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A REINSURANCE RISK MODEL WITH A THRESHOLD COVERAGE POLICY: THE GERBER–SHIU PENALTY FUNCTION

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Abstract

We consider a Cramér–Lundberg insurance risk process with the added feature of reinsurance. If an arriving claim finds the reserve below a certain threshold γ, or if it would bring the reserve below that level, then a reinsurer pays part of the claim. Using fluctuation theory and the theory of scale functions of spectrally negative Lévy processes, we derive expressions for the Laplace transform of the time to ruin and of the joint distribution of the deficit at ruin and the surplus before ruin. We specify these results in much more detail for the threshold set-up in the case of proportional reinsurance.

Keywords: Spectrally negative Lévy process; scale function; ruin probability; claim refraction

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1. Introduction

Let $X(t)$ be the surplus at time t of the classical Cramér–Lundberg risk process,

$$X(t) = u + ct - \sum_{i=1}^{N_t} Z_i. \tag{1.1}$$

In this model the company earns premium at a fixed rate c, the claim arrival process $\{N_t : t \geq 0\}$ is a Poisson process at rate λ, $\{Z_i : i = 1, 2, \ldots\}$ are the successive claim amounts indexed by their appearance and are independent and identically distributed (i.i.d.) positive random variables, and $u = X(0)$.

In such a model it is of interest to study the distribution of the time to ruin, the joint distribution of the time to ruin, the deficit at ruin, and the surplus before ruin. For a comprehensive overview of the state of the art of the classsical Cramér–Lundberg model, see Asmussen and Albrecher (2010).

In the last decade the classical Cramér–Lundberg model in (1.1) was modified to capture dividend payments to shareholders. Under the threshold dividend policy, dividends at rate $\tilde{c} < c$ are paid whenever the reserve is above a threshold γ. This process has a 'bend' at γ: it is called a refracted Lévy risk process; see Dickson and Drekic (2006), Gerber and Shiu (2006), Lin and

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Pavlova (2006) and Zhang *et al.* (2006). Wan (2007) considered the more general model where the compound Poisson risk model is perturbed by a Brownian motion.

Kyprianou and Loeffen (2010) considered such a state-dependent premium rate model for the general spectrally negative Lévy risk process. They used fluctuation theory and the theory of scale functions for spectrally negative Lévy processes to obtain the Laplace transform of the exit time, of the time to ruin, and of the joint probability for the surplus before and at ruin. Loeffen (2015) has recently presented more elegant analysis to obtain the results of this earlier paper with Kyprianou.

In order to reduce risk, the insurer insures part of the risk: the insurer pays a premium to the reinsurer who then pays a part of each claim. Motivated by the threshold dividend policy, we consider the reinsurance threshold policy: the insurer pays a constant premium to the reinsurer and the reinsurer pays part of the claim that falls below a threshold γ.

We apply Loeffen's (2015) methods to obtain quantities of interest for the reinsurance model. Assume that the company has a reinsurance contract which we now describe via some function $I(x)$, where $I(0) = 0$, $I(x) \le x$, and $I(x)$ is nondecreasing in x. The reinsurance pays part of the claim when the claim is below a given threshold γ. Let $\tilde{I}(x, y)$ denote the part that the insurer pays for a claim of size x occurring when the reserve level is y. The reinsurer pays $x - \tilde{I}(x, y)$, where $\tilde{I}(x, y)$ is given by

$$\tilde{I}(x, y) = \begin{cases} x & \text{if } y > \gamma,\ x \le y - \gamma, \\ y - \gamma + I(x - (y - \gamma)) & \text{if } y > \gamma,\ x \ge y - \gamma, \\ I(x) & \text{if } y \le \gamma. \end{cases}$$

Examples for $I(x)$ are $I(x) = \min(a, x)$ for a given constant a, and $I(x) = \alpha x$, $0 < \alpha < 1$.

Throughout we will not specify the reserve level y in $\tilde{I}(x, y)$ but it will be clear from the context. We consider the following risk process. The premium rate, the claim arrival process and the claim amounts are as for (1.1). When an arrival of a claim of size x finds the reserve below γ, the insurer pays only $I(x)$. When a claim of size x finds the reserve at level $y > \gamma$ and $x > y - \gamma$, the insurer pays $y - \gamma + I(x - (y - \gamma))$, i.e. he/she pays only that part of the claim that falls below γ. Denote the reserve level at time t under this policy by $U(t)$ or U_t.

In a companion paper (Boxma *et al.* (2016)) we analyze a risk process with state-dependent premium rate and state-dependent claim payments assuming a barrier dividend policy. Under this policy all the premium income is paid as dividends when the reserve level exceeds a barrier b. In that paper we applied different tools to find the distribution of the deficit at ruin and the amount of dividends until ruin. In the present paper we consider a special case of state-dependent claim payments and consider the expected discounted time to ruin and the joint distribution of the deficit at ruin and the reserve just before ruin.

The paper is organized as follows. In Section 2 we introduce some notation and a few identities related to exit times of spectrally negative Lévy processes; these play a crucial role in the remainder of the paper. In Section 3 we present expressions for the Laplace transform of the exit time from an upper barrier, the time to ruin and the joint probability for the surplus before and at ruin for general $I(x)$. In Section 4 these results are specified in much more detail for the case of proportional reinsurance, i.e. $I(x) = \alpha x$.

2. Notation

Above level γ the process U behaves as a risk process X_1, with premium rate c and i.i.d. claims distributed as Z_1 with distribution F_1, arriving according to a Poisson process at rate λ.

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Below the level γ, U behaves as a risk process X_0, with premium rate c and i.i.d. claims distributed as Z_0 with distribution F_0, arriving according to a Poisson process at rate λ, where $F_0(x) = \mathbb{P}(I(Z_1) \leq x)$.

When the process $U > \gamma$ it evolves as X_1 and when $U < \gamma$ it evolves as X_0.

For $i = 0, 1$ and $s \geq 0$, let $\widetilde{F}_i(s) = \mathbb{E}[\mathrm{e}^{-sZ_i}]$ and

$$\psi_i(s) = \mathbb{E}[\mathrm{e}^{sX_i(1)}] = cs - \lambda + \lambda \widetilde{F}_i(s).$$

Then define $\Phi_i(v) = \sup\{y \geq 0 \colon \psi_i(y) = v\}$.

Definition 2.1. For a given spectrally negative Lévy process X, with Laplace exponent ψ, and $q \geq 0$, there is a unique q-scale function associated with X, $W^{(q)} \colon \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for $x < 0$, and on $(0, \infty)$, $W^{(q)}$ is the unique continuous function with Laplace transform over $\operatorname{Re}(\beta) \geq 0$,

$$\int_0^\infty \mathrm{e}^{-\beta x} W^{(q)}(x) \, \mathrm{d}x = \frac{1}{\psi(\beta) - q}. \tag{2.1}$$

Denote $W^{(0)}$ by W. Consider the spectrally negative Lévy risk processes X_i, with q-scale function $W_i^{(q)}$, $i = 0, 1$. For $b > 0$, introduce the first passage times of X_i,

$$\tau_{i,b}^+ = \inf\{t > 0 \colon X_i(t) \geq b\} \quad \text{and} \quad \tau_{i,a}^- = \inf\{t > 0 \colon X_i(t) < a\}.$$

For the process U, define the first passage times

$$\kappa_b^+ = \inf\{t > 0 \colon U_t \geq b\} \quad \text{and} \quad \kappa_a^- = \inf\{t > 0 \colon U_t < a\}.$$

Denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the conditional probability and expectation given $X(0) = x$. In the sequel we apply the following identities from Kyprianou (2006, Chapter 8).

Theorem 2.1. *Let X be a spectrally negative Lévy process.*

(i) *For $q \geq 0$ and $x \leq b$,*

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \tau_0^-\}}] = \frac{W^{(q)}(x)}{W^{(q)}(b)}. \tag{2.2}$$

(ii) *For $q \geq 0$, $b > 0$, and $0 \leq x \leq b$,*

$$\int_0^\infty \mathrm{e}^{-qt} \mathbb{P}_x(X_t \in \mathrm{d}y,\ t < \tau_b^+ \wedge \tau_0^-) \, \mathrm{d}t = \bigg(\frac{W^{(q)}(x) W^{(q)}(b-y)}{W^{(q)}(b)} - W^{(q)}(x-y)\bigg) \mathrm{d}y. \tag{2.3}$$

3. The Laplace transform of the time to ruin for general $I(x)$

To derive the Laplace transform of the time to ruin, we first find some other quantities. Let $b > \gamma$ and $B \subset \mathbb{R}$. For $0 \leq x \leq b$, define

$$V^{(q)}(x, \gamma, b, B) = \int_{t=0}^\infty \mathrm{e}^{-qt} \mathbb{P}_x(U_t \in B,\ t < \kappa_0^- \wedge \kappa_b^+) \, \mathrm{d}t,$$

so $V^{(q)}(x, \gamma, b, B)$ is the discounted time that the process U_t spends in B, given $U_0 = x$.

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Proposition 3.1. (i) *For* $\gamma \le x < b$, $V^{(q)}(x,\gamma,b,B)$ *is equal to*

$$\begin{aligned}&\int_{y\in B\cap[\gamma,b]}\bigg(\frac{W_1^{(q)}(x-\gamma)W_1^{(q)}(b-y)}{W_1^{(q)}(b-\gamma)}-W_1^{(q)}(x-y)\bigg)\,\mathrm{d}y\\&\quad+\int_{y=0}^{b-\gamma}\int_{\theta\ge y}\bigg(\frac{W_1^{(q)}(x-\gamma)W_1^{(q)}(b-\gamma-y)}{W_1^{(q)}(b-\gamma)}-W_1^{(q)}(x-\gamma-y)\bigg)\lambda\,\mathrm{d}F_1(\theta)\,\mathrm{d}y\\&\quad\times\bigg[\int_{z\in B\cap[0,\gamma)}\bigg(\frac{W_0^{(q)}(\gamma-I(\theta-y))W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)}-W_0^{(q)}(\gamma-I(\theta-y)-z)\bigg)\,\mathrm{d}z\\&\qquad+\frac{W_0^{(q)}(\gamma-I(\theta-y))}{W_0^{(q)}(\gamma)}V^{(q)}(\gamma,\gamma,b,B)\bigg].\end{aligned}\tag{3.1}$$

(ii) *For* $0 < x < \gamma$, $V^{(q)}(x,\gamma,b,B)$ *is equal to*

$$\int_{y\in B\cap[0,\gamma)}\bigg(\frac{W_0^{(q)}(x)W_0^{(q)}(\gamma-y)}{W_0^{(q)}(\gamma)}-W_0^{(q)}(x-y)\bigg)\,\mathrm{d}y+\frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)}V^{(q)}(\gamma,\gamma,b,B).\tag{3.2}$$

Proof. (i) For $\gamma \le x < b$, $V^{(q)}(x,\gamma,b,B)$ is equal to

$$\int_{t=0}^{\infty}\mathrm{e}^{-qt}\int_{y\in B\cap[\gamma,b]}\mathbb{P}_{x-\gamma}(X_1(t)\in\mathrm{d}y-\gamma,\ t<\tau_{1,0}^-\wedge\tau_{1,b-\gamma}^+)\,\mathrm{d}t\tag{3.3}$$

$$+\int_{t=0}^{\infty}\mathrm{e}^{-qt}\int_{y=\gamma}^{b}\int_{\theta\ge y-\gamma}\mathbb{P}_{x-\gamma}(X_1(t)\in\mathrm{d}y-\gamma,\ t<\tau_{1,0}^-\wedge\tau_{1,b-\gamma}^+)\lambda\,\mathrm{d}F_1(\theta)\,\mathrm{d}t\tag{3.4}$$

$$\times\bigg[\int_{s=0}^{\infty}\int_{z\in B\cap[0,\gamma)}\mathrm{e}^{-qs}\mathbb{P}_{(\gamma-I(\theta-(y-\gamma)))}(X_0(s)\in\mathrm{d}z,\ s<\tau_{0,0}^-\wedge\tau_{0,\gamma}^+)\,\mathrm{d}s\tag{3.5}$$

$$+\mathbb{E}_{\gamma-I(\theta-(y-\gamma))}(\mathrm{e}^{-q\tau_{0,\gamma}^+},\mathbf{1}_{\{\tau_{0,\gamma}^+<\tau_{0,0}^-\}})V^{(q)}(\gamma,\gamma,b,B)\bigg].\tag{3.6}$$

Above γ, $U(t)$ behaves as $X_1(t)$: equation (3.3) describes the discounted time that U (or X_1) spends in $B\cap[\gamma,b]$ before it down-crosses the level γ.

Equation (3.4) describes the discounted time that $U(t)$ exceeds γ until it down-crosses γ before hitting b. It is the same as the discounted time that $X_1-\gamma$ exceeds 0 before hitting $b-\gamma$.

Below γ, U evolves as X_0. Thus, (3.5) is the expected discounted time that X_0 is in $B\cap[0,\gamma)$ given it started at $\gamma-I(\theta-(y-\gamma))$.

Thus, (3.4) multiplied by (3.5) is the discounted time that U spends in $B\cap[0,\gamma)$ from the moment it down-crosses γ until it exits $[0,\gamma)$. Similarly, (3.4) multiplied by (3.6) is the expected discounted time that U spends in B from the moment the process first hits γ after the first down-crossing of the level γ.

Using the scale function as in (2.3), and with $W_i^{(q)}(x)$ the scale function associated with X_i for $i=0,1$, we obtain (3.1).

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(ii) For $0 < x < \gamma$, similar arguments to (i) show that $V^{(q)}(x, \gamma, b, B)$ is equal to

$$\int_{t=0}^{\infty} \mathrm{e}^{-qt} \int_{y\in B\cap[0,\gamma)} \mathbb{P}_x(X_0(t) \in \mathrm{d}y,\ \mathbf{1}_{\{t<\tau_{0,0}^{-}\wedge\tau_{0,\gamma}^{+}\}})\,\mathrm{d}t + \int_{t=0}^{\infty} \mathrm{e}^{-qt} \int_{y\in[0,\gamma)} \mathbb{P}_x(X_0(t) \in \mathrm{d}y,\ \mathbf{1}_{\{t<\tau_{0,\gamma}^{+}<\tau_{0,0}^{-}\}})\,\mathrm{d}t\, V^{(q)}(\gamma, \gamma, b, B). \tag{3.7}$$

Applying (2.2) and (2.3) we obtain (3.2).

To find $V^{(q)}(\gamma, \gamma, b, B)$ put $x = \gamma$ in (3.1) and solve the equation that results. □

Next we obtain

$$V^{(q)}(x, \gamma, B) = \int_{t=0}^{\infty} \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in B,\ t < \kappa_0^{-})\,\mathrm{d}t = \lim_{b\to\infty} V^{(q)}(x, \gamma, b, B).$$

For a Lévy process $X(t)$ with Lévy exponent ψ and adapted to a σ-field $\mathcal{F}_t$, let

$$M_t(\beta) = \mathrm{e}^{\beta X(t)-\psi(\beta)t}$$

be the Wald martingale associated with X. For real $\beta \geq 0$, define the measure $\mathbb{P}^{\beta}$ for $\mathcal{A} \in \mathcal{F}_t$ by

$$\mathbb{P}^{\beta}(\mathcal{A}) = \mathbb{E}[\mathrm{e}^{M_t(\beta)}\,\mathbf{1}_{\mathcal{A}}].$$

Let $W_{(\beta)}^{(q)}$ denote the scale function associated with X under $\mathbb{P}^{\beta}$. Apply Kyprianou (2006, Chapter 8.2) or Kuznetsov *et al.* (2013, Equation (53)) to obtain

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x}W_{(\Phi(q))}(x) = \mathrm{e}^{\Phi(q)x}\frac{1}{\psi'_{\Phi(q)}(0+)}\mathbb{P}_x^{\Phi(q)}(\underline{X}(\infty) \geq 0), \tag{3.8}$$

where $\underline{X}(t) = \inf_{s\leq t} X(s)$. Under the measure $\mathbb{P}_x^{\Phi(q)}$, X drifts to ∞.

Proposition 3.2. (i) *For $x \geq \gamma$, $V^{(q)}(x, \gamma, B)$ is equal to*

$$\begin{aligned}
&\int_{y\in B\cap[\gamma,\infty)} (W_1^{(q)}(x-\gamma)\mathrm{e}^{-\Phi_1(q)(y-\gamma)} - W_1^{(q)}(x-y))\mathrm{d}y \\
&\quad + \int_{y=\gamma}^{\infty}\int_{\theta\geq y-\gamma}\Big(W_1^{(q)}(x-\gamma)\mathrm{e}^{-\Phi_1(q)(y-\gamma)} - W_1^{(q)}(x-y)\Big)\lambda\,\mathrm{d}F_1(\theta)\,\mathrm{d}y \\
&\quad \times\Bigg[\int_{z\in B\cap[0,\gamma)}\Bigg(\frac{W_0^{(q)}(\gamma - I(\theta+\gamma-y))W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)} \\
&\qquad\qquad - W_0^{(q)}(\gamma - I(\theta+\gamma-y)-z)\Bigg)\mathrm{d}z \\
&\qquad + \frac{W_0^{(q)}(\gamma - I(\theta+\gamma-y))}{W_0^{(q)}(\gamma)}V^{(q)}(\gamma,\gamma,B)\Bigg].
\end{aligned} \tag{3.9}$$

(ii) *For $0 < x < \gamma$, $V^{(q)}(x, \gamma, B)$ is equal to*

$$\int_{y\in B\cap[0,\gamma)}\Bigg(\frac{W_0^{(q)}(x)W_0^{(q)}(\gamma-y)}{W_0^{(q)}(\gamma)} - W_0^{(q)}(x-y)\Bigg)\mathrm{d}y + \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)}V^{(q)}(\gamma,\gamma,B). \tag{3.10}$$

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(iii) *Then* $V^{(q)}(\gamma,\gamma,B)$ *is equal to*

$$\bigg(1-\int_{y=\gamma}^{\infty}\int_{\theta\geq y-\gamma} W_1^{(q)}(0)\mathrm{e}^{-\Phi_1(q)(y-\gamma)}\lambda\,\mathrm{d}F_1(\theta)\frac{W_0^{(q)}(\gamma-I(\theta+\gamma-y))}{W_0^{(q)}(\gamma)}\,\mathrm{d}y\bigg)^{-1}$$

$$\times\bigg(\int_{y\in B\cap(\gamma,\infty]} W_1^{(q)}(0)\mathrm{e}^{-\Phi_1(q)(y-\gamma)}\mathrm{d}y$$

$$+\bigg[\int_{y=\gamma}^{\infty}\int_{\theta\geq y-\gamma} W_1^{(q)}(0)\mathrm{e}^{-\Phi_1(q)(y-\gamma)}\lambda\,\mathrm{d}F_1(\theta)\,\mathrm{d}y \tag{3.11}$$

$$\times\int_{z\in B\cap(0,\gamma]}\bigg(\frac{W_0^{(q)}(\gamma-I(\theta+\gamma-y))W_0^{(q)}(\gamma-z)}{W_0^{(q)}(\gamma)} \tag{3.12}$$

$$-W_0^{(q)}(\gamma-I(\theta+\gamma-y)-z)\bigg)\mathrm{d}z\bigg]\bigg).$$

Proof. (i) and (ii) Applying (3.8) to X_1 shows that, for $\gamma\leq y<b$,

$$\frac{W_1^{(q)}(b-y)}{W_1^{(q)}(b-\gamma)}=\frac{\mathrm{e}^{\Phi_1(q)(b-y)}\mathbb{P}_{b-y}^{\Phi_1(q)}(\underline{X}_1(\infty)\geq 0)}{\mathrm{e}^{\Phi_1(q)(b-\gamma)}\mathbb{P}_{b-\gamma}^{\Phi_1(q)}(\underline{X}_1(\infty)\geq 0)}\to\mathrm{e}^{-\Phi_1(q)(y-\gamma)}\quad\text{as } b\to\infty. \tag{3.13}$$

Thus, taking the limit as $b\to\infty$ in (3.1) leads to (3.9). Similarly, taking the limit in (3.2) leads to (3.10).

(iii) We obtain $V^{(q)}(\gamma,\gamma,B)$ by substituting $x=\gamma$ in (3.9) . □

To obtain the Laplace transform of the time to ruin, we introduce $\mathcal{E}_q$, a random variable exponentially distributed on $\mathbb{R}_+$ with mean $1/q$. Then the Laplace transform of the time to ruin is given by

$$\begin{aligned}\mathbb{E}_x(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\infty\}})&=\mathbb{P}_x(\mathcal{E}_q>\kappa_0^-)=1-\mathbb{P}_x(\mathcal{E}_q\leq\kappa_0^-)\\&=1-q\int_0^{\infty}\mathrm{e}^{-qt}\mathbb{P}_x(U_t\in[0,\infty))\,\mathrm{d}t\\&=1-qV^{(q)}(x,\gamma,[0,\infty)).\end{aligned}$$

Consider next the Gerber–Shiu penalty function; this is a nonnegative function of the deficit at ruin $|U_{\kappa_0^-}|$ and the surplus just before ruin $U_{\kappa_{0-}^-}$. For h, a nonnegative function, define

$$m(x,q)=\mathbb{E}_x[\mathrm{e}^{-q\kappa_0^-}h(U_{\kappa_{0-}^-},|U_{\kappa_0^-}|)].$$

Proposition 3.3. *We have*

$$\begin{aligned}m(x,q)=&\int_{y=0}^{\gamma}V^{(q)}(x,\gamma,\mathrm{d}y)\int_{\theta\geq y}\lambda\,h(y,\theta-y)\,\mathrm{d}F_0(\theta)\\&+\int_{y=\gamma}^{\infty}V^{(q)}(x,\gamma,\mathrm{d}y)\int_{I(\theta-(y-\gamma))>\gamma}\lambda\,h(y,I(\theta-(y-\gamma)-\gamma))\,\mathrm{d}F_1(\theta).\end{aligned}$$

Proof. The discounted time in $(y,y+\mathrm{d}y)$ is $V(x,\gamma,\mathrm{d}y)$. When $y<\gamma$ ruin occurs when the downwards jump θ exceeds y, in which case the deficit at ruin is $\theta-y$. When the surplus just before ruin is above γ, ruin occurs when the part that the insurer pays $I(\theta-(y-\gamma))$ exceeds γ. Since $I(x)\leq x$, $I(\theta-(y-\gamma))\geq\gamma$ implies that $\theta\geq y$. In this case the deficit is $I(\theta-(y-\gamma))-\gamma$. □

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4. Results for the $I(x) = \alpha x$ case

In this section we consider the case that $I(x) = \alpha x$, $0 < \alpha < 1$. We obtain simpler expressions, especially for (3.1)–(3.11), obtaining expressions involving only one integral instead of two. We apply a fundamental identity introduced by Loeffen (2015). Throughout we use an index 1 for quantities related to the risk process $X_1(t)$ with premium rate c and claim distribution F_1, and an index 0 for quantities related to the risk process $X_0(t)$ with premium rate c and claim distribution F_0, where $F_0(x) = F_1(x/\alpha)$.

Let $\mathcal{A}_j$ be the generator of X_j for $j = 0, 1$. Let h be a locally bounded function satisfying the smoothness and boundedness conditions (i), (ii), and (iv) of Definition 1 in Loeffen (2015),

$$\mathcal{A}_j h(x) = ch'_-(x) + \int_0^\infty [h(x-\theta) - h(x)]\lambda \, \mathrm{d}F_j(\theta), \tag{4.1}$$

where h'_- denotes the left derivative of h. Also, when $I(x) = \alpha x$, $\widetilde{F}_1(s) = \widetilde{F}_0(s/\alpha)$, thus

$$\psi_1(s) = \mathbb{E}[\mathrm{e}^{sX_1(1)}] = \psi_0\left(\frac{s}{\alpha}\right) - \frac{cs}{\alpha}(1-\alpha). \tag{4.2}$$

This section is organized as follows. In Section 4.1 we apply Loeffen's (2015) result and establish in Proposition 4.1 a key identity which is applied in the remainder of the section. In Section 4.2 we obtain an expression for the discounted time that $U_t \in B$ before exiting $[0, b]$, where $B \subset \mathbb{R}$. In Sections 4.3 and 4.4 we obtain expressions for the potential measure for U. Section 4.5 presents the Laplace transform of the time to ruin, and in Section 4.6 we derive the ruin probability. In Section 4.7 we present an expression for the Gerber–Shiu penalty function and the joint probability of the surplus before and at ruin.

4.1. A key identity

The following key identity is a consequence of Loeffen's (2015) Theorem 2 (see also Equation (19) in that paper).

Proposition 4.1. *When $I(x) = \alpha x$, for $x \in [\gamma, b)$,*

$$\begin{aligned}
&\mathbb{E}_x\big[\mathrm{e}^{-q\tau^-_{1,\gamma}} W_0^{(q)}(\alpha X_1(\tau^-_{1,\gamma}) + (1-\alpha)\gamma)\,\mathbf{1}_{\{\tau^-_{1,\gamma} < \tau^+_{1,b}\}}\big] \\
&\quad = W_0^{(q)}(\alpha x + \gamma(1-\alpha)) - \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_0^{(q)}(\alpha b + \gamma(1-\alpha)) \\
&\qquad - (1-\alpha)c \int_\gamma^b W_0^{(q)\prime}(\alpha y + \gamma(1-\alpha)) \bigg[\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-y) \\
&\qquad\qquad - W_1^{(q)}(x-y)\bigg]\,\mathrm{d}y.
\end{aligned}$$

Proof. By Loeffen's (2015) Theorem 2 with $\sigma = 0$,

$$\begin{aligned}
&\mathbb{E}_x\big(\mathrm{e}^{-q\tau^-_{1,\gamma}} W_0^{(q)}(\alpha X_1(\tau^-_{1,\gamma}) + (1-\alpha)\gamma)\,\mathbf{1}_{\{\tau^-_{1,\gamma} < \tau^+_{1,b}\}}\big) \\
&\quad = W_0^{(q)}(\alpha x + \gamma(1-\alpha)) - \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_0^{(q)}(\alpha b + \gamma(1-\alpha))
\end{aligned}$$

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$$+\int_{\gamma}^{b}(\mathcal{A}_1-q)W_0^{(q)}(\alpha y+(1-\alpha)\gamma)\bigg[\frac{W_1^{(q)}(x-\gamma+q)}{W_1^{(q)}(b-\gamma)}W_1^{(q)}(b-y) - W_1^{(q)}(x-y)\bigg]\mathrm{d}y.$$

We now show that

$$(\mathcal{A}_1-q)W_0^{(q)}(\alpha y+(1-\alpha)\gamma)=-c(1-\alpha)W_0^{(q)\prime}(\alpha y+\gamma(1-\alpha)). \tag{4.3}$$

By (4.1), the left-hand side above is equal to

$$\alpha c W_0^{(q)\prime}(\alpha y+\gamma(1-\alpha)) + \int_0^{\infty}[W_0^{(q)}(\alpha(y-\theta)+\gamma(1-\alpha))\lambda\,\mathrm{d}F_1(\theta)-(\lambda+q)W_0^{(q)}(\alpha y+\gamma(1-\alpha))]. \tag{4.4}$$

Equation (4.4) is defined for $y\geq -\gamma(1-\alpha)/\alpha$. To prove (4.3) we take Laplace transforms of both sides of (4.3) and show that they are equal; we find

$$\begin{aligned}\int_{-\gamma(1-\alpha)/\alpha}^{\infty}\mathrm{e}^{-sy}W_0^{(q)\prime}(\alpha y+\gamma(1-\alpha))\mathrm{d}y &= \frac{1}{\alpha}\mathrm{e}^{s\gamma(1-\alpha)/\alpha}\int_{z=0}^{\infty}\mathrm{e}^{-sz/\alpha}W_0^{(q)\prime}(z)\,\mathrm{d}z\\ &= \frac{1}{\alpha}\mathrm{e}^{s\gamma(1-\alpha)/\alpha}\bigg(-\frac{1}{c}+\frac{s}{\alpha(\psi_0(s/\alpha)-q)}\bigg),\end{aligned} \tag{4.5}$$

where the first equality above comes from using integration by parts and in the second equality we apply (2.1) and the identity $W_0^{(q)}(0)=1/c$ (Kyprianou (2006, Lemma 8.6)). Similarly, by (2.1),

$$\int_{-\gamma(1-\alpha)/\alpha}^{\infty}\mathrm{e}^{-sy}W_0^{(q)}(\alpha y+\gamma(1-\alpha))\,\mathrm{d}y=\mathrm{e}^{s\gamma(1-\alpha)/\alpha}\frac{1}{\alpha(\psi_0(s/\alpha)-q)}. \tag{4.6}$$

Integration by parts and change of variables now yields

$$\begin{aligned}\lambda\int_{-\gamma(1-\alpha)/\alpha}^{\infty}&\mathrm{e}^{-sy}\int_{\theta=0}^{\infty}(W_0^{(q)}(\alpha(y-\theta)+\gamma(1-\alpha))\,\mathrm{d}F_1(\theta)\\ &=\lambda\mathrm{e}^{s\gamma(1-\alpha)/\alpha}\int_{z=0}^{\infty}\mathrm{e}^{-sz}\int_{\theta=0}^{z}W_0^{(q)}(\alpha(z-\theta))\,\mathrm{d}F_1(\theta)\\ &=\frac{\lambda}{\alpha}\mathrm{e}^{s\gamma(1-\alpha)/\alpha}\int_{\theta=0}^{\infty}\mathrm{e}^{-s\theta}\,\mathrm{d}F_1(\theta)\int_{z=0}^{\infty}\mathrm{e}^{-sz/\alpha}W_0^{(q)}(z)\,\mathrm{d}z\\ &=\mathrm{e}^{s\gamma(1-\alpha)/\alpha}\frac{\lambda\widetilde{F}_1(s)}{\alpha(\psi_0(s/\alpha)-q)}\\ &=\mathrm{e}^{s\gamma(1-\alpha)/\alpha}\frac{\lambda\widetilde{F}_0(s/\alpha)}{\alpha(\psi_0(s/\alpha)-q)}.\end{aligned} \tag{4.7}$$

In the last line we applied (2.1) and used $Z_0=\alpha Z_1$. From (4.4)–(4.7) we conclude that the Laplace transform of $(\mathcal{A}_1-q)W_0^{(q)}(\alpha x+\gamma(1-\alpha))$ is equal to

$$\mathrm{e}^{s\gamma(1-\alpha)/\alpha}\bigg[-1+\frac{1}{\alpha}+(1-\alpha)\frac{cs}{\alpha^2(\psi_0(s/\alpha)-q)}\bigg]=\frac{-c(1-\alpha)}{\alpha}\bigg[-\frac{1}{c}+\frac{s}{\alpha(\psi_0(s/\alpha)-q)}\bigg].$$

Thus, (4.5) yields (4.3). □

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For $x \geq \gamma$, define

$$w_{\alpha}^{(q)}(x, z) = W_0^{(q)}(\alpha x - z + \gamma(1-\alpha)) + (1-\alpha)c \int_{\gamma}^{x} W_1^{(q)}(x-y) W_0^{(q)\prime}(\alpha y - z + \gamma(1-\alpha))\,\mathrm{d}y, \tag{4.8}$$

and, for $x < \gamma$, define

$$w_{\alpha}^{(q)}(x, z) = W_0^{(q)}(x - z). \tag{4.9}$$

Then, for $\gamma \leq x < b$,

$$\mathbb{E}_x\big[\mathrm{e}^{-q\tau_{1,\gamma}^-} W_0(\alpha x + (1-\alpha)\gamma)\,\mathbf{1}_{\{\tau_{1,\gamma}^- < \tau_{1,b}^+\}}\big] = w_{\alpha}^{(q)}(x, 0) - \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} w_{\alpha}^{(q)}(b, 0). \tag{4.10}$$

4.2. Exit time for U

Let $\gamma < b$. Define $p(x, \gamma, b, q) := \mathbb{E}_x(\mathrm{e}^{-q\kappa_b^+}\mathbf{1}_{\{\kappa_b^+ < \kappa_0^-\}} \mid U_0 = x)$.

Proposition 4.2. *When $I(x) = \alpha x$,*

$$p(x, \gamma, b, q) = \frac{w_{\alpha}^{(q)}(x, 0)}{w_{\alpha}^{(q)}(b, 0)}. \tag{4.11}$$

Proof. Let $\gamma \leq x < b$. Either b is reached before γ or the process down-crosses γ before reaching b. Then $X_1(\tau_{1,\gamma}^-)$ is the state of X_1 after down-crossing γ. Since the insurer pays $\alpha(\gamma - X_1(\tau_{1,\gamma}^-))$ (instead of $(\gamma - X_1(\tau_{1,\gamma}^-))$), the state of the process U after undershooting γ is

$$\gamma - \alpha(\gamma - X_1(\tau_{1,\gamma}^-)) = \alpha X_1(\tau_{1,\gamma}^-) + (1-\alpha)\gamma.$$

Hence, the discounted time until reaching b before ruin is equal to the sum of the discounted time to reach γ before ruin, and the discounted time to reach b before ruin starting at γ. Applying the strong Markov property at γ and using (2.2) and (4.8), $p(x, \gamma, b, q)$ is equal to

$$\begin{aligned}
&\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} + \frac{\mathbb{E}_x\big[\mathrm{e}^{-q\tau_{1,\gamma}^-} W_0^{(q)}\big(\alpha X(\tau_{1,\gamma}^-) + (1-\alpha)\gamma\big)\,\mathbf{1}_{\{\tau_{1,\gamma}^- < \tau_{1,b}^+\}}\big]}{W_0^{(q)}(\gamma)} p(\gamma, \gamma, b, q) \\
&\quad = \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} + \frac{w_{\alpha}^{(q)}(x, 0) - (W_1^{(q)}(x-\gamma)/W_1^{(q)}(b-\gamma)) w_{\alpha}^{(q)}(b, 0)}{W_0^{(q)}(\gamma)} p(\gamma, \gamma, b, q).
\end{aligned} \tag{4.12}$$

We now find $p(\gamma, \gamma, b, q)$ by substituting $x = \gamma$ in (4.12) and applying (4.10), i.e.

$$\begin{aligned}
p(\gamma, \gamma, b, q) &= \frac{W_1^{(q)}(0)}{W_1^{(q)}(b-\gamma)} \\
&\quad + \frac{w_{\alpha}^{(q)}(\gamma, 0) - (W_1^{(q)}(0)/W_1^{(q)}(b-\gamma)) w_{\alpha}^{(q)}(b, 0)}{W_0^{(q)}(\gamma)} p(\gamma, \gamma, b, q).
\end{aligned} \tag{4.13}$$

Since $w_{\alpha}^{(q)}(\gamma, 0) = W_0^{(q)}(\gamma)$, we obtain

$$p(\gamma, \gamma, b, q) = \frac{W_0^{(q)}(\gamma)}{w_{\alpha}^{(q)}(b, 0)} = \frac{w_{\alpha}^{(q)}(\gamma, 0)}{w_{\alpha}^{(q)}(b, 0)}. \tag{4.14}$$

Substituting (4.14) in (4.12) we obtain (4.11) for the $\gamma \leq x < b$ case.

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For $0 < x < \gamma$, the strong Markov property with (2.2) and (4.14) yields

$$p(x,\gamma,b,q) = \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} p(\gamma,\gamma,b,q) = \frac{W_0^{(q)}(x)}{w_\alpha^{(q)}(b,0)} = \frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)}, \tag{4.15}$$

i.e. (4.11) again holds. □

4.3. $V^{(q)}(x,\gamma,b,B)$

Recall (see Proposition 3.1) the discounted time the process is in B before exiting $(0,b)$,

$$V^{(q)}(x,\gamma,b,B) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in B,\ t < \kappa_0^- \wedge \kappa_b^+)\,\mathrm{d}t.$$

Proposition 4.3. *When $I(x) = \alpha x$,*

$$V^{(q)}(x,\gamma,b,B) = \int_{y\in B\cap[\gamma,b)} \bigg(\frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)} W_1^{(q)}(b-y) - W_1^{(q)}(x-y)\bigg)\,\mathrm{d}y$$
$$+ \int_{y\in B\cap[0,\gamma)} \bigg(\frac{w_\alpha^{(q)}(x,0)}{w_\alpha^{(q)}(b,0)} w_\alpha^{(q)}(b,y) - w_\alpha^{(q)}(x,y)\bigg)\,\mathrm{d}y. \tag{4.16}$$

Proof. Consider first the $\gamma \le x < b$ case. By (2.3), $V^{(q)}(x,\gamma,b,B)$ is equal to

$$\int_{y\in B\cap[\gamma,b)} \bigg(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-y) - W_1^{(q)}(x-y)\bigg)\,\mathrm{d}y \tag{4.17}$$

$$+ \int_{y\in B\cap(0,\gamma]} \mathbb{E}_x\bigg[\mathrm{e}^{-q\tau_{1,\gamma}^-}\mathbf{1}_{\{\tau_{1,\gamma}^- < \tau_{1,b}^+\}}\bigg(\frac{W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + (1-\alpha)\gamma))}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma - y)$$
$$- W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + (1-\alpha)\gamma - y)\bigg)\bigg]\,\mathrm{d}y \tag{4.18}$$

$$+ \frac{\mathbb{E}_x[\mathrm{e}^{-q\tau_{1,\gamma}^-}\mathbf{1}_{\{\tau_{1,\gamma}^- < \tau_{1,b}^+\}} W_0^{(q)}(\alpha X_1(\tau_{1,\gamma}^-) + \gamma(1-\alpha))]}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma,\gamma,b,B). \tag{4.19}$$

On the right-hand side above, (4.17) is the discounted time spent in $B \cap [\gamma, b)$ until exit from $[\gamma, b)$, and (4.18) is the discounted time spent in $B \cap (0,\gamma)$ until exit from $(0,\gamma)$. Since the process starts at $x > \gamma$ it first enters $(0,\gamma)$ at $\tau_{1,\gamma}^-$ and then falls to $\alpha X_1(\tau_{1,\gamma}^-) + (1-\alpha)\gamma)$ (partial coverage by the reinsurer prevents fall to $X_1(\tau_{1,\gamma}^-)$). Thus, (4.18) follows from (2.3). Substituting (4.10) in (4.18) and (4.19) shows that $V^{(q)}(x,\gamma,b,B)$ is equal to

$$\int_{y\in B\cap[\gamma,b)} \bigg(\frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} W_1^{(q)}(b-y) - W_1^{(q)}(x-y)\bigg)\mathrm{d}y$$

$$+ \int_{y\in B\cap(0,\gamma]} \bigg(\frac{w_\alpha^{(q)}(x,0) - (W_1^{(q)}(x-\gamma)/W_1^{(q)}(b-\gamma))w_\alpha^{(q)}(b,0)}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma-y)$$
$$- \bigg(w_\alpha^{(q)}(x,y) - \frac{W_1^{(q)}(x-\gamma)}{W_1^{(q)}(b-\gamma)} w_\alpha^{(q)}(b,y)\bigg)\bigg)\,\mathrm{d}y \tag{4.20}$$

$$+ \frac{w_\alpha^{(q)}(x,0) - (W_1^{(q)}(x-\gamma)/W_1^{(q)}(b-\gamma))w_\alpha^{(q)}(b,0)}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma,\gamma,b,B). \tag{4.21}$$

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Substituting $x = \gamma$ and recalling that $w_\alpha^{(q)}(\gamma, y) = W_0^{(q)}(\gamma - y)$ for $0 \le y < \gamma$, shows that $V^{(q)}(\gamma, \gamma, b, B)$ is equal to

$$\frac{W_0^{(q)}(\gamma)}{w_\alpha(b,0)} \int_{y \in B\cap[\gamma,b)} W_1^{(q)}(b-y)\mathrm{d}y + \int_{y\in B\cap(0,\gamma]} \biggl[\frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)} w_\alpha^{(q)}(b,y) - W_0^{(q)}(\gamma - y)\biggr]\mathrm{d}y. \tag{4.22}$$

Substituting (4.22) in (4.21) yields (4.16) for $\gamma \le x < b$.

For $x < \gamma$ we find, by substituting (4.22) in (3.2), that $V^{q)}(x, \gamma, b, B)$ is equal to

$$\begin{aligned}
&\int_{y\in B\cap[0,\gamma)} \biggl[\frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma - y) - W_0^{(q)}(x-y)\biggr]\mathrm{d}y + \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} V^{(q)}(\gamma,\gamma,b,B) \\
&\quad = \int_{y\in B\cap[0,\gamma)} \biggl[\frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} W_0^{(q)}(\gamma - y) - W_0^{(q)}(x-y)\biggr]\mathrm{d}y \\
&\qquad + \frac{W_0^{(q)}(x)}{W_0^{(q)}(\gamma)} \biggl[\frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)} \int_{y\in B\cap[\gamma,b)} W_1^{(q)}(b-y)\,\mathrm{d}y \\
&\qquad\qquad + \int_{y\in B\cap[0,\gamma]} \biggl(-W_0^{(q)}(\gamma-y) + \frac{W_0^{(q)}(\gamma)}{w_\alpha^{(q)}(b,0)} w_\alpha^{(q)}(b,y)\biggr)\,\mathrm{d}y\biggr] \\
&\quad = \int_{y\in B\cap[0,\gamma)} \biggl[\frac{W_0^{(q)}(x)}{w_\alpha^{(q)}(b,0)} w_\alpha^{(q)}(b,y) - W_0^{(q)}(x-y)\biggr]\,\mathrm{d}y \\
&\qquad + \frac{W_0^{(q)}(x)}{w_\alpha^{(q)}(b,0)} \int_{y\in B\cap[\gamma,b)} W_1^{(q)}(b-y)\,\mathrm{d}y.
\end{aligned} \tag{4.23}$$

Since $w_\alpha^{(q)}(x,z) = W_0^{(q)}(x-z)$ for $x < \gamma$, (cf. (4.9)), we obtain a similar expression to (4.16), completing the proof. □

4.4. Limit results for $b \to \infty$

In this subsection we obtain (cf. before (3.8))

$$V^{(q)}(x,\gamma,B) = \lim_{b\to\infty} V^{(q)}(x,\gamma,b,B) = \mathbb{E}_x\biggl[\int_{t=0}^{\infty} \mathrm{e}^{-qt}\,\mathbf{1}_{(U_t\in B,\,t<\kappa_0^-)}\,\mathrm{d}t\biggr]. \tag{4.24}$$

Proposition 4.4. *When $I(x) = \alpha x$, for all $x \ge 0$, $V^{(q)}(x, \gamma, \mathrm{d}y)$ is equal to the sum*

$$\begin{aligned}
&\biggl[\frac{w_\alpha^{(q)}(x,0)}{c(1-\alpha)A}\mathrm{e}^{-\Phi_1(q)y} - W_1^{(q)}(x-y)\biggr]\mathbf{1}_{\{y\in[\gamma,\infty)\}}\,\mathrm{d}y \\
&\quad + \biggl[\frac{w_\alpha^{(q)}(x,0)}{A}\int_{z=\gamma}^{\infty} \mathrm{e}^{-\Phi_1(q)z} W_0^{(q)\prime}(\alpha z - y + \gamma(1-\alpha))\,\mathrm{d}z - w_\alpha^{(q)}(x,y)\biggr]\mathbf{1}_{\{y\in[0,\gamma)\}}\,\mathrm{d}y,
\end{aligned} \tag{4.25}$$

where

$$A = \int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y} W_0^{(q)\prime}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y. \tag{4.26}$$

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Proof. Setting $x = b$ in (4.8) shows that $w_\alpha^{(q)}(b, z)$ is equal to

$$W_0^{(q)}(\alpha b - z + \gamma(1-\alpha)) + c(1-\alpha)\int_\gamma^b W_1^{(q)}(b-y)W_0^{(q)\prime}(\alpha y - z + \gamma(1-\alpha))\,\mathrm{d}y. \quad (4.27)$$

Implicit in (3.8) is the definition $W_{(\Phi(q))}(x) = \mathrm{e}^{-\Phi(q)x}W^{(q)}(x)$. Now applying (3.8) to (4.27) shows that

$$\frac{w_\alpha^{(q)}(b,z)}{W_1^{(q)}(b)} = \frac{\mathrm{e}^{\Phi_0(q)(\alpha b - z + \gamma(1-\alpha))}W_{(\Phi_0(q))}(\alpha b - z + \gamma(1-\alpha))}{\mathrm{e}^{\Phi_1(q)b}W_{(\Phi_1(q))}(b)} \quad (4.28)$$

$$+ c(1-\alpha)\int_\gamma^b \frac{\mathrm{e}^{\Phi_1(q)(b-y)}W_{(\Phi_1(q))}(b-y)}{\mathrm{e}^{\Phi_1(q)b}W_{(\Phi_1(q))}(b)}W_0^{(q)\prime}(\alpha y - z + \gamma(1-\alpha))\,\mathrm{d}y. \quad (4.29)$$

Because $X_0(1) > X_1(1)$, $\psi_0(s) > \psi_1(s)$, and $\Phi_0(q) < \Phi_1(q)$. By Kuznetsov *et al.* (2013, Section 3.1), $W_{(\Phi(q))}(+\infty) = 1/\psi'\big(\Phi(q)\big)$. Thus,

$$\mathrm{e}^{[\Phi_0(q)\alpha - \Phi_1(q)]b} \to 0 \quad \text{as } b \to \infty$$

and the limit as $b \to \infty$ of the expression in (4.28) is 0. The limit of (4.29) is :

$$\frac{\int_\gamma^b W_1^{(q)}(b-y)W_0^{(q)\prime}(\alpha y - z + \gamma(1-\alpha))\mathrm{d}y}{W_1^{(q)}(b)} \to \int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)\prime}(\alpha y - z + \gamma(1-\alpha))\mathrm{d}y \quad \text{as } b \to \infty. \quad (4.30)$$

By (3.13) and (4.26)–(4.30),

$$\frac{W_1^{(q)}(b-y)}{w_\alpha^{(q)}(b,0)} = \frac{W_1^{(q)}(b-y)}{W_1^{(q)}(b)}\bigg/\frac{w_\alpha^{(q)}(b,0)}{W_1^{q)}(b)} \to \frac{\mathrm{e}^{-\Phi_1(q)y}}{c(1-\alpha)A} \quad \text{as } b \to \infty, \quad (4.31)$$

and

$$\frac{w_\alpha^{(q)}(b,y)}{w_\alpha^{(q)}(b,0)} = \frac{w_\alpha^{(q)}(b-y)}{W_1^{(q)}(b)}\bigg/\frac{w_\alpha^{(q)}(b,0)}{W_1^{q)}(b)} \to \frac{\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)\prime}(\alpha y - z + \gamma(1-\alpha))\mathrm{d}y}{A} \quad \text{as } b \to \infty. \quad (4.32)$$

Letting $b \to \infty$ in (4.16), from (4.31) and (4.32), it follows that $V^{(q)}(x, \gamma, B)$ is equal to

$$\int_{y\in B\cap[\gamma,\infty)}\bigg[\frac{w_\alpha^{(q)}(x,0)}{c(1-\alpha)A}\mathrm{e}^{-\Phi_1(q)y} - W_1^{(q)}(x-y)\bigg]\mathrm{d}y$$
$$+ \int_{y\in B\cap[0,\gamma)}\bigg[\frac{w_\alpha^{(q)}(x,0)}{A}\int_{z=\gamma}^\infty \mathrm{e}^{-\Phi_1(q)z}W_0^{(q)\prime}(\alpha z - y + \gamma(1-\alpha))\mathrm{d}z - w_\alpha^{(q)}(x,y)\bigg]\mathrm{d}y.$$

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Thus, (4.25) is proved for all $x \geq 0$:

$$V^{(q)}(x, \gamma, \mathrm{d}y) = \left[\frac{w_\alpha^{(q)}(x, 0)}{c(1-\alpha)A}\mathrm{e}^{-\Phi_1(q)y} - W_1^{(q)}(x-y)\right]\mathbf{1}_{\{y\in[\gamma,\infty)\}}\,\mathrm{d}y + \left[\frac{w_\alpha^{(q)}(x, 0)}{A}\int_{z=\gamma}^{\infty}\mathrm{e}^{-\Phi_1(q)z}W_0^{(q)\prime}(\alpha z - y + \gamma(1-\alpha))\mathrm{d}z - w_\alpha^{(q)}(x, y)\right]\mathbf{1}_{\{y\in[0,\gamma)\}}\,\mathrm{d}y. \tag{4.33}$$

□

4.5. The Laplace transform of the time to ruin

Proposition 4.5. *When $I(x) = \alpha x$, the Laplace transform of the time to ruin, $\mathcal{L}_x(q) = \mathbb{E}_x[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\infty\}}]$, depends on $x \geq$or$<\gamma$ as follows:*

(i) *for $x \geq \gamma$, $\mathcal{L}_x(q)$ is equal to*

$$1 + q\int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y + qc(1-\alpha)\int_{y=\gamma}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y - q\,\frac{w_\alpha^{(q)}(x, 0)\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y}{\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)\prime}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y}; \tag{4.34}$$

(ii) *for $x < \gamma$, $\mathcal{L}_x(q)$ is equal to*

$$1 - q\,\frac{W_0^{(q)}(x)\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y}{A} + q\int_{z=0}^{\gamma} W_0^{(q)}(x-z)\,\mathrm{d}z. \tag{4.35}$$

Proof. Let $\mathcal{E}(q)$ be an exponentially distributed random variable with parameter q. Then

$$\mathcal{L}_x(q) = \mathbb{E}_x[\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\infty\}}] \tag{4.36}$$

$$= \mathbb{P}_x[\mathcal{E}(q) > \kappa_0^-]$$

$$= 1 - \mathbb{P}_x[U_s > 0,\ s < \mathcal{E}(q)]$$

$$= 1 - q\int_0^\infty \mathrm{e}^{-qt}\,\mathbf{1}_{\{(U_s\in(0,\infty)),\,0<s<t)\}}\,\mathrm{d}t$$

$$= 1 - q\int_{y=0}^\infty V^{(q)}(x, \gamma, \mathrm{d}y). \tag{4.37}$$

In the last equality of (4.37) we applied (4.24). To find the last integral we have to integrate (4.25) between 0 and ∞. Observe that

$$\int_{y=0}^{\gamma}\int_{z=\gamma}^{\infty}\mathrm{e}^{-\Phi_1(q)z}W_0^{(q)\prime}(\alpha z - y + \gamma(1-\alpha))\,\mathrm{d}z\,\mathrm{d}y$$

$$= \int_{z=\gamma}^{\infty}\mathrm{e}^{-\Phi_1(q)z}\int_{y=0}^{\gamma}W_0^{(q)\prime}(\alpha z - y + \gamma(1-\alpha))\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_{z=\gamma}^{\infty}\mathrm{e}^{-\Phi_1(q)z}W_0^{(q)}(\alpha z + \gamma(1-\alpha))\,\mathrm{d}z - \int_{z=\gamma}^{\infty}\mathrm{e}^{-\Phi_1(q)z}W_0^{(q)}(\alpha(z-\gamma))\,\mathrm{d}z, \tag{4.38}$$

and

$$\int_{z=\gamma}^{\infty} \mathrm{e}^{-\Phi_1(q)z} W_0^{(q)}(\alpha(z-\gamma))\,\mathrm{d}z = \frac{\mathrm{e}^{-\Phi_1(q)\gamma}}{\alpha} \int_{z=0}^{\infty} \mathrm{e}^{-(\Phi_1(q)/\alpha)z} W_0^{(q)}(z)\,\mathrm{d}z$$
$$= \frac{\mathrm{e}^{-\Phi_1(q)\gamma}}{\alpha(\psi_0(\Phi_1(q)/\alpha)-q)}$$
$$= \frac{\mathrm{e}^{-\Phi_1(q)\gamma}}{c(1-\alpha)\Phi_1(q)}, \tag{4.39}$$

where in the last equality we applied (4.2). Thus, by (4.25), (4.38), and (4.39),

$$\begin{aligned}\int_{y=0}^{\infty} V^{(q)}(x,\gamma,\mathrm{d}y) &= w_\alpha^{(q)}(x,0)\frac{\mathrm{e}^{-\Phi_1(q)\gamma}}{\Phi_1(q)c(1-\alpha)A} - \int_{\gamma}^{\infty} W_1^{(q)}(x-z)\,\mathrm{d}z \\ &\quad + \frac{w_\alpha^{(q)}(x,0)\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y} W_0^{(q)}(\alpha y+\gamma(1-\alpha))\,\mathrm{d}y}{A} \\ &\quad - \frac{w_\alpha^{(q)}(x,0)\mathrm{e}^{-\Phi_1(q)\gamma}}{\Phi_1(q)Ac(1-\alpha)} - \int_{z=0}^{\gamma} w_\alpha^{(q)}(x,z)\,\mathrm{d}z.\end{aligned} \tag{4.40}$$

We check (i); let $x \geq \gamma$. Substitute (4.8) for $w_\alpha^{(q)}(x,z)$ in (4.40), which is simplified by adding and subtracting $\int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y$. This yields

$$\begin{aligned}\mathcal{L}_x(q) &= 1 - q\bigg[-\int_\gamma^\infty W_1^{(q)}(x-z)\,\mathrm{d}z + \frac{w_\alpha^{(q)}(x,0)\int_\gamma^\infty e^{-\Phi_1(q)y} W_0^{(q)}(\alpha y+\gamma(1-\alpha))\,\mathrm{d}y}{A} \\ &\qquad - \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))\,\mathrm{d}z \\ &\qquad - c(1-\alpha)\int_{y=\gamma}^{x} W_1^{(q)}(x-y)\int_{z=0}^{\gamma} W_0^{(q)\prime}(\alpha y - z+\gamma(1-\alpha))\,\mathrm{d}z\,\mathrm{d}y\bigg] \\ &= 1 - q\bigg[-\int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y - \int_\gamma^\infty W_1^{(q)}(x-z)\,\mathrm{d}z \\ &\qquad + \frac{w_\alpha^{(q)}(x,0)\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y} W_0^{(q)}(\alpha y+\gamma(1-\alpha))\,\mathrm{d}y}{A} \\ &\qquad + \int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y - \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z+\gamma(1-\alpha))\,\mathrm{d}z \\ &\qquad - c(1-\alpha)\int_{y=\gamma}^{x} W_1^{(q)}(x-y)\int_{z=0}^{\gamma} W_0^{(q)\prime}(\alpha y - z+\gamma(1-\alpha))\,\mathrm{d}z\,\mathrm{d}y\bigg] \\ &= 1 + q\bigg[\int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y + \int_\gamma^\infty W_1^{(q)}(x-z)\,\mathrm{d}z \\ &\qquad - \frac{w_\alpha^{(q)}(x,0)\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y} W_0^{(q)}(\alpha y+\gamma(1-\alpha))\,\mathrm{d}y}{A} \\ &\qquad - \int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y + \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z+\gamma(1-\alpha))\,\mathrm{d}z\end{aligned}$$

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$$+ c(1-\alpha)\int_{y=\gamma}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha y+\gamma(1-\alpha))\,\mathrm{d}y$$
$$- c(1-\alpha)\int_{y=\gamma}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha(y-\gamma))\,\mathrm{d}y\Big]$$
$$= 1 + q\int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y$$
$$- q\frac{w_\alpha^{(q)}(x,0)\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)}(\alpha y+\gamma(1-\alpha))\mathrm{d}y}{A}$$
$$+ qc(1-\alpha)\int_{y=\gamma}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha y+\gamma(1-\alpha))\,\mathrm{d}y + q\mathcal{C}(x), \tag{4.41}$$

where

$$\mathcal{C}(x) = \int_\gamma^\infty W_1^{(q)}(x-z)\,\mathrm{d}z - \int_0^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y$$
$$+ \int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))\,\mathrm{d}z$$
$$- c(1-\alpha)\int_{y=\gamma}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha(y-\gamma))\,\mathrm{d}y.$$

Now $W_1^{(q)}(v) = 0$ for $v < 0$ so, for the first integral in $\mathcal{C}(x)$, we can write

$$\int_\gamma^\infty W_1^{(q)}(x-z)\,\mathrm{d}z = \int_0^{x-\gamma} W_1^{(q)}(y)\,\mathrm{d}y,$$

and for the last two,

$$\int_{z=0}^{\gamma} W_0^{(q)}(\alpha x - z + \gamma(1-\alpha))\,\mathrm{d}z = \int_{y=\alpha(x-\gamma)}^{\alpha x+\gamma(1-\alpha)} W_0^{(q)}(y)\,\mathrm{d}y,$$
$$\int_{y=\gamma}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha(y-\gamma))\,\mathrm{d}y = \int_{y=0}^{x-\gamma} W_1^{(q)}(x-\gamma-y)W_0^{(q)}(\alpha y)\,\mathrm{d}y.$$

This enables us to write $\mathcal{C}(x) = g(x-\gamma)$, where

$$g(x) = \int_0^x W_1^{(q)}(y)\,\mathrm{d}y - \int_0^{\alpha x} W_0^{(q)}(y)\,\mathrm{d}y - c(1-\alpha)\int_{y=0}^{x} W_1^{(q)}(x-y)W_0^{(q)}(\alpha y)\,\mathrm{d}y.$$

For the Laplace transform $\int_0^\infty \mathrm{e}^{-sx}g(x)\,\mathrm{d}x$, we obtain

$$\int_0^\infty \mathrm{e}^{-sx}g(x)\,\mathrm{d}x = \frac{1}{s(\psi_1(s)-q)} - \frac{1}{s(\psi_0(s/\alpha)-q)} - \frac{c(1-\alpha)/\alpha}{(\psi_0(s/\alpha)-q)(\psi_1(s)-q)}.$$

Applying (4.2) we conclude that the last expression is equal to 0. Thus, $g(x) = 0$ and (4.34) follows from (4.41).

(ii) For $x < \gamma$, from (4.40) or directly from (4.25), it follows that $\mathcal{L}_x(q)$ is equal to

$$1 - q\frac{W_0^{(q)}(x)\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)}(\alpha y+\gamma(1-\alpha))\,\mathrm{d}y}{A} + q\int_{z=0}^{\gamma} W_0^{(q)}(x-z)\,\mathrm{d}z,$$

completing the proof. □

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4.6. Ruin probability

In this section we find explicitly the ruin probability $\varpi_x := \mathbb{P}_x(\kappa_0^- < \infty)$. We do this by recalling the Laplace transform in Proposition 4.5 and appealing to monotone convergence:

$$\varpi_x = \lim_{q\downarrow 0} \mathcal{L}_x(q) = \lim_{q\downarrow 0} \mathbb{E}_x[\mathrm{e}^{-q\kappa_0^-} \mathbf{1}_{\{\kappa_0^- < \infty\}}].$$

Proposition 4.6. *Let $I(x) = \alpha x$. When $\psi_1'(0) \le 0$, the ruin probability $\varpi_x = 1$ for all x. When $\psi_1'(0) > 0$ and $\beta := \alpha\psi_1'(0)/[1 - c(1-\alpha)W_0(\gamma)]$, we have*

(i) *if $x > \gamma$ then*

$$\varpi_x = 1 - \bigg(W_0(\alpha x + \gamma(1-\alpha)) + c(1-\alpha)\int_\gamma^x W_1(x-y)W_0'(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y\bigg)\beta; \tag{4.42}$$

(ii) *if $x \le \gamma$ then*

$$\varpi_x = 1 - W_0(x)\beta. \tag{4.43}$$

Proof. (i) Irrespective of $\psi_1'(0)$, we obtain the ruin probability ϖ_x for $x \ge \gamma$ by evaluating the limit of (4.34) as $q \downarrow 0$. The limit of the last two terms in the first line of (4.34) is 0.

To evaluate the limit of the last expression in (4.34) observe that

$$\begin{aligned} &q\frac{w_\alpha^{(q)}(x,0)\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y}{\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)\prime}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y} \\ &\quad= qw_\alpha^{(q)}(x,0)\frac{1}{\alpha}\mathrm{e}^{\Phi_1(q)\gamma(1-\alpha)/\alpha}\frac{\int_0^\infty \mathrm{e}^{-\Phi_1(q)y/\alpha}W_0^{(q)}(y)\,\mathrm{d}y - \int_0^\gamma \mathrm{e}^{-\Phi_1(q)y/\alpha}W_0^{(q)}(y)\,\mathrm{d}y}{\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)\prime}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y} \\ &\quad= w_\alpha^{(q)}(x,0)\frac{1}{\alpha}\mathrm{e}^{\Phi_1(q)\gamma(1-\alpha)/\alpha}\frac{q/[\psi_0(\Phi_1(q)/\alpha) - q] - q\int_0^\gamma \mathrm{e}^{-\Phi_1(q)y/\alpha}W_0^{(q)}(y)\,\mathrm{d}y}{\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)\prime}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y} \\ &\quad= w_\alpha^{(q)}(x,0)\frac{1}{\alpha}\mathrm{e}^{\Phi_1(q)\gamma(1-\alpha)/\alpha}\frac{q/\delta\Phi_1(q) - q\int_0^\gamma \mathrm{e}^{-\Phi_1(q)y/\alpha}W_0^{(q)}(y)\,\mathrm{d}y}{\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)\prime}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y}, \end{aligned} \tag{4.44}$$

where we have used (4.2) in the last line of (4.44). If $\psi_1'(0) < 0$ then $\Phi_1(0) > 0$, the limit of the last expression is 0 and thus the ruin probability is 1. If $\psi_1'(0) \ge 0$ then $\Phi_1(0) = 0$, in which case

$$\lim_{q\downarrow 0}\frac{q}{\Phi_1(q)} = \psi_1'(0) = \frac{\psi_0'(0)}{\alpha} - \frac{1}{\alpha}c(1-\alpha).$$

Next, consider the denominator in (4.44) where integration by parts yields

$$\begin{aligned} A &= \int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y}W_0^{(q)\prime}(\alpha y + \gamma(1-\alpha))\,\mathrm{d}y \\ &= \frac{1}{\alpha}\mathrm{e}^{\Phi_1(q)\gamma(1-\alpha)/\alpha}\int_\gamma^\infty \mathrm{e}^{-\Phi_1(q)y/\alpha}W_0^{(q)\prime}(y)\,\mathrm{d}y \end{aligned} \tag{4.45}$$

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$$= \frac{1}{\alpha}\mathrm{e}^{\Phi_1(q)\gamma(1-\alpha)/\alpha}\biggl[-W_0^{(q)}(\gamma)\mathrm{e}^{-\Phi_1(q)\gamma/\alpha} + \frac{\Phi_1(q)}{\alpha}\int_0^\infty \mathrm{e}^{-\Phi_1(q)y/\alpha}W_0^{(q)}(y)\,\mathrm{d}y - \frac{\Phi_1(q)}{\alpha}\int_0^\gamma \mathrm{e}^{-\Phi_1(q)y/\alpha}W_0^{(q)}(y)\,\mathrm{d}y\biggr]$$

$$= \frac{1}{\alpha}\mathrm{e}^{\Phi_1(q)\gamma(1-\alpha)/\alpha}\biggl[-W_0^{(q)}(\gamma)\mathrm{e}^{-\Phi_1(q)\gamma/\alpha} + \frac{\Phi_1(q)}{\alpha}\frac{1}{c(1-\alpha)\Phi_1(q)/\alpha} - \frac{\Phi_1(q)}{\alpha}\int_0^\gamma \mathrm{e}^{-\Phi_1(q)y/\alpha}W_0^{(q)}(y)\,\mathrm{d}y\biggr], \tag{4.46}$$

where in the last equality we have applied (2.1) and (4.2). The limit of the denominator of (4.44) as $q \downarrow 0$ is

$$\frac{1}{\alpha}\biggl(-W_0(\gamma) + \frac{1}{c(1-\alpha)}\biggr). \tag{4.47}$$

Thus, when $\Phi_1(0) = 0$, substituting (4.8) in (4.44), and applying (4.8) and (4.47), we conclude that the ruin probability for $x > \gamma$ is as in (4.42).

(ii). Taking the limit of (4.35) as $q \downarrow 0$ in the $x < \gamma$ case shows that the ruin probability is 1 when $\psi_1'(0) < 0$ and otherwise is given by (4.43). □

4.7. Gerber–Shiu penalty function

As in Proposition 3.3 let $-U_{\kappa_0^-}$ be the deficit at ruin and $U_{\kappa_{0-}^-}$ the surplus just before ruin. We want to find for a nonnegative function $h(x, y)$ the Gerber–Shiu penalty function

$$m(x, q) = \mathbb{E}_x[\mathrm{e}^{-q\kappa_0^-}h(U_{\kappa_{0-}^-}, |U_{\kappa_0^-}|) \mid U_0 = x].$$

Proposition 4.7. *When $I(x) = \alpha x$, the Gerber–Shiu penalty function $m(x, q)$ is equal to*

$$\begin{aligned}
&\int_{y=0}^{\gamma} V^{(q)}(x,\gamma,\mathrm{d}y)\int_{z=0}^\infty h(y,z)\lambda\,\mathrm{d}F_0(y+z)\\
&\quad + \int_{y=\gamma}^{\infty} V^{(q)}(x,\gamma,\mathrm{d}y)\int_{z=0}^\infty h(y,z)\lambda\,\mathrm{d}F_1\biggl(y-\gamma+\frac{\gamma+z}{\alpha}\biggr)\\
&\quad = \int_{y=0}^{\gamma}\biggl(\frac{w_\alpha^{(q)}(x,0)}{A}\int_{s=\gamma}^\infty \mathrm{e}^{-\Phi_1(q)s}W_0^{(q)\prime}(\alpha s - y + \gamma(1-\alpha))\,\mathrm{d}s - w_\alpha^{(q)}(x,y)\biggr)\\
&\qquad\qquad \times \int_{z=0}^\infty h(y,z)\lambda\,\mathrm{d}F_0(y+z)\,\mathrm{d}y\\
&\qquad + \int_\gamma^\infty\biggl(\frac{w_\alpha^{(q)}(x,0)}{c(1-\alpha)A}\mathrm{e}^{-\Phi_1(q)y} - W_1^{(q)}(x-y)\biggr)\int_{z=0}^\infty \lambda\,\mathrm{d}F_1(y-\gamma+\frac{\gamma+z}{\alpha})h(y,z)\,\mathrm{d}y.
\end{aligned} \tag{4.48}$$

Proof. This is the same as for the more general Proposition 3.3, where now we substitute from (4.25) for $V^{(q)}(x, \gamma, \mathrm{d}y)$. □

As an application of the Gerber–Shiu penalty function, we derive the joint distribution of the reserve just before ruin and the deficit at ruin when $\psi_1'(0) \ge 0$, i.e. when $\Phi_1(0) = 0$. Take sets $C, D \subset (0, \infty)$, and let $h^*(y, z) = \mathbf{1}_{(y\in C,\, z\in D)}$. Then to find $\mathbb{P}_x(U_{\kappa_{0-}^-} \in C,\ |U_{\kappa_0^-}| \in D)$,

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substitute h^* in (4.48) and take the limit as $q \downarrow 0$. By (4.46), $\lim_{q\downarrow 0} A$ is given by (4.47). Similarly,

$$\lim_{q\downarrow 0}\int_{s=\gamma}^{\infty} \mathrm{e}^{-\Phi_1(q)s} W_0^{(q)\prime}(\alpha s - y + \gamma(1-\alpha))\,\mathrm{d}s = \frac{1 - c(1-\alpha)W_0(\gamma - y)}{\alpha c(1-\alpha)}.$$

Let $w_\alpha(x, y) = w_\alpha^{(0)}(x, y)$. Since $\Phi_1(0) = 0$, substituting h^* in (4.48) and taking the limit $q \downarrow 0$ leads to

$$\begin{aligned}
&\mathbb{P}_x(U_{\kappa_{0-}^-} \in C,\ |U_{\kappa_0^-}| \in D) \\
&\quad= \int_{y\in C\cap(\gamma,\infty)} \bigg(\frac{\alpha w_\alpha(x,0)}{1 - c(1-\alpha)W_0(\gamma)} - W_1(x-y)\bigg)\Pi_1\bigg(y - \gamma + \frac{\gamma + D}{\alpha}\bigg)\,\mathrm{d}y \\
&\qquad+ \int_{y\in B\cap(0,\gamma)} \bigg(w_\alpha(x,0)\frac{1 - c(1-\alpha)W_0(\gamma - y)}{1 - c(1-\alpha)W_0(\gamma)} - w_\alpha(x,y)\bigg)\Pi_0(y + D)\,\mathrm{d}y,
\end{aligned}$$

where $\Pi_i(B)$ is the Lévy measure of the set B; in the compound Poisson case $\Pi_i(B) = \lambda F_i(B) = \lambda\mathbb{P}(Z^i \in B)$ when Z^i has distribution F_i, $i = 0, 1$.

5. Conclusions

In this paper we studied a compound Poisson risk process in the case that claims are ‘refracted’, i.e. only a part of the claim is paid when the reserve is less than γ. We obtained expressions for the Laplace transform of the exit time from an upper barrier, the time to ruin, and the joint probability for the surplus before and at ruin, for a general function $I(x)$ as defined in the Introduction. We obtained relatively simple expressions for the special but important case that $I(x) = \alpha x$. In this case the results have the same flavour as for spectrally negative refracted Lévy processes, where the premium income rate when the reserve exceeds γ is αc; Kyprianou and Loeffen (2010) studied this case.

We analyzed the model for the compound Poisson risk process; the same analysis holds for the more general spectrally negative bounded variation Lévy risk process.

It would be worthwhile to consider more general reinsurance policies, for example $I(x) = \min(a, x)$, where a is a positive constant.

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