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DISCRETE CHOICE AND COMPLEX DYNAMICS IN DETERMINISTIC OPTIMIZATION PROBLEMS

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This paper shows that complex dynamics arises naturally in deterministic discrete choice problems. In particular, it shows that if the objective function of a maximization problem can be written as a function of a sequence of discrete variables, and if the (maximized) value function is strictly increasing in an exogenous variable, then for almost all values of the exogenous variable, any optimal path exhibits aperiodic dynamics. This result is applied to a maximization problem with indivisible durable goods, as well as to a Ramsey model with an indivisible consumption good. In each model, it is shown that optimal dynamics is almost always complex. These results are illustrated with various numerical examples.

Keywords: Discrete Choice, Complex Dynamics, Chaos, Indivisible Goods

1. INTRODUCTION

Discrete choice problems abound in economic decision-making. Most manufactured products, real estate, and works of art are indivisible. Choices regarding education, occupation, marriage, etc. are discrete in nature. It is not surprising that over the past decades, dynamic discrete choice models have gained considerable popularity in empirical studies [e.g, Keane and Wolpin (2009); Aguirregabiria and Mira (2010)].

In sharp contrast to the popularity of these models, there have been very few developments regarding deterministic, dynamic discrete choice problems since Kamihigashi (2000a, 2000b).[1] The purpose of this paper is to reinforce the point made in our earlier work that complex dynamics arises rather naturally in deterministic discrete choice models.[2] In particular, we show that if the objective function of a maximization problem can be expressed as a function of a sequence of endogenous discrete variables, and if the value function (or maximized value of the objective function) is strictly increasing in an exogenous variable, then for almost all values of the exogenous variable, any optimal path exhibits aperiodic dynamics.

This result generalizes the similar result shown in Kamihigashi (2000b) for a life-cycle model with an indivisible consumption good. Another similar result was

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shown in Kamihigashi (2000a) for an optimization problem with indivisible labor. Whereas these previous results were shown for specific models, the result shown in this paper is extremely general and can be applied to a wide range of economic problems. For example, we apply it to an optimization problem with indivisible durable goods as well as to a Ramsey model with an indivisible consumption good, and show that in these models, optimal paths almost always exhibit complex dynamics. These results are illustrated with various numerical examples.

This paper is related to two branches of the literature on the optimal dynamics of deterministic models. First, because discrete choice problems can be regarded as models with nonconvexities, this paper adds to the literature on the discrete-time dynamics of optimal growth with nonconvexities, which was initiated by the seminal paper of Dechert and Nishimura (1983).[3] Second, this paper is also related to the literature on the possibility of chaos in optimal growth models with low discounting [e.g., Nishimura and Yano (1995); Nishimura et al. (1994)], because our result shows that the optimal dynamics of a discrete choice model can be complex for any discount factor.[4]

The rest of the paper is organized as follows. Section 2 establishes our general result. Section 3 illustrates this result with a model with an indivisible consumption good as well as one with indivisible durable goods. Section 4 studies a Ramsey model with an indivisible consumption good.

2. THE GENERAL RESULT

Let $n \in \mathbf{N}$. Let $\lambda(A)$ denote the Lebesgue measure of the (Lebesgue) measurable set $A \subset \mathbf{R}$, and $\lambda^n(B)$ the Lebesgue measure of the measurable set $B \subset \mathbf{R}^n$. For $j = 1, \ldots, n$, let I_j be an interval in $\mathbf{R}$ with nonempty interior. Each I_j need not be closed and need not be bounded. Let $X = \prod_{j=1}^{n} I_j$. Let K be a nowhere dense (thus countable) subset of $\mathbf{R}^m$ with $m \in \mathbf{N}$. Define K^∞ to be the set of sequences in K: $K^\infty = \{\{c_t\}_{t=0}^\infty : \forall t \in \mathbf{Z}_+, c_t \in K\}$. Let C be a correspondence from X to K^∞; i.e., $\forall x \in X, C(x) \subset K^\infty$. Define $D = \cup_{x \in X} C(x)$. Let $w : D \to \mathbf{R}$. We assume the following.

Assumption 1. For each $x \in X$, $\max_{c \in C(x)} w(c)$ exists in $\mathbf{R}$.

For $x \in X$, define

$$v(x) = \max_{c \in C(x)} w(c), \tag{1}$$

$$C^*(x) = \operatorname*{argmax}_{c \in C(x)} w(c). \tag{2}$$

We use the following definitions: a function $g : \mathbf{R}^n \to \mathbf{R}$ is *strictly increasing* if $g(x) < g(y)$ whenever $x < y$;[5] a sequence $\{z_t\}_{t=0}^\infty$ (in K or $\mathbf{R}^n$) is *periodic* if there exists $i \in \mathbf{N}$ such that $\forall t \in \mathbf{Z}_+, z_{t+i} = z_t$; a sequence $\{y_t\}$ is *eventually periodic* if there exists a periodic sequence $\{z_t\}$ such that $\exists T \in \mathbf{Z}_+, \forall t \geq T, y_t = z_t$; a sequence $\{y_t\}$ is *asymptotically periodic* if there exists a periodic sequence $\{z_t\}$ such that $\|y_t - z_t\| \to 0$ as $t \uparrow \infty$, where $\|\cdot\|$ is any equivalent norm; and

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a sequence $\{y_t\}$ is *asymptotically aperiodic* if it is not asymptotically periodic. Because K is nowhere dense, a sequence in K is asymptotically aperiodic if and only if it is not eventually periodic.

We are ready to state the main result of this paper:

THEOREM 1. *Suppose that* $v : X \to \mathbf{R}$ *is strictly increasing.*[6] *Then for almost all* $x \in X$ *(with respect to Lebesgue measure), any sequence* $\{c_t\} \in C^*(x)$ *is asymptotically aperiodic. In particular, there exists a measurable set* $Z \subset X$ *such that* (a) $\lambda^n(Z) = 0$ *and* (b) *for each* $x \in X \setminus Z$*, any sequence* $\{c_t\} \in C^*(x)$ *is asymptotically aperiodic.*

Proof. See Appendix A. ■

This result considerably generalizes Kamihigashi (2000b, Theorem 2). The proof of Theorem 1 shows that the set of eventually periodic sequences in K is countable. The assumption that v is strictly increasing ensures that for each $\{c_t\} \in K^\infty$, the set of $x \in X$ with $v(x) = w(\{c_t\})$ has measure zero. Because the set of eventually periodic sequences in K is countable, it follows that the set of $x \in X$ with $v(x) = w(\{c_t\})$ for some eventually periodic sequence $\{c_t\} \in K^\infty$ has measure zero.

The only role of the value function in the proof is to ensure that for each $\{c_t\} \in K^\infty$, the set of $x \in X$ with $v(x) = w(\{c_t\})$ has measure zero. Thus the conclusions of Theorem 1 hold for an arbitrary correspondence C^* from X to K^∞ such that for each $\{c_t\} \in K^\infty$, the set of $x \in X$ with $\{c_t\} \in C^*(x)$ has measure zero. If C^* is given by (2), this can be ensured by assuming that each level set of v has measure zero:

COROLLARY 1. *Suppose that* $v : X \to \mathbf{R}$ *is measurable. Suppose further that*

$$\forall a \in \mathbf{R}, \quad \lambda^n(\{x \in X : v(x) = a\}) = 0. \tag{3}$$

Then the conclusions of Theorem 1 hold.

Proof. See Appendix A. ■

3. EXAMPLES

3.1. Rational vs. Irrational Numbers

To better understand Theorem 1, consider the following rather trivial maximization problem:

$$\max_{\{c_t\}_{t=0}^\infty} \quad \sum_{t=0}^\infty 10^{-t} c_t, \tag{4}$$

$$\text{s.t.} \quad \sum_{t=0}^\infty 10^{-t} c_t \le x, \tag{5}$$

$$\forall t \in \mathbf{Z}_+, \quad c_t \in \{0, 1, 2, \ldots, 9\}. \tag{6}$$

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We assume that $x \in X \equiv [0, 10]$. It is immediate from (6) that the value function $v(x)$ for this problem satisfies $v(x) \leq x$. Because x has a decimal representation $x = c_0.c_1c_2c_3\ldots$ satisfying (6) with equality, we also have $v(x) \geq x$. Hence $v(x) = x$, and $C^*(x)$ can be identified with the decimal representations of x. Clearly v is strictly increasing, so that Theorem 1 applies.

Recall that a real number has an eventually periodic decimal representation if and only if it is rational. Let Z be the set of rational numbers in X. Because Z is countable, we have $\lambda(Z) = 0$. If $x \in X \setminus Z$, then x is irrational, and the decimal representation of x is asymptotically aperiodic. It follows that Z has properties (a) and (b) in Theorem 1.

3.2. Indivisible Consumption Goods

Consider the maximization problem

$$\max_{\{c_t, s_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{7}$$

$$\text{s.t. } \forall t \in \mathbf{Z}_+, \quad c_t + s_t = x_t, \tag{8}$$

$$x_{t+1} = R s_t + y, \tag{9}$$

$$x_{t+1} \geq 0, \tag{10}$$

$$c_t \in \{0, \delta, 2\delta, 3\delta, \ldots\}, \tag{11}$$

$$x_0 = x \geq 0 \text{ given}, \tag{12}$$

where $u : \mathbf{R}_+ \to \mathbf{R}$ is the utility function, which is assumed to be strictly increasing; $\beta \in (0, 1)$ is the discount factor; c_t is the consumption of the indivisible good in period t; x_t is wealth at the beginning of period t; s_t is saving in period t; and $R > 1$ and $y \geq 0$ are the gross interest rate and income in each period, respectively. Constraint (12) means that the consumption good can be purchased only in multiples of δ. As $\delta \downarrow 0$, this problem approaches the standard problem with $c_t \geq 0$ instead of (11).[7]

In this model the consumption good is indivisible, whereas wealth is perfectly divisible. It is necessary for wealth to be a continuous variable as long as the gross interest rate R is an arbitrary real number strictly greater than one. In most cases it seems reasonable to assume that wealth is more divisible than goods, and our setup represents the extreme case in which wealth is perfectly divisible.

We say that a three dimensional sequence $\{c_t, s_t, x_{t+1}\}$ is a *feasible path from* x if it satisfies (8)–(12), and an *optimal path from* x if it solves the maximization problem (7)–(12); a sequence $\{c_t\}$ is a *feasible consumption path from* x if there exist sequences $\{s_t\}$ and $\{x_{t+1}\}$ such that $\{c_t, s_t, x_{t+1}\}$ is feasible from x. Optimal consumption paths, feasible wealth paths, and optimal wealth paths are defined

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similarly. We say that $x \geq 0$ is a *steady state* if the wealth path $\{x_{t+1}\}$ with $x_t = x$ for all $t \in \mathbf{N}$ is optimal from x.

Under regularity conditions, an optimal path from x exists and the value function $v(x)$ is finite for all $x \geq 0$. We assume these properties in what follows.

To see that v is strictly increasing, let $0 \leq x < x'$, and let $\{c_t, s_t, x_{t+1}\}$ be an optimal path from x. Note that $\{c_t\}$ is feasible also from x'. Thus $v(x) \leq v(x')$. Let $\{c'_t, s'_t, x'_{t+1}\}$ be the feasible path from x' with $\{c'_t\} = \{c_t\}$. It follows from (8) and (9) that

$$\forall t \in \mathbf{Z}_+, \quad x'_t - x_t = R^t(x - x'). \tag{13}$$

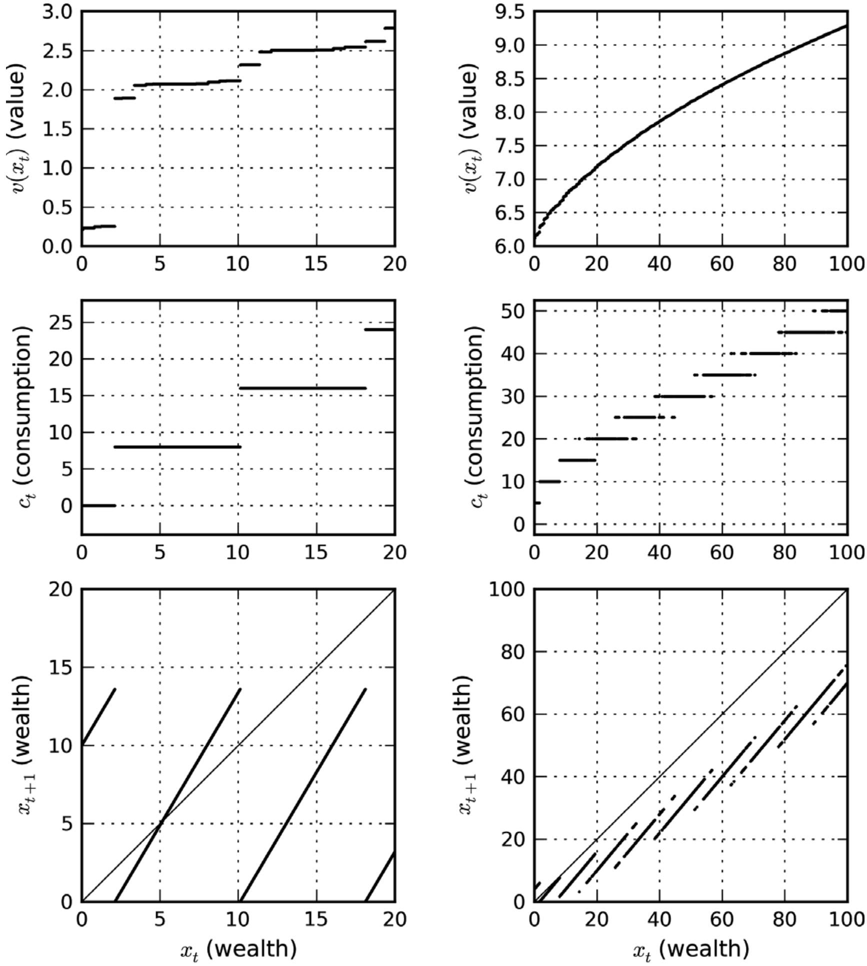
Because $R > 1$, we have $x'_t - x_t > \delta$ for sufficiently large t, which implies that it is feasible to increase c'_s for some large s without decreasing any c'_t with $t \neq s$. It follows that v is strictly increasing. Now by Theorem 1, for almost all $x \geq 0$, any optimal consumption path from x is asymptotically aperiodic. This implies that the corresponding wealth path is also asymptotically aperiodic.[8]

The preceding result is shown in Kamihigashi (2000b), where it is also shown that for β sufficiently small, the (optimal) policy function for wealth takes the form of a random number generator. The bottom plot in Figure 1a illustrates a policy function for wealth that takes the form of a linear congruential generator.[9] Recall, however, that Theorem 1 does not require β to be small. The bottom plot in Figure 1b shows a policy function for wealth with $\beta = 0.7$. In this case, although there is an overall tendency for wealth to decline toward zero, it keeps fluctuating near zero, which is not a steady state. Figure 2 illustrates optimal wealth paths in the two cases in Figure 1.

The assumption that $R > 1$ plays two important roles here. First, it is crucial to showing that v is strictly increasing; recall (13). Second, it implies that any steady state is locally unstable. This is particularly clear in Figure 1a. Indeed, whenever the policy function for wealth crosses the 45° line, it does so from below. This is because we have $x_{t+1} = R(x_t + y - c_t)$ by (8) and (9), and the policy function for wealth is continuous only where the policy function for consumption is constant, which implies that the slope of the former is $R > 1$ wherever it is continuous.

If $R < 1$, then v is *never* strictly increasing. This is illustrated in Figure 3a, which shows a piecewise constant value function. If $R < 1$, wealth shrinks to zero even if nothing is consumed. Thus consumption, being discrete, can take place only finitely many times. Hence any compact interval can be divided into finitely many subintervals according to optimal consumption paths. It is also interesting to observe that consumption is clearly not a monotone function of wealth in Figure 3a.

As discussed earlier, Theorem 1 applies as long as $R > 1$. This, however, does not mean that optimal dynamics is almost always complex. For example, if $y = 0$, the maximization problem here is an AK model, and endogenous growth is possible depending on the parameter values. Such a case is illustrated in Figure 3b,[10] where both consumption and wealth grow unboundedly and



(a) $\beta = 0.1$, $R = 1.7$, $y = 10$, $\delta = 8$
$u(c) = c^{0.3}$, 100,000 states

(b) $\beta = 0.7$, $R = 1.2$, $y = 10$, $\delta = 5$
$u(c) = c^{0.3}$, 100,000 states

FIGURE 1. Value functions and policy functions for consumption and wealth.

monotonically provided that initial wealth is strictly greater than the nonzero steady state, which is around 15.[11]

It is easy to see that the analysis can be extended to models with many consumption goods. For example, suppose that c_t consists of m goods, i.e., $c_t = (c_t^1, c_t^2, \ldots, c_t^m)$, and that the price of good i is given by p^i. We can then replace (8) and (11) with

$$\sum_{i=1}^{m} p^i c_t^i + s_t = x_t, \qquad \textbf{(14)}$$

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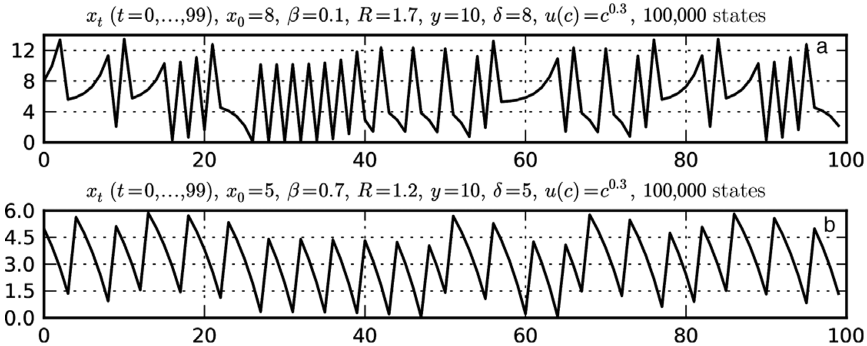

FIGURE 2. Optimal wealth paths.

$$\forall i \in \{1, \ldots, m\}, \quad c_t^i \in \{0, \delta^i, 2\delta^i, 3\delta^i, \ldots\}, \tag{15}$$

where $\delta^i > 0$ for each i. Constraint (15) means that good i can be purchased only in multiples of δ^i. Even in this setting, Theorem 1 applies in exactly the same way.

3.3. Indivisible Durable Goods

The analysis of the previous section can also be extended to models with indivisible durable goods. To be specific, consider the maximization problem

$$\max_{\{c_t, k_t, s_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(k_t), \tag{16}$$

$$\text{s.t. } \forall t \in \mathbf{Z}_+, \quad k_t = (1-\gamma)k_{t-1} + c_t, \tag{17}$$

$$c_t + s_t = x_t, \tag{18}$$

$$x_{t+1} = Rs_t + y, \tag{19}$$

$$c_t \in \{0, \delta, 2\delta, 3\delta, \ldots\}, \tag{20}$$

$$x_0 = x \geq 0, k_{-1} = k \geq 0 \text{ given}, \tag{21}$$

where k_t is the stock of durable goods at the end of period t, and $\gamma \in (0, 1]$ is the depreciation rate of durable goods. We assume that the utility function $u : \mathbf{R}_+ \to \mathbf{R}$ is strictly increasing. The definitions and assumptions for the other variables and parameters are as in the previous section.

Note from (17) that k_t can be written as a function of $\{c_t\}$ and $k = k_{-1}$:

$$k_t = \sum_{i=0}^{t} (1-\gamma)^i c_{t-i} + (1-\gamma)^{t+1} k. \tag{22}$$

Hence the objective function can be expressed as a function of $\{c_t\}$ and k. Therefore, with k fixed, the maximization problem here takes the form of the right-hand side of (1).

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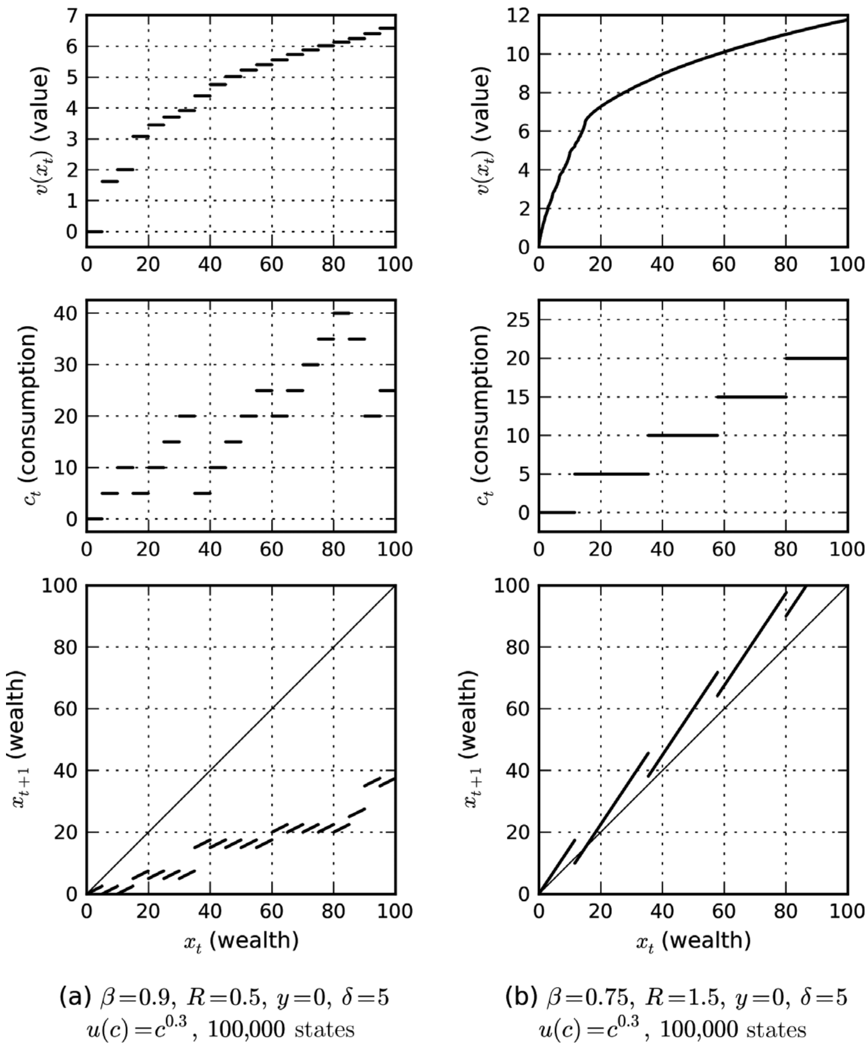

FIGURE 3. Value functions and policy functions for consumption and wealth.

Let $v(x, k)$ be the corresponding value function. Then the argument of the previous section shows that $v(x, k)$ is strictly increasing in x. Thus by Theorem 1, with k fixed, for almost all $x \geq 0$, any optimal consumption path from x is asymptotically aperiodic.

Although $v(x, k)$ is also strictly increasing in k, Theorem 1 cannot be used to show the same result in terms of k. This is because the theorem requires the objective function to be expressed entirely as a function of $\{c_t\}$. In the current setting the objective function always depends on k, so that the problem does not

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reduce to the right-hand side of (1) (with x fixed). To see why this is important, suppose that $u(k_t) = k_t$. In this case, if $\{c_t\}$ is optimal from k, then it is also optimal from any $k' \geq 0$ because the objective function is additively separable in $\{c_t\}$ and k; recall (22). If $\{c_t\}$ happens to be eventually periodic, then this means that there is an eventually periodic optimal consumption path from any $k' \geq 0$.

4. A RAMSEY MODEL WITH LINEAR UTILITY

As we mentioned in Section 3.2, an asymptotically aperiodic sequence can grow unboundedly and monotonically. In standard neoclassical (or Ramsey) models, however, unbounded growth is ruled out by technology constraints. For example, consider the maximization problem

$$\max_{\{c_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t c_t, \tag{23}$$

$$\text{s.t. } \forall t \in \mathbf{Z}_+, \quad c_t + x_{t+1} = f(x_t), \tag{24}$$

$$c_t \in H^\delta \equiv \begin{cases} \{0, \delta, 2\delta, 3\delta, \ldots\} & \text{if } \delta > 0, \\ \mathbf{R}_+ & \text{if } \delta = 0, \end{cases} \tag{25}$$

$$x_{t+1} \geq 0, \tag{26}$$

$$x_0 = x \geq 0 \text{ given}, \tag{27}$$

where x_t is the capital stock at the beginning of period t, and $f : \mathbf{R}_+ \to \mathbf{R}_+$ is the production function, which is assumed to be differentiable on $\mathbf{R}_{++}$, continuous, strictly increasing, and strictly concave, and to satisfy

$$f(0) = 0, \quad \lim_{x \downarrow 0} \beta f'(x) > 1, \quad \lim_{x \uparrow \infty} f'(x) < 1. \tag{28}$$

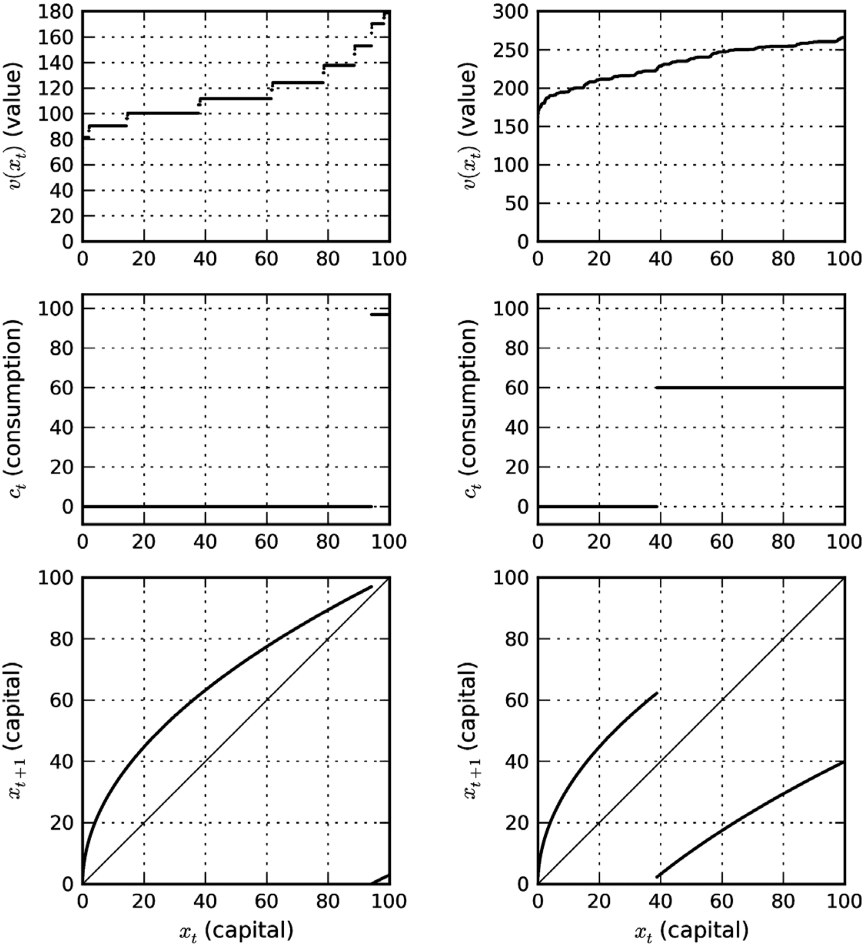
The last inequality rules out unbounded growth from any initial capital stock. Let $\overline{x} > 0$ be the maximum sustainable capital stock, which is given by $f(\overline{x}) = \overline{x}$. Note that any feasible capital path from $x \in X \equiv [0, \overline{x}]$ stays in X forever. From here on we restrict ourselves to capital paths in X (though this is not necessary).

Let x^* be the unique steady state of the model with $\delta = 0$: i.e., $\beta f'(x^*) = 1$. For $\delta \geq 0$ and $x \in X$, define

$$F^\delta(x) = \{y \geq 0 : f(x) - y \in H^\delta\}. \tag{29}$$

Let $v^\delta : \mathbf{R}_+ \to \mathbf{R}_+$ be the value function of the maximization problem (23)–(27).

Because f is nonlinear, it is not as easy as in Section 3.2 to show that v^δ is strictly increasing. In fact, it is not strictly increasing when δ is sufficiently

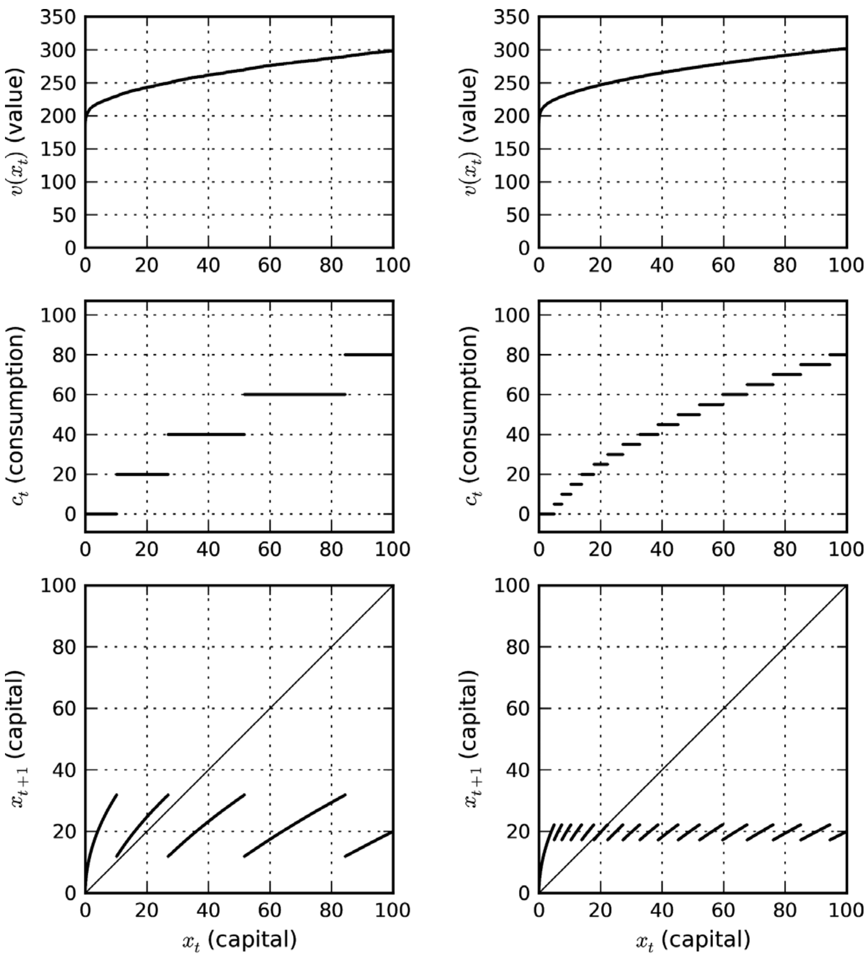


(a) $\beta = 0.9,\ \delta = 97,\ f(x) = 10x^{0.5}$
$u(c) = c$, 100,000 states

(b) $\beta = 0.9,\ \delta = 60,\ f(x) = 10x^{0.5}$
$u(c) = c$, 100,000 states

FIGURE 4. Value functions and policy functions for consumption and capital.

large. For example, if $\delta = \overline{x}$, we have $v^\delta(x) = 0$ for all $x \in [0, \overline{x})$. We can thus expect that v^δ is not strictly increasing when δ is close to $\overline{x}$. Figure 4a illustrates such a case. Intuitively, if δ is close to $\overline{x}$, an optimal capital path spends most of its time near $\overline{x}$, where the slope of the production function is strictly less than one. This means that two optimal capital paths from similar initial capital stocks tend to converge to each other and to have identical consumption paths. As δ decreases, however, v^δ starts appearing to be a strictly increasing function; see Figure 4b.[12]



(a) $\beta = 0.9,\ \delta = 20,\ f(x) = 10x^{0.5}$
$u(c) = c$, 100,000 states

(b) $\beta = 0.9,\ \delta = 5,\ f(x) = 10x^{0.5}$
$u(c) = c$, 100,000 states

FIGURE 5. Value functions and policy functions for consumption and capital.

If δ is small enough, an optimal capital path is expected to stay in a neighborhood of x^* in the long run, as in the model with $\delta = 0$. Because $f'(x^*) = 1/\beta > 1$, we can expect that the argument based on the assumption that $R > 1$ in Section 3.2 can be used to show that v^δ is strictly increasing. This is the idea of the next result. Figure 5 shows that the value function appears to be strictly increasing when δ is relatively small.

LEMMA 1. *There exists* $\overline{\delta} > 0$ *such that for all* $\delta \in (0, \overline{\delta}]$, v^δ *is strictly increasing.*

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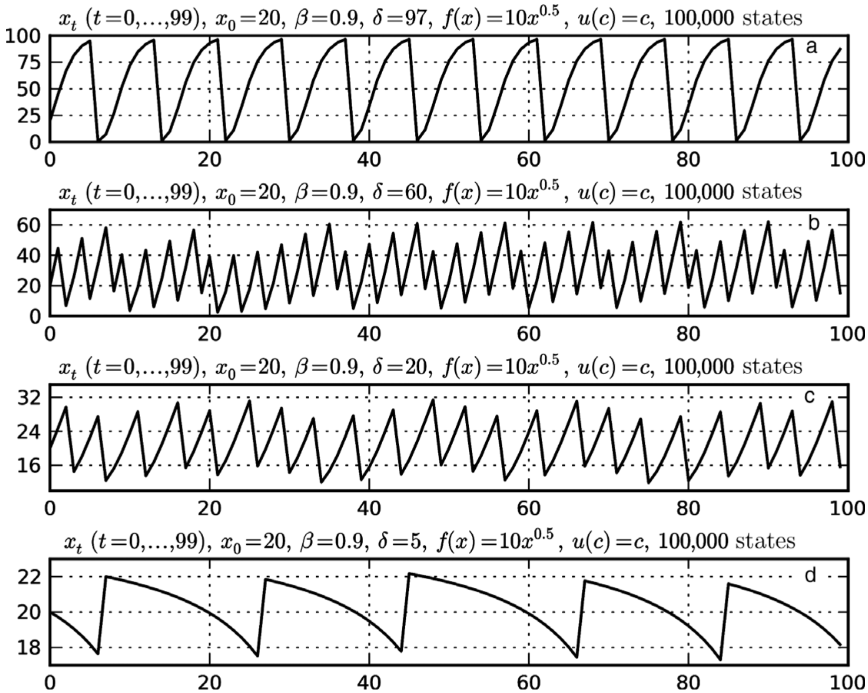

FIGURE 6. Optimal capital paths.

Proof. See Appendix B. ■

The following result is immediate from Lemma 1 and Theorem 1.[13]

PROPOSITION 1. *There exists* $\overline{\delta} > 0$ *such that for all* $\delta \in (0, \overline{\delta}]$, *for almost all* $x \in X$, *any optimal consumption and capital paths from* x *are bounded and asymptotically aperiodic.*

It is interesting to observe that a decrease in δ does not necessarily have a significant effect on v^δ. Although decreasing δ from 97 to 60 has a large effect on v^δ (Figure 4), the effect is much less dramatic when δ is decreased from 60 to 20 (Figures 4b and 5a), and it appears to be almost negligible when δ is decreased from 20 to 5 (Figure 5). On the other hand, a change in δ always has a comparable impact on the optimal policy functions for consumption and capital, as can be seen in Figures 4 and 5.

Figure 6 illustrates optimal capital paths in the four cases in Figures 4 and 5. Although the magnitude of fluctuations is directly related to δ, there appears to be no clear-cut relation between δ and the frequency of ups and downs.

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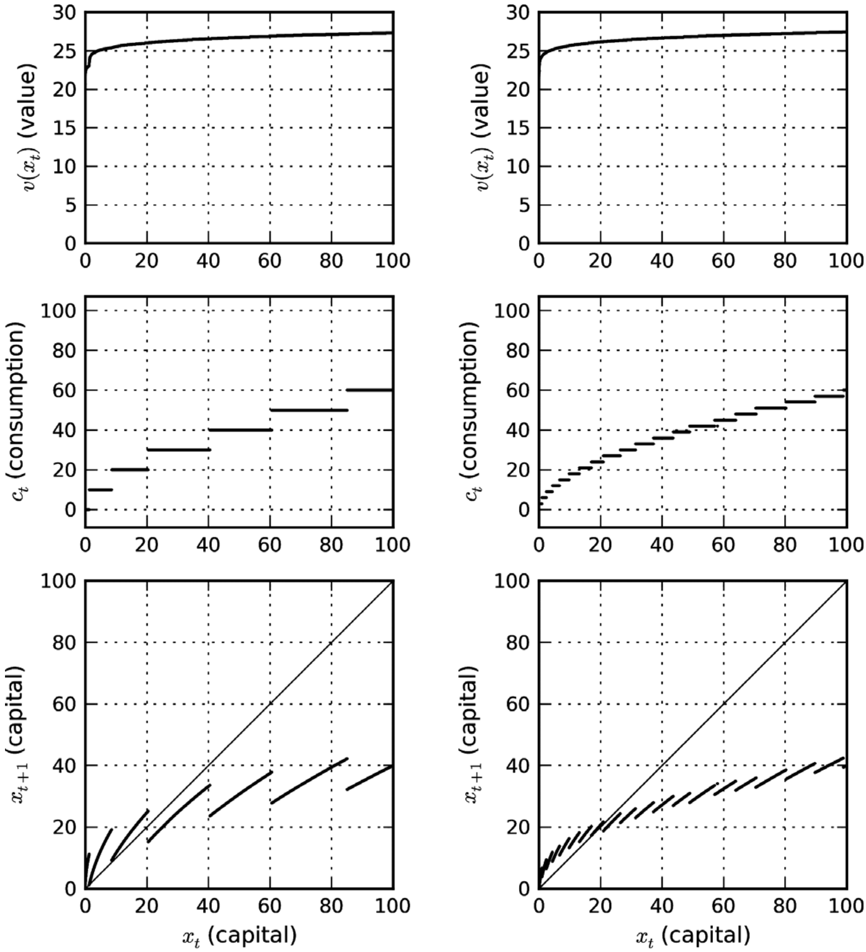

(a) $\beta = 0.9$, $\delta = 10$, $f(x) = 10x^{0.5}$
$u(c) = c^{0.3}$, 100,000 states

(b) $\beta = 0.9$, $\delta = 3$, $f(x) = 10x^{0.5}$
$u(c) = c^{0.3}$, 100,000 states

FIGURE 7. Value functions and policy functions for consumption and capital.

Finally, the idea of Lemma 1 seems to work even if the utility function is strictly concave. This point is illustrated with the policy functions for capital in Figure 7, which suggest that even if the utility function is strictly concave, as long as δ is sufficiently small, any optimal capital path eventually stays close to x^*, where the slope of the production function is strictly greater than one. If this is the case, v^δ can be shown to be strictly increasing using the argument based on (13). Figure 8 illustrates optimal capital paths in the cases in Figure 7. These paths roughly remain in the $\delta/2$-neighborhood of x^* (=20.25).

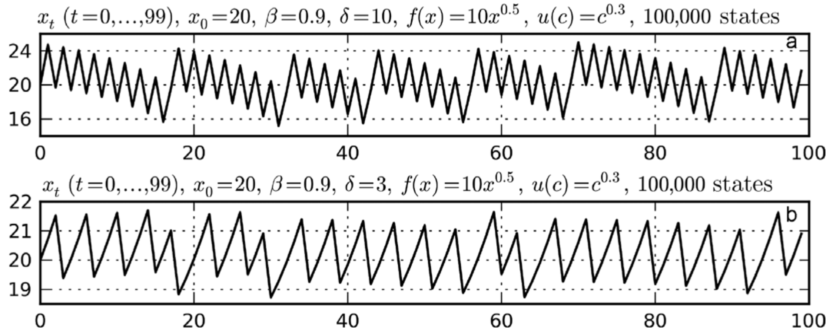

FIGURE 8. Optimal capital paths.

NOTES

1. See Aoki (1998), Verbrugge (2003), and Bischi et al. (2006) for examples of theoretical stochastic models involving discrete choices.
2. In this paper, the term "complex" means "aperiodic and bounded."
3. Their analysis was recently extended by Kamihigashi and Roy (2006, 2007) to models with nonsmooth technologies.
4. See Kamihigashi (2000a) for discussion on ergodic chaos in a discrete choice model.
5. Here the inequality $x < y$ means that $x^i \leq y^i$ for all $i = 1, \ldots, n$, and there is at least one i with $x^i < y^i$, where $x = (x^1, \ldots, x^n)$, etc.
6. Given our assumption on X, the value function v is also measurable by Chabrillac and Crouzeix (1987, Theorem 4). The proof of Lemma A.1 shows that we need only assume that v is nondecreasing and that there is at least one i such that $v(x^1, \ldots, x^n)$ is strictly increasing in x^i.
7. Essentially the same model is studied in Kamihigashi (2000b). Here we offer additional insight as well as outlining some of the basic arguments in Kamihigashi (2000b) to facilitate subsequent discussion.
8. Note that given any feasible path $\{c_t, s_t, x_{t+1}\}$, if $\{x_t\}$ is asymptotically periodic, then $\{c_t\}$ must be asymptotically periodic by (8) and (9). Hence if $\{c_t\}$ is asymptotically aperiodic, then $\{x_t\}$ must be asymptotically aperiodic.
9. See Kamihigashi (2000b) for discussion on random number generators. Each numerical example in this paper (except for Figure 3b) is obtained by solving the corresponding Bellman equation by modified policy iteration with 100,000 equally spaced grid points (or states).
10. In the case of Figure 3, the value and policy functions are computed for $x \in [0, 300]$ with 300,000 states, and these functions are plotted only for $x \in [0, 100]$ to reduce the effect of truncation.
11. Because $R > 1$, the steady state here is locally unstable, as discussed previously.
12. A more detailed computation of the value functions in Figures 4a and 4b suggests that the former is indeed piecewise constant whereas the latter is strictly increasing.
13. Recall also footnote 8. Proposition 1 does not follow from any argument in Kamihigashi (2000a, 2000b).
14. That is, $1\{v(x) = a\} = 1$ if $v(x) = a$, and $1\{v(x) = a\} = 0$ otherwise.
15. This result is adapted from Kamihigashi (2000b, Lemma 6). A similar argument is used in Kamihigashi (2000a, Lemma 5.14).

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APPENDIX A: PROOF OF THEOREM 1

LEMMA A.1. *Suppose that v is strictly increasing. Then for any $a \in \mathbf{R}$, we have $\lambda^n(V_a) = 0$, where $V_a = \{x \in X : v(x) = a\}$.*

Proof. Let $a \in \mathbf{R}$. Without loss of generality, we extend the domain of v to the entire $\mathbf{R}^n$ by defining $v(x) = -\infty$ for $x \in \mathbf{R}^n \setminus X$. Because $a \in \mathbf{R}$, this extension does not affect V_a. We have

$$\lambda^n(V_a) = \int 1\{v(x) = a\} d\lambda^n(x) \tag{A.1}$$

$$= \int \cdots \int 1\{v(x_1, x_2, \ldots, x_n) = a\} d\lambda(x_1) \cdots d\lambda(x_n), \tag{A.2}$$

where $1\{\cdot\}$ is the indicator function,[14] and (A.2) holds by the Tonelli–Fubini theorem [Dudley (2002, p. 139)]. Because v is strictly increasing, given $(x_2, \ldots, x_n) \in \mathbf{R}^{n-1}$, we have $1\{v(x_1, x_2, \ldots, x_n) = a\} = 0$ for almost all $x_1 \in \mathbf{R}$; thus

$$\int 1\{v(x_1, x_2, \ldots, x_n) = a\} d\lambda(x_1) = 0. \tag{A.3}$$

Substituting this into (A.2) yields the lemma. ■

LEMMA A.2. $\Pi \equiv \{\{c_t\} \in K^\infty : \{c_t\}$ *is eventually periodic*$\}$ *is countable.*[15]

Proof. Let $\Gamma = \cup_{i,j\in\mathbf{N}} K^i \times K^j$. Define $\Psi : \Pi \to \Gamma$ as follows: for $c \equiv \{c_t\} \in \Pi$, let i_c be the smallest $i \in \mathbf{N}$ such that $\{c_t\}_{t=i}^{\infty}$ is periodic; let j_c be the smallest $j \in \mathbf{N}$ such that $\forall t \geq i_c, c_t = c_{t+j}$; and define

$$\Psi(\{c_t\}) = (\{c_t\}_{t=0}^{i_c-1}, \{c_t\}_{t=i_c}^{i_c+j_c-1}) \in K^{i_c} \times K^{j_c} \subset \Gamma. \tag{A.4}$$

Clearly Ψ is one-to-one. Because the countable union of countable sets is countable, Γ is countable. Therefore Π is also countable. ■

To prove Theorem 1, suppose that v is strictly increasing. Define

$$A = \{a \in \mathbf{R} : \exists c \in \Pi \cap D, w(c) = a\}. \tag{A.5}$$

Because Π is countable by Lemma A.2, A is also countable. Define

$$Z = \cup_{a\in A} V_a = \{x \in X : v(x) \in A\}. \tag{A.6}$$

Because A is countable and the V_a are disjoint,

$$\lambda^n(Z) = \sum_{a\in A} \lambda^n(V_a) = 0, \tag{A.7}$$

where the second equality holds by Lemma A.1. Let $x \in X \setminus Z$ and $c \in C^*(x)$. We have $w(c) = v(x) \notin A$, which implies that $c \notin \Pi$ by (A.5). Thus c is asymptotically aperiodic. This establishes that Z has the desired properties.

To see Corollary 1, note that the second equality in (A.7) holds by (3). Thus the corollary follows by the above argument.

APPENDIX B: PROOF OF LEMMA 1

Note that for $\delta \geq 0$ and $x \in X$, we have the Bellman equation

$$v^\delta(x) = \max_{y\in F^\delta(x)} \{f(x) - y + \beta v^\delta(y)\} \tag{B.1}$$

$$= f(x) + \max_{y\in F^\delta(x)} \{\beta v^\delta(y) - y\}. \tag{B.2}$$

Let G^δ be the optimal policy correspondence:

$$G^\delta(x) = \operatorname*{argmax}_{y\in F^\delta(x)}\{f(x) - y + \beta v^\delta(y)\} = \operatorname*{argmax}_{y\in F^\delta(x)}\{\beta v^\delta(y) - y\}. \tag{B.3}$$

LEMMA B.1. $v^0 : \mathbf{R}_+ \to \mathbf{R}_+$ *is continuous and strictly concave. Furthermore, x^* is the unique solution to* $\max_{y\geq 0}[\beta v^0(y) - y]$*:*

$$\forall y \in \mathbf{R}_+ \setminus \{x^*\}, \quad \beta v^0(y) - y < \beta v^0(x^*) - x^*. \tag{B.4}$$

Proof. Standard arguments show that v^0 is continuous and strictly concave. Let $x_0 \geq x^*$. Then the capital path $\{x_{t+1}\}$ given by $x_t = x^*$ for all $t \in \mathbf{N}$ is feasible and satisfies the Euler

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equation and the transversality condition:

$$1 = \beta f'(x_{t+1}), \qquad \lim_{t\uparrow\infty} \beta^t f'(x_{t+1})x_{t+1} = 0. \tag{B.5}$$

By strict concavity, $\{x_{t+1}\}$ is the unique optimal capital path from x_0; thus $G^0(x_0) = \{x^*\}$, i.e., $x^* = \operatorname{argmax}_{y\in[0, f(x_0)]}\{\beta v^0(y) - y\}$. Because v^0 is strictly concave, $\beta v^0(y) - y$ is strictly decreasing in $y > f(x_0) > x^*$. Thus (B.4) holds. ■

Because the definitions of feasibility and optimality depend on δ, we make the dependence on δ explicit by saying that a path is δ-feasible, etc.

LEMMA B.2. *For any* $\delta > 0$, *we have*

$$\forall x \in X, \quad v^\delta(x) \geq v^0(x) - \delta/(1-\beta). \tag{B.6}$$

Proof. Let $\delta > 0$ and $x \in X$. Let $\{c_t\}$ be a 0-optimal consumption path from x. For $t \in \mathbf{Z}_+$, define $c_t^\delta = \max\{c \in H^\delta : c \leq c_t\}$. Then $\{c_t^\delta\}$ is δ-feasible from x, and $c_t^\delta \geq c_t - \delta$ for all $t \in \mathbf{Z}_+$. We have

$$v^\delta(x) \geq \sum_{t=0}^{\infty} \beta^t c_t^\delta \geq \sum_{t=0}^{\infty} \beta^t (c_t - \delta) = v^0(x) - \delta/(1-\beta). \tag{B.7}$$

Now (B.6) follows. ■

LEMMA B.3. *For any* $\hat{x} > x^*$, *there exists* $\overline{\delta} > 0$ *such that*

$$\forall \delta \in (0, \overline{\delta}], \forall x \in X, \forall y \in G^\delta(x), \quad y + \delta < \hat{x}. \tag{B.8}$$

Proof. Suppose that the lemma is false. Then there exist $\hat{x} > x^*$, $\{\delta_i\}_{i=1}^\infty \subset (0, \hat{x} - x^*)$ with $\delta_i \downarrow 0$, $\{x_i\}_{i=1}^\infty \subset X$, and $\{y_i\}_{i=1}^\infty \subset X$ such that

$$\forall i \in \mathbf{N}, \qquad y_i \in G^{\delta_i}(x_i), \quad y_i + \delta_i \geq \hat{x}. \tag{B.9}$$

Taking a subsequence, we may assume that y_i converges to some $y^* \in [\hat{x}, \overline{x}]$.

Let $i \in \mathbf{N}$. Because $v^0 \geq v^{\delta_i}$, we have

$$\beta v^0(y_i) - y_i \geq \beta v^{\delta_i}(y_i) - y_i. \tag{B.10}$$

Note from (B.9) that $y_i \geq \hat{x} - \delta_i > x^*$. Thus $\{y \in F^{\delta_i}(x_i) : y \geq x^*\} \neq \emptyset$. Define $\tilde{y}_i = \min\{y \in F^{\delta_i}(x_i) : y \geq x^*\} \in [x^*, x^* + \delta_i]$; note that $\tilde{y}_i \leq y_i$. We have

$$\beta v^{\delta_i}(y_i) - y_i \geq \beta v^{\delta_i}(\tilde{y}_i) - \tilde{y}_i \geq \beta[v^0(\tilde{y}_i) - \delta_i/(1-\beta)] - \tilde{y}_i, \tag{B.11}$$

where the second inequality uses Lemma B.2. It follows from (B.10) and (B.11) that

$$\beta v^0(y_i) - y_i \geq \beta[v^0(\tilde{y}_i) - \delta_i/(1-\beta)] - \tilde{y}_i. \tag{B.12}$$

Because $y_i \to y^* \geq \hat{x}$ and $\tilde{y}_i \to x^*$, letting $i \uparrow \infty$ in (B.12) and recalling the continuity of v^0, we obtain

$$\beta v^0(y^*) - y^* \geq \beta v^0(x^*) - x^*, \tag{B.13}$$

which contradicts Lemma B.1. This completes the proof. ■

For the rest of the proof, let $\hat{x} > x^*$ be such that $f'(\hat{x}) > 1$, and let $\overline{\delta} > 0$ satisfy (B.8).

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LEMMA B.4. *Let* $\delta \in (0, \bar{\delta}]$, $x \in X$, *and* $x' > x$. *Let* $\{c_t\}$ *be a* δ*-optimal consumption path from* x. *Then there exists a* δ*-feasible consumption path* $\{c'_t\}$ *from* x' *such that*

$$\text{(i) } \forall t \in \mathbf{Z}_+, c'_t \geq c_t,$$
$$\text{(ii) } \exists t \in \mathbf{Z}_+, c'_t > c_t. \quad \textbf{(B.14)}$$

Proof. Define $\{x'_t\}$ recursively by $x'_{t+1} = f(x'_t) - c_t$. Suppose that there exists no $\{c'_t\}$ satisfying (B.14). Then we must have

$$\forall t \in \mathbf{Z}_+, \quad x'_{t+1} < x_{t+1} + \delta < \hat{x}. \quad \textbf{(B.15)}$$

The second inequality holds by (B.8). The first inequality holds because if $x'_{s+1} \geq x_{s+1} + \delta$ for some $s \in \mathbf{Z}_+$, then c_s can be increased by δ without decreasing c_t with $t \neq s$. For $t \in \mathbf{N}$, by concavity of f and (B.15),

$$x'_{t+1} - x_{t+1} = f(x'_t) - f(x_t) \geq f'(x'_t)(x'_t - x_t) \geq f'(\hat{x})(x'_t - x_t). \quad \textbf{(B.16)}$$

Because $f'(\hat{x}) > 1$, it follows that $x'_t - x_t \to \infty$, contradicting (B.15). ■

Lemma 1 now follows from Lemma B.4.

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