Assume that, when $a \leqslant b \leqslant c$, the inequality

$$h_a + h_b + h_c \leqslant 9r + ar\left(\sqrt{\frac{a}{c}} - \sqrt{\frac{c}{a}}\right)^2$$

holds.

Let $a = b = 1$ and $c = x$, where $1 \leqslant x < 2$ (to satisfy the triangle inequality). Then, using $\frac{h_a}{r} = \frac{a + b + c}{a}$ and so on, the inequality is equivalent to

$$3 + \frac{2}{x} + 2(x + 1) \leqslant 9 + \alpha\left(x - 2 + \frac{1}{x}\right)$$

$$\Leftrightarrow (2 - \alpha)\left(x + \frac{1}{x}\right) \leqslant 2(2 - \alpha)$$

$$\Leftrightarrow (2 - \alpha)\frac{(x - 1)^2}{x} \leqslant 0.$$

Since this is true for all $1 \leqslant x < 2$, we have $\alpha \geqslant 2$, and so 2 is the best possible constant.

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106.29 An improvement on the Garfunkel-Bankoff inequality*

In a triangle ABC, the semi-perimeter, circumradius and inradius are denoted by s, R and r respectively. In [1] Garfunkel proposed the following inequality as an open problem

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \qquad (1)$$

This was first proved by Bankoff in [2], and is known as the Garfunkel-Bankoff inequality. It has received considerable attention from researchers in the field of geometrical inequalities and has motivated a number of papers providing various generalisations and analogue, such as [3] and the references in it.

In this Note, we give a sharpened version of (1), which appears as a corollary to a theorem. The proof of the theorem relies on

$$s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}, \qquad (2)$$

which is described in [4, 5.10] as 'the fundamental inequality of a triangle'.

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Theorem

$$s^2 \leqslant \frac{4R^3(4R + r)^2}{16R^3 - 8R^2r + Rr^2 - 2r^3}. \tag{3}$$

Equality holds if, and only if, the triangle is equilateral.

Proof

By (2) it is sufficient to prove

$$2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - r)} \leqslant \frac{4R^3(4R + r)^2}{16R^3 - 8R^2r + Rr^2 - 2r^3}. \tag{4}$$

Putting $t = \frac{r}{R}$, we have $0 < t \leqslant \frac{1}{2}$, and (4) is equivalent to

$$2 + 10t - t^2 + 2(1 - 2t)\sqrt{1 - 2t} \leqslant \frac{4(4 + t)^2}{16 - 8t + t^2 - 2t^3}.$$

This is true since

$$\left[4(4 + t)^2 - (2 + 10t - t^2)(16 - 8t + t^2 - 2t^3)\right]^2$$
$$-\left[2(1 - 2t)\sqrt{1 - 2t}(16 - 8t + t^2 - 2t^3)\right]^2$$
$$= 4t^{10} + 44t^9 + 177t^8 + 364t^7 + 504t^6 + 144t^5 - 768t^4 + 256t^3$$
$$= t^3(t^3 + 4t^2 + 8t + 16)(4 + t)^2(1 - 2t)^2 \geqslant 0,$$

which is obviously true for $0 < t \leqslant \frac{1}{2}$.

Corollary

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + \frac{1}{4R}\left(\frac{r}{R}\right)^2(R - 2r). \tag{5}$$

Proof: Using the well-known identities

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$$

and

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} = \frac{(4R + r)^2}{s^2} - 2$$

this is equivalent to (3).

By Euler's inequality $R \geqslant 2r$, (5) is stronger than (1), and equivalent if, and only if, the triangle is equilateral.

In [5], M. Lukarevski and D. S. Marinescu gave a refinement of Kooi's inequality

$$s^2 \leqslant \frac{R(4R + r)^2}{2(2R - r)} - \frac{r^2(R - 2r)}{4R}. \tag{6}$$

We point out that (5) is stronger than (6), since $R \geqslant 2r$ and

$$\frac{R(4R+r)^2}{2(2R-r)} - \frac{r^2(R-2r)}{4R} - \frac{4R^3(4R+r)^2}{16R^3 - 8R^2r + Rr^2 - 2r^3}$$
$$= \frac{r^3(R-2r)\left(48R^3 - 8R^2r + 5Rr^2 - 2r^3\right)}{4R(2R-r)\left(16R^3 - 8R^2r + Rr^2 - 2r^3\right)} \geqslant 0.$$

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106.30 Threshold functions and the birthday paradox

Our goal is to illustrate the idea of a threshold function in the context of the birthday paradox. We do this by exploring the asymptotics of binomial coefficients.

To begin, we consider the limit definition of the exponential function. It is well known that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any real constant x. But what happens when x grows with n? For what functions $x = x(n)$ is it the case that $\left(1 + \frac{x}{n}\right)^n$ behaves asymptotically like e^x? More generally, given a function x, what adjustment factor $A_n(x)$ is needed so that $\left(1 + \frac{x}{n}\right)^n \sim A_n(x)e^x$, where we write $f(n) \sim g(n)$ to denote that $\lim_{n\to\infty} f(n)/g(n) = 1$?

For fixed x and n, by taking logs and using the Taylor expansion for $\ln(1 + z)$, we have

$$\left(1 + \frac{x}{n}\right)^n = \exp\left[n \ln\left(1 + \frac{x}{n}\right)\right]$$
$$= \exp\left[n\left(\frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \dots\right)\right], \text{ if } 0 \leqslant x < n,$$