Taylor's Cubics associated with a Triangle in Non-Euclidean Geometry.

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§1. The famous theorem of the pedal line of a triangle in ordinary geometry can be stated as follows:—"Given a triangle ABC and a point P such that the feet of the perpendiculars X, Y, Z, dropped from P on the sides of the triangle, are collinear, then the locus of P is the circumcircle." In non-euclidean geometry this locus is not a circle or even a curve of the second degree, but a cubic; and in both cases the envelope of the line XYZ is a curve of the third class. The explanation of the inconsistency in ordinary geometry is that the complete locus consists of the circumcircle together with the straight line at infinity.*

§2. Mr F. G. Taylor has investigated for ordinary geometry a similar locus arising from the condition that the lines AX, BY, CZ should be concurrent. He has found that the locus of P is a cubic curve circumscribing the triangle and passing also through the orthocentre, the in-centres and other remarkable points; and further, that if p is the point of concurrence of AX, BY, CZ, the locus of p is also a cubic. These form two associated cubics connected with the triangle. It is proposed in this paper to investigate the two loci in non-euclidean geometry. The investigation will be found to throw further light on the euclidean case, showing incidently that certain points which function with different rôles become separated out into distinct points in non-euclidean geometry.

* See the author's paper, "The pedal line of the triangle in non-euclidean geometry." *Proc. Intern. Cong. Math.* 5 (Cambridge, 1912), ii., 93–101.

§ 3. In order to make the investigation quite general, taking the given triangle as triangle of reference, we shall suppose the equation of the absolute to be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

and we shall use the capital letters, A, B, C, F, G, H, with their usual meanings, so that

$$A = bc - f^2, \qquad F = gh - af,$$
$$B = ca - g^2, \qquad G = hf - bg,$$
$$C = ab - h^2, \qquad H = fg - ch.$$

Denote the vertices of the triangle by α, β, γ, and those of the absolute polar triangle by α^*, β^*, γ^*, so that the latter points are the poles of the sides of the triangle $\alpha\beta\gamma$ with respect to the absolute, and their coordinates are

$$\alpha^* \equiv (A,\ H,\ G)$$
$$\beta^* \equiv (H,\ B,\ F)$$
$$\gamma^* \equiv (G,\ F,\ C).$$

The triangles $\alpha\beta\gamma$ and $\alpha^*\beta^*\gamma^*$ are in perspective, the centre of perspective being their common *orthocentre*

$$O \equiv \left(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}\right) \equiv (GH,\ HF,\ FG)$$

and the axis of perspective the *orthaxis*

$$\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0.$$

§ 4. Let $P \equiv (x_1, y_1, z_1)$ be any point in the plane of the triangle. The equation of the perpendicular from P on $\beta\gamma$, *i.e.* the line $P\alpha^*$, is

$$\begin{vmatrix} x & y & z \\ A & H & G \\ x_1 & y_1 & z_1 \end{vmatrix} = 0,$$

and this cuts $\beta\gamma$ in

$$X \equiv (0,\ Hx_1 - Ay_1,\ Gx_1 - Az_1).$$

Similarly

$$Y \equiv (Hy_1 - Bx_1,\ 0,\ Fy_1 - Bz_1),$$
$$Z \equiv Gz_1 - Cx_1,\ Fz_1 - Cy_1,\ 0).$$

We shall find it convenient to use the following abbreviations:

$$Gz - Cx \equiv l \equiv (\beta\gamma^*), \quad Hx - Ay \equiv m \equiv (\gamma\alpha^*), \quad Fy - Bz \equiv n \equiv (\alpha\beta^*),$$
$$Hy - Bx \equiv l' \equiv (\gamma\beta^*), \quad Fz - Cy \equiv m' \equiv (\alpha\gamma^*), \quad Gx - Az \equiv n' \equiv (\beta\alpha^*),$$

thus expressing that l is the line $\beta\gamma^*$, etc.

If αX, βY, γZ are concurrent in (x, y, z), we have

$$\frac{y}{z} = \frac{Hx_1 - Ay_1}{Gx_1 - Az_1} = \frac{m}{n'}, \quad \frac{z}{x} = \frac{Fy_1 - Bz_1}{Hy_1 - Bx_1} = \frac{n}{l'}, \quad \frac{x}{y} = \frac{Gz_1 - Cx_1}{Fz_1 - Cy_1} = \frac{l}{m'} \quad \ldots(1)$$

Hence, eliminating x, y, z and dropping the suffixes, we find the equation of the locus of P.

$$l\,m\,n = l'\,m'\,n'$$

or

$$(Hx - Ay)(Fy - Bz)(Gz - Cx) = (Gx - Az)(Hy - Bx)(Fz - Cy) \ldots (2)$$

which represents a cubic curve passing through the nine points of intersection of the three lines $\beta\gamma^*$, $\gamma\alpha^*$, $\alpha\beta^*$ with the three lines $\gamma\beta^*$, $\alpha\gamma^*$, $\beta\alpha^*$, *i.e.* through the points

$$\alpha, \beta, \gamma, \alpha^*, \beta^*, \gamma^*, \begin{pmatrix}\beta\gamma^*\\ \gamma\beta^*\end{pmatrix} \equiv A', \begin{pmatrix}\gamma\alpha^*\\ \alpha\gamma^*\end{pmatrix} \equiv B', \begin{pmatrix}\alpha\beta^*\\ \beta\alpha^*\end{pmatrix} \equiv C'.$$

It is also easily seen by elementary geometry that the locus passes through the orthocentre O, the four circumcentres C, C_1, C_2, C_3, and the four in-centres I, I_1, I_2, I_3.

§ 5. The coordinates of A', B', C' are

$$A' \equiv (GH, BG, CH), \quad B' \equiv (AF, HF, CH), \quad C' \equiv (AF, BG, FG)$$

and $\alpha A'$, $\beta B'$, $\gamma C'$ are concurrent in

$$S \equiv (AF, BG, CH),$$

which also lies on the curve. S is the isogonal conjugate of O (see § 7).

It will be noticed that A', B', C' are the meeting points of lines drawn through pairs of vertices of the triangle perpendicular to the intervening side, and in ordinary geometry S would be the circumcentre. It can also be proved that each side of the triangle $A'B'C'$ is perpendicular to the corresponding altitude of the triangle $\alpha\beta\gamma$.

Also α^*A', β^*B', γ^*C', the perpendiculars from A', B', C' on the sides of the triangle, are concurrent in the point

$$O' \equiv (GH + AF,\ HF + BG,\ FG + CH),$$

which also lies on the curve; and O, S, O' are collinear.

In the euclidean case S is the mid-point of OO', but this is not true in non-euclidean geometry. We shall still, however, call pairs of points in which lines through S cut the cubic *opposite* points.

§ 6. The equation of the cubic may be written in the form

$$\begin{vmatrix} x^2 & A & (GH+AF)\,x \\ y^2 & B & (HF+BG)\,y \\ z^2 & C & (FG+CH)\,z \end{vmatrix} = 0$$

whence we see that if (x, y, z) lies on the curve so does $\left(\dfrac{A}{x}, \dfrac{B}{y}, \dfrac{C}{z}\right)$, and these pairs of points are collinear with O'. What is the relation between such pairs of points?

§ 7. *Isogonal and Isotomic Conjugates.*

If P is any point, and lines $\alpha X'$, $\beta Y'$, $\gamma Z'$ are drawn such that $\angle X'\alpha\gamma = \beta\alpha P$, $Y'\beta\alpha = \gamma\beta P$, $Z'\gamma\beta = \alpha\gamma P$, the three lines $\alpha X'$, $\beta Y'$, $\gamma Z'$ are concurrent in a point P', the *isogonal conjugate* of P.

To find the isogonal conjugate of the point $P \equiv (x_1, y_1, z_1)$: The equation of αP is $yz_1 = zy_1$. Let $Q \equiv (x_2, y_2, z_2)$ be the isogonal conjugate of P. The equation of αQ is $yz_2 - zy_2 = 0$. Equating the cosines of the angles $\beta\alpha P$ and $Q\alpha\gamma$ we have

$$\frac{Cy_1 - Fz_1}{\sqrt{C}\,\sqrt{Bz_1^2 + Cy_1^2 - 2\,Fy_1z_1}} = \frac{Bz_2 - Fy_2}{\sqrt{B}\,\sqrt{Bz_2^2 + Cy_2^2 - 2\,Fy_2z_2}},$$

which leads to *

$$Cy_1y_2 = Bz_1z_2,$$

whence

$$x_2 : y_2 : z_2 = \frac{A}{x_1} : \frac{B}{y_1} : \frac{C}{z_1}.$$

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* An alternative equation resulting from this equation is

$$Cy_1y_2 + Bz_1z_2 = 2\,Fz_1y_2.$$

This corresponds to the condition that $\angle\ \gamma\alpha Q = \beta\alpha P$.

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Hence the points (x, y, z) and $\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right)$ are isogonal conjugates.

Again, if αP, βP, γP cut the opposite sides of the triangle in X, Y, Z, and $X'\gamma = \beta X$, $Y'\alpha = \gamma Y$, $Z'\beta = \alpha Z$, the three lines $\alpha X'$, $\beta Y'$, $\gamma Z'$ are concurrent in a point Q, which is called the *isotomic conjugate* of P. As before, let the coordinates of P be (x_1, y_1, z_1) and of Q (x_2, y_2, z_2). Then the coordinates of X are $(0, y_1, z_1)$ and of X' $(0, y_2, z_2)$. Equating the cosines of βX and $X'\gamma$ we have

$$\frac{by_1 + fz_1}{\sqrt{b}\ \sqrt{by_1^2 + cz_1^2 + 2fy_1z_1}} = \frac{cz_2 + fy_2}{\sqrt{c}\ \sqrt{by_2^2 + cz_2^2 + 2fy_2z_2}},$$

which leads to

$$by_1y_2 = cz_1z_2,$$

whence

$$x_2 : y_2 : z_2 = \frac{1}{ax_1} : \frac{1}{by_1} : \frac{1}{cz_1}.$$

Hence the points (x, y, z) and $\left(\frac{1}{ax}, \frac{1}{by}, \frac{1}{cz}\right)$ are isotomic conjugates.

In a similar way, if XYZ is any line cutting the sides of the triangle in X, Y, Z, and X', Y', Z' are points on the sides of the triangle such that $\angle X'\alpha\beta = \gamma\alpha X$, $Y'\beta\gamma = \alpha\beta Y$, $Z'\gamma\alpha = \beta\gamma Z$, then X', Y', Z' are collinear, and the lines XYZ and $X'Y'Z'$ are *isogonal conjugate lines*. If (ξ, η, ζ) are the line-coordinates of XYZ, the line-coordinates of the isogonal conjugate are $\left(\frac{1}{A\xi}, \frac{1}{B\eta}, \frac{1}{C\zeta}\right)$. Also, if $X'\beta = \gamma X$, $Y'\gamma = \alpha Y$, $Z'\alpha = \beta Z$, then the line $X'Y'Z'$ is the *isotomic conjugate* of XYZ, and if (ξ, η, ζ) are the line-coordinates of XYZ, the line-coordinates of the isotomic conjugate are $\left(\frac{a}{\xi}, \frac{b}{\eta}, \frac{c}{\zeta}\right)$

§ 8. Hence we have the result: *Every line through the point O' cuts the P-cubic in pairs of points which are isogonal conjugates.*

There are four isogonal self-conjugate points, for putting $x : y : z = \frac{A}{x} : \frac{B}{y} : \frac{C}{z}$ we find $x : y : z = \pm\sqrt{A} : \pm\sqrt{B} : \pm\sqrt{C}$.

These four points are the in-centres; and since they lie on the locus we have the result that I, I_1, I_2, I_3 are the points of contact of tangents from O'. We deduce further, since four real tangents can be drawn from a point on the curve to the curve, that the cubic is not unicursal but bipartite, and O' lies on the "serpentine" branch. The four points I are said to form a *tetrad* on the P-cubic, and O' is their common *tangential.*

§ 9. *The p-cubic.*

Returning to equations (1) of § 4, by eliminating x_1, y_1, z_1 we get the equation

$$\begin{vmatrix} Gy - Hz & Az & -Ay \\ -Bz & Hz - Fx & Bx \\ Cy & -Cx & Fx - Gy \end{vmatrix} = 0, \quad \ldots\ldots\ldots\ldots (3)$$

which represents the locus of the point p. This is a cubic curve passing through α, β, γ, but not through α^*, β^*, γ^*.

The equation may also be written in the form

$$\begin{vmatrix} ax^2 & Fx & 1 \\ by^2 & Gy & 1 \\ cz^2 & Hz & 1 \end{vmatrix} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3')$$

from which we see that if (x, y, z) lies on the p-cubic so does the point $\left(\frac{1}{ax}, \frac{1}{by}, \frac{1}{cz}\right)$, *i.e.* the isotomic conjugate of (x, y, z), and these points are collinear with the point

$$O_1 \equiv \left(\frac{F}{a}, \frac{G}{b}, \frac{H}{c}\right),$$

which also lies on the p-cubic. O_1 is the isotomic conjugate of the orthocentre. It follows that the p-cubic also passes through O. We have then the theorem: *Every line through O_1 cuts the p-cubic in a pair of isotomic conjugates.*

§ 10. *Every line through O' cuts the two cubics in pairs of corresponding points.*

For if O', (x, y, z), and (x_1, y_1, z_1) are collinear

$$\begin{vmatrix} GH + AF & x & x_1 \\ HF + BG & y & y_1 \\ FG + CH & z & z_1 \end{vmatrix} = 0.$$

Now, (x_1, y_1, z_1) being any point on the P-cubic, if (x, y, z) is the corresponding point on the p-cubic, we have from equation (1)

$$x : y : z = ll' : l'm' : ln.$$

Substituting in the determinant and expanding we have

$$ll'(Gn - Hm') + l'm'(Hl - Fn') + ln(Fm - Gl') = F(lmn - l'm'n') = 0,$$

which proves the theorem.

§ 11. On the two cubics there are four self-corresponding points, viz., putting $x_1, y_1, z_1 = x, y, z$ in equations (1) we have

$$Gxy = Hzx,\ Hyz = Fxy,\ Fzx = Gyz,$$

which are satisfied only by (1, 0, 0), (0, 1, 0), (0, 0, 1) and $Fx = Gy = Hz$.

Hence * the four self-corresponding points are α, β, γ, and the orthocentre O.

§ 12. The points on the p-cubic which correspond to the circumcentres on the P-cubic are the four centroids, G, G_1, G_2, G_3. These are also the four isotomic self-conjugate points, and their coordinates are therefore $\left(\pm\frac{1}{\sqrt{a}}, \pm\frac{1}{\sqrt{b}}, \pm\frac{1}{\sqrt{c}}\right)$. Hence, by the theorem of § 9, G, G_1, G_2, G_3 are the points of contact of tangents from O_1 to the p-cubic. Hence the p-cubic is also bipartite and O_1 lies on the "serpentine" branch. The four points G form a tetrad on the p-cubic and O_1 is their common tangential.

§ 13. If L, M, N, L_1, M_1, N_1, etc., are the points of contact of the inscribed circles with the sides of the triangle, then AL, BM, CN are concurrent in a point i, AL_1, BM_1, CN_1 in i_1, and so on, and the points i, i_1, i_2, i_3 are the points on the p-cubic which correspond to the points I, I_1, I_2, I_3 on the P cubic. They are obviously the points of contact of tangents from O' to the p-cubic. From equations (1) we find the coordinates of the points i to be

$$\frac{1}{F - \sqrt{BC}},\ \frac{1}{G - \sqrt{CA}},\ \frac{1}{H - \sqrt{AB}},$$

* The point O' lies on both cubics, but is not a self-corresponding point. The p-correspondent of O' is O_1, and the P-correspondent of O' is T', the tangential of O' (see § 16).

while those of i_1, i_2, i_3 are obtained by changing the sign of $\sqrt{A}$, $\sqrt{B}$, $\sqrt{C}$ in succession. Further, the lines AL_1, BM_2, CN_3 are concurrent in a point n, which in ordinary geometry is called the Nagel point of the triangle, and whose coordinates are

$$\frac{1}{F+\sqrt{BC}},\quad \frac{1}{G+\sqrt{CA}},\quad \frac{1}{H+\sqrt{AB}}.$$

It may be verified by substitution in equation (3) that this point also lies on the p-cubic. Since

$$BC-F^2=\Delta a,\quad CA-G^2=\Delta b,\quad AB-H^2=\Delta c,$$

we see that the Nagel point is the isotomic conjugate of i. The Nagel point is just one of four, like the in-centres. We have three other concurrent triads, viz. (AL, BM_3, CN_2) concurrent in a point n_1 whose coordinates are given by

$$(F+\sqrt{BC})\,x=(G-\sqrt{CA})\,y=(H-\sqrt{AB})\,z,$$

and which is the isotomic conjugate of i_1, and (AL_3, BM, CN_1), (AL_2, BM_1, CN) concurrent respectively in n_2 and n_3 the isotomic conjugates of i_2 and i_3.

The points on the P-cubic which correspond to n, n_1, n_2, n_3 on the p-cubic can be found from equations (1): viz., for n

$$x=\sqrt{A}\,(-F\sqrt{A}+G\sqrt{B}+H\sqrt{C}+\sqrt{ABC}),$$

y and z being written down by cyclic permutation of the letters, and similar coordinates for n_1, n_2, n_3 obtained by changing successively the sign of $\sqrt{A}$, $\sqrt{B}$, and $\sqrt{C}$.

If we write these coordinates in the form

$$x=-2AF+\sqrt{A}\,(F\sqrt{A}+G\sqrt{B}+H\sqrt{C}+\sqrt{ABC}),$$

we see that these points are the opposites of I, I_1, I_2, I_3. This is a particular example of a general theorem, viz.: *If P, P' are a pair of opposites on the P-cubic, their correspondents on the p-cubic are a pair of isotomic conjugates.* (See § 19.)

§ 14. In ordinary geometry this theorem is fairly obvious. Let p and p' be isotomic conjugates on the p-cubic, so that $\beta X'=X\gamma$, etc., and P, P' the corresponding points on the P-cubic. Let S be the mid-point of PP'. Then, if perpendiculars SL, SM, SN are drawn upon the sides of the triangle, $X'L=LX$, therefore $\beta L=L\gamma$, etc., and L, M, N are the mid-points of the sides. Hence the segment PP' is always bisected at the circumcentre of the triangle, *i.e.* P and P' are opposite points.

Further, still in euclidean geometry, if P, P' are any pair of points, not necessarily on the P-cubic, such that, if X, X', etc., are the feet of the perpendiculars on the sides of the triangle, $\beta X' = X\gamma$, etc., then PP' is always bisected at the fixed point, the centre of the circumcircle.

In the non-euclidean case, it can be proved that the join of two points P, P', which are related in this way, does not pass through the fixed point S unless the two points lie on the P-cubic.*

§ 15. Consider any line through one of the circumcentres C. The feet of the perpendiculars D, E, F, from C on the sides of the triangle, are the mid-points of the sides. Then if P ... is a series of points on the line through C, we have the following homographic ranges:

$$\begin{aligned}(CP\ldots) \overline{\wedge}_{\alpha^*} (DX\ldots) &\overline{\wedge} (DX'\ldots)\\ \overline{\wedge}_{\beta^*} (EY\ldots) &\overline{\wedge} (EY'\ldots)\\ \overline{\wedge}_{\gamma^*} (FZ\ldots) &\overline{\wedge} (FZ'\ldots)\end{aligned}$$

Hence we have the homographic pencils

$$\alpha^* (DX'\ldots) \overline{\wedge} \beta^* (EY'\ldots) \overline{\wedge} \gamma^* (FZ'\ldots).$$

Hence the points of intersection of corresponding rays, P' ..., lie on a conic through α^*, β^*, γ^*. Since α^*, β^*, γ^* are not collinear, this conic does not in general, as in euclidean geometry, split up into two straight lines. Therefore P' ... are not in general collinear. If P ... lie on CD, then P' ... also lie on CD, and in

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* The coordinates of P', the opposite of $P \equiv (x, y, z)$, are found to be

$$x' = AF(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + \Delta fz(ax + hy + gz) - \Delta Ayz,$$

the values of y' and z' being written down by cyclic permutation of all the letters. If (p, q, r) is any point, the join of opposites will pass through this fixed point if

$$\Sigma p[\phi(x, y, z)(CHy - BGz) + \Delta yz(Gy - Hz) + \Delta x(Bz^2 - Cy^2)] = 0,$$

i.e. if (x, y, z) lies on a certain cubic. Similarly, the join of a pair of isogonal or isotomic conjugates will pass through a fixed point only if the points lie on a certain cubic.

this case the conic breaks up into the two straight lines: CD through α^* and $\beta^* \gamma^*$. PP' in this case, of course, always passes through C, but not through S, since S does not lie on CD. There is just one exceptional case in which PP' also passes throngh S, viz., when P, P' coincide with C. The circumcentres C, C_1, C_2, C_3 are self-opposite points, and the tangents at these points to the P-cubic all pass through S. The coordinates of the circumcentres may be found from equations (1) by putting for x, y, z the values of the coordinates of the centroids, since the centroids are the points on the p-cubic which correspond to the circumcentres on the P-cubic. The coordinates of C are thus found to be

$$x = A\sqrt{a} + H\sqrt{b} + G\sqrt{c},$$
$$y = H\sqrt{a} + B\sqrt{b} + F\sqrt{c},$$
$$z = G\sqrt{a} + F\sqrt{b} + C\sqrt{c},$$

and those of C_1, C_2, C_3 are found by changing the sign of $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ respectively.

§ 16. Since O_1 is the isotomic conjugate of O, the P-correspondent of O_1 is the opposite of O, *i.e.* O'. O' lies also on the p-cubic, and its P-correspondent, which we shall denote by T', is found from equations (1); viz. T' is the point

$$x = AF(-AF^2 + BG^2 + CH^2 - ABC) + 2ABC(GH + AF),$$

y and z being written down by cyclic permutation of the letters. $O'T'$ is the tangent to the p-cubic at O'.

The isogonal conjugate of O' is T, the tangential of O'.

$$T \equiv \left(\frac{A}{GH + AF}, \frac{B}{HF + BG}, \frac{C}{FG + CH}\right).$$

We may write the coordinates of T'

$$x = 2A(HF + BG)(FG + CH) - AF(2FGH - ABC + \Sigma AF^2),$$

which proves that S, T, T' are collinear; therefore T' is the opposite of T. In euclidean geometry T' is the tangential of O; but this is not the case in non-euclidean geometry, for if it were then S would be its own tangential, whereas it is the tangential of the centroids. The p-correspondent of T is the isotomic conjugate of O' (§ 13), and therefore its coordinates are $1/a(GH + AF)$, etc.

§ 17. *Quadric Transformations.*

If (x, y, z) and (x', y', z') are corresponding points on the P- and the p- cubics, we have

$$(1)\ \frac{y'}{z'}=\frac{Hx-Ay}{Gx-Az}=\frac{m}{n'} \qquad (2)\ \frac{z'}{x'}=\frac{Fy-Bz}{Hy-Bx}=\frac{n}{l'}$$

$$(3)\ \frac{x'}{y'}=\frac{Gz-Cx}{Fz-Cy}=\frac{l}{m'}.$$

If (x, y, z) is *any* point, not on the P-cubic, these three equations cannot be simultaneously satisfied, but suppose we take two of the relations (2) and (3), then we have

$$\frac{x'}{ll'}=\frac{y'}{l'm'}=\frac{z'}{nl} \quad \ldots\ldots\ldots\ldots (4)$$

These represent a quadric transformation from the P-plane to the p-plane, such that to any straight line $px'+qy'+rz'=0$ in the p-plane corresponds a conic $pll'+ql'm'+rnl=0$ in the P-plane. The conics of this system, depending on only two parameters, all pass through three fixed points, viz.

$$\begin{pmatrix} l=0 \\ l'=0 \end{pmatrix} \equiv \begin{pmatrix} \beta\gamma^* \\ \gamma\beta^* \end{pmatrix} \equiv A', \ \begin{pmatrix} l=0 \\ m'=0 \end{pmatrix} \equiv \gamma^*, \text{ and } \begin{pmatrix} l'=0 \\ n=0 \end{pmatrix} \equiv \beta^*.$$

The conics therefore form a net, and the transformation is a (1, 1) quadric point transformation.

Two other similar transformations are obtained by taking (3), (1) and (1), (2). The conics of these systems pass through α^*, γ^*, B' and α^*, β^*, C' respectively.

Similarly, we have a set of three quadric transformations by expressing x, y, z in terms of x', y', z'; viz., from (2) and (3) we get

$$\begin{aligned} -Bz'x+(Hz'-Fx_1)\,y+Bx'z &= 0, \\ Cy'x-Cx'y \qquad +(Fx'-Gy')\,z &= 0, \end{aligned}$$

whence

$$\begin{aligned} x:y:z=(BC-F^2)\,x^2 &- GHy'z'+HFz'x'+FGx'y' \\ &: B\,(Gy'z'+Fz'x'+Cx'y') \\ &: C\,(-Hy'z'+Bz'x'+Fx'y') \quad \ldots\ldots\ldots (5). \end{aligned}$$

The net of conics in this case passes through the three fixed points

$$(0,\ 1,\ 0)\equiv\beta, \quad (0,\ 0,\ 1)\equiv\gamma, \quad \text{and} \left(-\frac{1}{a},\ \frac{1}{h},\ \frac{1}{g}\right)\equiv A_1.$$

This gives us a set of three points of some importance lying on the p cubic, viz.

$$A_1 \equiv \left(-\frac{1}{a}, \frac{1}{h}, \frac{1}{g}\right), \qquad B_1 \equiv \left(\frac{1}{h}, -\frac{1}{b}, \frac{1}{f}\right),$$

$$C_1 \equiv \left(\frac{1}{g}, \frac{1}{f}, -\frac{1}{c}\right).$$

In ordinary geometry these points become confounded with the external centroids. αA_1, βB_1, γC_1 pass through the point (f, g, h).

§ 18. Let a line in the P-plane cut the P-cubic in P, Q, R. Then the line PQR it transformed by the transformation (5) into a conic through A_1, β^*, γ^* and cutting the p-cubic in three other points, p, q, r, the p-correspondents of P, Q, R.

Keeping P fixed, and therefore p fixed, vary Q and R. Then qr passes through a fixed point s. To find s take $Q = O'$, so that R is the isogonal conjugate of P. Then $q = O_1$, and r is the isotomic conjugate of s. Hence P and s are so related that the isogonal conjugate of P is the P-correspondent of the isotomic conjugate of s: s is called the *cross-correspondent* of P. Then we have the theorem: *If the join of two points Q, R on the P-cubic passes through a fixed point P on the P-cubic, the join of the p-correspondents of Q, R passes through a fixed point on the p-cubic, the c.c. of P.*

If $Q = R$ so that PQ is a tangent to the P-cubic, $q = r$ and sq is a tangent to the p-cubic at q. Hence if P_1, P_2, P_3, P_4 are the points of contact of tangents from P to the P-cubic, the corresponding points p_1, p_2, p_3, p_4 on the p-cubic are the points of contact of tangents from the c.c. of P.

To every line through a point P on the P-cubic, which cuts the P-cubic in Q, R, correspond two lines through any given point v on the p-cubic, joining v to the corresponding points q, r. But if v is the c.c. of P the points q, r are collinear with s, and there is only one line through s corresponding to the given line through P; and *vice versa*. Hence there is a (1, 1) correspondence between lines joining P to the points of the P-cubic and the lines joining s to the corresponding points of the p-cubic. Hence these pencils are homographic. In particular the cross-ratio of the pencil formed by the tangents from P to the P-cubic is equal to that of the pencil formed by the tangents from s to the p-cubic, *i.e.* the two cubics have the same cross-ratio.

§ 19. The c.c. of S is O_1, for the isogonal conjugate of S is O, the p-correspondent of O is O, and the isotomic conjugate of O is O_1.

Hence if the join of two points on the P-cubic passes through S, the join of their p-correspondents passes through O_1, *i.e. the p-correspondents of a pair of opposites on the P-cubic are a pair of isotomic conjugates.*

Since O' is the opposite of O, its p-correspondent is O_1, the isotomic conjugate of O.

The point O' is its own cross-correspondent, and is the only self-cross-corresponding point. Suppose the point P is its own cross-correspondent. Let P' be the isogonal conjugate of P, and p' the p-correspondent of P'; then p' is the isotomic conjugate of P. Hence $PO'O_1P'p'$ are collinear. Now O' is the P-correspondent of O_1, therefore $O'O_1$ is a tangent to the P-cubic at O' and cuts the p-cubic in one other point, the isotomic conjugate of O'. Hence P, which lies on both cubics, must coincide with O'.

§ 20 Some further properties of the loci will be merely mentioned, as the proofs which Mr Taylor has given for the euclidean case are projective, and can be applied to the non-euclidean case.

Every conic through α, β, γ, O *cuts the p-cubic in a pair of isotomic conjugates, the P-cubic in two points which are isogonal conjugates of a pair of opposites, and the two cubics in two pairs of cross-correspondents.*

The following are two general theorems for cubics:—

(1) *U, V are two fixed points on a cubic, and a line through U cuts the cubic in P and Q. If PV, QV cut the cubic in P', Q', then $P'Q'$ meets the cubic in a fixed point.*

We have the following collinear triads, in rows and columns:

$$\begin{matrix} P & Q & U \\ V & V & F \\ P' & Q' & \end{matrix}$$

where F is the tangential of V, and therefore a fixed point. Hence $P'Q'$ meets the cubic in the fixed point where it is met by the line UF.

(2) *U is a fixed point on a cubic, and P, Q, R, S are a tetrad of points on the cubic with common tangential F. If PU, QU, RU, SU meet the cubic again in P', Q', R', S', these four points also have a common tangential.*

Let T be the tangential of U, and F' the tangential of P'. Since P, U, P' are collinear their tangentials F, T, F' are collinear; but F, T are fixed, therefore F' is fixed and the same for the four points P, Q, R, S.

From (1) and the theorem of § 18 we deduce that: *If the join of two points on the P-cubic meets the curve in a fixed point, the join of their cross-correspondents meets the p-cubic in a fixed point.*

Further, from (1), *the joins of isogonal conjugates of opposites on the P-cubic pass through the fixed point T'.* For $U = S$ and $V = O'$. F, the tangential of O', is T (§ 16), and ST cuts the cubic in T', the P-correspondent of O' on the p-cubic.

Also, *the joins of opposites of isogonal conjugates on the P-cubic pass through a fixed point.* $U = O'$ and $V = S$. If S' is the tangential of S, the fixed point is the third point of the triad $O'S'$. In euclidean geometry $S' = S$ and the fixed point is O.

From (2) it follows that *if four points $PQRS$ on the P-cubic form a tetrad, so also do their p-correspondents, and if $PQRS$ form a tetrad on the p-cubic, so also do their isotomic conjugates.*

From the theorem in § 18: if four points on the P-cubic form a tetrad so also do their p-correspondents, we deduce further that their cross-correspondents also form a tetrad.

§ 21. In conclusion, let us glance at the reciprocal problems connected with the one we have been considering.

In the reciprocal problem we start with a line L. Let L cut the sides of the triangle $\alpha\beta\gamma$ in X, Y, Z. Let α^*X cut $\beta^*\gamma^*$ in X', β^*Y cut $\gamma^*\alpha^*$ in Y' γ^*Z cut $\alpha^*\beta^*$ in Z'. Then we have to consider the envelope of the line L when X, Y, Z are collinear. In this way we get two associated curves of the third class, the envelope of L and the envelope of $X'Y'Z'$, whose properties are the exact reciprocals of the properties of the two associated cubics. They are the absolute polar reciprocals of the two cubics. In euclidean geometry the points X, Y, Z are always collinear since they lie on the line at infinity.

We may also interchange the two triangles $\alpha\beta\gamma$ and $\alpha^*\beta^*\gamma^*$. Then we get two associated cubics again. Now, since the circumcentres and the incentres of the triangle $\alpha\beta\gamma$ are respectively the incentres and the circumcentres of the triangle $\alpha^*\beta^*\gamma^*$, and the two triangles have the same orthocentre, the P-cubic for the triangle $\alpha^*\beta^*\gamma^*$ passes through α^*, β^*, γ^*, α, β, γ, O, I, I_1, I_2, I_3, C, C_1, C_2, C_3, and therefore coincides with the P-cubic for the triangle $\alpha\beta\gamma$. The two points S and O exchange rôles. The p-cubic for the triangle $\alpha^*\beta^*\gamma^*$, however, differs from that for the triangle $\alpha\beta\gamma$, since it passes through α^*, β^*, γ^*, but not through α, β, γ.