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A PROBLEM OF COMPLETE INTERSECTIONS

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Let X be a non-singular projective surface in $\boldsymbol{P}_k^3$ (k an algebraically closed field of characteristic 0) and C an irreducible curve, which is a set-theoretically complete intersection in X; is it true that C is actually a complete intersection in X ?

In this paper we give a positive answer even in a more general hypothesis.

We note that a similar question does not arise for a variety X with $\dim X \neq 2$. In fact Lefschetz theorem says that, if X is a non-singular projective variety which is a complete intersection in $\boldsymbol{P}_k^N$ and such that $\dim X \geqslant 3$, *any* positive divisor on X is a complete intersection in X.

On the other hand, if X is a non-singular conic in $\boldsymbol{P}_k^2$ and P a point on X, then P is a set-theoretically complete intersection but not a complete intersection in X.

As to the surfaces, it is a well known fact that on a "general" surface of degree $\geqslant 4$ in $\boldsymbol{P}_k^3$ *any* curve is a complete intersection, but there are surfaces whose Picard group is different from $\boldsymbol{Z}$ (e.g. non-singular quadric and cubic surfaces) (see [4]).

Nevertheless no example is known of an irreducible curve on a non-singular surface in $\boldsymbol{P}_k^3$, which is a set-theoretically complete intersection in X, but not a complete intersection in X (see [1]), and in fact we are going to prove that such an example cannot exist.

For this we make use of the techniques developed by Grothendieck to prove Lefschetz theorem (see [2] and [3]).

We now state the following

THEOREM. *Let k be an algebraically closed field of characteristic* 0 *and let $X \subset \boldsymbol{P}_k^N$ be a non-singular projective surface, which is a complete intersection. If C is an irreducible curve on X, which is a set-theoreti-*

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cally complete intersection in X, then C is actually a complete intersection in X.

Proof. We shall give the proof in several steps.

Step 1. $\text{Pic}(\boldsymbol{P}^N) \simeq \text{Pic}(\widehat{\boldsymbol{P}^N})$, where $\boldsymbol{P}^N$ stands for $\boldsymbol{P}_k^N$ and $\widehat{\boldsymbol{P}^N}$ denotes the formal completion of $\boldsymbol{P}^N$ along X.
The proof is in [3] Ch. IV (essentially Th. 1.5 and Th. 3.1).

Step 2. X is projectively normal.
The proof is in [6] n. 77, 78, p. 272–273.

Step 3. $\text{Pic}(X)$ is a finitely generated group.
Indeed $H^1(X, \mathcal{O}_X) = 0$ (see [6] n. 78 p. 273–274), hence $\text{Pic}^0(X)$ is just a point and, calling $NS(X)$ the Neron-Severi group of X, we get $\text{Pic}(X) = NS(X)$ which is finitely generated by classical results.

Step 4. $\text{Pic}(X)$ is torsion-free, hence by step 3 $\text{Pic}(X)$ is a finitely generated free group.
Let $\mathcal{T}$ be the sheaf of ideals defining X and call X_n the scheme $(X, \mathcal{O}_{\boldsymbol{P}^N}/\mathcal{T}^n)$. We can use the exact sequences

$$0 \to \mathcal{T}^{n-1}/\mathcal{T}^n \to (\mathcal{O}_{\boldsymbol{P}}/\mathcal{T}^n)^* \to (\mathcal{O}_{\boldsymbol{P}}/\mathcal{T}^{n-1})^* \to 0$$

where $*$ denotes the multiplicative group of units and the first map sends x to $1 + x$ (for more details see [3] Ch. 4 p. 179 and [2] Exp II p. 124). We get long exact sequences

$$(1) \qquad \begin{aligned} \cdots \longrightarrow H^1(\boldsymbol{P}^N, \mathcal{T}^{n-1}/\mathcal{T}^n) \longrightarrow \text{Pic}(X_n) \xrightarrow{\varphi_n} \text{Pic}(X_{n-1}) \longrightarrow \\ \longrightarrow H^2(\boldsymbol{P}^N, \mathcal{T}^{n-1}/\mathcal{T}^n) \longrightarrow \cdots \end{aligned}$$

But $\mathcal{T}^{n-1}/\mathcal{T}^n \simeq \bigoplus_i \mathcal{O}_X(m_i)$ for suitable integers m_i (see [3] proof of coroll. 3.1. p. 180). Hence $H^1(\boldsymbol{P}^N, \mathcal{T}^{n-1}/\mathcal{T}^n) = H^1(X, \mathcal{T}^{n-1}/\mathcal{T}^n) = 0$ (see [6] n. 78 p. 273–274).

On the other hand $H^2(\boldsymbol{P}^N, \mathcal{T}^{n-1}/\mathcal{T}^n)$ is a vector space over a field of characteristic 0, hence torsion-free. If T_n denotes the torsion subgroup of $\text{Pic}(X_n)$ and $T = T_1$, we get $T_n = T_{n-1} = \cdots = T$. Hence $T = \text{Tors}(\varprojlim \text{Pic}(X_n)) = \text{Tors}\,\text{Pic}(\widehat{\boldsymbol{P}^N}) = \text{Tors}\,\text{Pic}(\boldsymbol{P}^N) = 0$.

Step 5. $\varprojlim \text{Pic}(X_n) \simeq \text{Pic}(X_{n_0}) \quad$ for $n_0 \gg 0$.
From the proof of step 4 we get that $\text{Pic}(X_n) \simeq \boldsymbol{Z}^{\rho_n}(\rho_n = \text{rank}(\text{Pic}(X_n)))$, the canonical map $\text{Pic}(X_n) \xrightarrow{\varphi_n} \text{Pic}(X_{n-1})$ is injective, and $\text{coker}\,\varphi_n$ is torsion-free. Hence via φ_n $\text{Pic}(X_n)$ is a direct factor subgroup of $\text{Pic}(X_{n-1})$ and therefore φ_n must be an isomorphism for n large.

Step 6. $[\mathcal{O}_X(1)]$ belongs to a basis of the free group Pic (X). If $\mathscr{L}$ is an invertible sheaf on a scheme, we call $[\mathscr{L}]$ its class in the Picard group. It is well-known that Pic $(\boldsymbol{P}^N) \simeq \boldsymbol{Z}$ is generated by $[\mathcal{O}_{\boldsymbol{P}^N}(1)]$; since by the previous steps we can write the following exact sequence

$$\boldsymbol{Z} \simeq \text{Pic}\,(\boldsymbol{P}^N) \simeq \text{Pic}\,(\widehat{\boldsymbol{P}^N}) \simeq \text{Pic}\,(X_{n_0}) \xrightarrow{\varphi_{n0}} \cdots \longrightarrow \text{Pic}\,(X) \simeq \boldsymbol{Z}^\rho$$

where the maps are canonical, the composite map from Pic $(\boldsymbol{P}^N)$ to Pic (X) sends $[\mathcal{O}_{\boldsymbol{P}^N}(1)]$ to $[\mathcal{O}_X(1)]$ and, since Pic (X_n) is a direct factor subgroup of Pic (X_{n-1}), we are through.

Step 7. If $\mathscr{L}$ is an invertible sheaf on X, q, n integers and $[q\mathscr{L}] = [\mathcal{O}_X(n)]$, then there exists an integer r such that $n = qr$ and $[\mathscr{L}] = [\mathcal{O}_X(r)]$.

Indeed, by step 6, $[\mathcal{O}_X(1)]$ belongs to a basis of Pic (X); let $[\mathcal{O}_X(1)], [\mathscr{L}_2], [\mathscr{L}_3], \cdots, [\mathscr{L}_\rho]$ be such a basis, then $[\mathscr{L}] = r[\mathcal{O}_X(1)] + \sum_i r_i[\mathscr{L}_i]$ hence $[q\mathscr{L}] = [\mathcal{O}_X(qr)] + \sum_i [r_i q\mathscr{L}_i]$. But $[q\mathscr{L}] = [\mathcal{O}_X(n)]$ and therefore $qr = n, r_i = 0$.

Step 8 (conclusion). Let C be an irreducible curve on X, which is a set-theoretically complete intersection in X, and let $\mathcal{O}_X(C)$ be the associated invertible sheaf. Then $\mathcal{O}_X(qC) \simeq \mathcal{O}_X(n)$ or, which is the same, $[q\mathcal{O}_X(C)] = [\mathcal{O}_X(n)]$. By step 7 we get $[\mathcal{O}_X(C)] = [\mathcal{O}_X(r)]$; combining with step 2 we are done.

COROLLARY. *Let k be an algebraically closed field of characteristic 0 and let A be the homogeneous coordinate ring of a non-singular projective surface which is a complete intersection in $\boldsymbol{P}_k^N$. Then if A is almost factorial, A is factorial.*

Proof. We recall that a ring A is called almost factorial ("fastfaktoriell" in German) if A is a Krull domain and the divisor class group $C(A)$ is torsion (see [7]) and that for investigating $C(A)$ it is sufficient to consider homogeneous ideals (see [5] n° 2). Let now $\mathfrak{P}$ be a homogeneous prime ideal of height 1. Since A is almost factorial, $\mathfrak{P} = \sqrt{(F)}$, F being a suitable homogeneous element. The irreducible curve associated to $\mathfrak{P}$ is therefore a set-theoretically complete intersection, hence a complete intersection by the theorem, and so $\mathfrak{P}$ is principal.

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