Proceedings of the Royal Society of Edinburgh, **149**, 691–718, *2019*
DOI:10.1017/prm.2018.45

General sharp weighted Caffarelli–Kohn–Nirenberg inequalities

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(MS received 20 October 2016; accepted 7 March 2017)

In this paper, we will use optimal mass transport combining with suitable transforms to study the sharp constants and optimizers for a class of the Gagliardo–Nirenberg and Caffarelli–Kohn–Nirenberg inequalities. Moreover, we will investigate these inequalities with and without the monomial weights $x_1^{A_1} \cdots x_N^{A_N}$ on $\mathbb{R}^N$.

Keywords: Gagliardo–Nirenberg inequalities; Caffarelli–Kohn–Nirenberg inequalities; best constants; extremal functions; mass transport; duality

2010 *Mathematics Subject Classification:* Primary 26D10; 46E35

1. Introduction and main results

Functional and Geometric inequalities have many applications in geometry, the theory of functions of a complex variable, the calculus of variations, embedding theorems of function spaces, *a priori* estimates for solutions of differential equations, and so on. They, together with their sharp constants and extremal functions, take a crucial part in geometric analysis, partial differential equations and other branches of modern mathematics.

Among those inequalities, the Caffarelli–Kohn–Nirenberg (CKN) inequalities are one of the most important and interesting ones. They were first introduced in 1984 by Caffarelli, Kohn and Nirenberg in their celebrated work [14]:

Theorem A. *There exists a positive constant $C = C(N, r, p, q, \gamma, \alpha, \beta)$ such that for all $u \in C_0^\infty(\mathbb{R}^N)$:*

$$\left\| |x|^\gamma u \right\|_r \leqslant C \left\| |x|^\alpha |\nabla u| \right\|_p^a \left\| |x|^\beta u \right\|_q^{1-a} \tag{1.1}$$

where

$$p, q \geqslant 1, \ r > 0, \ 0 \leqslant a \leqslant 1$$

$$\frac{1}{p} + \frac{\alpha}{N}, \ \frac{1}{q} + \frac{\beta}{N}, \ \frac{1}{r} + \frac{\gamma}{N} > 0 \quad \textit{where}$$

$$\gamma = a\sigma + (1-a)\beta$$
$$\frac{1}{r} + \frac{\gamma}{N} = a\left(\frac{1}{p} + \frac{\alpha - 1}{N}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{N}\right),$$

and

$$0 \leqslant \alpha - \sigma \quad \textit{if } a > 0 \quad \textit{and}$$
$$\alpha - \sigma \leqslant 1 \quad \textit{if } a > 0 \quad \textit{and} \quad \frac{1}{p} + \frac{\alpha - 1}{N} = \frac{1}{r} + \frac{\gamma}{N}.$$

If we perform the following change of exponents as in [45]:

$$\alpha = -\frac{\mu}{p},\ \beta = -\frac{\theta}{q},\ \gamma = -\frac{s}{r},$$

then we obtain the following equivalent form:

$$\left(\int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{|x|^s}\right)^{1/r} \leqslant C \left(\int_{\mathbb{R}^N} |\nabla u|^p \frac{\mathrm{d}x}{|x|^\mu}\right)^{a/p} \left(\int_{\mathbb{R}^N} |u|^q \frac{\mathrm{d}x}{|x|^\theta}\right)^{(1-a)/q} \tag{CKN}$$

where

$$a = \frac{[(N-\theta)r - (N-s)q]p}{[(N-\theta)p - (N-\mu-p)q]r}.$$

It is worth noting that many well-known and important inequalities such as Gagliardo–Nirenberg inequalities, Sobolev inequalities, Hardy–Sobolev (HS) inequalities, Nash's inequalities, and so on, are just the special cases of the CKN inequalities. For example, when $s = \mu = \theta = 0$ and $a = 1$, we recover the well-known Sobolev inequality:

$$\left(\int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x\right)^{1/p^*} \leqslant S(N,p) \left(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x\right)^{1/p} \tag{1.2}$$

where $p^* = ((Np)/(N-p))$. This inequality has important applications in many areas of mathematics and there is a vast literature. For $p = 1$, it has been known that the Sobolev inequality is equivalent to the classical Euclidean isoperimetric inequality. When $p > 1$, the best constant $S(N,p)$ was found in the works of Aubin [5] and Talenti [43] using rather classical tools such as Schwarz rearrangement, and solution of a particular one-dimensional problem, the Bliss inequality. The case $p = 2$ was investigated more deeply by Beckner in [7] due to its conformal invariance.

When $a = 1,\ \mu = 0,\ 0 \leqslant s \leqslant p < N$ and $r = p^*(s) = ((N-s)/(N-p))p$, the CKN inequality becomes the HS inequality that is the interpolation of the Sobolev inequality and the Hardy inequality:

$$\left(\int_{\mathbb{R}^N} |u|^{p^*(s)} \frac{\mathrm{d}x}{|x|^s}\right)^{1/p^*(s)} \leqslant HS(N,p,s) \left(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x\right)^{1/p}. \tag{1.3}$$

In this situation, in [34] Lieb applied the symmetrization arguments to study (1.3) in the case $p = 2$ and gave the best constants and explicit optimizers. The study of the best constant $HS(N, p, s)$ and extremal functions for the inequalities (1.3) in the general range goes back at least to Ghoussoub and Yuan in [29]: The maximizers for the HS inequality when $0 \leqslant s < p < N$ are the functions

$$u_{c,\lambda}(x) = c\left(\lambda + |x|^{((p-s)/(p-1))}\right)^{-((N-p)/(p-s))} \quad \text{for some } c \neq 0, \lambda > 0. \tag{1.4}$$

Actually, $u_{c,\lambda}$ (after rescaling) is the only positive radial solutions of

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \frac{u^{p^*(s)-1}}{|x|^s} \quad \text{on } \mathbb{R}^N.$$

When $a = 1$ and $0 < \mu, s < N$, the CKN inequalities do not contain the interpolation term. There are great efforts to investigate the sharp constants, existence/nonexistence and symmetry/symmetry breaking of extremizers in this situation, especially, when $p = 2$. See [16, 17, 44], among others. For instance, in [17], Chou and Chu considered the case $p = 2$ and $\mu/2 \leqslant s/r \leqslant \mu/2 + 1$ and provided the best constants and explicit optimizers. In [44], Wang and Willem studied the compactness of all maximizing sequences up to dilations in the spirit of Lions [35–38]. In [16], Catrina and Wang investigated the class of $p = 2$ and $\mu < 0$ and established the attainability/unattainability and symmetry breaking of extremal functions. Caldiroli and Musina studied the symmetry breaking of extremals for the CKN inequalities in a non-Hilbertian setting in [15]. In a recent paper [23], Dolbeault, Esteban and Loss studied the characterization of the optimal symmetry breaking region in HS inequalities with $p = 2$. As a consequence, maximizers and best constants are calculated in the symmetry region. Their result solves a long-standing conjecture on the optimal symmetry range. We also mention here that the situation is different when $s = p$, namely, the Hardy inequality. The best constant, in this case, is $HS(N, p, p) = ((p)/(N - p))$ and is never achieved. Hence, it is natural to study the improved Hardy inequalities where we can try to add missing terms to the right-hand side of the Hardy inequalities, see [1, 11, 12, 25, 26, 41], just to name a few.

In the case $0 < a < 1$ and $s = \theta = \mu = 0$, we obtain the non-weighted CKN inequality, namely the Gagliardo–Nirenberg inequality. This inequality has been studied at length by many authors, see for example, [9, 16, 21, 22, 30], to mention just a few. Especially, for very particular classes, the best constant and the maximizers for the Gagliardo–Nirenberg inequality are provided explicitly by Del Pino and Dolbeault in [19, 20]. Indeed, in the special class $r = p((q - 1)/(p - 1))$, Del Pino and Dolbeault proved that the maximizers for the Gagliardo–Nirenberg inequality have the form $\gamma(1 + \delta|x - \overline{x}|^{p/p-1})^{-((p-1)/(q-p))}$ while in the case $q = p((r - 1)/(p - 1))$, the optimizers are $\gamma(1 - \delta|x - \overline{x}|^{p/p-1})_+^{-((p-1)/(r-p))}$, for some $\gamma \in \mathbb{R}$, $\delta > 0$ and $\overline{x} \in \mathbb{R}^N$. See also [2, 3] where Agueh gave a proof by studying a p-Laplacian type equation and by transforming the unknown of the equation via some change of functions. We also cite [18] where Cordero–Erausquin, Nazaret, and Villani set up a beautiful link between optimal transportation and certain Sobolev inequalities and Gagliardo–Nirenberg inequalities.

https://doi.org/10.1017/prm.2018.45 Published online by Cambridge University Press

Now, let $C : \mathbb{R}^N \to \mathbb{R}^+$ be even, strictly convex function. We suppose that C is q- homogeneous, that is there exists $q > 1$ such that

$$C(\lambda x) = \lambda^q C(x) \quad \forall \lambda \geqslant 0, \quad \forall x \in \mathbb{R}^N.$$

Then C^*, the Legendre transform of C, defined by

$$C^*(x) = \sup_y \{\langle x, y\rangle - C(y)\},$$

is even, strictly convex function and is p-homogeneous with $p = q/q - 1$. The following result has been proved by Agueh–Ghoussoub–Kang [4] (see also Chapter 13 in the book [28]):

Theorem B. *Let $F : [0, \infty) \to \mathbb{R}$ be a differentiable function on $(0, \infty)$ with $F(0) = 0$ and $x \longmapsto x^N F(x^{-N})$ convex and nonincreasing, and set $G_F(x) = (1 - N)F(x) + NxF'(x)$ Let $\psi : [0, \infty) \to \mathbb{R}$. be a differentiable function chosen in such a way that*

$$\psi(0) = 0 \quad \text{and} \quad \left|\psi^{1/p}(F' \circ \psi)'\right| = 1.$$

If Ω is an open, bounded and convex subset of $\mathbb{R}^N$, we consider the following two extremal problems:

$$D_\infty = \sup\left\{-\int_\Omega [F(\rho) + C\rho]\mathrm{d}x; \rho \in \mathcal{P}_a(\Omega)\right.$$
$$\left.= \left\{\rho : \Omega \to \mathbb{R}; \rho \geqslant 0 \quad \text{and} \quad \int_\Omega \rho(x)\mathrm{d}x = 1\right\}\right\}$$

and

$$P_\infty = \inf\left\{\int_\Omega [C^*(-\nabla f) - G_F \circ \psi(f)]\mathrm{d}x; f \in C_0^\infty(\Omega), \int_\Omega \psi(f) = 1\right\}$$

Then, we have $D_\infty \leqslant P_\infty$. Also, if there exists $\overline{f}$ that satisfies

$$-(F' \circ \psi)'\left(\overline{f}\right) \nabla \overline{f}(x) = \nabla C(x) \quad a.e.$$

then $D_\infty = P_\infty$, D_∞ is attained at $\overline{f}$, and P_∞ is attained at $\overline{\rho} = \psi(\overline{f})$. Moreover, f solves

$$\begin{cases} \operatorname{div}(\nabla C^*(-\nabla f)) - (G_F \circ \psi)'(f) = \lambda \psi'(f) & \text{in } \Omega \\ \nabla C^*(-\nabla f) \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

for some $\lambda \in \mathbb{R}$, and $\overline{\rho}$ is a stationary solution of

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \operatorname{div}(\rho \nabla (F'(\rho) + C)) & \text{in } (0.\infty) \times \Omega \\ \rho \nabla (F'(\rho) + C) \cdot \nu = 0 & \text{on } (0.\infty) \times \partial\Omega. \end{cases}$$

It is worth noting that theorem B can be used to derive the duality associated with the Sobolev inequality and the Gagliardo–Nirenberg inequalities [18], as well as the duality between ground state solutions of some semilinear equations and the stationary solutions of Fokker–Planck equations [28].

The Sobolev inequalities with monomial weights have been also studied intensively recently. For instance, in [6], the authors used the stereographic projection combining with the Curvature-Dimension condition to prove the following Sobolev inequality with monomial weight: for $a \geqslant 0$, $N+a>2$, there exists $S(N,a)>0$ such that for all smooth, compactly supported function f on $\mathbb{R}^{N-1}\times\mathbb{R}_+$:

$$\left[\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}_+}|u(x)|^{((2(N+a))/(N+a-2))}\,x_N^a\mathrm{d}x\right]^{((N+a-2)/((2N+a)))} \leqslant S(N,a)\left[\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}_+}|\nabla u(x)|^2\,x_N^a\mathrm{d}x\right]^{1/2}.$$

The best constant $S(N,a)$ was also calculated explicitly in [6]. In [42], V.H. Nguyen employed the mass transport approach to re-prove and extend the above result. Moreover, he also studied the best constants and extremal functions for the Gagliardo–Nirenberg inequalities and logarithmic Sobolev inequalities with the weight x_N^a and with the arbitrary norm. In [13], Cabré and Ros-Oton used the Alexandroff-Bakelman-Pucci method to investigate the isoperimetric inequalities with monomial weights and then applied a version of the weighted Schwarz rearrangement to establish the Sobolev, Morrey and Trudinger inequalities. Optimal Trudinger–Moser inequalities and Hardy inequalities were also studied recently in [31, 32].

1.1. Main results

In this paper, we will derive some general results about the weighted CKN inequalities. First, we consider the following class of the CKN inequalities:

$$1<p\leqslant p+\mu<N,\quad \theta=\frac{N\mu}{N-p}<N, \tag{1.5}$$
$$a=\frac{[(N-\theta)r-(N-\theta)s]p}{[(N-\theta)p-(N-\mu-p)s]r}=\frac{[Nr-Ns]p}{[Np-(N-p)s]r}.$$

Denote $D^{p,s}_{\mu,\theta}(\mathbb{R}^N)$ the completion of the space of smooth compactly supported functions with the norm

$$\left(\int_{\mathbb{R}^N}pC^*(\nabla u)\frac{\mathrm{d}x}{C(x)^{\mu/q}}\right)^{1/p}+\left(\int_{\mathbb{R}^N}|u|^s\frac{\mathrm{d}x}{C(x)^{\theta/q}}\right)^{1/s},$$

and set

$$CKN(N,\mu,p,s,r)=\sup_{u\in D^{p,s}_{\mu,\theta}(\mathbb{R}^N)}\frac{\left(\int_{\mathbb{R}^N}|u|^r\,\frac{\mathrm{d}x}{C(x)^{\theta/q}}\right)^{1/r}}{\left(\int_{\mathbb{R}^N}pC^*(\nabla u)\frac{\mathrm{d}x}{C(x)^{\mu/q}}\right)^{a/p}\left(\int_{\mathbb{R}^N}|u|^s\,\frac{\mathrm{d}x}{C(x)^{\theta/q}}\right)^{1-a/s}}.$$

Then, we will prove that

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THEOREM 1.1. *Assume* (1.5).

1/ *If* $r = p((s-1)/(p-1))$ *and* $p < r < ((Np))/(N-p))$, *then* CKN (N, μ, p, s, r) *is achieved by maximizers of the form*

$$V_0(x) = \gamma' \left(1 + \delta' C(x)^{((N-p-\mu)/(N-p))}\right)^{-((p-1)/(s-p))}$$
$$\textit{for some } \gamma' \in \mathbb{R}, \delta' > 0.$$

2/ *If* $1 < s = p((r-1)/(p-1)) < p$ *and* $r > 2 - ((1)/(p))$, $CKN(N, \mu, p, s, r)$ *is achieved by maximizers of the form*

$$V_0(x) = \gamma'(1 - \delta' C(x)^{((N-p-\mu)/(N-p))})_+^{-((p-1)/(r-p))} \quad \textit{for some } \gamma' \in \mathbb{R}, \delta'>0.$$

REMARK 1.1. Theorem 1.1 was first set up under the regular Euclidean norm in [**33**]. In the same paper, the authors also proved the similar result for arbitrary norm on $\mathbb{R}^N$.

We also consider the following class of the weighted CKN inequalities on $\Omega = \mathbb{R}^{N-1} \times \mathbb{R}_+$:

$$1 < p \leqslant p + \mu < N + b, \quad \theta = \frac{(N+b)\mu}{N+b-p} < N+b, \tag{1.6}$$
$$a = \frac{p(N+b)[r-s]}{r[p(N+b) - s(N+b-p)]}, b \geqslant 0.$$

Let $\omega(x) = x_N^b$ on Ω, denote $D^{p,s}_{\mu,\theta}(\Omega, \omega)$ the completion of the space of smooth compactly supported functions with the norm

$$\left(\int_\Omega pC^*(\nabla u)\omega \frac{\mathrm{d}x}{C(x)^{\mu/q}}\right)^{1/p} + \left(\int_\Omega |u|^s \omega \frac{\mathrm{d}x}{C(x)^{\theta/q}}\right)^{1/s},$$

and set

$$CKN(N, \mu, p, s, r, b) = \sup_{u \in D^{p,s}_{\mu,\theta}(\Omega,\omega)}$$
$$\times \frac{\left(\int_\Omega |u|^r \omega((\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1/r}}{\left(\int_\Omega pC^*(\nabla u)\omega((\mathrm{d}x)/(C(x)^{\mu/q}))\right)^{a/p} \left(\int_\Omega |u|^s \omega((\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1-a/s}}.$$

Then, we will also prove that

THEOREM 1.2. *Assume* (1.6).

1/ *If* $p < r = p((s-1)/(p-1)) < (((N+b)p)/(N+b-p))$ *and, then* CKN (N, μ, p, s, r, b) *is achieved by maximizers of the form*

$$V_0(x) = \gamma'(1 + \delta' C(x)^{((N+b-p-\mu)/(N+b-p))})^{-((p-1)/(s-p))}$$
$$\textit{for some } \gamma' \in \mathbb{R}, \delta' > 0.$$

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2/ *If* $1 < s = p((r-1)/(p-1)) < p$ *and* $r > 2 - ((1)/(p))$, $CKN(N, \mu, p, s, r, b)$ *is achieved by maximizers of the form*

$$V_0(x) = \gamma'(1 - \delta' C(x)^{((N+b-p-\mu)/(N+b-p))})_+^{-((p-1)/(r-p))} \tag{1.7}$$

$$\textit{for some } \gamma' \in \mathbb{R}, \delta' > 0. \tag{1.8}$$

In the special case where $C(\cdot) = 1/q|\cdot|^q$, $C^*(\cdot) = 1/p|\cdot|^p$ where $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^N$, we get the following result as a consequence:

THEOREM 1.3. *Assume* (1.6).

1/ *If* $p < r = p((s-1)/(p-1)) < (((N+b)p)/(N+b-p))$ *and, then* CKN (N, μ, p, s, r, b) *is achieved by maximizers of the form*

$$V_0(x) = \gamma' \left(1 + \delta' |x|^{q((N+b-p-\mu)/(N+b-p))}\right)^{-((p-1)/(s-p))}$$
$$\textit{for some } \gamma' \in \mathbb{R}, \delta' > 0.$$

2/ *If* $1 < s = p((r-1)/(p-1)) < p$ *and* $r > 2 - 1/p$, $CKN(N, \mu, p, s, r, b)$ *is achieved by maximizers of the form*

$$V_0(x) = \gamma' \left(1 - \delta' |x|^{q((N+b-p-\mu)/(N+b-p))}\right)_+^{-((p-1)/(r-p))}$$
$$\textit{for some } \gamma' \in \mathbb{R}, \delta' > 0.$$

Here

$$CKN(N, \mu, p, s, r, b) = \sup_{u \in D^{p,s}_{\mu,\theta}(\Omega, \omega)} \times \frac{\left(\int_\Omega |u|^r x_N^b ((\mathrm{d}x)/(|x|^\theta))\right)^{1/r}}{\left(\int_\Omega |\nabla u|^p x_N^b ((\mathrm{d}x)/(|x|^\mu)\right)^{a/p} \left(\int_\Omega |u|^s x_N^b ((\mathrm{d}x)/(|x|^\theta))\right)^{1-a/s}}.$$

It is surprising that although the weights in the above theorems are not radial, the optimizers are. It is also worth mentioning that in [**13**], Cabré and Ros-Oton pointed out an interesting fact that the monomial weights are not radially symmetric but still Euclidean balls centred at the origin solve the monomial weighted isoperimetric problems.

Our paper is organized as follows: Preliminaries and some helpful lemmata will be provided in § 2. In § 3, we will establish the Gagliardo–Nirenberg and Caffarelli–Kohn Nirenberg inequalities and present the proof of Theorem 1.1. The Gagliardo–Nirenberg and Caffarelli–Kohn Nirenberg inequalities with monomial weight x_N^b and the proof of Theorem 1.2 will be studied in § 4. Finally, inequalities with general monomial weights will be stated in § 5.

2. Preliminaries and some useful lemmata

Let $C : \mathbb{R}^N \to \mathbb{R}^+$ be even, strictly convex function. We suppose that C is q-homogeneous, that is, there exists $q > 1$ such that

$$C(\lambda x) = \lambda^q C(x) \quad \forall \lambda \geqslant 0, \quad \forall x \in \mathbb{R}^N. \tag{2.1}$$

Then C^*, the Legendre transform of C, defined by

$$C^*(x) = \sup_y \{\langle x, y\rangle - C(y)\},$$

is even, strictly convex function and is $p-$homogeneous with $p = q/q - 1$.

We have that $\langle X, Y\rangle \leqslant C^*(X) + C(Y)$ for all X, Y. Hence $\langle X, Y\rangle \leqslant \lambda^p C^*(X) + \lambda^{-q} C(Y)$ for all $\lambda > 0$, X, Y. Minimizing the right-hand side with respect to λ gives the Cauchy–Schwarz inequality

$$X \cdot Y \leqslant [qC(Y)]^{1/q} [pC^*(X)]^{1/p}.$$

By Young's inequality, we have

$$X \cdot Y \leqslant [qC(Y)]^{1/q} [pC^*(X)]^{1/p} \leqslant C^*(x) + C(y).$$

Hence, we also have

$$[pC^*(X)]^{1/p} = \sup_Y \frac{X \cdot Y}{[qC(Y)]^{1/q}}.$$

In other words,

$$C^*(X) = \sup_Y \frac{|X \cdot Y|^p}{p[qC(Y)]^{p/q}}$$

We will assume that for all $x \in \mathbb{R}^N$, there exists a unique vector x^* such that

$$x \cdot x^* = qC(x) \quad \text{and} \quad C^*(x^*) = (q-1)\, C(x) = \frac{q}{p} C(x).$$

In other words, for all $x \in \mathbb{R}^N$, there exists a unique vector x^* such that the equality in the Cauchy–Schwarz inequality happens.

Noting that from (2.1), we get that $C(\cdot)$ is differentiable a.e. We will assume that the gradient of $C(\cdot)$ at $x \in \mathbb{R}^N$ is the unique vector x^*.

Example 2.1. The functions that we have in mind are $C(x) = 1/q|x|^q$ and $C^*(x) = 1/p|x|^p$ with $|\cdot|$ is the regular Euclidean norm on $\mathbb{R}^N$. Another example is the pair $C(x) = 1/q\|x\|^q$ and $C^*(x) = 1/p\|x\|_*^p$, where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^N$ and

$$\|X\|_* = \sup_{\|Y\| \leqslant 1} X \cdot Y.$$

The following lemma was observed recently in [**24**, **33**]. For the completeness, we will provide the proof here.

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LEMMA 2.1. *We have*

$$|x \cdot \nabla u(x)| = [qC(x)]^{1/q}[pC^*(\nabla u)]^{1/p} \quad \textit{for a.e. } x \in \mathbb{R}^N,$$

if and only if u is C-radial, that is, $u(x) = u(y)$ *when* $C(x) = C(y)$.

Proof. If u is C-radial, then recalling that $\nabla(C(\cdot))(x) = x^*$, we have

$$\frac{\partial u}{\partial x_j}(x) = u'(C(x))x_j^*.$$

Hence,

$$C^*(\nabla u) = C^*(u'(C(x))x^*) = |u'(C(x))|^p C^*(x^*) = |u'(C(x))|^p \frac{q}{p} C(x)$$

and

$$[qC(x)]^{1/q}[pC^*(\nabla u)]^{1/p} = [qC(x)]^{1/q}[|u'(C(x))|^p qC(x)]^{1/p} = |u'(C(x))|qC(x).$$

Also,

$$|x \cdot \nabla u(x)| = \left|\sum_{j=1}^N x_j \frac{\partial u}{\partial x_j}(x)\right| = |u'(\|x\|)| \left|\sum_{j=1}^N x_j x_j^*\right| = |u'(\|x\|)|\, qC(x).$$

Now, if for all $x \in \mathbb{R}^N$:

$$|x \cdot \nabla u(x)| = [qC(x)]^{1/q}[pC^*(\nabla u)]^{1/p},$$

then $\nabla u(x)$ has the same direction with x^*. That is we can find a function $f(x)$ such that $\nabla u(x) = f(x)x^*$. Now let a and b be two points on the C-sphere with radius $r > 0$. That is $C(a) = C(b) = r$. We connect a and b by a piecewise smooth curve $r(t)$ on the sphere, that is, $C(r(t)) = r$ and $C(r(0)) = a$, $C(r(1)) = b$. Then we have

$$\nabla u(r(t)) = f(r(t))(r(t))^*.$$

Using that fact that $C(r(t)) = r$ for all t, we get

$$(r(t))^* \cdot \nabla r(t) = 0.$$

Hence

$$\int_0^1 \nabla u(r(t)) \cdot \nabla r(t) \mathrm{d}t = \int_0^1 f(r(t))(r(t))^* \cdot \nabla r(t) \mathrm{d}t = 0.$$

In other words,

$$u(b) - u(a) = u(C(r(1))) - u(C(r(0))) = 0.$$

□

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Let $d > 0$. We now consider the quasi-conformal mapping type of transform $L_{N,d} : \mathbb{R}^N \to \mathbb{R}^N$:

$$L_{N,d}(x) = C(x)^d x.$$

The Jacobian matrix of this function $L_{N,d}$ is

$$\mathbf{J}_{\mathbf{L_{N,d}}} = C(x)^d \mathbb{I}_N + A$$

where

$$\mathbf{A} = \begin{pmatrix} dC(x)^{d-1}x_1x_1^* & dC(x)^{d-1}x_1x_2^* & \dots & dC(x)^{d-1}x_1x_N^* \\ dC(x)^{d-1}x_2x_1^* & dC(x)^{d-1}x_2x_2^* & \dots & dC(x)^{d-1}x_2x_N^* \\ \vdots & \vdots & \ddots & \vdots \\ dC(x)^{d-1}x_Nx_1^* & dC(x)^{d-1}x_Nx_2 & \dots & dC(x)^{d-1}x_Nx_N^*. \end{pmatrix}$$

It is obvious that $rank(A) = 1$ and

$$tr(A) = dC(x)^{d-1}x \cdot x^* = dqC(x)^d$$

Hence, its characteristic polynomial is

$$\det(\lambda \mathbb{I}_N - A) = \lambda^N - dqC(x)^d \lambda^{N-1}.$$

Choosing $\lambda = -C(x)^d$, we get

$$\det(J_{L_{N,d}}) = (-1)^N \det(-C(x)^d \mathbb{I}_N - A) = (1 + dq)C(x)^{Nd}.$$

Hence, we have

$$\det(J_{L_{N,d}}) = (1 + dq)C(x)^{Nd}. \tag{2.2}$$

We now define mappings $D_{N,d,p}$ with $p > 1$ by

$$D_{N,d,p}u(x) := \left(\frac{1}{1+dq}\right)^{((p-1)/(p))} u(L_{N,d}(x)) = \left(\frac{1}{1+dq}\right)^{((p-1)/(p))} u(C(x)^d x). \tag{2.3}$$

We also define $D^{-1}_{N,d,p}$

$$D^{-1}_{N,d,p}u = v \quad \text{if } u = D_{N,d,p}v.$$

Then, in [**33**], the following result has been established:

Lemma 2.2. (1) *For continuous function f, we have*

$$\int_{\mathbb{R}^N} \frac{f((((1)/(1+\mathrm{d}q)))^{((p-1)/(p))}u(x))}{C(x)^t} \mathrm{d}x = (1 + \mathrm{d}q) \int_{\mathbb{R}^N} \frac{f(D_{N,d,p}u(x))}{C(x)^{t(\mathrm{d}q+1)-Nd}} \mathrm{d}x.$$

In particular, we obtain that $u \in L^s(((dx)/(C(x)^t)))$ if and only if $D_{N,d,p}u \in L^s(((dx)/(C(x)^{t(dq+1)-Nd})))$.

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(2) *For smooth functions u :*

$$\int_{\mathbb{R}^N} \frac{C^*(\nabla D_{N,d,p}u(x))}{C(x)^{(qd+1)\mu+pd-Nd}}\mathrm{d}x \leqslant \int_{\mathbb{R}^N} \frac{C^*(\nabla u(y))}{C(y)^{\mu}}\mathrm{d}y.$$

The equality occurs if and only if u is C-radially symmetric.

We now will set up the similar results on half-spaces $\mathbb{R}^{N-1}\times\mathbb{R}_+$ with weight x_N^b, $b\geqslant 0$:

LEMMA 2.3. *Let* $b\geqslant 0$.

(1) *For continuous function f, we have*

$$\int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{f((((1)/(1+dq)))^{p-1/p}u(x))}{C(x)^t}x_N^b\mathrm{d}x = (1+\mathrm{d}q)\int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{f(D_{N,d,p}u(x))}{C(x)^{t(dq+1)-(N+b)d}}x_N^b\mathrm{d}x.$$

In particular, we obtain that $u\in L^s(((x_N^b dx)/(C(x)^t)))$ *if and only if* $D_{N,d,p}u\in L^s(((x_N^b dx)/(C(x)^{t(dq+1)-(N+b)d})))$.

(2) *For smooth functions u :*

$$\int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{C^*(\nabla D_{N,d,p}u(x))}{C(x)^{(qd+1)\mu+pd-(N+b)d}}x_N^b\mathrm{d}x \leqslant \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{C^*(\nabla u(y))}{C(y)^{\mu}}y_N^b dy.$$

The equality occurs if and only if u is C-radially symmetric.

Proof. (1) We will first show

$$\int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{f(D_{N,d,p}u(x))}{C(x)^{t(\mathrm{d}q+1)-(N+b)d}}x_N^b\mathrm{d}x = \frac{1}{1+dq}\int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{f((1/1+dq)^{p-1/p}u(y))}{C(y)^t}y_N^b dy.$$

Using change of variables $y_i = C(x)^d x_i$, $i=1,2,\ldots,N$, we have

$$dy = \det(J_{L_{N,d}})dx = (1+dq)C(x)^{Nd}dx,$$

and

$$dx = \frac{1}{(1+dq)}\frac{dy}{C(y)^{((Nd)/(dq+1))}}.$$

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Hence

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{f(D_{N,d,p}u(x))}{C(x)^{t(dq+1)-(N+b)d}} x_N^b \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{f((((1)/(1+dq)))^{((p-1)/(p))} u(C(x)^d x))}{C(x)^{t(dq+1)-Nd}} C(x)^{bd} x_N^b \mathrm{d}x \\
&\quad = \frac{1}{1+dq} \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{f(((1)/(1+dq))^{((p-1)/(p))} u(y))}{C(y)^{((t(dq+1)-Nd)/(dq+1))}} y_N^b \frac{\mathrm{d}y}{C(y)^{((Nd)/(dq+1))}} \\
&\quad = \frac{1}{1+dq} \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{f(((1)/(1+dq))^{((p-1)/(p))} u(y))}{C(y)^t} y_N^b \mathrm{d}y.
\end{aligned}$$

(2) Now we begin to consider the gradient of $D_{N,d,p}u$. After calculations, we have

$$\begin{pmatrix} \dfrac{\partial D_{N,d,p}u}{\partial x_1}(x) \\ \dfrac{\partial D_{N,d,p}u}{\partial x_2}(x) \\ \vdots \\ \dfrac{\partial D_{N,d,p}u}{\partial x_N}(x) \end{pmatrix} = \nabla D_{N,d,p}u(x) = \left(\frac{1}{1+dq}\right)^{p-1/p} \nabla(u(C(x)^d x))$$

$$= \left(\frac{1}{1+dq}\right)^{p-1/p} J_{L_{N,d}}^T \begin{pmatrix} \dfrac{\partial u}{\partial x_1}(C(x)^d x) \\ \dfrac{\partial u}{\partial x_2}(C(x)^d x) \\ \vdots \\ \dfrac{\partial u}{\partial x_N}(C(x)^d x) \end{pmatrix}.$$

Hence we have

$$\frac{\partial u(C(x)^d x)}{\partial x_i} = \left(C\,(x)^d\, \frac{\partial u}{\partial x_i}(C(x)^d x) + A_i\right),$$

for $i = 1, 2, \ldots N$, where

$$A_i := \sum_{j=1}^{N} dC(x)^{d-1} x_i^* x_j \frac{\partial u}{\partial x_j}(C(x)^d x).$$

$$C^*(X) = \sup \frac{|X \cdot Y|^p}{p[qC(Y)]^{p/q}}$$

Hence, we obtain

$$
\begin{aligned}
&C^*(\nabla D_{N,d,p}u(x)) \\
&= C^*\left(\left(\frac{1}{1+dq}\right)^{p-1/p}\nabla(u(C(x)^d x))\right) = \left(\frac{1}{1+dq}\right)^{p-1} C^*\left(\nabla(u(C(x)^d x))\right) \\
&= \left(\frac{1}{1+dq}\right)^{p-1}\sup_y\left\{\frac{\left(\nabla(u(C(x)^d x))\cdot y\right)^p}{p[qC(y)]^{p/q}}\right\} \\
&= \left(\frac{1}{1+dq}\right)^{p-1}\sup_y\left\{\frac{\left[\sum_{i=1}^N \left[C(x)^d((\partial u)/(\partial x_i))(C(x)^d x)y_i + A_i y_i\right]\right]^p}{p[qC(y)]^{p/q}}\right\}.
\end{aligned}
$$

The first term is easy to compute:

$$
\begin{aligned}
I_1 &= \sum_{i=1}^N C(x)^d \frac{\partial u}{\partial x_i}(C(x)^d x)y_i \\
&= C(x)^d \nabla u(C(x)^d x)\cdot y \\
&\leqslant C(x)^d[qC(y)]^{1/q}[pC^*(\nabla u(C(x)^d x))]^{1/p}
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality

$$
X\cdot Y \leqslant [qC(Y)]^{1/q}[pC^*(X)]^{1/p},
$$

we can estimate the second term:

$$
\begin{aligned}
I_2 &= \sum_{i=1}^N A_i y_i \\
&= \sum_{i=1}^N\sum_{j=1}^N dC(x)^{d-1}x_i^* x_j \frac{\partial u}{\partial x_j}(C(x)^d x)y_i \\
&= dC(x)^{d-1}\sum_{i=1}^N x_i^* y_i \sum_{j=1}^N x_j \frac{\partial u}{\partial x_j}(C(x)^d x) \\
&\leqslant dC(x)^{d-1}\left|x^*\cdot y\right|\left|x\cdot\nabla u(C(x)^d x)\right| \\
&\leqslant dC(x)^{d-1}[qC(y)]^{1/q}[pC^*(x^*)]^{1/p}[qC(x)]^{1/q}[pC^*(\nabla u(C(x)^d x))]^{1/p} \\
&\leqslant dC(x)^{d-1}[qC(y)]^{1/q}[qC(x)]^{1/p}[qC(x)]^{1/q}[pC^*(\nabla u(C(x)^d x))]^{1/p} \\
&\leqslant qdC(x)^d[qC(y)]^{1/q}[pC^*(\nabla u(C(x)^d x))]^{1/p}
\end{aligned}
$$

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Therefore,

$$\sup_y \left\{ \frac{\left[\sum_{i=1}^{N}\left[C(x)^d((\partial u)/(\partial x_i))(C(x)^d x)y_i + A_i y_i\right]\right]^p}{p\,[qC(y)]^{p/q}} \right\}$$
$$\leqslant \sup_y \left\{ \frac{[(1+qd)]^p C(x)^{pd}[qC(y)]^{p/q} pC^*(\nabla u(C(x)^d x))}{p[qC(y)]^{p/q}} \right\}$$
$$= [(1+qd)]^p C(x)^{pd} C^*(\nabla u(C(x)^d x)).$$

In conclusion, we get

$$C^*(\nabla D_{N,d,p}u(x)) \leqslant (1+qd)C(x)^{pd}C^*(\nabla u(C(x)^d x)).$$

Using the change of variables again, we get

$$\int_{\mathbb{R}^N} \frac{C^*(\nabla u(y))}{C(y)^\mu} y_N^b \mathrm{d}y = \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{C^*(\nabla u(C(x)^d x))}{C(C(x)^d x)^\mu}(1+dq)C(x)^{(N+b)d} x_N^b \mathrm{d}x$$
$$\geqslant \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{C^*(\nabla D_{N,d,p}u(x))}{C(x)^{(qd+1)\mu}C(x)^{pd}} C(x)^{(N+b)d} x_N^b \mathrm{d}x$$
$$= \int_{\mathbb{R}^{N-1}\times\mathbb{R}_+} \frac{C^*(\nabla D_{N,d,p}u(x))}{C(x)^{(qd+1)\mu+pd-(N+b)d}} x_N^b \mathrm{d}x.$$

Finally, it is easy to check that the equalities hold if and only if the equality in the Cauchy–Schwarz inequality occurs. It means that u is C-radially symmetric. □

Now, let μ and ν be two Borel probability measures on $\mathbb{R}^N$ such that μ is absolutely continuous with respect to Lebesgue measure. Then by the results of Brenier [**10**] and McCann [**39**], there exists a convex function φ such that

$$\int b(y)\mathrm{d}\nu(y) = \int b(\nabla\varphi(x))\mathrm{d}\mu(x)$$

for every bounded or positive, Borel function b. Moreover, $\nabla\varphi$ is uniquely determined $d\mu$ almost everywhere. If μ and ν are absolutely continuous with densities F and G, then

$$\int b(y)G(y)\mathrm{d}y = \int b(\nabla\varphi(x))F(x)\mathrm{d}x$$

for every bounded or positive, Borel function b. If φ is of class C^2, then φ solves the Monge–Ampère equation

$$F(x) = G(\nabla\varphi(x))\det(D^2\varphi(x)). \tag{2.4}$$

Actually, by a result of McCann in [**40**], without further assumptions on F and G beyond integrability, (2.4) holds $F(x)dx$ a.e. Then, we have the following lemma in [**42**]:

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LEMMA 2.4. *Let $b \geqslant 0$, $1 \neq \gamma \geqslant 1 - 1/N + b$ and $\omega(x) = x_N^b$ $\forall x \in \Omega$. Let F and G be two nonnegative functions on $\Omega = \mathbb{R}^{N-1} \times (0, \infty)$ with $\int_\Omega F\omega \mathrm{d}x = \int_\Omega G\omega \mathrm{d}x = 1$, and such that F^γ is C^1 on Ω and F, G are compactly supported on $\overline{\Omega}$. Then if $\nabla\varphi$ is the Brenier map pushing $F\omega dx$ toward to $G\omega dx$, then we have*

$$\frac{1}{1-\gamma}\int_\Omega G^\gamma \omega \mathrm{d}x \leqslant \frac{1-(N+b)(1-\gamma)}{1-\gamma}\int_\Omega F^\gamma \omega \mathrm{d}x - \int_\Omega \nabla F^\gamma \cdot \nabla\varphi\omega \mathrm{d}x.$$

3. Gagliardo–Nirenberg inequalities and CKN inequalities

Now, for $\alpha \geqslant 0$, we define

$$U_{\alpha,p}(x) = (\sigma_{\alpha,p} + (\alpha-1)qC(x))_+^{((1)/(1-\alpha))}$$

where $\sigma_{\alpha,p}$ is chosen such that $\|U_{\alpha,p}\|_{\alpha p} = 1$. We will first prove that

THEOREM 3.1. *Let $N \geqslant 2$, $p > 1$ and $0 < \alpha \neq 1$ be such that $\alpha(N-p) \leqslant N$. Let f and g be such that $\|g\|_{\alpha p} = 1$ and $\|f\|_{\alpha p} = 1$. Then, for all $\mu > 0$,*

$$\begin{aligned}&\frac{\alpha p}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |g|^{\alpha(p-1)+1} - \frac{\mu^q}{q}\int qC(y)|g(y)|^{\alpha p}\mathrm{d}y\\&\quad\leqslant \frac{1}{p\mu^p}\int pC^*(\nabla f) + \frac{\alpha p - N(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |f|^{\alpha(p-1)+1}.\end{aligned}$$

When $\mu = q^{1/q}$, then the equality happens if $f = g = U_{\alpha,p}$. As a consequence, we have that for $\alpha > 1$:

$$\begin{aligned}&\frac{\left[\int C^*(\nabla f)\right]^{a/p}\left[\int |f|^{\alpha(p-1)+1}\right]^{\frac{1-a}{\alpha(p-1)+1}}}{\|f\|_{\alpha p}}\\&\quad\geqslant \frac{\left[\int C^*(\nabla U_{\alpha,p})\right]^{a/p}\left[\int |U_{\alpha,p}|^{\alpha(p-1)+1}\right]^{\frac{1-a}{\alpha(p-1)+1}}}{\|U_{\alpha,p}\|_{\alpha p}},\end{aligned}$$

with

$$a = \frac{N(\alpha-1)}{\alpha[Np-(\alpha p+1-\alpha)(N-p)]}.$$

Also, for $\alpha < 1$:

$$\frac{\left[\int C^*(\nabla f)\right]^{a/p}\|f\|_{\alpha p}^{1-a}}{\left[\int |f|^{\alpha(p-1)+1}\right]^{((1)/(\alpha(p-1)+1))}} \geqslant \frac{\left[\int C^*(\nabla U_{\alpha,p})\right]^{a/p}\|U_{\alpha,p}\|_{\alpha p}^{1-a}}{\left[\int |U_{\alpha,p}|^{\alpha(p-1)+1}\right]^{((1/\alpha(p-1)+1))}},$$

with

$$a = \frac{N(1-\alpha)}{(\alpha p+1-\alpha)[N-\alpha(N-p)]}.$$

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Proof. Applying Theorem B with

$$\rho = |g|^{\alpha p}, F(x) = \frac{\alpha p}{1-\alpha} x^{((\alpha(p-1)+1)/(\alpha p))}, \psi(x) = \left(\frac{1}{\alpha p-(\alpha-1)}\right)^{\alpha p} |x|^{\alpha p},$$
$$c(\cdot) = [\alpha p-(\alpha-1)]\mu^q C(\cdot), c^*(\cdot) = \frac{1}{[\alpha p-(\alpha-1)]^{p/q}\mu^p} C^*(\cdot),$$

we get

$$-\int_\Omega [F(\rho)+c\rho]\mathrm{d}x \leqslant \int_\Omega [c^*(-\nabla f) - G_F \circ \psi(f)]\mathrm{d}x.$$

We will now check that $x \longmapsto x^N F(x^{-N})$ is convex and nonincreasing. Indeed,

$$\begin{aligned} l(x) = x^N F(x^{-N}) &= x^N \frac{\alpha p}{1-\alpha} x^{-N((\alpha(p-1)+1)/(\alpha p))} \\ &= \frac{\alpha p}{1-\alpha} x^{N\alpha-1/\alpha p}, \end{aligned}$$

so

$$\begin{aligned} l'(x) &= \frac{\alpha p}{1-\alpha} N \frac{\alpha-1}{\alpha p} x^{N((\alpha-1)/(\alpha p))-1} = -N x^{N((\alpha-1)/(\alpha p))-1} < 0 \quad \text{for } x \in (0,\infty), \\ l''(x) &= -N\left(N\frac{\alpha-1}{\alpha p}-1\right) x^{N((\alpha-1)/(\alpha p))-2} \\ &= N\left(\frac{\alpha p+N-N\alpha}{\alpha p}\right) x^{N((\alpha-1)/(\alpha p))-2} > 0 \\ &\quad \text{for } x \in (0,\infty) \quad \text{if } \alpha(N-p) \leqslant N. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{\alpha p}{\alpha-1}\int_\Omega |g|^{\alpha(p-1)+1} - [\alpha p-(\alpha-1)]\mu^q \int_\Omega C(y)|g(y)|^{\alpha p}\mathrm{d}y \\ &\leqslant \frac{[\alpha p-(\alpha-1)]}{\mu^p[\alpha p-(\alpha-1)]^p}\int_\Omega C^*(-\nabla([\alpha p-(\alpha-1)]f)) \\ &- \frac{\alpha p-N(\alpha-1)}{(1-\alpha)[\alpha p-(\alpha-1)]^{\alpha p-(\alpha-1)}}\int |[\alpha p-(\alpha-1)]f|^{\alpha(p-1)+1}. \end{aligned}$$

Equivalently,

$$\begin{aligned} &\frac{\alpha p}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |g|^{\alpha(p-1)+1} - \frac{\mu^q}{q}\int qC(y)|g(y)|^{\alpha p}\mathrm{d}y \\ &\leqslant \frac{1}{p\mu^p}\int pC^*(\nabla f) + \frac{\alpha p-N(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |f|^{\alpha(p-1)+1}. \end{aligned}$$

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Moreover, it can be checked, using theorem B or by direct calculation, that when $\mu = q^{1/q}$, then

$$\sup_{\|g\|_{\alpha p}=1} \left\{ \frac{\alpha p}{(\alpha-1)[\alpha p-(\alpha-1)]} \int |g|^{\alpha(p-1)+1} - \int qC(y)|g(y)|^{\alpha p} \mathrm{d}y \right\}$$
$$= \inf_{\|f\|_{\alpha p}=1} \left\{ \frac{1}{pq^{p/q}} \int pC^*(\nabla f) + \frac{\alpha p - N(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]} \int |f|^{\alpha(p-1)+1} \right\}$$

and $U_{\alpha,p}$ is the optimizer in both variational problems.

For $\alpha > 1$, we have that for all $f : \|f\|_{\alpha p} = 1 :$

$$\frac{1}{pq^{p/q}} \int pC^*(\nabla f) + \frac{\alpha p - N(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]} \int |f|^{\alpha(p-1)+1}$$
$$\geqslant \frac{\alpha p}{(\alpha-1)[\alpha p-(\alpha-1)]} \int |U_{\alpha,p}|^{\alpha(p-1)+1} - \int qC(y)|U_{\alpha,p}(y)|^{\alpha p} \mathrm{d}y := M.$$

The equality happens when $f = U_{\alpha,p}$. This means that for all f :

$$\frac{1}{pq^{p/q}} \int pC^*\left(\nabla \frac{f}{\|f\|_{\alpha p}}\right) + \frac{\alpha p - N(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]} \int \left|\frac{f}{\|f\|_{\alpha p}}\right|^{\alpha(p-1)+1} \geqslant M$$

that is,

$$\frac{1}{pq^{p/q}} \frac{\int pC^*(\nabla f)}{\|f\|_{\alpha p}^p} + \frac{\alpha p - N(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]} \frac{\int |f|^{\alpha(p-1)+1}}{\|f\|_{\alpha p}^{\alpha(p-1)+1}} \geqslant M$$

and so for $\lambda > 0$:

$$\lambda^p \lambda^{N/\alpha} \frac{1}{pq^{p/q}} \frac{\int pC^*(\nabla f)}{\|f\|_{\alpha p}^p} + \frac{\alpha p - N(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]} \frac{\lambda^{((N[\alpha(p-1)+1])/(\alpha p))}}{\lambda^N}$$
$$\frac{\int |f|^{\alpha(p-1)+1}}{\|f\|_{\alpha p}^{\alpha(p-1)+1}} \geqslant M.$$

Now, if we optimize with respect to $\lambda > 0$, we get that for the optimal choice of $\lambda = \lambda_{opt}$:

$$\frac{\left[\int C^*(\nabla f)\right]^{a/p} \left[\int |f|^{\alpha(p-1)+1}\right]^{((1-a)/(\alpha(p-1)+1))}}{\|f\|_{\alpha p}}$$
$$\geqslant \frac{\left[\int C^*(\nabla U_{\alpha,p,\lambda_{opt}})\right]^{a/p} \left[\int |U_{\alpha,p,\lambda_{opt}}|^{\alpha(p-1)+1}\right]^{((1-a)/(\alpha(p-1)+1))}}{\|U_{\alpha,p,\lambda_{opt}}\|_{\alpha p}}$$
$$= \frac{\left[\int C^*(\nabla U_{\alpha,p})\right]^{a/p} \left[\int |U_{\alpha,p}|^{\alpha(p-1)+1}\right]^{((1-a)/(\alpha(p-1)+1))}}{\|U_{\alpha,p}\|_{\alpha p}},$$

where $U_{\alpha,p,\lambda_{opt}} = U_{\alpha,p}(\lambda_{opt} x)$. Here we get by direct calculation from the scaling invariance argument that

$$a = \frac{N(\alpha-1)}{\alpha[Np-(\alpha p+1-\alpha)(N-p)]}.$$

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Similarly, when $\alpha < 1$, we obtain

$$\frac{\left[\int C^*(\nabla f)\right]^{a/p} \|f\|_{\alpha p}^{1-a}}{\left[\int |f|^{\alpha(p-1)+1}\right]^{((1)/(\alpha(p-1)+1))}} \geqslant \frac{\left[\int C^*(\nabla U_{\alpha,p})\right]^{a/p} \|U_{\alpha,p}\|_{\alpha p}^{1-a}}{\left[\int |U_{\alpha,p}|^{\alpha(p-1)+1}\right]^{((1)/(\alpha(p-1)+1))}},$$

with

$$a = \frac{N(1-\alpha)}{(\alpha p + 1 - \alpha)[N - \alpha(N-p)]}.$$

□

Remark 3.1. Using the ideas in [**19**, **20**], when $\alpha \to 1$, we can obtain the general optimal L^p-Euclidean logarithmic Sobolev inequality that was studied in [**27**].

We will now provide the proof for Theorem 1.1:

Proof of Theorem 1.1. Set

$$GN(N,p,s,r) = \sup_{u \in D_{0,0}^{p,s}(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |u|^r \mathrm{d}x\right)^{1/r}}{\left(\int_{\mathbb{R}^N} pC^*(\nabla u)\mathrm{d}x\right)^{a/p} \left(\int_{\mathbb{R}^N} |u|^s \mathrm{d}x\right)^{1-a/s}}.$$

1/ When $r = p((s-1)/(p-1))$ and $\theta = ((N\mu)/(N-p))$, we have from theorem 3.1 that $GN(N,p,s,r)$ is achieved by maximizers of the form

$$U_0(x) = \gamma(1 + \delta C(x))^{-((p-1))/(s-p))} \quad \text{for some } \gamma \in \mathbb{R}, \delta > 0.$$

Now, set $V_0 = D_{N,d,p}^{-1} U_0$ with $d = 1/q((\mu)/(N-p-\mu))$. Then we will show that V_0 is a maximizer of $CKN(N,\mu,p,s,r)$. Indeed, by lemma 2.2, we get

$$\int_{\mathbb{R}^N} \frac{C^*(\nabla u(y))}{C(y)^{\mu/q}} \mathrm{d}y \geqslant \int_{\mathbb{R}^N} C^*(\nabla D_{N,d,p} u(x))\mathrm{d}x$$

$$\left(\frac{1}{1+dq}\right)^{s((p-1)/(p))} \int_{\mathbb{R}^N} |u|^s \frac{\mathrm{d}x}{C(x)^{\theta/q}} = (1 + \mathrm{d}q) \int_{\mathbb{R}^N} |D_{N,d,p} u(x)|^s \mathrm{d}x$$

$$\left(\frac{1}{1+dq}\right)^{r((p-1)/(p))} \int_{\mathbb{R}^N} |u|^r \frac{\mathrm{d}x}{C(x)^{\theta/q}} = (1 + \mathrm{d}q) \int_{\mathbb{R}^N} |D_{N,d,p} u(x)|^r \mathrm{d}x$$

$$\int_{\mathbb{R}^N} \frac{C^*(\nabla V_0(y))}{C(y)^{\mu/q}} \mathrm{d}y = \int_{\mathbb{R}^N} C^*(\nabla U_0(x))\mathrm{d}x$$

$$\left(\frac{1}{1+dq}\right)^{s((p-1)/(p))} \int_{\mathbb{R}^N} |V_0|^s \frac{\mathrm{d}x}{C(x)^{\theta/q}} = (1 + \mathrm{d}q) \int_{\mathbb{R}^N} |U_0(x)|^s \mathrm{d}x$$

$$\left(\frac{1}{1+dq}\right)^{r((p-1)/(p))} \int_{\mathbb{R}^N} |V_0|^r \frac{\mathrm{d}x}{C(x)^{\theta/q}} = (1 + \mathrm{d}q) \int_{\mathbb{R}^N} |U_0(x)|^r \mathrm{d}x.$$

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Hence,

$$\frac{\left(\int_{\mathbb{R}^N} |u|^r((\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1/r}}{\left(\int_{\mathbb{R}^N} pC^*(\nabla u)((\mathrm{d}x)/(C(x)^{\mu/q}))\right)^{a/p}\left(\int_{\mathbb{R}^N} |u|^s((\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{((1-a)/(s))}}$$
$$\leqslant \left(\frac{(1+dq)^{(1+rp-1/p)1/r}}{(1+dq)^{(1+s((p-1)/(p)))((1-a)/(s))}}\right)$$
$$\times \frac{\left(\int_{\mathbb{R}^N} |D_{N,d,p}u|^r \mathrm{d}x\right)^{1/r}}{\left(\int_{\mathbb{R}^N} pC^*(\nabla D_{N,d,p}u)\mathrm{d}x\right)^{a/p}\left(\int_{\mathbb{R}^N} |D_{N,d,p}u|^s \mathrm{d}x\right)^{((1-a)/(s))}}$$
$$\leqslant \left(\frac{(1+dq)^{(1+r((p-1)/(p)))1/r}}{(1+\mathrm{d}q)^{(1+s((p-1)/(p)))((1-a)/(s))}}\right)$$
$$\times \frac{\left(\int_{\mathbb{R}^N} |U_0|^r \mathrm{d}x\right)^{1/r}}{\left(\int_{\mathbb{R}^N} pC^*(\nabla U_0)\mathrm{d}x\right)^{a/p}\left(\int_{\mathbb{R}^N} |U_0|^s \mathrm{d}x\right)^{((1-a)/(s))}}$$
$$= \frac{\left(\int_{\mathbb{R}^N} |V_0|^r((\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1/r}}{\left(\int_{\mathbb{R}^N} pC^*(\nabla V_0)((\mathrm{d}x)/(C(x)^{\mu/q}))\right)^{a/p}\left(\int_{\mathbb{R}^N} |V_0|^s((\mathrm{dx})/(C(x)^{\theta/q}))\right)^{((1-a)/(s))}}.$$

Hence, we could conclude that $CKN(N,\mu,p,s,r)$ can be attained by optimizers of the form

$$V_0 = D^{-1}_{N,d,p}U_0 = \gamma'(1+\delta' C(x)^{((1)/(qd+1))})^{-((p-1)/(s-p))} \quad \text{for some } \gamma' \in \mathbb{R}, \delta' > 0$$

Simiarly, when $\theta = ((N\mu)/(N-p))$, if $s = p((r-1)/(p-1))$ and $r > 2 - 1/p$, $CKN(N,\mu,p,q,r)$ is achieved by maximizers of the form

$$V_0(x) = \gamma'(1-\delta' C(x)^{((1)/(qd+1))})_+^{-((p-1)/(r-p))} \quad \text{for some } \gamma' \in \mathbb{R}, \delta' > 0.$$

□

4. Sobolev inequality, Gagliardo–Nirenberg inequalities and CKN inequalities with weight $\omega(x) = x_N^b$ on $\mathbb{R}^{N-1} \times \mathbb{R}_+$

Let $\Omega = \mathbb{R}^{N-1} \times (0,\infty)$, $b \geqslant 0$ with $1 < p < N+b$. We introduce the function

$$V_{p,b}(x) = (\sigma_{p,b} + qC(x))_+^{-((N+b-p)/(p))},$$

where $\sigma_{p,b}$ is chosen such that

$$\int_\Omega V_{p,b}(x)^{p_b^*}\omega(x)\mathrm{d}x = 1, \quad p_b^* = \frac{(N+b)p}{N+b-p}.$$

We will prove the following weighted Sobolev inequality:

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THEOREM 4.1. *Let* $b \geqslant 0$ *with* $1 < p < N + b$. *Let* f *and* g *be such that* $\int_\Omega |f|^{p_b^*}\omega \mathrm{d}x = \int_\Omega |g|^{p_b^*}\omega \mathrm{d}x = 1$, *then*

$$\frac{\int_\Omega |g|^{p_b^*(1-((1)/(N+b)))}\omega(x)\mathrm{d}x}{\left[\int_\Omega qC(y)|g(y)|^{p_b^*}\omega(y)\mathrm{d}y\right]^{1/q}} \leqslant \frac{p(N+b-1)}{(N+b)(N+b-p)}\left[\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right]^{1/p}. \quad (4.1)$$

The equality occurs when $f = g = V_{p,b}$.

As consequences, we obtain

1/

$$\sup_{\int_\Omega |g|^{p_b^*}\omega \mathrm{d}x=1} \frac{\int_\Omega |g|^{p_b^*(1-((1)/(N+b)))}\omega(x)\mathrm{d}x}{\left[\int_\Omega qC(y)|g(y)|^{p_b^*}\omega(y)\mathrm{d}y\right]^{1/q}} = \inf_{\int_\Omega |f|^{p_b^*}\omega \mathrm{d}x=1} \frac{p(N+b-1)}{(N+b)(N+b-p)}\left[\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right]^{1/p}.$$

2/

$$\frac{\left[\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right]^{1/p}}{\left[\int_\Omega |f|^{p_b^*}\omega \mathrm{d}x\right]^{1/p_b^*}} \geqslant \frac{\left[\int_\Omega pC^*(\nabla V_{p,b})\omega \mathrm{d}x\right]^{1/p}}{\left[\int_\Omega |V_{p,b}|^{p_b^*}\omega \mathrm{d}x\right]^{((1)/(p_b^*))}} = \frac{1}{S(N,b,p)}.$$

Proof. By standard arguments, we can assume that f, g are nonnegative, compactly supported on $\overline{\Omega}$ and f is in $C^1(\overline{\Omega})$. Using lemma 2.4 with $F = |f|^{p_b^*}$, $G = |g|^{p_b^*}$ and $\gamma = 1 - 1/N + b$, we get

$$\frac{1}{1-\gamma}\int_\Omega G^\gamma \omega \mathrm{d}x \leqslant \frac{1-(N+b)(1-\gamma)}{1-\gamma}\int_\Omega F^\gamma \omega \mathrm{d}x - \int_\Omega \nabla F^\gamma \cdot \nabla\varphi\omega \mathrm{d}x.$$

That is,

$$(N+b)\int_\Omega |g|^{p_b^*(1-((1)/(N+b)))}\omega \mathrm{d}x \leqslant -\frac{p(N+b-1)}{N+b-p}\int_\Omega |f|^{((p_b^*)/(q))}\nabla f \cdot \nabla\varphi\omega \mathrm{d}x.$$

By the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\int_\Omega |g|^{p_b^*(1-((1)/(N+b)))}\omega \mathrm{d}x \\
&\leqslant \frac{p(N+b-1)}{(N+b)(N+b-p)}\left[\int_\Omega qC(|f|^{((p_b^*)/(q))}\nabla\varphi)\omega \mathrm{d}x\right]^{1/q}\left[\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right]^{1/p} \\
&= \frac{p(N+b-1)}{(N+b)(N+b-p)}\left[\int_\Omega q|f|^{p_b^*}C(\nabla\varphi)\omega \mathrm{d}x\right]^{1/q}\left[\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right]^{1/p}.
\end{aligned}$$

Hence, by the definition of mass transportation:

$$\begin{aligned}
&\int_\Omega |g|^{p_b^*(1-((1/(N+b)))}\omega \mathrm{d}x \\
&\leqslant \frac{p(N+b-1)}{(N+b)(N+b-p)}\left[\int_\Omega qC(y)|g(y)|^{p_b^*}\omega(y)\mathrm{d}y\right]^{1/q}\left[\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right]^{1/p}.
\end{aligned}$$

When $f = g = V_{p,b}$, $\nabla\varphi(x) = x$. Also, in this case, by lemma 2.1, the equality in the Cauchy–Schwarz inequality happens. Hence the equality in (4.1) occurs. Hence, we also obtain the duality principle:

$$\sup_{\int_\Omega |g|^{p_b^*}\omega \mathrm{d}x=1} \frac{\int_\Omega |g|^{p_b^*(1-((1)/(N+b)))}\omega(x)\mathrm{d}x}{\left[\int_\Omega qC(y)|g(y)|^{p_b^*}\omega(y)\mathrm{d}y\right]^{1/q}}$$
$$= \inf_{\int_\Omega |f|^{p_b^*}\omega \mathrm{d}x=1} \frac{p(N+b-1)}{(N+b)(N+b-p)} \left[\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right]^{1/p}.$$

Next, if we proceed as in the proof of Theorem 3.1, we have the general weighted Sobolev inequality:

$$\frac{\left[\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right]^{1/p}}{\left[\int_\Omega |f|^{p_b^*}\omega \mathrm{d}x\right]^{((1)/(p_b^*))}} \geqslant \frac{\left[\int_\Omega pC^*(\nabla V_{p,b})\omega \mathrm{d}x\right]^{1/p}}{\left[\int_\Omega |V_{p,b}|^{p_b^*}\omega \mathrm{d}x\right]^{((1)/(p_b^*))}}.$$

□

Remark 4.1. Theorem 4.1 has been set up in [**13**] under the regular Euclidean norm and was extended to the arbitrary norm in [**42**].

Now, let $b \geqslant 0$, $p > 1$, $0 < \alpha \neq 1$, $\alpha(N+b-p) \leqslant N+b$, and $\omega(x) = x_N^b$ $\forall x \in \Omega$. Again, we introduce the function

$$U_{\alpha,p,b}(x) = (\sigma_{\alpha,p,b} + (\alpha-1)qC(x))_+^{1/1-\alpha},$$

where $\sigma_{\alpha,p,b}$ is chosen such that

$$\int_\Omega U_{\alpha,p,b}(x)^{\alpha p}\omega(x)\mathrm{d}x = 1.$$

Then we will next show that

Theorem 4.2. *Let $b \geqslant 0$, $p > 1$ and $0 < \alpha \neq 1$ be such that $\alpha(N+b-p) \leqslant N+b$. Let f and g be such that $\int_\Omega |f|^{\alpha p}\omega(x)\mathrm{d}x = \int_\Omega |g|^{\alpha p}\omega(x)\mathrm{d}x = 1$. Then, for all $\mu > 0$,*

$$\frac{\alpha p}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |g|^{\alpha(p-1)+1}\omega - \frac{\mu^q}{q}\int qC(y)|g(y)|^{\alpha p}\omega \mathrm{d}y$$
$$\leqslant \frac{1}{p\mu^p}\int pC^*(\nabla f)\omega + \frac{\alpha p-(N+b)(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |f|^{\alpha(p-1)+1}\omega.$$

When $\mu = q^{1/q}$, then the equality happens if $f = g = U_{\alpha,p,b}$. As a consequence

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1/ If $\alpha > 1$, then

$$\frac{\left(\int_\Omega pC^*(\nabla f)\omega(x)\mathrm{d}x\right)^{a/p}\left[\int_\Omega |f|^{\alpha(p-1)+1}\omega(x)\mathrm{d}x\right]^{((1-a)/(\alpha(p-1)+1))}}{\left[\int_\Omega |f|^{\alpha p}\omega(x)\mathrm{d}x\right]^{1/\alpha p}}$$
$$\geqslant \frac{\left(\int_\Omega pC^*(\nabla U_{\alpha,p,b})\omega(x)\mathrm{d}x\right)^{a/p}\left[\int_\Omega |U_{\alpha,p,b}|^{\alpha(p-1)+1}\omega(x)\mathrm{d}x\right]^{((1-a)/(\alpha(p-1)+1))}}{\left[\int_\Omega |U_{\alpha,p,b}|^{\alpha p}\omega(x)\mathrm{d}x\right]^{1/\alpha p}}$$

with

$$a = \frac{(N+b)(\alpha-1)}{\alpha\left[(N+b)p-(\alpha p+1-\alpha)(N+b-p)\right]}.$$

2/ For $\alpha < 1$:

$$\frac{\left(\int_\Omega pC^*(\nabla f)\omega(x)\mathrm{d}x\right)^{a/p}\left[\int_\Omega |f|^{\alpha p}\omega(x)\mathrm{d}x\right]^{1-a/\alpha p}}{\left[\int_\Omega |f|^{\alpha(p-1)+1}\omega(x)\mathrm{d}x\right]^{((1)/(\alpha(p-1)+1))}}$$
$$\geqslant \frac{\left(\int_\Omega pC^*(\nabla U_{\alpha,p,b})\omega(x)\mathrm{d}x\right)^{a/p}\left[\int_\Omega |U_{\alpha,p,b}|^{\alpha p}\omega(x)\mathrm{d}x\right]^{((1-a)/(\alpha p))}}{\left[\int_\Omega |U_{\alpha,p,b}|^{\alpha(p-1)+1}\omega(x)\mathrm{d}x\right]^{((1)/(\alpha(p-1)+1))}},$$

with

$$a = \frac{(N+b)(1-\alpha)}{(\alpha p+1-\alpha)[N+b-\alpha(N+b-p)]}.$$

Proof. We can assume that f, g are nonnegative, compactly supported functions on $\overline{\Omega}$, and f is $C^1(\overline{\Omega})$. Let $\nabla\varphi$ be the Brenier map pushing $|f|^{\alpha p}\omega(x)dx$ forward to $|g|^{\alpha p}\omega(x)dx$. Using lemma 2.4 with $F = |f|^{\alpha p}$, $G = |g|^{\alpha p}$ and $\gamma = ((\alpha(p-1)+1)/(\alpha p))$, we get

$$\begin{aligned}&\frac{1}{1-((\alpha(p-1)+1)/(\alpha p))}\int_\Omega |g|^{\alpha(p-1)+1}\omega\mathrm{d}x \\ &\leqslant \frac{1-(N+b)\left(1-((\alpha(p-1)+1)/(\alpha p))\right)}{1-((\alpha(p-1)+1)/(\alpha p))}\int_\Omega |f|^{\alpha(p-1)+1}\omega\mathrm{d}x \\ &\quad -\int_\Omega \nabla\left(|f|^{\alpha(p-1)+1}\right)\cdot\nabla\varphi\omega\mathrm{d}x.\end{aligned} \tag{4.2}$$

Then, we get $1 \neq \gamma \geqslant 1-1/N+b$. Indeed

$$\gamma = \frac{\alpha(p-1)+1}{\alpha p} \geqslant 1-\frac{1}{N+b}$$

is equivalent to

$$\alpha(N+b-p) \leqslant N+b.$$

By the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
&\frac{\alpha p}{\alpha-1}\int_\Omega |g|^{\alpha(p-1)+1}\omega \mathrm{d}x\\
&\leqslant \frac{\alpha p-(N+b)(\alpha-1)}{\alpha-1}\int_\Omega |f|^{\alpha(p-1)+1}\omega \mathrm{d}x-(\alpha(p-1)+1)\\
&\times\int_\Omega |f|^{\alpha(p-1)}\nabla f\cdot\nabla\varphi\omega \mathrm{d}x\\
&\leqslant \frac{\alpha p-(N+b)(\alpha-1)}{\alpha-1}\int_\Omega |f|^{\alpha(p-1)+1}\omega \mathrm{d}x \qquad (4.3)\\
&+(\alpha(p-1)+1)\left[\frac{1}{p\mu^p}\int_\Omega pC^*(\nabla f)\omega(x)\mathrm{d}x+\frac{\mu^q}{q}\int_\Omega qC(\nabla\varphi)|f|^{\alpha p}\omega \mathrm{d}x\right]\\
&=\frac{\alpha p-(N+b)(\alpha-1)}{\alpha-1}\int_\Omega |f|^{\alpha(p-1)+1}\omega \mathrm{d}x\\
&+(\alpha(p-1)+1)\left[\frac{1}{p\mu^p}\int_\Omega pC^*(\nabla f)\omega(x)\mathrm{d}x+\frac{\mu^q}{q}\int_\Omega qC(y)|g|^{\alpha p}\omega \mathrm{d}x\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\alpha p}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |g|^{\alpha(p-1)+1}\omega-\frac{\mu^q}{q}\int qC(y)|g(y)|^{\alpha p}\omega \mathrm{d}y\\
&\leqslant \frac{1}{p\mu^p}\int pC^*(\nabla f)\omega+\frac{\alpha p-(N+b)(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |f|^{\alpha(p-1)+1}\omega.
\end{aligned}
$$

When $f=g=U_{\alpha,p,b}$, then $\nabla\varphi(x)=x$. Then we can check that the equality of (4.2) happens. Also, when $f=g=U_{\alpha,p,b}$ and $\mu=q^{1/q}$, since $f=U_{\alpha,p,b}$ is C-radial, by lemma 2.1, we also have that the equality of (4.3) happens. Hence, in this case, we get the following dual principle:

$$
\begin{aligned}
&\sup_{\int_\Omega |g|^{\alpha p}\omega(x)\mathrm{d}x=1}\left[\frac{\alpha p}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |g|^{\alpha(p-1)+1}\omega-\int qC(y)|g(y)|^{\alpha p}\omega \mathrm{d}y\right]\\
&=\inf_{\int_\Omega |f|^{\alpha p}\omega(x)\mathrm{d}x=1}\left[\frac{1}{pq^{p/q}}\int pC^*(\nabla f)\omega+\frac{\alpha p-(N+b)(\alpha-1)}{(\alpha-1)[\alpha p-(\alpha-1)]}\int |f|^{\alpha(p-1)+1}\omega\right].
\end{aligned}
$$

By the same method as in the proof of Theorem 3.1, from this dual principle, we also obtain Statements 1/ and 2/. □

REMARK 4.2. Again using the ideas in [**19**, **20**] and the techniques in [**8**, **42**], in the limiting case $\alpha\to 1$, we can obtain the following general optimal weighted L^p-Euclidean logarithmic Sobolev inequality: Let $p>1$, then for any $f\in W^{1,p}(\Omega,\omega)$ such that $\int_\Omega |f|^p\omega \mathrm{d}x=1$, then

$$
\int_\Omega |f|^p\ln(|f|^p)\omega \mathrm{d}x\leqslant \frac{N+b}{p}\ln\left[\mathcal{L}_{N,b}\int_\Omega pC^*(\nabla f)\omega \mathrm{d}x\right],
$$

https://doi.org/10.1017/prm.2018.45 Published online by Cambridge University Press

where the sharp constant $\mathcal{L}_{N,b}$ can be computed from

$$\int_\Omega |\overline{f}|^p \ln(|\overline{f}|^p)\omega \mathrm{d}x = \frac{N+b}{p} \ln\left[\mathcal{L}_{N,b}\int_\Omega pC^*(\nabla \overline{f})\omega \mathrm{d}x\right],$$

with

$$\overline{f}(x) = \tau e^{-qC(x)}, |\tau|^{-p} = \int_\Omega \mathrm{e}^{-pqC(x)}\omega \mathrm{d}x.$$

Now, we are ready to prove Theorem 1.2:

Proof of Theorem 1.2. Define

$$GN(N,p,s,r,b) = \sup_{u\in D^{p,s}_{0,0}(\Omega,\omega)} \frac{\left(\int_\Omega |u|^r\omega \mathrm{d}x\right)^{1/r}}{\left(\int_\Omega pC^*(\nabla u)\omega \mathrm{d}x\right)^{a/p}\left(\int_\Omega |u|^s\omega \mathrm{d}x\right)^{1-a/s}}.$$

1/When $r = p((s-1)/(p-1))$ and $\theta = ((N\mu)/(N-p))$, we have from Theorem 3.1 that $GN(N,p,s,r,b)$ is achieved by maximizers of the form

$$U_0(x) = \gamma(1+\delta C(x))^{-((p-1)/(s-p))} \quad \text{for some } \gamma \in \mathbb{R}, \delta > 0.$$

Now, set $V_0 = D^{-1}_{N,d,p}U_0$ with $d = 1/q((\mu)/(N+b-p-\mu))$. Then we will show that V_0 is a maximizer of $CKN(N,\mu,p,s,r,b)$. Indeed, by lemma 2.3, we get

$$\int_\Omega \frac{C^*(\nabla u(y))}{C(y)^{\mu/q}}\omega(y)\mathrm{d}y \geqslant \int_\Omega C^*(\nabla D_{N,d,p}u(x))\omega(x)\mathrm{d}x$$

$$\left(\frac{1}{1+dq}\right)^{s((p-1)/(p))}\int_\Omega |u|^s\frac{\omega(x)\mathrm{d}x}{C(x)^{\theta/q}} = (1+dq)\int_\Omega |D_{N,d,p}u(x)|^s\omega(x)\mathrm{d}x$$

$$\left(\frac{1}{1+dq}\right)^{r((p-1)/(p))}\int_\Omega |u|^r\frac{\omega(x)\mathrm{d}x}{C(x)^{\theta/q}} = (1+dq)\int_\Omega |D_{N,d,p}u(x)|^r\omega(x)\mathrm{d}x$$

$$\int_\Omega \frac{C^*(\nabla V_0(y))}{C(y)^{\mu/q}}\omega(y)\mathrm{d}y = \int_\Omega C^*(\nabla U_0(x))\omega(x)\mathrm{d}x$$

$$\left(\frac{1}{1+dq}\right)^{s((p-1)/(p))}\int_\Omega |V_0|^s\frac{\omega(x)\mathrm{d}x}{C(x)^{\theta/q}} = (1+\mathrm{d}q)\int_\Omega |U_0(x)|^s\omega(x)\mathrm{d}x$$

$$\left(\frac{1}{1+dq}\right)^{r((p-1)/(p))}\int_\Omega |V_0|^r\frac{\omega(x)\mathrm{d}x}{C(x)^{\theta/q}} = (1+dq)\int_\Omega |U_0(x)|^r\omega(x)\mathrm{d}x.$$

Hence,

$$\frac{\left(\int_\Omega |u|^r((\omega(x)\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1/r}}{\left(\int_\Omega pC^*(\nabla u)((\omega(x)\mathrm{d}x)/(C(x)^{\mu/q}))\right)^{a/p}\left(\int_\Omega |u|^s((\omega(x)\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1-a/s}}$$
$$\leqslant \left(\frac{(1+dq)^{(1+r((p-1)/(p)))1/r}}{(1+dq)^{(1+s((p-1)/(p)))((1-a)/(s))}}\right)$$
$$\times \frac{\left(\int_\Omega |D_{N,d,p}u|^r\omega(x)\mathrm{d}x\right)^{1/r}}{\left(\int_\Omega pC^*(\nabla D_{N,d,p}u)\omega(x)\mathrm{d}x\right)^{a/p}\left(\int_\Omega |D_{N,d,p}u|^s\omega(x)\mathrm{d}x\right)^{1-a/s}}$$
$$\leqslant \left(\frac{(1+dq)^{(1+r((p-1)/(p)))1/r}}{(1+dq)^{(1+s((p-1)/(p)))((1-a)/(s))}}\right)$$
$$\times \frac{\left(\int_\Omega |U_0|^r\omega(x)\mathrm{d}x\right)^{1/r}}{\left(\int_\Omega pC^*(\nabla U_0)\omega(x)\mathrm{d}x\right)^{a/p}\left(\int_\Omega |U_0|^s\omega(x)\mathrm{d}x\right)^{1-a/s}}$$
$$= \frac{\left(\int_\Omega |V_0|^r((\omega(x)\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1/r}}{\left(\int_\Omega pC^*(\nabla V_0)((\omega(x)\mathrm{d}x)/(C(x)^{\mu/q}))\right)^{a/p}\left(\int_\Omega |V_0|^s((\omega(x)\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1-a/s}}.$$

Hence, we could conclude that $CKN(N,\mu,p,s,r,b)$ can be attained by optimizers of the form

$$V_0 = D_{N,d,p}^{-1}U_0 = \gamma'\left(1+\delta' C(x)^{1/qd+1}\right)^{-((p-1)/(s-p))} \quad \text{for some } \gamma' \in \mathbb{R}, \delta' > 0$$

Simiarly, when $\theta = (N+b)\mu/N+b-p$, if $s = p((r-1)/(p-1))$ and $r > 2-1/p$, $CKN(N,\mu,p,q,r,b)$ is achieved by maximizers of the form

$$V_0(x) = \gamma'(1-\delta' C(x)^{1/qd+1})_+^{-((p-1)/(r-p))} \quad \text{for some } \ \gamma' \in \mathbb{R}, \delta' > 0.$$

□

5. CKN inequalities with weight $\omega(x) = x_1^{A_1}\cdots x_N^{A_N}$

Let $\omega(x) = x_1^{A_1}\cdots x_N^{A_N}$ with $A_1 \geqslant 0, \ldots, A_N \geqslant 0$. Set

$$A = A_1 + \cdots + A_N$$
$$\mathbb{R}_*^N = \left\{(x_1,\ldots,x_N) \in \mathbb{R}^N : x_i > 0 \quad \text{whenever } A_i > 0\right\}.$$

Let

$$1 < p \leqslant p+\mu < N+A, \quad \theta = \frac{(N+A)\mu}{N+A-p} < N+A, \tag{5.1}$$
$$a = \frac{p(N+A)[r-s]}{r[p(N+A)-s(N+A-p)]},$$

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and set

$CKN(N,\mu,p,s,r,\omega) =$

$$\sup_{u\in D^{p,s}_{\mu,\theta}(\mathbb{R}^N_*,\omega)} \frac{\left(\int_{\mathbb{R}^N_*}|u|^r\omega((\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1/r}}{\left(\int_{\mathbb{R}^N_*}pC^*(\nabla u)\omega((\mathrm{d}x)/(C(x)^{\mu/q}))\right)^{a/p}\left(\int_{\mathbb{R}^N_*}|u|^s\omega((\mathrm{d}x)/(C(x)^{\theta/q}))\right)^{1-a/s}}.$$

Then we can set up the following result

THEOREM 5.1. *Assume* (5.1). *1/ If* $p<r=p((s-1)/(p-1))<(((N+A)p)/(N+A-p))$ *and, then* $CKN(N,\mu,p,s,r,\omega)$ *is achieved by maximizers of the form*

$$V_0(x)=\gamma'(1+\delta' C(x)^{((N+A-p-\mu)/(N+A-p))})^{-((p-1)/(s-p))} \quad \textit{for some } \gamma'\in\mathbb{R},\delta'>0.$$

2/ If $1<s=p((r-1)/(p-1))<p$ *and* $r>2-((1)/(p))$, $CKN(N,\mu,p,s,r,\omega)$ *is achieved by maximizers of the form*

$$V_0(x)=\gamma'(1-\delta' C(x)^{((N+A-p-\mu)/(N+A-p))})_+^{-((p-1)/(r-p))} \quad \textit{for some } \gamma'\in\mathbb{R},\delta'>0.$$

We will need the following lemmata:

LEMMA 5.1. *Let* $1\neq\gamma\geqslant 1-1/N+A$. *Let* F *and* G *be two nonnegative functions on* $\mathbb{R}^N_*$ *with* $\int_{\mathbb{R}^N_*}F\omega dx=\int_{\mathbb{R}^N_*}G\omega \mathrm{d}x=1$, *and such that* F^γ *is* C^1 *on* $\mathbb{R}^N_*$ *and* F, G *are compactly supported on* $\overline{\mathbb{R}^N_*}$. *Then if* $\nabla\varphi$ *is the Brenier map pushing* $F\omega dx$ *toward to* $G\omega dx$, *then we have*

$$\frac{1}{1-\gamma}\int_{\mathbb{R}^N_*}G^\gamma\omega \mathrm{d}x\leqslant\frac{1-(N+A)(1-\gamma)}{1-\gamma}\int_{\mathbb{R}^N_*}F^\gamma\omega \mathrm{d}x-\int_{\mathbb{R}^N_*}\nabla F^\gamma\cdot\nabla\varphi\omega \mathrm{d}x.$$

LEMMA 5.2. (1) *For continuous function* f, *we have*

$$\int_{\mathbb{R}^N_*}\frac{f((((1)/(1+dq)))^{((p-1)/(p))}u(x))}{C(x)^t}\omega \mathrm{d}x = (1+dq)\int_{\mathbb{R}^N_*}\frac{f(D_{N,d,p}u(x))}{C(x)^{t(dq+1)-(N+A)d}}\omega \mathrm{d}x.$$

In particular, we obtain that $u\in L^s(((\omega dx)/(C(x)^t)))$ *if and only if* $D_{N,d,p}u\in L^s(((\omega dx)/(C(x)^{t(dq+1)-(N+A)d})))$.

(2) *For smooth functions* u :

$$\int_{\mathbb{R}^N_*}\frac{C^*(\nabla D_{N,d,p}u(x))}{C(x)^{(qd+1)\mu+pd-(N+A)d}}\omega \mathrm{d}x\leqslant\int_{\mathbb{R}^N_*}\frac{C^*(\nabla u(y))}{C(y)^\mu}\omega \mathrm{d}y.$$

The equality occurs if and only if u *is* C*-radially symmetric.*

The proof of Lemma 5.1 could be found in [**42**] while using the same approach as in the proof of Lemma 2.3, we can derive lemma 5.2. Now the proof of Theorem 5.1 is similar to that of theorem 1.2 and will be omitted.

https://doi.org/10.1017/prm.2018.45 Published online by Cambridge University Press

Acknowledgements

The author would like to thank Professor Nassif Ghoussoub for many useful comments and suggestions. Research of this work was partially supported by the PIMS-Math Distinguished Post-doctoral Fellowship from the Pacific Institute for the Mathematical Sciences

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