q} \left(\int_a^y v_2^{1-p_2'}\right)^{r_2/q'} v_2^{1-p_2'}(y)\,\mathrm{d}y\right)^{s/r_2} \times \left(\int_a^x v_1^{1-p_1'}\right)^{s/r_2'} v_1^{1-p_1'}(x)\,\mathrm{d}x\right)^{1/s}, \tag{2.6}$$

$$A_{(2.7)} := \left(\int_a^b \left(\int_x^b \left(\int_y^b w\right)^{r_1/q} \left(\int_a^y v_1^{1-p_1'}\right)^{r_1/q'} v_1^{1-p_1'}(y)\,\mathrm{d}y\right)^{s/r_1} \times \left(\int_a^x v_2^{1-p_2'}\right)^{s/r_1'} v_2^{1-p_2'}(x)\,\mathrm{d}x\right)^{1/s}. \tag{2.7}$$

Proof. As in the proof of Theorem 2.3, one has $C_{(2.1)} \simeq B_1 + B_2$, where B_1 and B_2 are defined in the same way as in that proof. Next, Theorem 1.1 (ii) with

$$\alpha := \frac{p_1}{q}, \qquad \beta := \frac{r_2}{q},$$
$$u(y) := \left(\int_a^y v_1^{1-p_1'}\right)^{-p_1/q'} v_1^{p_1'/r_1}(y),$$
$$z(x) := \left(\int_x^b w\right)^{r_2/q} \left(\int_a^x v_2^{1-p_2'}\right)^{r_2/q'} v_2^{1-p_2'}(x)$$

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gives $B_1 \simeq A_{(2.6)}$, and Theorem 1.2 (ii) with

$$\alpha := \frac{p_1}{q}, \qquad \beta := \frac{r_2}{q},$$
$$u(y) := \left(\int_x^b w\right)^{-p_1/q} \left(\int_a^y v_1^{1-p_1{}'}\right)^{-p_1/q'} v_1^{p_1{}'/r_1}(y),$$
$$z(x) := \left(\int_a^x v_2^{1-p_2{}'}\right)^{r_2/q'} v_2^{1-p_2{}'}(x)$$

gives $B_2 \simeq A_{(2.7)}$. □

3. Equivalent conditions

The 'A-conditions' from the previous section have more equivalent forms. This can be observed simply by comparing the conditions we obtained with those from [**1**]. We are going to make this comparison and even prove the equivalences of the conditions directly.

Proposition 3.1. *In the setting from Theorem 2.2, it holds that* $A_{(2.3)} \simeq A_{(2.2)} + A_{(2.5)}$.

Proof. For all $x \in (a,b)$ integration by parts (cf. [**17**, Lemma, p. 176]) yields

$$\left(\int_x^b \left(\int_y^b w\right)^{r_2/p_2} \left(\int_a^y v_2^{1-p_2{}'}\right)^{r_2/p_2{}'} w(y)\,\mathrm{d}y\right)^{1/r_2}$$
$$\simeq \left(\int_x^b w\right)^{1/q} \left(\int_a^x v_2^{1-p_2{}'}\right)^{1/p_2{}'}$$
$$+ \left(\int_x^b \left(\int_y^b w\right)^{r_2/q} \left(\int_a^y v_2^{1-p_2{}'}\right)^{r_2/q'} v_2^{1-p_2{}'}(y)\,\mathrm{d}y\right)^{1/r_2}.$$

Multiplying both sides by $(\int_a^x v_1^{1-p_1{}'})^{1/p_1{}'}$, we show that $A_{(2.3)} \simeq A_{(2.2)} + A_{(2.5)}$ holds even pointwise, i.e. without the supremum over x. □

Proposition 3.2. *In the setting from Theorem 2.3, it holds that*

$$A_{(2.4)} + A_{(2.5)} \simeq A_{(2.2)} + A_{(2.4)} + A_{(2.5)} \simeq A_{(2.3)} + A^*_{(2.5)}, \tag{3.1}$$

where

$$A^*_{(2.5)} := \sup_{a<x<b} \left(\int_a^x v_2^{1-p_2{}'}\right)^{1/p_2{}'} \left(\int_x^b \left(\int_y^b w\right)^{r_1/p_1} \left(\int_a^y v_1^{1-p_1{}'}\right)^{r_1/p_1{}'} w(y)\,\mathrm{d}y\right)^{1/r_1}.$$

Proof. The second equivalence in (3.1) holds pointwise for $x \in (a,b)$ by partial integration. The fact that we proved that $C_{(2.1)} \simeq A_{(2.4)} + A_{(2.5)}$, while in [**1**, Theorem 3] it was proved that $C_{(2.1)} \simeq A_{(2.2)} + A_{(2.4)} + A_{(2.5)}$, gives an indirect proof of the first equivalence in (3.1).

A simple direct proof of the inequality $A_{(2.2)} \lesssim A_{(2.4)} + A_{(2.5)}$ can be obtained by employing the same idea as that from [**7**, Lemma 2.2]. It goes as follows. For each $x \in (a,b)$ there exists $y(x) \in (a,x)$ such that

$$\int_a^{y(x)} v_1^{1-p_1{}'} = \int_{y(x)}^x v_1^{1-p_1{}'} = \frac{1}{2}\int_a^x v_1^{1-p_1{}'}.$$

Now we get

$$\begin{aligned}
&\left(\int_x^b w\right)^{1/q}\left(\int_a^x v_1^{1-p_1{}'}\right)^{1/p_1{}'}\left(\int_a^x v_2^{1-p_2{}'}\right)^{1/p_2{}'}\\
&\quad\simeq \left(\int_x^b w\right)^{1/q}\left(\int_a^{y(x)} v_1^{1-p_1{}'}\right)^{1/p_1{}'}\left(\int_a^x v_2^{1-p_2{}'}\right)^{1/p_2{}'}\\
&\quad= \left(\int_x^b w\right)^{1/q}\left(\int_a^{y(x)} v_1^{1-p_1{}'}\right)^{1/p_1{}'}\left(\int_a^{y(x)} v_2^{1-p_2{}'} + \int_{y(x)}^x v_2^{1-p_2{}'}\right)^{1/p_2{}'}\\
&\quad\simeq \left(\int_x^b w\right)^{1/q}\left(\int_{y(x)}^x v_1^{1-p_1{}'}\right)^{1/p_1{}'}\left(\int_a^{y(x)} v_2^{1-p_2{}'}\right)^{1/p_2{}'}\\
&\qquad\quad + \left(\int_x^b w\right)^{1/q}\left(\int_a^{y(x)} v_1^{1-p_1{}'}\right)^{1/p_1{}'}\left(\int_{y(x)}^x v_2^{1-p_2{}'}\right)^{1/p_2{}'}\\
&\quad\simeq \left(\int_x^b w\right)^{1/q}\left(\int_{y(x)}^x \left(\int_{y(x)}^t v_1^{1-p_1{}'}\right)^{r_1/q'} v_1^{1-p_1{}'}(t)\,\mathrm{d}t\right)^{1/r_1}\left(\int_a^{y(x)} v_2^{1-p_2{}'}\right)^{1/p_2{}'}\\
&\qquad\quad + \left(\int_x^b w\right)^{1/q}\left(\int_a^{y(x)} v_1^{1-p_1{}'}\right)^{1/p_1{}'}\left(\int_{y(x)}^x \left(\int_{y(x)}^t v_2^{1-p_2{}'}\right)^{r_2/q'} v_2^{1-p_2{}'}(t)\,\mathrm{d}t\right)^{1/r_2}\\
&\quad\leqslant \left(\int_a^{y(x)} v_2^{1-p_2{}'}\right)^{1/p_2{}'}\left(\int_{y(x)}^x \left(\int_t^b w\right)^{r_1/q}\left(\int_a^t v_1^{1-p_1{}'}\right)^{r_1/q'} v_1^{1-p_1{}'}(t)\,\mathrm{d}t\right)^{1/r_1}\\
&\quad\leqslant \left(\int_a^{y(x)} v_1^{1-p_1{}'}\right)^{1/p_1{}'}\left(\int_{y(x)}^x \left(\int_t^b w\right)^{r_2/q}\left(\int_a^t v_2^{1-p_2{}'}\right)^{r_2/q'} v_2^{1-p_2{}'}(t)\,\mathrm{d}t\right)^{1/r_2}\\
&\quad\leqslant A_{(2.5)} + A_{(2.4)}.
\end{aligned}$$

Taking the supremum over $x \in (a,b)$, we obtain $A_{(2.2)} \lesssim A_{(2.4)} + A_{(2.5)}$. Observe that this inequality does not hold pointwise in x, rather only with the supremum. □

Proposition 3.3. *In the setting from Theorem 2.4, it holds that*

$$A_{(2.6)} + A_{(2.7)} \simeq A^* + A_{(2.6)} + A_{(2.7)}, \tag{3.2}$$

where

$$A^* := \left(\int_a^b \left(\int_x^b w\right)^{s/p_1+s/p_2} w(x)\left(\int_a^x v_1^{1-p_1{}'}\right)^{s/p_1{}'}\left(\int_a^x v_2^{1-p_2{}'}\right)^{s/p_2{}'}\mathrm{d}x\right)^{1/s}.$$

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Moreover, it holds that $A_{(2.7)} \simeq A^*_{(2.7)}$, *where*

$$A^*_{(2.6)} := \left(\int_a^b \left(\int_x^b \left(\int_y^b w \right)^{r_2/q} \left(\int_a^y v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{s/p_1} \times \left(\int_a^x v_1^{1-p_1{}'} \right)^{s/p_1{}'} \left(\int_x^b w \right)^{r_2/q} \left(\int_a^x v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \right)^{1/s},$$

and $A_{(2.7)} \simeq A^*_{(2.7)}$, *where* $A^*_{(2.7)}$ *is defined analogously to* $A^*_{(2.6)}$ *with the indices* 1 *and* 2 *switched.*

Proof. The equivalence $A_{(2.6)} \simeq A^*_{(2.6)}$ follows directly by integration by parts. Theorem 2.4 yields $C_{(2.1)} \simeq A_{(2.6)} + A_{(2.7)}$, while [**1**, Theorem 4] gives $C_{(2.1)} \simeq A^* + A_{(2.6)} + A_{(2.7)}$, and hence (3.2) is true.

However, we will also provide a direct proof of (3.2). Obviously, we just need to prove that $A^* \lesssim A_{(2.6)} + A_{(2.7)}$. First, integrating by parts we get

$$\begin{aligned}(A^*)^s &\simeq \int_a^b \left(\int_x^b w \right)^{s/q} \left(\int_a^x v_1^{1-p_1{}'} \right)^{s/p_1{}'} \left(\int_a^x v_2^{1-p_2{}'} \right)^{s/r_1{}'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \\ &\quad + \int_a^b \left(\int_x^b w \right)^{s/q} \left(\int_a^x v_2^{1-p_2{}'} \right)^{s/p_2{}'} \left(\int_a^x v_1^{1-p_1{}'} \right)^{s/r_2{}'} v_1^{1-p_1{}'}(x)\,\mathrm{d}x \\ &=: B_3 + B_4.\end{aligned}$$

Now we prove that $B_3 \lesssim A_{(2.6)} + A^*_{(2.7)}$. The idea resembles that of [**8**, Theorem 3.1]. We may suppose that for all $\varepsilon \in (0, b-a)$ it holds that $\int_a^{a+\varepsilon} v_2^{1-p_2{}'} < \infty$, otherwise all the terms B_3, $A_{(2.6)}$, $A^*_{(2.7)}$ become infinite. We also assume that $\int_a^b v_2^{1-p_2{}'} = \infty$ (if this is not satisfied, then the following part of the proof needs only minor changes). Now, for $k \in \mathbb{Z}$ let $x_k \in (a,b)$ be such that $\int_a^{x_k} v_2^{1-p_2{}'} = 2^k$, and let $y_k \in [x_k, x_{k+1}]$ be such that

$$\sup_{y \in [x_k, x_{k+1}]} \left(\int_y^b w \right)^{s/q} \left(\int_a^y v_1^{1-p_1{}'} \right)^{s/p_1{}'} = \left(\int_{y_k}^b w \right)^{s/q} \left(\int_a^{y_k} v_1^{1-p_1{}'} \right)^{s/p_1{}'}.$$

Now we can write

$$\begin{aligned}B_3 &= \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} \left(\int_a^x v_2^{1-p_2{}'} \right)^{s/r_1{}'} v_2^{1-p_2{}'}(x) \left(\int_x^b w \right)^{s/q} \left(\int_a^x v_1^{1-p_1{}'} \right)^{s/p_1{}'} \mathrm{d}x \\ &\leqslant \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} \left(\int_a^x v_2^{1-p_2{}'} \right)^{s/r_1{}'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \sup_{y \in [x_k, x_{k+1}]} \left(\int_y^b w \right)^{s/q} \left(\int_a^y v_1^{1-p_1{}'} \right)^{s/p_1{}'} \\ &\lesssim \sum_{k \in \mathbb{Z}} 2^{ks/p_2{}'} \left(\int_{y_k}^b w \right)^{s/q} \left(\int_a^{y_k} v_1^{1-p_1{}'} \right)^{s/p_1{}'}\end{aligned}$$

$$
\begin{aligned}
&\simeq \sum_{k\in\mathbb{Z}} 2^{ks/p_2{}'}\biggl(\int_{y_k}^b w\biggr)^{s/q}\biggl(\int_{y_{k-4}}^{y_k} v_1^{1-p_1{}'}\biggr)^{s/p_1{}'}\\
&\qquad + \sum_{k\in\mathbb{Z}} 2^{ks/p_2{}'}\biggl(\int_{y_k}^b w\biggr)^{s/q}\biggl(\int_{a}^{y_{k-4}} v_1^{1-p_1{}'}\biggr)^{s/p_1{}'}\\
&=: B_5 + B_6.
\end{aligned}
$$

Observe that for all $k \in \mathbb{Z}$ it holds that

$$
2^k \leqslant \int_a^{y_k} v_2^{1-p_2{}'} \leqslant 2^{k+1}, \qquad 2^{k-1} \leqslant \int_{y_{k-2}}^{y_k} v_2^{1-p_2{}'} \leqslant 2^{k+1}.
$$

Hence,

$$
\begin{aligned}
B_5 &\lesssim \sum_{k\in\mathbb{Z}} \int_{x_{k-6}}^{x_{k-4}} \biggl(\int_a^x v_2^{1-p_2{}'}\biggr)^{s/r_1{}'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \biggl(\int_{y_k}^b w\biggr)^{s/q}\biggl(\int_{y_{k-4}}^{y_k} v_1^{1-p_1{}'}\biggr)^{s/p_1{}'}\\
&\simeq \sum_{k\in\mathbb{Z}} \int_{x_{k-6}}^{x_{k-4}} \biggl(\int_a^x v_2^{1-p_2{}'}\biggr)^{s/r_1{}'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x\\
&\qquad\qquad \times \biggl(\int_{y_k}^b w\biggr)^{s/q}\biggl(\int_{y_{k-4}}^{y_k}\biggl(\int_{y_{k-4}}^{y} v_1^{1-p_1{}'}\biggr)^{r_1/q'} v_1^{1-p_1{}'}(y)\,\mathrm{d}y\biggr)^{s/r_1}\\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{x_{k-6}}^{x_{k-4}} \biggl(\int_a^x v_2^{1-p_2{}'}\biggr)^{s/r_1{}'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x\\
&\qquad\qquad \times \biggl(\int_{y_{k-4}}^{y_k}\biggl(\int_{y}^b w\biggr)^{r_1/q}\biggl(\int_{a}^{y} v_1^{1-p_1{}'}\biggr)^{r_1/q'} v_1^{1-p_1{}'}(y)\,\mathrm{d}y\biggr)^{s/r_1}\\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{x_{k-6}}^{x_{k-4}} \biggl(\int_a^x v_2^{1-p_2{}'}\biggr)^{s/r_1{}'} v_2^{1-p_2{}'}(x)\\
&\qquad\qquad \times \biggl(\int_{x}^{y_k}\biggl(\int_{y}^b w\biggr)^{r_1/q}\biggl(\int_{a}^{y} v_1^{1-p_1{}'}\biggr)^{r_1/q'} v_1^{1-p_1{}'}(y)\,\mathrm{d}y\biggr)^{s/r_1}\mathrm{d}x\\
&\leqslant 2(A_{(2.7)})^s.
\end{aligned}
$$

Next, we have to estimate B_6. First, for any $k \in \mathbb{Z}$ it holds that

$$
\begin{aligned}
2^{ks/p_2{}'} &\lesssim \int_{y_{k-4}}^{y_{k-2}} \biggl(\int_{y_{k-4}}^x v_2^{1-p_2{}'}\biggr)^{r_2/q'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x\\
&\qquad\qquad \times \biggl(\int_{y_{k-2}}^{y_k}\biggl(\int_{y_{k-2}}^{y} v_2^{1-p_2{}'}\biggr)^{r_2/q'} v_2^{1-p_2{}'}(y)\,\mathrm{d}y\biggr)^{s/p_1}\\
&\leqslant \int_{y_{k-4}}^{y_{k-2}} \biggl(\int_{y_{k-2}}^{y_k}\biggl(\int_{a}^{y} v_2^{1-p_2{}'}\biggr)^{r_2/q'} v_2^{1-p_2{}'}(y)\,\mathrm{d}y\biggr)^{s/p_1}\\
&\qquad\qquad\qquad \times \biggl(\int_a^x v_2^{1-p_2{}'}\biggr)^{r_2/q'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x
\end{aligned}
$$

$$\leqslant \int_{y_{k-4}}^{y_{k-2}} \left(\int_x^{y_k} \left(\int_a^y v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{s/p_1} \times \left(\int_a^x v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x$$

$$\leqslant \int_{y_{k-4}}^{y_k} \left(\int_x^b \left(\int_a^y v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{s/p_1} \left(\int_a^x v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x.$$

Therefore,

$$\begin{aligned} B_6 &\lesssim \sum_{k\in\mathbb{Z}} \int_{y_{k-4}}^{y_k} \left(\int_x^b \left(\int_a^y v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{s/p_1} \left(\int_a^x v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(x)\,\mathrm{d}x \\ &\qquad \times \left(\int_{y_k}^b w \right)^{s/q} \left(\int_a^{y_{k-4}} v_1^{1-p_1{}'} \right)^{s/p_1{}'} \\ &\leqslant \sum_{k\in\mathbb{Z}} \int_{y_{k-4}}^{y_k} \left(\int_x^b \left(\int_y^b w \right)^{r_2/q} \left(\int_a^y v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(y)\,\mathrm{d}y \right)^{s/p_1} \\ &\qquad \times \left(\int_x^b w \right)^{r_2/q} \left(\int_a^x v_2^{1-p_2{}'} \right)^{r_2/q'} v_2^{1-p_2{}'}(x) \left(\int_a^x v_1^{1-p_1{}'} \right)^{s/p_1{}'} \mathrm{d}x \\ &\leqslant 4(A^*_{(2.6)})^s. \end{aligned}$$

At this point we have proved that $(B_3)^{1/s} \lesssim A_{(2.6)} + A^*_{(2.7)}$. In exactly the same way, only switching the indices 1 and 2, one proves that $(B_4)^{1/s} \lesssim A_{(2.7)} + A^*_{(2.6)}$. Using all the estimates we have collected, we get

$$A^* \simeq (B_3)^{1/s} + (B_4)^{1/s} \lesssim A_{(2.6)} + A_{(2.7)} + A^*_{(2.6)} + A^*_{(2.7)} \simeq A_{(2.6)} + A_{(2.7)},$$

which is what we wanted to show. □

4. Further results

In this final part we give examples of various further problems that may be successfully treated by the iteration method.

The following notation will be used: unless otherwise specified, $\mathscr{M}$ denotes the cone of all (extended) real-valued measurable functions on a suitable measure space $(\mathcal{R}, \mu)$. For $f \in \mathscr{M}$, the symbol f^* denotes the *non-increasing rearrangement* of f, and

$$f^{**}(t) := \frac{1}{t} \int_0^t f^* \quad \text{for } t \in (0, \mu(\mathcal{R}))$$

(see [**2**] for details). If u is a weight on $(0, \mu(\mathcal{R}))$, then we define

$$f_u^{**}(t) := \left(\int_0^t u \right)^{-1} \int_0^t f^* u.$$

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For definitions of *rearrangement-invariant (r.i.) spaces* and *r.i. lattices*, see, for example, [**2**, **4**, **9**].

If $0 < p < \infty$ and u, v are weights on $(0, \mu(\mathcal{R}))$, the *weighted Lorentz 'spaces'* $\Lambda^p(v)$, $\Gamma^p(v)$ and $\Gamma^p_u(v)$ are defined as

$$\begin{aligned}\Lambda^p(v) &:= \{f \in \mathscr{M};\ \|f\|_{\Lambda^p(v)} := \|f^*\|_{L^p(v)} < \infty\},\\ \Gamma^p(v) &:= \{f \in \mathscr{M};\ \|f\|_{\Gamma^p(v)} := \|f^{**}\|_{L^p(v)} < \infty\},\\ \Gamma^p_u(v) &:= \{f \in \mathscr{M};\ \|f\|_{\Gamma^p_u(v)} := \|f^{**}_u\|_{L^p(v)} < \infty\}.\end{aligned}$$

Here, of course, the $L^p(v)$-space consists of functions over $(0, \mu(\mathcal{R}))$.

If X, Y are r.i. spaces (lattices), we say that X *is embedded into* Y, and write $X \hookrightarrow Y$, if there exists $C \in (0, \infty)$ such that for all $f \in X$ it holds that $\|f\|_Y \leqslant C\|f\|_X$.

4.1. Multilinear Hardy operator

The iteration method may be obviously extended for a multilinear Hardy operator H_n defined by

$$H_n(f_1, \dots, f_n)(t) := \prod_{i=1}^{n} H_1 f_i(t)$$

for $f_i \in \mathscr{M}_+$, $i = 1, \dots, n$, and $t \in (a, b)$. In this case we obtain the following recursive formula for the norm of H_n:

$$\begin{aligned}&\|H_n\|_{L^{p_1}(v_1)\times\cdots\times L^{p_n}(v_n)\to L^q(w)}\\ &\qquad = \sup_{\substack{f_i \in \mathscr{M}_+\\ i=1,\dots,n}} \frac{(\int_a^b (H_{n-1}(f_1, \dots, f_{n-1})(t))^q (H_1 f_n(t))^q w(t)\,\mathrm{d}t)^{1/q}}{\prod_{i=1}^{n-1} \|f_i\|_{L^{p_i}(v_i)} \|f_n\|_{L^{p_n}(v_n)}}\\ &\qquad = \sup_{f_n \in \mathscr{M}_+} \frac{\|H_{n-1}\|_{L^{p_1}(v_1)\times\cdots\times L^{p_{n-1}}(v_{n-1})\to L^q(w(H_1 f_n)^q)}}{\|f_n\|_{L^{p_n}(v_n)}}.\end{aligned}$$

In this way one can deduce the conditions on the weights and exponents under which $H_n\colon L^{p_1}(v_1) \times \cdots \times L^{p_n}(v_n) \to L^q(w)$, using only the knowledge of the conditions for $H_1\colon L^p(v) \to L^q(w)$. During the process there is no need for a method harder than changing the order of suprema, and using Fubini's theorem and L^p-duality.

4.2. Other product-based operators

Clearly, the idea above applies to any operator T such that

$$T(f_1, \dots, f_n) = \prod_{i=1}^{n} T_i f_i, \tag{4.1}$$

where T_i are certain other operators. Using the iteration method, we might be able to get conditions for boundedness $T\colon X_1 \times \cdots \times X_n \to X$ from the conditions for $T_i\colon Y_i \to Z_i$, where X, X_i, Y_i, Z_i are some suitable spaces (or even more general structures, for example, r.i. lattices). Simple examples of such operators T include products of the 'dual Hardy' operators, or products of a mixture of Hardy operators, 'dual Hardy' operators, Hardy-type integral or supremal operators with kernels, etc.

4.3. 'Multidimensional' Hardy operators involving non-increasing rearrangement

Let K be a weight (kernel). Define the Hardy-type operator $\mathcal{H}_{1,K}$ and its 'dual version' $\mathcal{H}'_{1,K}$ by

$$\mathcal{H}_{1,K}f(t) := \int_0^t f^*(s)K(s)\,\mathrm{d}s, \qquad \mathcal{H}'_{1,K}f(t) := \int_t^\infty f^*(s)K(s)\,\mathrm{d}s$$

for any $f \in \mathscr{M}$. If $K \equiv 1$, we simply write $\mathcal{H}_1 := \mathcal{H}_{1,K}$ and $\mathcal{H}'_1 := \mathcal{H}'_{1,K}$. Let us note that these operators are in general not linear.

Consider the operator $\mathcal{H}_2$ constructed as

$$\mathcal{H}_2(f,g)(t) := \mathcal{H}_1 f(t)\mathcal{H}_1 g(t) = \int_0^t f^*(s)\,\mathrm{d}s \int_0^t g^*(s)\,\mathrm{d}s.$$

This operator is obviously a special case of T from (4.1). In [**12**], the iteration method was used to characterize boundedness $\mathcal{H}_2\colon \Lambda^{p_1}(v_1) \times \Lambda^{p_2}(v_2) \to L^q(w)$, i.e. to produce weighted bilinear Hardy inequalities for non-increasing functions.

Let us take yet another Hardy-type operator $\widetilde{\mathcal{H}}_2$, defined by

$$\widetilde{\mathcal{H}}_2(f,g)(t) := \int_0^t f^*(s)g^*(s)\,\mathrm{d}s,$$

and study its boundedness $\widetilde{\mathcal{H}}_2\colon \Lambda^{p_1}(v_1) \times \Lambda^{p_2}(v_2) \to L^q(w)$. (The same idea may be used if the Λ-spaces are replaced by other appropriate structures.) Observe that $\widetilde{\mathcal{H}}_2(f,g)(t) = \mathcal{H}_{1,g^*}(f)$. We get

$$\begin{aligned} \|\widetilde{\mathcal{H}}_2\|_{\Lambda^{p_1}(v_1)\times\Lambda^{p_2}(v_2)\to L^q(w)} &= \sup_{g\in\mathscr{M}} \frac{1}{\|g\|_{\Lambda^{p_2}(v_2)}} \sup_{f\in\mathscr{M}} \frac{\|\int_0^{\bullet} f^*g^*\|_{L^q(w)}}{\|f\|_{\Lambda^{p_1}(v_1)}} \\ &= \sup_{g\in\mathscr{M}} \frac{\|\operatorname{id}\|_{\Lambda^{p_1}(v_1)\to\Gamma^q_{g^*}(\psi)}}{\|g\|_{\Lambda^{p_2}(v_2)}}. \end{aligned} \tag{4.2}$$

Here $\psi(t) := w(t)(\int_0^t g^*)^q$. We may now use the known characterization of the embedding $\Lambda^{p_1}(v_1) \hookrightarrow \Gamma^q_{g^*}(\psi)$ (see, for example, [**6**]). This embedding is also, in other words, equivalent to the $\Lambda^{p_1}(v_1) \to L^q(w)$ boundedness of the operator $\mathcal{H}_{1,g^*}$. Anyway, the optimal constant $\|\operatorname{id}\|_{\Lambda^{p_1}(v_1)\to\Gamma^q_{g^*}(\psi)}$ usually takes the form of a sum of the $L^\alpha(\varphi)$-norms of $\mathcal{H}_{1,K}(g)$, $\mathcal{H}'_{1,K}(g)$ or supremal variants of these operators. Here K, α and φ depend on the original parameters p, q, v_1, v_2, w. Hence, in the next phase, (4.2) will dissolve into a sum of factors

$$\sup_{g\in\mathscr{M}} \frac{\|\mathcal{H}_{1,K}(g)\|_{L^\alpha(\varphi)}}{\|g\|_{\Lambda^{p_2}(v_2)}},$$

or similar ones. Then we again use suitable existing characterizations of boundedness of $\mathcal{H}_{1,K}$, $\mathcal{H}'_{1,K}$ or, if needed, some supremal variants of those operators. In this way, the desired estimate on $\|\widetilde{\mathcal{H}}_2\|_{\Lambda^{p_1}(v_1)} \times \Lambda^{p_2}(v_2) \to L^q(w)$ will be obtained. The required boundedness characterizations for $\mathcal{H}'_{1,K}$ may be found in [**5**]. Corresponding conditions for other Hardy-type operators (for example, the supremal ones) may be derived using the reduction theorems presented in [**5**]. The boundedness conditions for $\mathcal{H}_{1,K}$ are, as already mentioned, listed in [**6**].

In a similar way, higher-order operators like $\mathcal{H}_n$, $\widetilde{\mathcal{H}}_n$, etc., constructed analogously to their $n = 2$ cases, may be treated. It is, however, worth noting that the complexity of the involved expressions grows rapidly with increasing n. Proofs involving general-weight cases using the iteration method may thus become very technical.

4.4. General product-type operator in a Γ-space

Let, for simplicity, $\mathscr{M}$ denote the cone of real-valued Lebesgue-measurable functions on $\mathbb{R}^n$. Motivated by [**16**], we now consider an arbitrary operator P mapping $\mathscr{M} \times \mathscr{M}$ into $\mathscr{M}$ and such that the inequality

$$\int_0^t (P(f,g))^*(s)\,\mathrm{d}s \leqslant \int_0^t f^*(s)g^*(s)\,\mathrm{d}s \tag{4.3}$$

holds for all $f, g \in \mathscr{M}$ and $t > 0$. The simplest example of such an operator is the ordinary product operator $P(f,g) := fg$ (see [**2**, p. 88]).

Let X_1, X_2 be r.i. spaces (or lattices) of functions defined over $\mathbb{R}^n$. It is now easy to find conditions for the boundedness $P\colon X_1 \times X_2 \to \Gamma^q(w)$. By (4.3), one gets

$$C_{(4.4)} := \sup_{f,g\in\mathscr{M}} \frac{\|P(f,g)\|_{\Gamma^q(w)}}{\|f\|_{X_1}\|g\|_{X_2}} \leqslant \sup_{f,g\in\mathscr{M}} \frac{\|\widetilde{\mathcal{H}}_2(f,g)\|_{L^q(t\mapsto t^{-q}w(t))}}{\|f\|_{X_1}\|g\|_{X_2}}. \tag{4.4}$$

The problem of finding an upper bound for $C_{(4.4)}$ hence reduces into a certain boundedness question regarding the operator $\widetilde{\mathcal{H}}_2$, which was treated in the previous section.

The possibility of providing a lower bound for $C_{(4.4)}$ depends to a great extent on the ‘sharpness’ of (4.3). Let us here, for example, consider the simple operator $P(f,g) := fg$. It may easily be checked that if both f and g are positive and radially decreasing, then $\int_0^t (fg)^* = \int_0^t f^*g^*$, and therefore equality in (4.3) is attained for these functions. This in turn implies that the two suprema in (4.4) are equal. The substantial facts here are that X_1 and X_2 are r.i., and that every $f \in \mathscr{M}_i$ may be rearranged into a positive (non-negative) radially decreasing (non-increasing) function $h \in \mathscr{M}_i$ such that $f^* \equiv h^*$. For details of these ideas we refer the reader to [**9**–**11**].

A general product operator may be also defined in another way, as suggested by O’Neil in [**16**]. See the final remark in the section below for more details.

4.5. Convolution in a Γ-space

Again, let $\mathscr{M}$ stand for the cone of Lebesgue-measurable real-valued functions on $\mathbb{R}^n$. The convolution of $f \in \mathscr{M}$ and $g \in \mathscr{M}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y. \tag{4.5}$$

As shown in [**16**], the bilinear operator $T(f,g) := f * g$ satisfies the O'Neil convolution inequality

$$(T(f,g))^{**}(t) \leqslant \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s \int_0^t g^*(s)\,\mathrm{d}s + \int_t^\infty f^*(s)g^*(s)\,\mathrm{d}s \tag{4.6}$$

for all $f, g \in \mathscr{M}$ and all $t > 0$. Moreover, in the case of both f and g being positive and radially decreasing, the reverse inequality holds with a constant depending only on the dimension n (see [**9**, **11**, **16**]). Observe that the right-hand side of (4.6) is again composed of certain Hardy-type operators acting on f, g.

In [**9**–**11**], the following problem was studied: given that X is one of the spaces $\Lambda^p(v)$, $\Gamma^p(v)$ or the class $S^p(v)$ (see [**10**]), characterize the largest r.i. space Y such that the Young-type inequality

$$\|f * g\|_{\Gamma^q(w)} \leqslant C\|f\|_X\|g\|_Y$$

holds for all $f, g \in \mathscr{M}$. In particular, an r.i. space Y was found such that for every positive radially decreasing g it holds that

$$\sup_{f\in\mathscr{M}} \frac{\|f * g\|_{\Gamma^q(w)}}{\|f\|_X} \simeq \|g\|_Y. \tag{4.7}$$

In all the cases $X = \Lambda^p(v), \Gamma^p(v), S^p(v)$ it turns out that this (quasi-)norm $\|\cdot\|_Y$ may be expressed as $\|\cdot\|_Y \simeq \|\cdot\|_{Y_1} + \|\cdot\|_{Y_2}$ with Y_1 being a Γ-type space and Y_2 a K-type space. The latter type was defined in [**9**].

A related problem, which may be successfully approached using the iteration method and the above results, is stated as follows. Under which conditions does the inequality $\|f * g\|_{\Gamma^q(w)} \leqslant C\|f\|_{\Lambda^{p_1}(v_1)}\|g\|_{\Lambda^{p_2}(v_2)}$ hold for all $f, g \in \mathscr{M}$? In other words, one is being asked for a characterization of

$$\sup_{f,g\in\mathscr{M}} \frac{\|f * g\|_{\Gamma^q(w)}}{\|f\|_{X_1}\|g\|_{X_2}}, \tag{4.8}$$

where $X_1 = \Lambda^{p_1}(v_1)$ and $X_2 = \Lambda^{p_2}(v_2)$. In view of (4.7), we proceed as follows:

$$\sup_{g\in\mathscr{M}} \frac{\|g\|_Y}{\|g\|_{X_2}} = \sup_{\substack{g\in\mathscr{M}\\ g \text{ pos.rad.dec.}}} \frac{\|g\|_Y}{\|g\|_{X_2}} \simeq \sup_{f,g\in\mathscr{M}} \frac{\|f * g\|_{\Gamma^q(w)}}{\|f\|_{X_1}\|g\|_{X_2}},$$

where 'pos.rad.dec.' abbreviates 'positive and radially decreasing'. (Notice that X_2, Y are r.i., and thus the first two terms are indeed equal.) Since we know that in this case $\|\cdot\|_Y \simeq \|\cdot\|_\Gamma + \|\cdot\|_K$, the problem is reduced to finding the optimal constants for certain embeddings $\Lambda \hookrightarrow \Gamma$ and $\Lambda \hookrightarrow K$. Characterizations of $\Lambda \hookrightarrow \Gamma$ are well known (see, for example, [**4**, **6**]); the problem of $\Lambda \hookrightarrow K$ was studied in [**12**].

The same strategy may be used if we choose X_1, X_2 in (4.8) as any other combination of Λ, Γ or S, or even as other r.i. spaces.

Moreover, in [**16**] O'Neil proposed a fairly general definition of a convolution operator as a bilinear operator T satisfying

$$\left.\begin{aligned} \|T(f,g)\|_1 &\leqslant \|f\|_1\|g\|_1,\\ \|T(f,g)\|_\infty &\leqslant \|f\|_\infty\|g\|_1,\\ \|T(f,g)\|_\infty &\leqslant \|f\|_1\|g\|_\infty.\end{aligned}\right\} \tag{4.9}$$

He then attempted to prove that a bilinear operator is a convolution operator in this sense if and only if it satisfies (4.6) for all f, g. However, as pointed out by Yap [**18**], O'Neil's proof of this statement contains a minor flaw and it seems that it cannot be fixed without some additional assumptions on T. For example, assuming that

$$\left.\begin{gathered} T \text{ maps pairs of positive functions into a positive function, and}\\ \forall f, f_n, g \geqslant 0\colon [f_n \uparrow f \text{ a.e.} \implies T(f_n,g) \uparrow T(f,g) \text{ a.e.}],\end{gathered}\right\} \tag{4.10}$$

where 'a.e.' abbreviates 'almost everywhere', should overcome the problem. Despite these problems with technical details, O'Neil's proof idea is correct for the ordinary convolution operator (4.5), which indeed satisfies (4.6).

Anyway, our technique of estimating (4.8) works for any bilinear operator satisfying the inequality (4.6). Thus, it also applies to the class of operators satisfying the interpolation inequalities (4.9) and the additional conditions (4.10).

Besides this, O'Neil also suggested a definition of a general product operator P by means of conditions analogous to (4.9) (see [**16**]). For such operators the inequality (4.3) plays a similar role to that played by (4.6) for the general convolution operators. Again, it seems that the assumption of conditions like (4.10) is necessary to prove that this general product operator satisfies (4.3). That is why we defined the 'product operator' in the previous section by (4.3) and not in O'Neil's style by some interpolation-type inequalities. As in the case of convolution operators, we may still choose the latter approach with some careful corrections.

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