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Non-local effects in an integro-PDE model from population genetics

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In this paper, we study the following non-local problem:

$$\begin{cases} u_t = d\frac{1}{\rho}\nabla\cdot(\rho V\nabla u) + b(\bar{u}-u) + g(x)u^2(1-u) & \text{in } \Omega\times(0,\infty),\\ 0\leqslant u\leqslant 1 & \text{in } \Omega\times(0,\infty),\\ \nu\cdot V\nabla u = 0 & \text{on } \partial\Omega\times(0,\infty). \end{cases}$$

This model, proposed by T. Nagylaki, describes the evolution of two alleles under the joint action of selection, migration, and *partial panmixia – a non-local term*, for the *complete dominance* case, where $g(x)$ is assumed to change sign at least once to reflect the diversity of the environment. First, properties for general non-local problems are studied. Then, existence of non-trivial steady states, in terms of the diffusion coefficient d and the partial panmixia rate b, is obtained under different signs of the integral $\int_\Omega g(x)dx$. Furthermore, stability and instability properties for non-trivial steady states, as well as the trivial steady states $u \equiv 0$ and $u \equiv 1$ are investigated. Our results illustrate how the non-local term – namely, the partial panmixia – helps the migration in this model.

Key words: partial panmixia, non-local effects, non-trivial steady states, stability

1 Introduction

The aim of this paper is to study a genetic model of two alleles with partial panmixia in the complete dominance case.

To motivate our studies, first recall the model dealing with two types of genes (alleles) A_1, A_2 as follows:

$$\begin{cases} u_t = d\Delta u + g(x)u(1-u)[hu+(1-h)(1-u)] & \text{in } \Omega\times(0,\infty),\\ 0\leqslant u\leqslant 1 & \text{in } \Omega\times(0,\infty),\\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 1$, ν is the unit outer normal vector to $\partial\Omega$, $g(x)$ changes signs in Ω and $0 \leqslant h \leqslant 1$ is a constant. In this model, $u(x,t)$ represents the frequency of the allele A_1 at location x and time t. Thus, $1 - u(x,t)$ denotes that of the allele A_2 and therefore only solutions with $0 \leqslant u \leqslant 1$ are under consideration. Moreover, the allele A_1 has selective advantage in the region where $g(x) > 0$ and selective disadvantage where $g(x) < 0$. The assumption that $g(x)$ changes sign in Ω signifies that the environment is so heterogeneous that the selection changes its direction at least once in Ω. (See [3,15,23] for more details of the derivation for the model.)

The constant h represents the degree of dominance which plays an important role in determining the qualitative properties of solutions to equation (1.1). For the case without dominance, i.e. $0 < h < 1$, the existence of non-trivial steady states has been studied in [3] and [21]. Under the more stringent condition $1/3 \leqslant h \leqslant 2/3$, the uniqueness of non-trivial steady states was verified in [1,5] and [10]. However, the case with complete dominance, i.e. $h = 1$, which means that A_2 is completely dominant, or similarly $h = 0$, which implies that A_1 is completely dominant, seems more challenging. Because in this case the linearized problem at one of the trivial steady states always has zero as the principal eigenvalue, which makes it impossible to determine the local stability based on linearized analysis and different approaches are needed. It remained completely open until 2010 when progress was finally made in [12,19] concerning the existence and stabilities properties of non-trivial steady states; however, to this date, the uniqueness is still not resolved.

Furthermore, in the model (1.1), the term Δu represents population dispersal, which corresponds to the diffusion approximation for short-distance migration. From the assumption that most species have a small portion of long-distance migrants, recently partial panmixia is introduced as the limiting case of long-distance migration. See [16–18] and the references therein. More precisely, in [17], if $\rho(x) > 0$ on $\overline{\Omega}$ is the normalized population density, i.e. $\int_\Omega \rho(x)dx = 1$, and, for any function $P(x,t)$, its averages with respect to $\rho(x)$ is denoted by

$$\bar{P}(t) = \int_\Omega P(x,t)\rho(x)dx,$$

then global panmixia is represented by the term $B(\bar{u} - u)$, where B is the scaled panmictic rate. Some general results concerning the effect of incorporating partial panmixia into single-locus clines maintained by migration and selection were also established in [17]. These analyses have been extended in [11]. Among other things, in [11], the effect of the parameter B on the dynamics of the model with conservative migration

$$\begin{cases} u_t = \dfrac{1}{\rho}\nabla \cdot (\rho V \nabla u) + B(\bar{u} - u) + \lambda g(x) f(u) & \text{in } \Omega \times (0,\infty), \\ 0 \leqslant u \leqslant 1 & \text{in } \Omega \times (0,\infty), \\ \nu \cdot V\nabla u = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.2}$$

is studied for the case without dominance. Here, $V(x)$ denotes the $N \times N$ symmetric, positive definite, single-generation covariance matrix of local migration, the positive constant λ represents the selection intensity, and f satisfies

$$f \in C^2([0,1]),\ f(0) = f(1) = 0,\ f > 0 \text{ in } (0,1) \text{ and } f'(0) > 0 > f'(1). \tag{1.3}$$

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In this paper, we consider the complete dominance case with partial panmixia

$$\begin{cases} u_t = d\frac{1}{\rho}\nabla\cdot(\rho V\nabla u) + b(\bar{u}-u) + g(x)u^2(1-u) & \text{in } \Omega\times(0,\infty), \\ 0 \leqslant u \leqslant 1 & \text{in } \Omega\times(0,\infty), \\ v\cdot V\nabla u = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1.4}$$

where, in comparison to the model (1.2), $d > 0$ and $b > 0$ are the ratios of the local and non-local migration rates to the intensity of selection respectively. It is clear that the function $u^2(1-u)$ in the reaction term of model (1.4) does not satisfies (1.3).

Remark 1.1 *Notice that the choices of parameters in model (1.4) are different from those in (1.2). This is because we are interested in studying the interaction between local and non-local diffusion terms in this paper.*

In this paper, we will always assume that

(A) $\rho(x) > 0$ in $\overline{\Omega}$, $\int_\Omega \rho(x)dx = 1$, $\rho(x) \in C^{1,\alpha}(\bar{\Omega})$, for some $0 < \alpha < 1$. The matrix $V(x) = (v_{ij}(x))_{1\leqslant i,j\leqslant N}$ designates an $N\times N$ symmetric, positive definite matrix, uniformly with respect to x, with $v_{ij}(x) \in C^{1,\alpha}(\bar{\Omega})$, i.e. there exists a positive constant κ such that

$$\xi^\top\cdot V(x)\xi \geqslant \kappa|\xi|^2 \quad \text{for} \quad \forall\xi\in\mathbb{R}^N,\ x\in\Omega.$$

(A1) $g(x)\in C^\alpha(\bar{\Omega})$, and changes sign in Ω.

As a preliminary for our study of the model (1.4), in the first part of this paper, we consider a general non-local problem as follows:

$$\begin{cases} u_t = d\frac{1}{\rho}\nabla\cdot(\rho V\nabla u) + f(x,u,\bar{u}) & \text{in } \Omega\times(0,T), \\ v\cdot V\nabla u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

and always impose the conditions (A) and

(A2) $f(x,u,\bar{u})$ is C^α for $x\in\bar{\Omega}$ and C^1 in u and $\bar{u}$.

Several basic properties of (1.5) will be discussed. First, the local existence and uniqueness of (1.5), as well as backward uniqueness, will be established in Theorems 2.1 and 2.2 respectively. Then in Section 2.3, we will demonstrate that the comparison principle holds for problem (1.5) with $f_{\bar{u}} \geqslant 0$. We will also present an example, which shows that problem (1.5) might not admit comparison principles if the condition $f_{\bar{u}} \geqslant 0$ is violated. Finally, in Section 2.4, it is proved that if problem (1.5) admits a Lyapunov functional, then its ω-limit set $\omega[u_0]$ consists of only steady states. Without non-local terms, these results are classical for semi-linear parabolic equations, the proofs are also quite standard. The details will be included for the convenience of readers.

In the second part of this paper, we will focus on model (1.4). In fact, we will study a more general problem than (1.4) as follows:

$$\begin{cases} u_t = d\frac{1}{\rho}\nabla\cdot(\rho V\nabla u) + b(\bar{u}-u) + g(x)f(u) & \text{in } \Omega\times(0,\infty), \\ 0 \leqslant u \leqslant 1 & \text{in } \Omega\times(0,\infty), \\ \nu\cdot V\nabla u = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1.6}$$

where it is assumed that

(A3) $f \in C^1([0,1])$, $f(0) = f(1) = 0$, $f > 0$ in $(0,1)$.

It is worth pointing out that under assumption (A3), f could be degenerate of any order at 0 and/or 1.

Let $u^*(x)$ denote a steady state of (1.6) with $0 \leqslant u^* \leqslant 1$, by the maximum principle, it is easy to see that there are only two possibilities:

- $u^*(x) \equiv 0$ or $u^*(x) \equiv 1$, i.e. *a trivial steady state*;
- $0 < u^*(x) < 1$ in $\bar{\Omega}$. In this case, u^* must be non-constant, i.e. *a non-trivial steady state.*

First, the existence and non-existence properties of non-trivial steady states of (1.6) will be investigated. For existence, our main result can be stated in two cases depending on $\bar{g}$ as follows.

Theorem 1.1 *Suppose that (A), (A1) and (A3) hold.*

(i) $\bar{g} = 0$: *Then, (1.6) admits a stable non-trivial steady state for any* $d, b > 0$.

(ii) $\bar{g} \neq 0$: *Then, (1.6) admits a stable non-trivial steady state for* $d, b > 0$ *small.*

For non-existence, we present our main result in three cases depending on $\bar{g}$.

Theorem 1.2 *Suppose that (A), (A1) and (A3) hold.*

(i) $\bar{g} \neq 0$: *If f satisfies that*

$$\lim_{s\to 0^+}\frac{f(s)}{s^{k_1}} = a_1 > 0 \text{ for some } k_1 \geqslant 1, \text{ and} \tag{1.7}$$

$$\lim_{s\to 1^-}\frac{f(s)}{(1-s)^{k_2}} = a_2 > 0 \text{ for some } k_2 \geqslant 1, \tag{1.8}$$

then there does not exist non-trivial steady state of (1.6) for $d + b$ *large.*

(ii) $\bar{g} > 0$: *If f satisfies (1.8) and*

$$f'(s) \geqslant 0 \text{ in } (0,\delta_0) \text{ for some } \delta_0 > 0, \tag{1.9}$$

then (1.6) has no non-trivial steady state for $d + b$ *large.*

(iii) $\bar{g} < 0$: *If f satisfies (1.7) and*

$$f'(s) \leqslant 0 \text{ in } (1-\delta_0, 1) \text{ for some } \delta_0 > 0, \tag{1.10}$$

then (1.6) does not admit non-trivial steady state for $d + b$ *large.*

Remark 1.2 *Theorems 1.1 and 1.2 confirm our belief that partial panmixia seems to further enhance the effects of random dispersal; in fact, it seems that they play similar roles in this selection-migration model. However, the proofs are much more involved now and require new ideas. We will address these difficulties more specifically in Section 3.*

Next, we analyze the stability properties of the two trivial steady states $u \equiv 0$ and $u \equiv 1$ of (1.6). In the following, let $u(\cdot, t; u_0)$ denote the unique classical solution of (1.6) with initial value $u_0 \in C(\bar{\Omega})$. Since for problem (1.6), only solutions with values in $[0, 1]$ are in consideration, define

$$\mathcal{K} := \{u \in C(\bar{\Omega}) : \ 0 \leqslant u(x) \leqslant 1, \ \forall \ x \in \bar{\Omega}\}. \tag{1.11}$$

By maximum principle, it is easy to show that $u_0 \in \mathcal{K}$ implies that $u(\cdot, t; u_0) \in \mathcal{K}$. Therefore, we are mainly interested in the stability of the steady state u^* relative to $\mathcal{K}$. We say u^* is *stable* relative to $\mathcal{K}$ if for each $\epsilon > 0$ there is $\delta > 0$ such that for $u_0 \in \mathcal{K}$ and $\|u_0 - u^*\|_{L^\infty} \leqslant \delta$, the solution $\|u(\cdot, t; u_0) - u^*\|_{L^\infty} \leqslant \epsilon$ for all $t > 0$. It is said to be *unstable* if it is not stable. In particular, $u \equiv 0$ is stable relative to $\mathcal{K}$ if and only if $u \equiv 0$ is stable from above and similarly $u \equiv 1$ is stable relative to $\mathcal{K}$ if and only if $u \equiv 1$ is stable from below. Moreover, we say u^* is *linearly stable* if the linearized operator $\mathcal{L}$ of (1.6) at u^* only has eigenvalues with negative real parts. It is said to be *linearly unstable* if $\mathcal{L}$ has an eigenvalue with positive real part.

We discuss the stability of $u \equiv 0$ first. It is worth pointing out that, compared with model (1.2), the key feature in model (1.4) is that the function $u^2(1 - u)$ in the reaction term is degenerate at 0. Thus, the condition $f'(0) = 0$ will be imposed while studying the stability of $u \equiv 0$. The main result here is as follows.

Theorem 1.3 *Under the assumptions (A), (A1) and (A3), the following statements hold for problem (1.6).*

(i) $\bar{g} = 0$: *If (1.9) holds with* $|\{s \in (0, \delta_0) : \ f'(s) = 0\}| = 0$ *for some* $\delta_0 > 0$*, then* $u \equiv 0$ *is unstable for any* $d, b > 0$.

(ii) $\bar{g} > 0$: *Then,* $u \equiv 0$ *is unstable for any* $d, b > 0$ *provided that either (1.7) or (1.9) holds.*

(iii) $\bar{g} < 0$: *If* $f'(0) = 0$*, (1.7) holds and either (1.8) or (1.10) is satisfied, then* $u \equiv 0$ *is stable for any* $d, b > 0$.

The proof of Theorem 1.3 is based on the techniques devised in proving [12, Theorem 1.1] – a combination of variational method and degree theory.

About the stability of $u \equiv 1$, we present two results. The first one is a dual version of Theorem 1.3.

Theorem 1.4 *Under the assumptions (A), (A1) and (A3), the following statements hold for problem (1.6).*

(i) $\bar{g} = 0$: *If (1.10) holds with* $|\{s \in (1 - \delta_0, 1) : f'(s) = 0\}| = 0$ *for some* $\delta_0 > 0$*, then* $u \equiv 1$ *is unstable for any* $d, b > 0$.

(ii) $\bar{g} < 0$: *Then, $u \equiv 1$ is unstable for any $d, b > 0$ provided that either (1.8) or (1.10) holds.*

(iii) $\bar{g} > 0$: *If $f'(1) = 0$, (1.8) holds and either (1.7) or (1.9) is imposed, then $u \equiv 1$ is stable for any $d, b > 0$.*

The second result is more closely related to model (1.4), as the condition that $f'(1) < 0$ is imposed. To state the result, we denote the principal eigenvalue of the following non-local eigenvalue problem:

$$b(\bar{\phi} - \phi) + g(x)f'(1)\phi = \mu\phi \quad \text{in } \Omega, \tag{1.12}$$

by

$$\mu_0(b) = \sup_{\phi \in L^2(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[-b(\phi^2 - \bar{\phi}^2) + g(x)f'(1)\phi^2\right]\rho dx}{\int_\Omega \phi^2 \rho dx}; \tag{1.13}$$

and the principal eigenvalue of the "local" problem

$$\begin{cases} d\dfrac{1}{\rho}\nabla \cdot [\rho V \nabla\varphi] + g(x)f'(1)\varphi = \ell\varphi & \text{in } \Omega, \\ v \cdot V\nabla\psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.14}$$

by $\ell_0(d)$.

Theorem 1.5 *For problem (1.6), assume that (A), (A1), (A3) are valid and $f'(1) < 0$.*

(i) $\bar{g} \leqslant 0$: *$u \equiv 1$ is linearly unstable for all $d, b > 0$.*

(ii) $\bar{g} > 0$: *There exist $b^* > 0$ and $d^* > 0$ such that $u \equiv 1$ is linearly stable if either $d > 0$, $b \geqslant b^*$ or $d \geqslant d^*$, $b > 0$. In fact, b^*, d^* are the unique roots of $\mu_0(b) = 0$, $\ell_0(d) = 0$ respectively. Moreover, there exists a strictly decreasing, concave continuous function $D = D(b)$ in $[0, b^*]$ with $D(0) = d^*$ and $D(b^*) = 0$ such that for $0 < b < b^*$, $u \equiv 1$ is linearly stable if $d > D(b)$ and linearly unstable if $0 < d < D(b)$.*

Remark 1.3 *Notice that, the condition $f'(1) < 0$ in Theorem 1.5 is stronger than the conditions in (i) and (ii) of Theorem 1.4. Indeed when $\bar{g} \leqslant 0$, the instability of $u \equiv 1$ in (i) and (ii) of Theorem 1.4 can be improved to be linear instability as stated in (i) of Theorem 1.5. However, when $\bar{g} > 0$, the stability properties of $u \equiv 1$ for the case $f'(1) = 0$ in Theorem 1.4 (iii) are dramatically different from that of the case $f'(1) < 0$ in Theorem 1.5 (ii). Indeed, when $f'(1) = 0$, $u \equiv 1$ is always unstable. While when $f'(1) < 0$, the stability of $u \equiv 1$ changes as d and b vary and our result indicates that the local migration rate d and non-local migration rate b play similar roles in the change of stability.*

The stability analysis in Theorems 1.3–1.5 has many consequences. First, under certain conditions, when $\bar{g} = 0$, both $u \equiv 0$ and $u \equiv 1$ are unstable for all $d, b > 0$. Thus, (1.6) admits at least one stable steady state. This has been proved in Theorem 1.1 using variational method. Moreover, if $\bar{g} \neq 0$, some multiplicity results and global asymptotic behaviours of system (1.6) follow easily from Theorems 1.1–1.5 combined. For clarity, we now summarize these consequences in three theorems based on the signs of $\bar{g}$.

Theorem 1.6 *Suppose that (A), (A1), (A3) hold and* $\bar{g} = 0$*. Assume further that (1.9) holds with* $|\{s \in (0, \delta_0) : f'(s) = 0\}| = 0$ *and (1.10) holds with* $|\{s \in (1-\delta_0, 1) : f'(s) = 0\}| = 0$*. Then, (1.6) admits at least one non-trivial stable steady state for all* $d, b > 0$*.*

Theorem 1.7 *Suppose that (A), (A1), (A3) hold and* $\bar{g} < 0$*. In addition, assume that* $f'(0) = 0$*, condition (1.7) holds and either (1.8) or (1.10) is satisfied. Then for* d, b *both small, (1.6) admits at least two non-trivial steady states – one is stable and the other is unstable. Moreover,* $u \equiv 0$ *is globally stable for* $d + b$ *large.*

Theorem 1.8 *Suppose that (A), (A1), (A3) hold and* $\bar{g} > 0$*. The following statements hold.*

(i) *Assume that* $f'(1) = 0$*, condition (1.8) holds and either (1.7) or (1.9) is imposed. Then for* d, b *both small, (1.6) admits at least two non-trivial steady states – one is stable and the other is unstable. Moreover,* $u \equiv 1$ *is globally stable for* $d + b$ *large.*

(ii) *Assume that* $f'(1) < 0$ *and either (1.7) or (1.9) holds, then for* $0 \leqslant b < b^*$ *and* $0 < d < D(b)$*, there exists a non-trivial stable steady state of (1.6). Moreover,* $u \equiv 1$ *is globally stable for* $d + b$ *large.*

In summary, we wish to reiterate our earlier remark that the non-local partial panmixia seems to have similar *qualitative* effects as the local random diffusion. It would be interesting to estimate how *quantitatively* this non-local term affects this model. To be more specific, a natural question is whether the introduction of non-local partial panmixia is advantageous to the existence of non-trivial steady states.

This paper is organized as follows. Section 2 is devoted to the study of various properties of the general non-local problem (1.5), which will be needed in proving our main results. In Section 3, we focus on the genetic model (1.6). We will establish Theorems 1.1, 1.2 in Sections 3.1 and 3.2 respectively, Theorems 1.3–1.5 are proved in Section 3.3. Finally, in Section 3.4, short proofs of Theorems 1.6–1.8 will be included.

2 General results

In this section, we mainly consider the general non-local problem (1.5) and for simplicity denote $\mathcal{A}u \equiv d\frac{1}{\rho}\nabla \cdot (\rho V \nabla u)$.

2.1 Local existence and uniqueness

We first deal with local existence, uniqueness, as well as continuous dependence on initial data, of solutions to problem (1.5).

Theorem 2.1 *Suppose that (A) and (A2) hold and* $u_0 \in L^\infty(\Omega)$*. Then, the problem (1.5) has a unique solution* $u \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega} \times (0, T])$ *for* $T > 0$ *small. Moreover, for* u_0*,* $v_0 \in L^\infty(\Omega)$ *and* $1 < p \leqslant \infty$*, we have*

$$\|u(\cdot, T; u_0) - v(\cdot, T; v_0)\|_{L^p} \leqslant C\|u_0 - v_0\|_{L^p},$$

where C *depends on* p*,* T *and* Ω *only.*

Proof For convenience, the following notations are prepared. First, for $1 < p < \infty$, let $X = L^p(\Omega)$ and define, for $u \in X$,

$$\|u\|_X = \|u\|_{L^p} = \left(\int_\Omega |u|^p \rho dx\right)^{1/p}.$$

Then, the domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \{u \in W^{2,p}(\Omega) : v \cdot V\nabla u = 0 \text{ on } \partial\Omega\}.$$

Here,

$$W^{2,p}(\Omega) = \{u \in W^2(\Omega) \mid D^\beta u \in L^p(\Omega) \text{ for all } |\beta| \leqslant 2\},$$

is equipped with the norm

$$\|u\|_{W^{2,p}} = \left(\int_\Omega \sum_{|\beta|\leqslant 2} |D^\beta u|^p \rho dx\right)^{1/p},$$

where $\beta = (\beta_1, \ldots, \beta_n)$, β_i =integer$\geqslant 0$, $|\beta| = \sum \beta_i$. Next, let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators $\mathcal{P} : X \to X$, endowed with the norm

$$\|\mathcal{P}\|_{\mathcal{L}(X)} = \sup_{u\in X,\ \|u\|_X=1} \|\mathcal{P}u\|_X.$$

Given $u_0 \in L^\infty(\Omega)$, fix any $T > 0$ and set

$$R = 4 \sup_{0<t\leqslant T} \left\|e^{t\mathcal{A}}u_0\right\|_X, \quad M_0 = \sup_{0\leqslant t\leqslant T} \|e^{t\mathcal{A}}\|_{\mathcal{L}(X)}.$$

Also denote $Y = \{u \in C((0, t_0]; X) : \|u(\cdot, t)\|_X \leqslant R,\ \forall\ t \in (0, t_0]\}$, where $t_0 \in (0, T]$ will be chosen properly, and

$$\|u\|_{C((0,t_0];X)} = \sup_{0<s\leqslant t_0} \|u(\cdot, s)\|_X.$$

For $v \in Y$, define

$$\Gamma(v)(\cdot, t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}} f(\cdot, v(\cdot, s), \bar{v}(s))ds.$$

We claim that Γ *is a contraction mapping which maps* Y *into itself provided that* t_0 *is sufficiently small.*

Assume that $v_1, v_2 \in Y$. First, according to assumption (A2), there exists $L = L(R)$ such that

$$\|f(x, v_1, \bar{v}_1) - f(x, v_2, \bar{v}_2)\|_X \leqslant L\|v_1 - v_2\|_X. \tag{2.1}$$

Then, we have

$$\begin{aligned} &\|\Gamma(v_1) - \Gamma(v_2)\|_{C((0,t_0];X)} \\ &= \left\|\int_0^t e^{(t-s)\mathcal{A}}[f(\cdot, v_1(\cdot, s), \bar{v}_1(s)) - f(\cdot, v_2(\cdot, s), \bar{v}_2(s))]ds\right\|_{C((0,t_0];X)} \\ &\leqslant t_0 L M_0 \|v_1 - v_2\|_{C((0,t_0];X)} \leqslant \frac{1}{2}\|v_1 - v_2\|_{C((0,t_0];X)}, \end{aligned} \tag{2.2}$$

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if $t_0 \leqslant \frac{1}{2}(LM_0)^{-1}$. Thus, for $v \in Y$, according to (2.2),

$$\begin{aligned}
&\|\Gamma(v)\|_{C((0,t_0];X)} \\
&\leqslant \|\Gamma(v) - \Gamma(0)\|_{C((0,t_0];X)} + \|\Gamma(0)\|_{C((0,t_0];X)} \\
&\leqslant \frac{1}{2}\|v\|_{C((0,t_0];X)} + \left\| e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}} f(\cdot,0,0)ds \right\|_{C((0,t_0];X)} \\
&\leqslant \frac{1}{2}R + \frac{1}{4}R + t_0 M_0 \|f(\cdot,0,0)\|_X \leqslant R,
\end{aligned}$$

provided that t_0 is sufficiently small.

Hence for t_0 small, Γ has a unique fixed point in Y, i.e., there exists $u(\cdot,t) \in Y$ such that

$$u(\cdot,t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}} f(\cdot,u(\cdot,s),\bar{u}(s))ds. \tag{2.3}$$

Furthermore, by [13, Proposition 7.1.10], it follows that the problem (1.5) locally admits a unique solution $u(\cdot,t) \in W^{2,p}(\Omega)$, i.e., u satisfies (1.5) for $0 < t < t_0$ and

$$u \in C^1((0,t_0];X)) \cap C((0,t_0];D(\mathcal{A})) \cap C([0,t_0];X). \tag{2.4}$$

Fix any $0 < \delta < t_0$ and denote $Q_\delta = \Omega \times (\delta, t_0]$. Note that we already have obtained that $u \in W_p^{2,1}(Q_\delta)$ for any $p > 1$. Thus, by choosing p properly such that $\alpha = 2-(N+2)/p > 0$, we have $u \in C^{\alpha,\alpha/2}(\bar{Q}_\delta)$. Then, from parabolic regularity and assumptions (A), (A2), it follows that $u \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_\delta)$. Since $\delta > 0$ is arbitrary, we have $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega} \times (0,t])$ for $t > 0$ small.

It remains to verify the continuous dependence on initial data. Due to (2.1) and (2.3), it is standard to calculate that for $0 < t < t_0$,

$$\begin{aligned}
&\|u(\cdot,t) - v(\cdot,t)\|_X \\
&\leqslant \|e^{t\mathcal{A}}(u_0 - v_0)\|_X + \int_0^t \left\| e^{(t-s)\mathcal{A}} \left[f(\cdot,u(\cdot,s),\bar{u}) - f(\cdot,v(\cdot,s),\bar{v}) \right] \right\|_X ds \\
&\leqslant M_0\|(u_0 - v_0)\|_X + M_0 L t \|u - v\|_{C((0,t_0];X)}.
\end{aligned}$$

This implies that when $t_0 < \frac{1}{2M_0L}$,

$$\|u - v\|_{C((0,t_0];X)} \leqslant 2M_0\|(u_0 - v_0)\|_X.$$

For any fixed $T > 0$, as long as the solutions remain bounded in X, this process can be repeated finitely many times to achieve the desired conclusion.

Note that so far, we have assumed that $X = L^p(\Omega)$ for any $1 < p < \infty$. To show

$$\|u(\cdot,t;u_0) - v(\cdot,t;v_0)\|_{L^\infty} \leqslant C\|u_0 - v_0\|_{L^\infty},$$

one simply needs set $X = L^\infty(\Omega)$ and repeat the arguments above. We omit the details. □

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2.2 Backward uniqueness

To study backward uniqueness of problem (1.5), we consider

$$\begin{cases} u_t = \mathcal{A}u + f(x,u,\bar{u}) & \text{in } \Omega \times (0,t_0], \\ \nu \cdot V\nabla u = 0 & \text{on } \partial\Omega \times (0,t_0]. \end{cases} \tag{2.5}$$

Theorem 2.2 *Suppose that (A), (A2) hold and that $u_{t_0}(x)$ satisfies*

$$\mathcal{A}u_{t_0} + f(x,u_{t_0},\bar{u}_{t_0}) = 0.$$

Let $u(x,t)$ be a solution of (2.5) with $u(x,t_0) = u_{t_0}(x)$. Then, $u(x,t) \equiv u_{t_0}(x)$.

Proof Let $v(x,t) = u(x,t) - u_{t_0}(x)$. It is easy to see that

$$\begin{cases} v_t = \mathcal{A}v + b(x,t)\bar{v} + c(x,t)v & \text{in } \Omega \times (0,t_0], \\ \nu \cdot V\nabla v = 0 & \text{on } \partial\Omega \times (0,t_0], \\ v(x,t_0) = 0 & \text{in } \Omega, \end{cases}$$

where

$$b(x,t) = \frac{f(x,u,\bar{u}) - f(x,u,\bar{u}_{t_0})}{\bar{u} - \bar{u}_{t_0}}, \; c(x,t) = \frac{f(x,u,\bar{u}_{t_0}) - f(x,u_{t_0},\bar{u}_{t_0})}{u - u_{t_0}}.$$

Now introduce

$$\Lambda(t) = \frac{\int_\Omega [d\nabla v \cdot (V\nabla v) + v^2]\rho dx}{\int_\Omega v^2 \rho dx}.$$

Direct computation yields that

$$\begin{aligned} \frac{1}{2}\Lambda'(t) &= \frac{\int_\Omega [d\nabla v \cdot (V\nabla v_t) + vv_t]\rho dx}{\int_\Omega v^2\rho dx} - \Lambda(t)\frac{\int_\Omega vv_t\rho dx}{\int_\Omega v^2\rho dx} \\ &= -\frac{\int_\Omega v_t[\mathcal{A}v - v + \Lambda(t)v]\rho dx}{\int_\Omega v^2\rho dx} \\ &= -\frac{\int_\Omega [\mathcal{A}v - v + \Lambda(t)v + v - \Lambda(t)v + b\bar{v} + cv][\mathcal{A}v - v + \Lambda(t)v]\rho dx}{\int_\Omega v^2\rho dx} \\ &= -\frac{\int_\Omega [\mathcal{A}v - v + \Lambda(t)v]^2\rho dx}{\int_\Omega v^2\rho dx} + (\Lambda(t) - 1)\frac{\int_\Omega v[\mathcal{A}v - v + \Lambda(t)v]\rho dx}{\int_\Omega v^2\rho dx} \\ &\quad - \frac{\int_\Omega [b\bar{v} + cv][\mathcal{A}v - v + \Lambda(t)v]\rho dx}{\int_\Omega v^2\rho dx} \\ &\quad = -\frac{\int_\Omega [\mathcal{A}v - v + \Lambda(t)v]^2\rho dx}{\int_\Omega v^2\rho dx} - \frac{\int_\Omega [b\bar{v} + cv][\mathcal{A}v - v + \Lambda(t)v]\rho dx}{\int_\Omega v^2\rho dx} \\ &\quad \leqslant -\frac{1}{2}\frac{\int_\Omega [\mathcal{A}v - v + \Lambda(t)v]^2\rho dx}{\int_\Omega v^2\rho dx} + C\frac{\bar{v}^2 + \int_\Omega v^2\rho dx}{\int_\Omega v^2\rho dx} \leqslant \frac{K}{2}. \end{aligned} \tag{2.6}$$

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Suppose that the conclusion does not hold. Let t_1 be the time such that $\|v\|_{L^2} \neq 0$ for $0 < t < t_1$, while $\|v\|_{L^2} = 0$ for $t_1 \leqslant t \leqslant t_0$. Note that this must be the case due to the existence and uniqueness result in Theorem 2.1.

Now for $0 < t < t_1$, by (2.6), we have

$$\frac{1}{2}\frac{d}{dt}\ln\frac{1}{\int_\Omega v^2\rho dx} = -\frac{\int_\Omega vv_t\rho dx}{\int_\Omega v^2\rho dx} = -\frac{\int_\Omega v[\mathcal{A}v + b\bar{v} + cv]\rho dx}{\int_\Omega v^2\rho dx}$$
$$= \Lambda(t) - \frac{\int_\Omega v[v + b\bar{v} + cv]\rho dx}{\int_\Omega v^2\rho dx} \leqslant \Lambda(0) + Kt + K_0 \leqslant K_1,$$

where $K_1 = \Lambda(0) + Kt_1 + K_0$. Hence,

$$\frac{1}{2}\left(\ln\frac{1}{\int_\Omega v^2(\cdot,t_1)\rho dx} - \ln\frac{1}{\int_\Omega v^2(\cdot,0)\rho dx}\right) \leqslant K_1 t_1 < \infty.$$

However, the choice of t_1 implies that

$$\ln\frac{1}{\int_\Omega v^2(\cdot,t_1)\rho dx} = +\infty.$$

This is a contradiction. □

2.3 Comparison principle

Next, we present the comparison principle for the non-local problem (1.5).

Theorem 2.3 *In the problem (1.5), suppose that assumptions (A) and (A2) hold, $f_{\bar{u}} \geqslant 0$ and $u_0, v_0 \in L^\infty(\Omega)$ satisfy $u_0 \leqslant v_0$ and $u_0 \not\equiv v_0$, then $u(x,t;u_0) < v(x,t;v_0)$ whenever $t > 0$ and solutions $u(x,t;u_0)$, $v(x,t;v_0)$ both exist.*

Proof Setting $w(x,t) = u(x,t) - v(x,t)$, we have

$$\begin{cases} w_t = \mathcal{A}w + b(x,t)\bar{w} + c(x,t)w & \text{in } \Omega \times (0,T), \\ \nu \cdot V\nabla w = 0 & \text{on } \partial\Omega \times (0,T), \\ w(x,0) = u_0(x) - v_0(x) & \text{in } \Omega. \end{cases} \tag{2.7}$$

Here, we have assumed that solutions $u(x,t;u_0)$, $v(x,t;v_0)$ both exist in $(0,T]$, $T > 0$, and

$$b(x,t) = \frac{f(x,u,\bar{u}) - f(x,u,\bar{v})}{\bar{u} - \bar{v}} \geqslant 0, \; c(x,t) = \frac{f(x,u,\bar{v}) - f(x,v,\bar{v})}{u - v}.$$

Assume that $|c(x,t)| \leqslant K$ *for* $(x,t) \in \Omega \times (0,T)$. Setting $w_K(x,t) = e^{-Kt}w(x,t)$, we have

$$\begin{cases} (w_K)_t = \mathcal{A}w_K + b(x,t)\bar{w}_K + \tilde{c}(x,t)w_K & \text{in } \Omega \times (0,T), \\ \nu \cdot V\nabla w_K = 0 & \text{on } \partial\Omega \times (0,T), \\ w_K(x,0) = u_0(x) - v_0(x) & \text{in } \Omega, \end{cases} \tag{2.8}$$

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where $\tilde{c}(x,t) = c(x,t) - K \leqslant 0$. Since $\bar{w}_K(0) < 0$, there exists $\delta > 0$ small such that $\bar{w}_K(t) < 0$ for $0 < t \leqslant \delta$. Therefore,

$$(w_K)_t = \mathcal{A}w_K + b(x,t)\bar{w}_K + \tilde{c}(x,t)w_K \leqslant \mathcal{A}w_K + \tilde{c}(x,t)w_K$$

in $(x,t) \in \Omega \times (0,\delta)$.

Suppose that $(x_1,t_1) \in \bar{\Omega} \times (0,\delta]$ satisfies

$$w_K(x_1,t_1) = \sup_{x\in\bar{\Omega},0<t<\delta} w_K(x,t) \geqslant 0.$$

Then, the maximum principle and Hopf boundary lemma together yield a contradiction. Hence, $w_K(x,t) < 0$ in $\bar{\Omega} \times (0,\delta]$.

Therefore, without loss of generality, we may assume that $w_K(x,0) = u_0(x) - v_0(x) < 0$ in $\bar{\Omega}$. Let t_2 denote the time when $w_K(x,t) < 0$ for $0 \leqslant t < t_2$ and there exists $x_2 \in \bar{\Omega}$ such that $w_K(x_2,t_2) = 0$. Then since $b(x,t) \geqslant 0$, we have

$$(w_K)_t = \mathcal{A}w_K + b(x,t)\bar{w}_K + \tilde{c}(x,t)w_K \leqslant \mathcal{A}w_K + \tilde{c}(x,t)w_K$$

in $(x,t) \in \Omega \times (0,t_2]$. A similar contradiction can be derived due to the maximum principle and Hopf boundary lemma. Hence, $w_K(x,t) < 0$ for $0 < t < T$. □

Indeed, (1.5) might not enjoy comparison principles if the condition $f_{\bar{u}} \geqslant 0$ is not satisfied. For the rest of this sub-section, we will demonstrate that how a counterexample can be constructed if the condition $f_{\bar{u}} \geqslant 0$ is violated.

Example 2.1. Consider

$$\begin{cases} u_t = \Delta u - u + \dfrac{u^p}{\bar{u}^q} & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{2.9}$$

where $\Omega = B_1(0)$, $0 < p-1 < q$ and $p > \frac{n+2}{n}$. Problem (2.9) can be viewed as a special case for the shadow system of an activator–inhibitor model due to Gierer and Meinhardt [8,9], and we refer interested readers to [7] for properties of general shadow systems.

Let $u(x,t;u_0)$ denote the solution of (2.9). It is proved in [8,9] that blow-up solution exists for proper choice of initial value $u_0(x) \in C^1(\bar{\Omega})$. However, choosing $v_0(x) \equiv M_0$ with $M_0 = \sup_{x\in\Omega} u_0(x)$, then $u(x,t;v_0) = V(t)$ which satisfies

$$\begin{cases} V_t = -V + V^{p-q} & \text{in } (0,T), \\ V(0) = M_0. \end{cases}$$

The condition $p - q < 1$ guarantees that $u(x,t;v_0) = V(t)$ remains finite for all time. Hence, obviously, $u(x,t;u_0) \leqslant u(x,t;v_0)$ fails for some $t > 0$, i.e., problem (2.9) does not support comparison principles.

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Moreover, according to the arguments in [8], if in addition, $p > \frac{n}{n-2}$, then for $0 < \delta < \delta_0$, there exists $u_\delta(x)$ such that $u(x,t;u_\delta)$ blows up at some finite time T_δ and for $0 < t < T_\delta$

$$\frac{1}{|\Omega|}\int_\Omega u(x,t;u_\delta)dx = c + O(\delta^2),$$

where the positive constant c is independent of δ.

Now, a more interesting counter-example can be constructed as follows. Given any $\epsilon > 0$, assume that $k_\epsilon(s)$ satisfies

- $k_\epsilon(s)$ is a C^1 continuous function in $\mathbb{R}$;
- $k_\epsilon(s) = s^{-q}$ for $s \in (c-\epsilon, c+\epsilon)$;
- k_ϵ is non-decreasing in $(-\infty, c-2\epsilon) \cup (c+2\epsilon, \infty)$.

The existence of such functions is obvious. Then, consider the following system:

$$\begin{cases} u_t = \Delta u - u + k_\epsilon(\bar{u})u^p & \text{in } B_1(0) \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1(0) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } B_1(0). \end{cases} \tag{2.10}$$

This corresponds to the case that $f = -u + k_\epsilon(\bar{u})u^p$ and the condition $f_{\bar{u}} \geqslant 0$ is only violated in $(c-\epsilon, c+\epsilon)$, where ϵ can be arbitrarily small. However, we can always choose $\delta = \delta(\epsilon)$ small enough and $u_0(x) = u_\delta$ such that for $0 < t < T_\delta$

$$\frac{1}{|\Omega|}\int_\Omega u(x,t;u_\delta)dx = c + O(\delta^2) \in (c-\epsilon, c+\epsilon).$$

Hence, with this initial value, the solution of problem (2.10) coincides with that of (2.9), and based on the previous discussions, when $0 < p-1 < q$ and $p > \frac{n}{n-2}$, the solution of (2.10) blows up at finite time and thus comparison principle fails.

2.4 ω-limit set

In this section, we investigate properties of ω-limit set of the solution $u(x,t)$ to problem (1.5). For clarity, let Y denote a Banach space, then the ω-limit set of the solution $u(x,t)$ in Y is denoted by $\omega[u_0|Y]$ and defined as follows:

$$\omega[u_0|Y] = \left\{ z \in Y \mid z = \lim_{n\to\infty} u(\cdot, t_n) \quad \text{where} \quad \lim_{n\to\infty} t_n = \infty \right\}.$$

Before presenting the main result, we propose the following assumption:

(A4) There exist a constant $\gamma > 0$ and a functional

$$J[u](t) = \int_\Omega \left[\frac{1}{2} d\nabla u \cdot (V\nabla u) - F(x,u,\bar{u})\right] \rho dx,$$

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such that if $u(x,t)$ is a classical solution of problem (1.5), then

$$\frac{d}{dt}J[u](t) \leqslant -\gamma \int_\Omega u_t^2(x,t)\rho dx.$$

Here comes the main result in this section.

Theorem 2.4 *For problem (1.5), assume that (A), (A2) and (A4) are valid. If (1.5) admits a global solution $u(x,t)$ with*

$$\sup_{t>0} \|u(\cdot,t)\|_{L^p} \leqslant C_0 < \infty,$$

for some $p > N+1$, then the following statements hold.

(i) $\omega[u_0|L^\infty(\Omega)] \neq \emptyset$.

(ii) $\omega[u_0|L^\infty(\Omega)] = \omega[u_0|C^2(\bar{\Omega})]$.

(iii) $\omega[u_0|L^\infty(\Omega)]$ *consists of only steady states of (1.5).*

The following lemma collects some facts on the asymptotics of the heat semi-group under no-flux boundary conditions, which will be useful in the proof of Theorem 2.4.

Lemma 2.1 *Suppose that (A) holds. Let λ_1 denote the first non-zero eigenvalue of*

$$\begin{cases} -\mathcal{A}\psi = -d\frac{1}{\rho}\nabla\cdot(\rho V \nabla\psi) = \lambda\psi & \text{in } \Omega, \\ v\cdot V\nabla\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, we have the following estimates.

(i) *If $1 \leqslant q \leqslant p \leqslant \infty$, then for all $w \in L^q(\Omega)$ with $\bar{w} = 0$, we have*

$$\|e^{t\mathcal{A}}w\|_{L^p} \leqslant C_1\left(1 + t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-\lambda_1 t}\|w\|_{L^q} \quad \text{for all } t > 0.$$

(ii) *If $1 \leqslant q \leqslant p \leqslant \infty$, then*

$$\|\nabla e^{t\mathcal{A}}w\|_{L^p} \leqslant C_2\left(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-\lambda_1 t}\|w\|_{L^q} \quad \text{for all } t > 0$$

is true for all $w \in L^q(\Omega)$.

(iii) *If $2 \leqslant p < \infty$, then*

$$\|\nabla e^{t\mathcal{A}}w\|_{L^p} \leqslant C_3 e^{-\lambda_1 t}\|\nabla w\|_{L^p} \quad \text{for all } t > 0$$

holds for all $w \in W^{1,p}(\Omega)$.

This lemma can be verified by applying similar arguments as in [24, Lemma 1.3] with obvious modifications, since the key element of the proof, the point-wise estimates for Green's function of the problem $u_t = \mathcal{A}u$ with no-flux boundary conditions, has already been obtained in [14, Theorem 2.2]. We omit the details here.

The proof of Theorem 2.4 (i) is based on Lemma 2.1, while that of Theorem 2.4 (ii), (iii) is standard. We include the details here for the convenience of readers.

Proof of Theorem 2.4 (i) Due to parabolic regularity, we assume, without loss of generality, that $u_0 \in W^{1,p}(\Omega)$. Again, denote $X = L^p(\Omega)$. Recall that by (2.3), $u(x,t)$ satisfies

$$u(\cdot,t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}} f(\cdot,u(\cdot,s),\bar{u}(s))ds.$$

From (A2) and Lemma 2.1, it follows that

$$\begin{aligned}
\|\nabla u(\cdot,t)\|_X &\leqslant \|\nabla e^{t\mathcal{A}}u_0\|_X + \int_0^t \|\nabla e^{(t-s)\mathcal{A}} f(\cdot,u(\cdot,s),\bar{u}(s))\|_X ds \\
&\leqslant C_3\|\nabla u_0\|_X + C_2\int_0^t \left(1+(t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)}\|f(\cdot,u(\cdot,s),\bar{u}(s))\|_X ds \\
&\leqslant C_3\|u_0\|_{W^{1,p}} + \tilde{C}_2\left(\|f(\cdot,0,0)\|_X + LC_0\right),
\end{aligned}$$

where

$$\tilde{C}_2 = C_2\int_0^\infty \left(1+(t-s)^{-\frac{1}{2}}\right) e^{-\lambda_1(t-s)} ds < \infty.$$

This shows that $\|u(\cdot,t)\|_{W^{1,p}}$ is uniformly bounded in t and thus $\{u(\cdot,t)|\ t>0\}$ is relatively compact in $L^\infty(\Omega)$. It follows that $\omega[u_0|L^\infty(\Omega)] \neq \emptyset$.

Now we will prove (ii) and (iii) simultaneously. Assume that $\phi \in \omega[u_0|L^\infty(\Omega)]$. By definition, there exists a sequence $0 < t_1 < t_2 < \cdots \to \infty$ such that

$$\lim_{n\to\infty} u(\cdot,t_n) = \phi \quad \text{in } L^\infty.$$

First, we claim that $u(x,t_n+t)$ *is relatively compact in* $C((0,t_*];C^2(\bar{\Omega}))$. Note that the arguments in establishing (i) already indicate that for $t_* > 0$, $u(x,t_n+t)$ is uniformly bounded in $L^\infty(\Omega\times(0,t_*))$, so is $f(x,u(x,t_n+t),\bar{u}(t_n+t))$. Hence, by standard arguments involving parabolic regularity estimates, $u(x,t_n+t)$ is relatively compact in

$$C^{\alpha,\alpha/2}(\bar{\Omega}\times[\delta,t_*])\bigcap C([\delta,t_*];C^1(\bar{\Omega}))$$

for any given $\delta\in(0,t_*]$. This, together with (A2), yields that $f(x,u(x,t_n+t),\bar{u}(t_n+t))$ is bounded in $C^{\alpha,\alpha/2}(\bar{\Omega}\times[\delta,t_*])$. So it follows from global Schauder estimates [4, Theorem 6 in Chapter 3] that $u(x,t_n+t)$ is bounded in $C^{2+\alpha,1+\alpha/2}(\bar{\Omega}\times[\delta,t_*])$, hence relatively compact in $C([\delta,t_*];C^2(\bar{\Omega}))$. Therefore, the claim is valid since δ is arbitrary.

Next, let $h(x,t)$ denote the solution of problem (1.5) with initial value ϕ, i.e.

$$\begin{cases}
h_t = d\dfrac{1}{\rho}\nabla\cdot(\rho V\nabla h) + f(x,h,\bar{h}) & \text{in } \Omega\times(0,t_*), \\
\nu\cdot V\nabla h = 0 & \text{on } \partial\Omega\times(0,t_*), \\
h(x,0) = \phi(x) & \text{in } \Omega.
\end{cases} \tag{2.11}$$

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By Theorem 2.1, one sees that

$$\sup_{0<t\leqslant t_*}\|u(\cdot,t_n+t)-h(\cdot,t)\|_{L^\infty}\leqslant C(t_*)\|u(\cdot,t_n)-\phi(\cdot)\|_{L^\infty}\to 0\quad\text{as } n\to\infty.$$

Together with the claim, we have, given any $\delta\in(0,t_*]$ and any compact set $K\subset\Omega$,

$$\lim_{n\to\infty}\sup_{\delta\leqslant t\leqslant t_*}\|u(\cdot,t_n+t)-h(\cdot,t)\|_{C^1(\bar\Omega)}=0 \tag{2.12}$$

and

$$\lim_{n\to\infty}\sup_{\delta\leqslant t\leqslant t_*}\|u(\cdot,t_n+t)-h(\cdot,t)\|_{C^2(K)}=0. \tag{2.13}$$

On the other hand, according to (A4),

$$\int_{t_1+t_*}^{t_n+t_*}\int_\Omega u_t^2(x,\tau)dxd\tau\leqslant\frac{1}{\gamma}\left\{J[u](t_1+t_*)-J[u](t_n+t_*)\right\}.$$

Since $u(\cdot,t_n+t_*)$ is bounded in $C^1(\bar\Omega)$, $J[u](t_n+t_*)$ remains bounded as $n\to\infty$. It follows that

$$\int_{t_1+t_*}^{\infty}\int_\Omega u_t^2(x,\tau)dxd\tau<\infty,$$

which, together with (2.13), implies that for any $\delta\in(0,t_*)$ and any compact set $K\subset\Omega$,

$$\begin{aligned}
0&=\lim_{n\to\infty}\int_{t_n+\delta}^{t_n+t_*}\int_\Omega u_t^2(x,\tau)\rho dxd\tau\\
&=\lim_{n\to\infty}\int_{\delta}^{t_*}\int_\Omega u_t^2(x,t_n+\tau)\rho dxd\tau\\
&\geqslant\lim_{n\to\infty}\int_{\delta}^{t_*}\int_K u_t^2(x,t_n+\tau)\rho dxd\tau\\
&=\lim_{n\to\infty}\int_{\delta}^{t_*}\int_K\left[\mathcal{A}u(x,t_n+\tau)+f(x,u(x,t_n+\tau),\bar u(t_n+\tau))\right]^2\rho dxd\tau\\
&=\int_{\delta}^{t_*}\int_K\left[\mathcal{A}h(x,\tau)+f(x,h(x,\tau),\bar h(\tau))\right]^2\rho dxd\tau.
\end{aligned}$$

Therefore,

$$\mathcal{A}h(x,t)+f(x,h(x,t),\bar h(t))=0\quad\text{in}\ \ \Omega\times(0,t_*)$$

and thus $h_t(x,t)=0$ in $\Omega\times(0,t_*)$. Moreover, since $h(x,t)$ is a solution of problem (2.11) with initial value $\phi\in L^\infty(\Omega)$, similar to (2.4), one sees that

$$\lim_{t\to 0}h(\cdot,t)=\phi(\cdot)\quad\text{in}\ \ L^\infty(\Omega).$$

Thus, $h(x,t)\equiv\phi(x)$. This clearly implies that $\phi(x)$ is a steady state of (1.5), hence (iii) is proved. Furthermore, due to (2.12) and (2.13), we have

$$\lim_{n\to\infty}u(\cdot,t_n+t_*)=\phi\quad\text{in } C^1(\bar\Omega)\bigcap C^2(\Omega),$$

i.e., $\phi\in\omega[u_0|C^1(\bar\Omega)\bigcap C^2(\Omega)]$. (ii) follows immediately. □

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2.5 Further remarks

All the properties established in Section 2 so far can be easily extended to more general non-local models

$$\begin{cases} u_t = d\frac{1}{\rho}\nabla\cdot(\rho V\nabla u) + f(x,u,I(u)) & \text{in } \Omega\times(0,T), \\ \nu\cdot V\nabla u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.14}$$

where $I(u) = \int_\Omega \ell(u)\rho(x)dx$, with assumptions

(A2)′ $f(x,u,\xi)$ is C^α in $x \in \bar{\Omega}$, C^1 in u and ξ, and $\ell(u)$ is C^1 in u.

For clarity, we state these properties for model (2.14).

Theorem 2.5 (*Well posedness*) *Suppose that (A) and (A2)′ hold and $u_0 \in L^\infty(\Omega)$. Then, the problem (2.14) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}\times(0,t])$ with $0<\alpha<1$, for $t>0$ small. Moreover, for $u_0,\ v_0 \in L^\infty(\Omega)$,*

$$\|u(\cdot,t;u_0) - v(\cdot,t;v_0)\|_{L^\infty} \leqslant C\|u_0 - v_0\|_{L^\infty},$$

where C depends on p, t and Ω only.

Theorem 2.6 (*Backward uniqueness*) *Suppose that (A), (A2)′ hold and $u_{t_0}(x)$ satisfies*

$$d\frac{1}{\rho}\nabla\cdot(\rho V\nabla u_{t_0}) + f(x,u_{t_0},I(u_{t_0})) = 0.$$

Let $u(x,t)$ be a solution of

$$\begin{cases} u_t = d\frac{1}{\rho}\nabla\cdot(\rho V\nabla u) + f(x,u,I(u)) & \text{in } \Omega\times(0,t_0], \\ \nu\cdot V\nabla u = 0 & \text{on } \partial\Omega\times(0,t_0], \end{cases}$$

with $u(x,t_0) = u_{t_0}(x)$. Then $u(x,t) \equiv u_{t_0}(x)$.

Theorem 2.7 (*Comparison principle*) *In problem (2.14), suppose that (A) and (A2)′ hold, and either, $f_\xi \geqslant 0$ and $\ell'(u) \geqslant 0$, or, $f_\xi \leqslant 0$ and $\ell'(u) \leqslant 0$. Then for $u_0, v_0 \in L^\infty(\Omega)$ satisfying $u_0 \leqslant v_0$ and $u_0 \not\equiv v_0$, we have $u(x,t;u_0) < v(x,t;v_0)$ whenever $t>0$ and solutions $u(x,t;u_0)$, $v(x,t;v_0)$ both exist.*

Theorem 2.8 (*ω-limit set*) *For problem (2.14), assume that (A), (A2)′ are valid and there exist a constant $\gamma>0$ and a functional*

$$J[u](t) = \int_\Omega \left[\frac{1}{2}d\nabla u\cdot(V\nabla u) - F(x,u,I(u))\right]\rho dx,$$

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such that for all classical solutions $u(x,t)$ of problem (2.14), it holds that

$$\frac{d}{dt}J[u](t) \leqslant -\gamma \int_\Omega u_t^2(x,t)\rho dx.$$

Then if (2.14) admits a global solution $u(x,t)$ with

$$\sup_{t>0} \|u(\cdot,t)\|_{L^\infty} \leqslant C_0 < \infty,$$

we have the following statements:

(i) $\omega[u_0|L^\infty(\Omega)] \neq \emptyset$.

(ii) $\omega[u_0|L^\infty(\Omega)] = \omega[u_0|C^1(\bar{\Omega})\bigcap C^2(\Omega)]$.

(iii) $\omega[u_0|L^\infty(\Omega)]$ *consists of steady states of (2.14).*

The proofs of these theorems are the same as before – with obvious modifications, thus the details are omitted.

3 An integro-PDE model

3.1 Existence of non-trivial steady states

To prove Theorem 1.1, following the basic idea in [19] we look for a global minimizer of the following variational functional on $H^1(\Omega)$:

$$J[u] = \int_\Omega \left[\frac{d}{2}\nabla u \cdot (V\nabla u) + \frac{b}{2}u^2 - \frac{b}{2}\bar{u}^2 - g(x)F(u)\right]\rho dx, \tag{3.1}$$

where $F(u) = \int_0^u f(s)ds$ with $f(u) > 0$ for $u \in (0,1)$ and $f(u) = 0$ outside the interval $(0,1)$. In proving the existence of a convergent minimizing sequence, a main step is to construct first a *bounded minimizing sequence* $\{u_k\}_{k=1}^\infty$ *in* $H^1(\Omega)$. In the case without partial panmixia, i.e. $b = 0$, in [19], this step can be achieved rather quickly as we can always assume that $0 \leqslant u_k \leqslant 1$, since the non-linearity $f(u)$ has been modified to be 0 outside the interval $(0,1)$. However, when $b > 0$, the assumption $0 \leqslant u_k \leqslant 1$ can no longer be imposed directly and more care is needed in deriving this desired property.

Proof of Theorem 1.1 First, observe that, since $F(u)$ is bounded,

$$\beta = \inf\{J[u] : u \in H^1(\Omega)\} > -\infty.$$

We claim that *when $\bar{g} = 0$, neither $u \equiv 0$ nor $u \equiv 1$ is a global minimizer for any $d, b > 0$.* Obviously, $J[0] = 0$ and $J[1] = 0$. Now, choose $\phi \in H^1(\Omega)$ such that $\int_\Omega g\phi\rho dx > 0$ and fix $0 < c < 1$. We compute

$$\begin{aligned} & J[c+\epsilon\phi] \\ & = \int_\Omega \left[\frac{d}{2}\epsilon^2\nabla\phi\cdot(V\nabla\phi) + \frac{b}{2}(c+\epsilon\phi)^2 - \frac{b}{2}(c+\epsilon\bar{\phi})^2 - g(x)F(c+\epsilon\phi)\right]\rho dx \\ & = -\epsilon\left[f(c)\int_\Omega g(x)\phi\rho dx + O(\epsilon)\right] < 0 \end{aligned}$$

for ϵ small.

Next, we claim that *for $\bar{g} \neq 0$, neither $u \equiv 0$ nor $u \equiv 1$ is a global minimizer for $d, b > 0$ small.*

It is clear that $J[0] = 0$ and

$$J[1] = -\int_\Omega g(x)F(1)\rho dx = -\left[\int_{\Omega_+} g(x)F(1)\rho dx + \int_{\Omega_-} g(x)F(1)\rho dx\right],$$

where $\Omega_+ = \{x \in \Omega \mid g(x) > 0\}$, $\Omega_- = \{x \in \Omega \mid g(x) < 0\}$. For convenience, denote

$$\delta_1 = \int_{\Omega_+} g(x)F(1)\rho dx > 0 \;\text{ and }\; \delta_2 = -\int_{\Omega_-} g(x)F(1)\rho dx > 0.$$

Now for $d, b > 0$ small, we need find $\tilde{u} \in H^1(\Omega)$ such that

$$J[\tilde{u}] < \min\{J[0], J[1]\} = \min\{0, -\delta_1 + \delta_2\}.$$

Notice that $-\int_\Omega g(x)F(\chi_{\{g(x)>0\}})\rho dx = -\int_{\Omega_+} g(x)F(1)\rho dx = -\delta_1$, hence there exists $\tilde{u} \in H^1(\Omega)$ such that

$$-\int_\Omega g(x)F(\tilde{u})\rho dx < \min\left\{-\frac{\delta_1}{2}, -\delta_1 + \frac{\delta_2}{2}\right\}.$$

Moreover, there exists $\epsilon > 0$ small enough such that for $d, b < \epsilon$

$$\int_\Omega \left[\frac{d}{2}\nabla\tilde{u}\cdot(V\nabla\tilde{u}) + \frac{b}{2}\tilde{u}^2 - \frac{b}{2}\bar{\tilde{u}}^2\right]\rho dx < \min\left\{\frac{\delta_1}{2}, \frac{\delta_2}{2}\right\}.$$

Therefore, $J[\tilde{u}] < \min\{J[0], J[1]\} = \min\{0, -\delta_1 + \delta_2\}$, and our assertion is proved.

To proceed, we assume the following holds:

$$\beta = \inf\{J[u] : u \in H^1(\Omega)\} < \min\{J[0], J[1]\}.$$

Let $\{u_k\}_{k=1}^\infty$ be a minimizing sequence, i.e., $J[u_k]$ decreases to β. If *$\{u_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega)$*, then, by passing to subsequence if necessary,

$$u_{k_j} \rightharpoonup u \text{ weakly in } H^1(\Omega) \;\text{ and }\; u_{k_j} \to u \text{ strongly in } L^2(\Omega),$$

it follows that

$$\int_\Omega \nabla u\cdot(V\nabla u)\rho dx \leqslant \liminf_{k_j\to\infty}\int_\Omega \nabla u_{k_j}\cdot(V\nabla u_{k_j})\rho dx$$

and $u_{k_j} \to u$ a.e. in Ω. By Lebesgue dominated convergence theorem, we obtain that

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$J[u] \leqslant \beta$. Therefore, u is a global minimizer of $J[u]$ which is different from 0 and 1, and the existence of non-trivial steady state of (1.6) would then be established if $0 \leqslant u \leqslant 1$.

We now come to the new, key part of this proof. We will show that *there exists a minimizing sequence, still denoted by* $\{u_k\}_{k=1}^{\infty}$, *such that* $0 \leqslant u_k \leqslant 1$ *for all* $k \geqslant 1$. Then, from the definition of $J[u]$, *it is a bounded minimizing sequence in* $H^1(\Omega)$, *and, therefore the global minimizer u obtained from this minimizing sequence has the desired property that* $0 \leqslant u \leqslant 1$.

Again, we assume that $J[u_k] < \min\{J[0], J[1]\}$ for $k \geqslant 1$. Note that under this assumption, we always have

$$|\{x \in \Omega \mid 0 < u_k < 1\}| \neq 0.$$

As an intermediate step, we show that *there exists a minimizing sequence, still denoted by* $\{u_k\}_{k=1}^{\infty}$, *such that* $0 \leqslant \bar{u}_k \leqslant 1$ *for all* $k \geqslant 1$. First, if $|\{x \in \Omega \mid u_k < 0\}| \neq 0$ for some $k \geqslant 1$, then simply replace u_k by

$$\tilde{u}_k = \begin{cases} 0 & \text{if } u_k(x) < 0, \\ u_k(x) & \text{if } u_k(x) \geqslant 0. \end{cases}$$

We now show that $J[\tilde{u}_k] \leqslant J[u_k]$.

According to the definition of $J[u]$, we only need demonstrate that

$$\int_\Omega (\tilde{u}_k^2 - \bar{\tilde{u}}_k^2)\rho dx \leqslant \int_\Omega (u_k^2 - \bar{u}_k^2)\rho dx. \tag{3.2}$$

Denote $\Omega_+^k = \{x \in \Omega : u_k(x) > 0\}$, $\Omega_-^k = \{x \in \Omega : u_k(x) < 0\}$. Then, one has

$$\begin{aligned}
&\int_\Omega (u_k^2 - \bar{u}_k^2)\rho dx - \int_\Omega (\tilde{u}_k^2 - \bar{\tilde{u}}_k^2)\rho dx \\
&\quad = \int_{\Omega_-^k} u_k^2 \rho dx - \left(\int_{\Omega_+^k} u_k \rho dx + \int_{\Omega_-^k} u_k \rho dx\right)^2 + \left(\int_{\Omega_+^k} u_k \rho dx\right)^2 \\
&\quad = \int_{\Omega_-^k} u_k^2 \rho dx - \left(\int_{\Omega_-^k} u_k \rho dx\right)^2 - 2\int_{\Omega_+^k} u_k \rho dx \int_{\Omega_-^k} u_k \rho dx \\
&\quad \geqslant \int_{\Omega_-^k} u_k^2 \rho dx - \left(\int_{\Omega_-^k} u_k \rho dx\right)^2 \geqslant \int_{\Omega_-^k} u_k^2 \rho dx - \int_{\Omega_-^k} u_k^2 \rho dx \int_{\Omega_-^k} \rho dx \geqslant 0.
\end{aligned}$$

Thus, (3.2) holds and $J[\tilde{u}_k] \leqslant J[u_k]$ follows. Now we obtain a new minimizing sequence, still denoted by $\{u_k\}_{k=1}^{\infty}$ satisfying $u_k \geqslant 0$.

Now, if $0 \leqslant u_k \leqslant 1$ for all $k \geqslant 1$, then we are done. Otherwise, if $|\{x \in \Omega \mid u_k > 1\}| \neq 0$ for some $k \geqslant 1$, define

$$u_{k*} = \begin{cases} u_k & u_k \leqslant 1, \\ 1 & u_k > 1. \end{cases}$$

To show that $J[u_{k*}] < J[u_k]$, it suffices to verify that

$$\int_\Omega (u_{k*}^2 - \bar{u}_{k*}^2)\rho dx < \int_\Omega (u_k^2 - \bar{u}_k^2)\rho dx. \tag{3.3}$$

Similarly, set $B_k = \{x \in \Omega : u_k(x) > 1\}$ and $\eta_k(x) = u_k(x) - 1 > 0$ for $x \in B_k$. Direct computation yields that

$$\begin{aligned}
&\int_\Omega (u_k^2 - \bar{u}_k^2)\rho dx - \int_\Omega (u_{k*}^2 - \bar{u}_{k*}^2)\rho dx \\
&= \int_\Omega (u_k^2 - u_{k*}^2)\rho dx - (\bar{u}_k^2 - \bar{u}_{k*}^2) \\
&= \int_{B_k} \eta_k^2 \rho dx + 2\int_{B_k} \eta_k \rho dx - \left[\bar{u}_k^2 - \left(\bar{u}_k - \int_{B_k} \eta_k \rho dx\right)^2\right] \\
&= \int_{B_k} \eta_k^2 \rho dx + 2(1 - \bar{u}_k)\int_{B_k} \eta_k \rho dx + \left(\int_{B_k} \eta_k \rho dx\right)^2 \\
&= \int_{B_k} \eta_k^2 \rho dx + 2\int_\Omega (1 - u_k)\rho dx \int_{B_k} \eta_k \rho dx + \left(\int_{B_k} \eta_k \rho dx\right)^2 \\
&= \int_{B_k} \eta_k^2 \rho dx + 2\int_{\Omega \setminus B_k} (1 - u_k)\rho dx \int_{B_k} \eta_k \rho dx - \left(\int_{B_k} \eta_k \rho dx\right)^2 > 0.
\end{aligned}$$

Hence, (3.3) follows and we obtain a non-trivial global minimizer.

Finally, it is standard to show the existence of a stable non-trivial global minimizer and the details will be included in Appendix A for completeness. □

3.2 Non-existence of non-trivial steady states

In this sub-section, we will demonstrate the non-existence of non-trivial steady states of problem (1.6) provided that $d+b$ is large. Similar to [19, Theorem 1.2], it seems natural to assume that the conclusion of Theorem 1.2 does not hold and then derive a contradiction by limiting arguments. However, due to the presence of the non-local term, not only the case that $d \to 0$ but also the case that $b \to \infty$ requires extra care and one difficulty in handling the latter case is the lack of elliptic regularity.

Proof of Theorem 1.2 We assume that under conditions in (i), (ii) and (iii) respectively, there exist sequences $\{d_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ with

$$\lim_{k\to\infty} (d_k + b_k) = \infty$$

such that for $d = d_k$ and $b = b_k$, problem (1.6) admits a non-trivial steady state, denoted by $u_k(x)$, i.e., $u_k(x)$ satisfies

$$\begin{cases} d\dfrac{1}{\rho}\nabla \cdot [\rho V \nabla u] + b(\bar{u} - u) + g(x)f(u) = 0 & \text{in } \Omega, \\ \nu \cdot V\nabla u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

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with $d = d_k$ and $b = b_k$. First of all, multiplying the first equation in (3.4) by $u_k\rho$ and integrating it over Ω, we get

$$\int_\Omega \left[d_k \nabla u_k \cdot (V\nabla u_k) + b_k(u_k - \bar{u}_k)^2\right] \rho dx = \int_\Omega g(x) f(u_k) u_k \rho dx \leqslant C.$$

This implies that $u_k \to c$ a.e. as $k \to \infty$, where $c \in [0, 1]$ is a constant.

Now we treat the three cases (i), (ii), and (iii) separately.

(i) First, assume that $0 < c < 1$. Multiplying the first equation in (3.4) by ρ and integrating it, we have $\int_\Omega g(x) f(u_k)\rho dx = 0$, which yields that $\bar{g} = 0$ by letting $k \to \infty$. This is a contradiction.

Secondly, assume that $c = 0$. Since $\lim_{k\to\infty}(d_k + b_k) = \infty$, we will consider the following two cases separately:

Case 1. $\lim_{k\to\infty} b_k = \infty$;
Case 2. there exists $B > 0$ such that $b_k \leqslant B$ and $\lim_{k\to\infty} d_k = \infty$.

In *Case 1*, set $v_k = u_k/\bar{u}_k$. Then, v_k satisfies

$$\begin{cases} d_k \dfrac{1}{\rho} \nabla \cdot (\rho V \nabla v_k) + b_k(1 - v_k) + g(x)\dfrac{f(u_k)}{u_k} v_k = 0 & \text{in } \Omega, \\ \nu \cdot V\nabla v_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

We claim that *there exists* $K > 0$ *such that for* $k > K$, $0 \leqslant v_k \leqslant 2$ *in* $\bar{\Omega}$.

Let $v_k(z_k) = \max_{\bar{\Omega}} v_k(x)$. Then at $x = z_k$, we have

$$b_k(1 - v_k(z_k)) + g(z_k)\frac{f(u_k(z_k))}{u_k(z_k)} v_k(z_k) \geqslant 0,$$

which implies that $b_k(v_k(z_k) - 1) \leqslant c v_k(z_k)$ for some constant $c > 0$. This gives that $(b_k - c)v_k(z_k) < b_k$. Hence, since $\lim_{k\to\infty} b_k = \infty$, the assertion follows.

From (3.5), we easily obtain that

$$\int_\Omega \left[d_k \nabla v_k \cdot (V\nabla v_k) + b_k(v_k - 1)^2\right] \rho dx = \int_\Omega g(x)\frac{f(u_k)}{u_k} v_k^2 \rho dx \leqslant C.$$

Thus one sees that, by passing to a subsequence if necessary, $\lim_{k\to\infty} v_k = 1$ a.e. in Ω. Multiplying (3.5) by $\rho(x)(\bar{u}_k)^{-k_1+1}$ and integrating it directly, we have

$$\int_\Omega g(x)\frac{f(u_k)}{(u_k)^{k_1}}(v_k)^{k_1} \rho dx = 0.$$

Again by letting $k \to \infty$, assumption (1.7) and Lebesgue dominated convergence theorem yield that $\bar{g} = 0$ since $\lim_{k\to\infty} u_k = 0$ and $\lim_{k\to\infty} v_k = 1$ a.e. in Ω. This is a contradiction.

In *Case 2*, set $w_k(x) = u_k(x)/\|u_k\|_{L^\infty}$. Then, w_k satisfies

$$\begin{cases} d_k \dfrac{1}{\rho} \nabla \cdot (\rho V \nabla w_k) + b_k(\bar{w}_k - w_k) + g(x)\dfrac{f(u_k)}{u_k} w_k = 0 & \text{in } \Omega, \\ \nu \cdot V\nabla w_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

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By assumption (A1) and elliptic regularity, we have $\|w_k\|_{C^{2,\alpha}(\bar{\Omega})}$ is uniformly bounded and thus there exist $w \in C^2(\bar{\Omega})$, and a subsequence of $\{w_k\}_{k=1}^{\infty}$, still denoted by $\{w_k\}_{k=1}^{\infty}$, such that $w_k \to w$ in $C^2(\bar{\Omega})$. Hence, clearly w satisfies

$$\begin{cases} \frac{1}{\rho}\nabla \cdot (\rho V \nabla w) = 0 & \text{in } \Omega, \\ v \cdot V\nabla w = 0 & \text{on } \partial\Omega, \\ \|w\|_{L^\infty} = 1. \end{cases}$$

This implies that $w \equiv 1$ in Ω. Then similarly, multiplying the first equation in (3.6) by $\rho(x)(\|u_k\|_{L^\infty})^{-k_1+1}$ and integrating it, we obtain that

$$\int_\Omega g(x)\frac{f(u_k)}{(u_k)^{k_1}}(w_k)^{k_1}\rho dx = 0,$$

which, due to assumption (1.7), implies that $\bar{g} = 0$ by letting $k \to \infty$. This is a contradiction.

Finally, assume that $c = 1$. Set $\tilde{u}_k = 1 - u_k$, then, based on assumption (1.8), a contradiction can be derived in a similar fashion as in the case $c = 0$ by studying the problem satisfied by $\tilde{u}_k$, and this completes the proof of (i).

(ii) According to assumptions in (ii), it is easy to see that the cases $0 < c < 1$ and $c = 1$ can be handled by applying the same approaches in the proof of (i) to derive contradictions.

It remains to deduce a contradiction provided that $c = 0$. Similar to the proof of (i), the following two cases will be considered:

Case 1. $\lim_{k\to\infty} b_k = \infty$;
Case 2. there exists $B > 0$ such that $b_k \leqslant B$ and $\lim_{k\to\infty} d_k = \infty$.

In *Case 1*, first notice that we only have $u_k \to 0$ a.e. as $k \to \infty$. Then, it implies that $\bar{u}_k \to 0$ as $k \to \infty$. Moreover, same as the arguments in *Case 1* in the proof of (i), we can derive that there exists $K > 0$ such that for $k > K$, $0 \leqslant u_k \leqslant 2\bar{u}_k$ in $\bar{\Omega}$. Hence, it follows that there exists $K_1 > 0$ such that

$$\|u_k\|_{L^\infty} < \delta_0 \text{ for } k > K_1. \tag{3.7}$$

For $k > K_1$, dividing the equation (3.4) for u_k by $f(u_k)/\rho$ and integrating, we have

$$\int_\Omega \left[d_k \frac{\frac{1}{\rho}\nabla \cdot (\rho V \nabla u_k)}{f(u_k)} + \frac{b_k(\bar{u}_k - u_k)}{f(u_k)} + g(x) \right] \rho dx = 0,$$

which is equivalent to

$$\int_\Omega \left[d_k \frac{\nabla u_k \cdot (V\nabla u_k)}{f^2(u_k)} f'(u_k) + \frac{b_k(\bar{u}_k - u_k)}{f(u_k)} - \frac{b_k(\bar{u}_k - u_k)}{f(\bar{u}_k)} \right] \rho dx + \bar{g} = 0.$$

This implies that

$$d_k \int_\Omega \frac{f'(u_k)}{f^2(u_k)} \nabla u_k \cdot (V\nabla u_k)\rho dx + b_k \int_\Omega \frac{f'(\xi_k)}{f(u_k)f(\bar{u}_k)}(\bar{u}_k - u_k)^2 \rho dx + \bar{g} = 0,$$

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where $\xi_k \in (0, \delta_0)$. This implies that $\bar{g} \leqslant 0$ since $f'(s) \geqslant 0$ in $(0, \delta_0)$. This contradicts the assumption that $\bar{g} > 0$.

In *Case 2*, since $b_k \leqslant B$ and $\lim_{k\to\infty} d_k = \infty$, by elliptic regularity, we have $\{u_k\}_{k=1}^{\infty}$ is bounded in $C^{2,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$. Moreover, observe that $u_k \to 0$ a.e. as $k \to \infty$, thus it is routine to derive that $\|u_k\|_{L^\infty} \to 0$ as $k \to \infty$. Then following the arguments in *Case 1*, a contradiction can be derived.

(iii) Set $\tilde{u}_k = 1 - u_k$ and consider the problem satisfied by $\tilde{u}_k$. The proof in (ii) can be applied to yield a contradiction. This completes the proof. □

3.3 Local stability of $u \equiv 0$ and $u \equiv 1$

This section is devoted to the proofs of Theorems 1.3–1.5, which are about the stability/instability properties of model (1.6) at $u \equiv 0$ and $u \equiv 1$. For clarity, we point out that in this section, conditions (A), (A1) and (A3) are always assumed.

The proofs of Theorem 1.3 (i) and (ii) are divided into the following two lemmas. We first make use of an energy functional to show that under the assumption $\bar{g} \geqslant 0$, when $u \equiv 0$ is isolated, it must be unstable. Then in the second lemma, verify that the conditions in Theorem 1.3 (i) and (ii) guarantee the isolation of $u \equiv 0$ and thus Theorem 1.3 (i) and (ii) are proved.

Lemma 3.1 *Assume that $\bar{g} \geqslant 0$ and $u \equiv 0$ is an isolated equilibrium of (1.6), then $u \equiv 0$ is unstable for all $d > 0$, $b > 0$.*

Proof As in (3.1), define

$$J[u] = \int_\Omega \left[\frac{d}{2} \nabla u \cdot (V \nabla u) + \frac{b}{2} u^2 - \frac{b}{2} \bar{u}^2 - g(x) F(u) \right] \rho dx,$$

where $F(u) = \int_0^u f(s)ds$ with $f(u) > 0$ for $u \in (0, 1)$ and $f(u) = 0$ outside the interval $(0, 1)$. Direct computation yields that $\frac{d}{dt} J[u](t) = -\int_\Omega u_t^2 \rho dx \leqslant 0$, thus, $J[u(\cdot, t; u_0)]$ is decreasing in t. Moreover, by Theorems 2.3 and 2.4, for any $u_0 \in \mathcal{K}$, the ω-limit set $\omega[u_0]$ is non-empty and consists of equilibria of (1.6). Since $u \equiv 0$ is an isolated equilibrium of (1.6), there exists $\ell > 0$ such that $u \equiv 0$ is the only equilibrium in $\{u \in \mathcal{K} : \|u\|_{L^\infty} \leqslant \ell\}$, where $\mathcal{K}$ is defined in (1.11). Then, we will consider two cases separately.

Case 1: $\bar{g} > 0$. For any $0 < \delta < 1$, letting $u_0 = \delta$, we have

$$J[u_0] = -F(\delta)\bar{g} < 0 = J[0].$$

This means that there exist a sequence $\{t_k\}_{k=1}^{\infty}$ and $u_\infty \in C^2(\Omega)$ such that

$$u(\cdot, t_k; u_0) \to u_\infty \in \omega[u_0] \text{ in } L^\infty,$$

where $u_\infty \neq 0$ is an equilibrium. Thus $\|u_\infty\|_{L^\infty} > \ell$, which implies that for k large, $\|u(\cdot, t_k; u_0)\|_{L^\infty} > \ell/2$. Therefore, $u \equiv 0$ is unstable.

Case 2: $\bar{g} = 0$. We may choose $0 < \phi \in C^1(\bar{\Omega})$ such that $\int_\Omega g\phi\rho dx > 0$. Let $u_0 = c + \epsilon\phi$, where $c \in (0,1)$ and $\epsilon > 0$ is small. Then,

$$
\begin{aligned}
J[u_0] &= J[c+\epsilon\phi] \\
&= \int_\Omega \left[\frac{d}{2}\epsilon^2 \nabla\phi \cdot (V\nabla\phi) + \frac{b}{2}(c+\epsilon\phi)^2 - \frac{b}{2}(c+\epsilon\bar{\phi})^2 - g(x)F(c+\epsilon\phi)\right]\rho dx \\
&= -\epsilon\left[f(c)\int_\Omega g(x)\phi\rho dx + O(\epsilon)\right] < 0
\end{aligned}
$$

for $\epsilon > 0$ sufficiently small. Hence for any $0 < \delta < 1$, we can find $u_0 \in \mathcal{K}$ such that $\|u_0\|_{L^\infty} < \delta$ and $\omega[u_0]$ does not contain 0. Then similar to *Case 1*, it can be proved that $u \equiv 0$ is unstable.

In summary, if $\bar{g} \geqslant 0$, then for any $d, b > 0$, $u \equiv 0$ is unstable provided that it is an isolated equilibrium. □

Thanks to Lemma 3.1, to prove (i) and (ii) of Theorem 1.3, it suffices to show that $u \equiv 0$ is an isolated equilibrium of (1.6) when the assumptions in (i) and (ii) of Theorem 1.3 are imposed. This is verified in the following lemma.

Lemma 3.2 *For problem (1.6), $u \equiv 0$ is an isolated equilibrium for all $d, b > 0$, if one of the following statements is valid:*

(a) $\bar{g} = 0$ *and condition (1.9) holds with* $|\{s \in (0,\delta_0) : f'(s) = 0\}| = 0$*;*

(b) $\bar{g} > 0$, *and (1.7) holds;*

(c) $\bar{g} > 0$, *and (1.9) holds.*

Proof Suppose, for contradiction, that $u \equiv 0$ is not an isolated equilibrium of (1.6) for some $d, b > 0$. Then there exists a sequence of equilibria $\{u_k\}_{k=1}^\infty$ with $0 < u_k < 1$ and $\lim_{k\to\infty}\|u_k\|_{L^\infty} = 0$. We will discuss how to obtain contradictions under assumptions in (a), (b) and (c) respectively.

First, assume that (a) is valid. Recall that u_k satisfies

$$
\begin{cases}
d\dfrac{1}{\rho}\nabla\cdot(\rho V\nabla u) + b(\bar{u} - u) + g(x)f(u) = 0 & \text{in } \Omega, \\
\nu \cdot \nabla u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{3.8}
$$

where $f(u) > 0$ since $0 < u < 1$. Dividing the equation by $f(u_k)/\rho$ and integrating it, we have

$$
\int_\Omega \left[d\frac{\nabla u_k \cdot (V\nabla u_k)}{f^2(u_k)} f'(u_k) + \frac{b(\bar{u}_k - u_k)^2}{f(u_k)f(\bar{u}_k)} f'(\xi_k)\right]\rho dx + \bar{g} = 0, \tag{3.9}
$$

where $\xi_k(x) \in [\min\{u_k(x), \bar{u}_k\}, \max\{u_k(x), \bar{u}_k\}]$. Since $\lim_{k\to\infty}\|u_k\|_{L^\infty} = 0$, there exists $K > 0$ such that for $k > K$, $\|u_k\|_{L^\infty} < \delta_0$, then a contradiction arises immediately from the assumptions and (3.9).

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Next assume (b) holds. Dividing the equation (3.8) for u_k by $(u_k)^{k_1}/\rho$ and integrating it, we get

$$\int_\Omega \left[dk_1 \frac{\nabla u_k \cdot (V\nabla u_k)}{(u_k)^{k_1+1}} + b\frac{(\bar{u}_k)^{k_1} - (u_k)^{k_1}}{(u_k)^{k_1}(\bar{u}_k)^{k_1}}(\bar{u}_k - u_k) + g(x)\frac{f(u_k)}{(u_k)^{k_1}} \right] \rho dx = 0.$$

This indicates that

$$\int_\Omega g(x)\frac{f(u_k)}{(u_k)^{k_1}}\rho dx \leqslant 0.$$

Since $\lim_{k\to\infty} \|u_k\|_{L^\infty} = 0$ and (1.7) holds, by letting $k \to \infty$, we have $a_1\bar{g} \leqslant 0$. This is a contradiction.

At the end, assume that (c) is true. Since $\lim_{k\to\infty} \|u_k\|_{L^\infty} = 0$, the left-hand side of (3.9) is strictly positive for large k. This is impossible. □

We summarize that, as mentioned earlier, Theorem 1.3 (i) and (ii) follow immediately from Lemmas 3.1 and 3.2. By considering the equation satisfied by $1-u$, Theorem 1.4 (i) and (ii) can be proved in the same way. Since similarly Theorem 1.4 (iii) is the dual version of Theorem 1.3 (iii), we only demonstrate the proof of Theorem 1.3 (iii) as follows. Degree theory is employed to handle Theorem 1.3 (iii). We argue by contradiction and suppose that for problem (1.6) with some $d = D$, $u \equiv 0$ is unstable. This, together with Theorem 1.4 (i), implies the existence of non-trivial steady state with $d = D$. Then, note that a steady state of (1.6) satisfies

$$u - \mathcal{H}(d,u) = u - \left(-\frac{1}{\rho}\nabla\cdot(\rho V\nabla) + \Lambda\mathcal{I}\right)^{-1}\left[\frac{b}{d}(\bar{u}-u) + \frac{1}{d}g(x)f(u) + \Lambda u\right] = 0.$$

The conditions in Theorem 1.3 (iii) ensure that non-trivial steady states of (1.6) are uniformly bounded away from both $u \equiv 0$ and $u \equiv 1$ for $d \geqslant D$. Thus, the degree of $\mathcal{I} - \mathcal{H}(d,\cdot)$ is well defined for $d \geqslant D$ in certain set of functions bounded away from $u \equiv 0$ and $u \equiv 1$. On the one side, prove that the degree of $\mathcal{I} - \mathcal{H}(d,\cdot)$ at 0 is 1 at $d = D$. On the other side, Theorem 1.2 indicates that the degree of $\mathcal{I} - \mathcal{H}(d,\cdot)$ at 0 is 0 for d large. This contradicts to homotopy invariance.

Proof of Theorem 1.3 (iii) Now we focus on the case that $\bar{g} < 0$. For clarity, the proof will be divided into three steps.

Step 1. We claim that *for any* $D > 0$, *there exists* $\delta_1 = \delta_1(D) > 0$ *such that*

(1) $\min_{\bar{\Omega}} u \geqslant \delta_1$ *for any non-trivial steady state of (1.6) with* $d \geqslant D$ *if (1.7) holds.*

(2) $\max_{\bar{\Omega}} u \leqslant 1 - \delta_1$ *for any non-trivial steady state of (1.6) with* $d \geqslant D$ *provided that either (1.8) or (1.10) is imposed.*

Suppose that (1) is not true, then there exist $\{u_k\}_{k=1}^\infty$ and $\{d_k\}_{k=1}^\infty$ satisfying $\lim_{k\to\infty} \min_{\bar{\Omega}} u_k = 0$ and $d_k \geqslant D$, where u_k is a steady state of (1.6) with $d = d_k$, i.e., u_k satisfies

$$\begin{cases} d_k\dfrac{1}{\rho}\nabla\cdot(\rho V\nabla u_k) + b(\bar{u}_k - u_k) + g(x)f(u_k) = 0 & \text{in } \Omega, \\ \nu\cdot V\nabla u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

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By elliptic regularity, there exist a subsequence of $\{u_k\}_{k=1}^{\infty}$, still denoted by $\{u_k\}_{k=1}^{\infty}$, $u_0 \in C^2(\bar{\Omega})$ and $\lambda_0 \in [0, 1/D]$ such that

$$1/d_k \to \lambda_0 \ \text{ and } \ u_k \to u_0 \text{ in } C^2(\bar{\Omega}).$$

Hence, u_0 satisfies

$$\begin{cases} \dfrac{1}{\rho}\nabla \cdot (\rho V \nabla u_0) + \lambda_0 b(\bar{u}_0 - u_0) + \lambda_0 g(x) f(u_0) = 0 & \text{in } \Omega, \\ \nu \cdot V \nabla u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\min_{\bar{\Omega}} u_0 = 0$. Due to the Maximum principle, we have $u_0 \equiv 0$, which tells us $u_k \to 0$ in $C^2(\bar{\Omega})$.

Next, consider the problem satisfied by $\tilde{u}_k = u_k / \|u_k\|_{L^\infty}$ as follows:

$$\begin{cases} d_k \dfrac{1}{\rho}\nabla \cdot (\rho V \nabla \tilde{u}_k) + b(\bar{\tilde{u}}_k - \tilde{u}_k) + g(x) \dfrac{f(u_k)}{u_k} \tilde{u}_k = 0 & \text{in } \Omega, \\ \nu \cdot V \nabla \tilde{u}_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

We may assume that $\tilde{u}_k \to \tilde{u}_0$ in $C^2(\bar{\Omega})$ by passing to a subsequence if necessary, where $\tilde{u}_0$ satisfies

$$\begin{cases} \dfrac{1}{\rho}\nabla \cdot (\rho V \nabla \tilde{u}_0) + \lambda_0 b(\bar{\tilde{u}}_0 - \tilde{u}_0) = 0 & \text{in } \Omega, \\ \nu \cdot V \nabla \tilde{u}_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $f'(0) = 0$ is used. This implies that $\tilde{u}_0$ equals a constant. While $\max_{\bar{\Omega}} \tilde{u}_0 = 1$, we have $\tilde{u}_0 \equiv 1$.

Now, back to (3.10), we easily obtain that

$$\int_\Omega g(x) \frac{f(u_k)}{(u_k)^{k_1}} (\tilde{u}_k)^{k_1} \rho dx = 0,$$

which, by letting $k \to \infty$, immediately yields that $\bar{g} = 0$ due to condition (1.7). A contradiction arises and (1) is proved.

As for the proof of (2), suppose that (2) is not true, then there exist $\{u_k\}_{k=1}^{\infty}$ and $\{d_k\}_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty} \max_{\bar{\Omega}} u_k = 1$ and $d_k \geqslant D$, where u_k is a steady state of (1.6) with $d = d_k$. Similar to (1), we can show that $u_k \to 1$ in $C^2(\bar{\Omega})$ by passing to a subsequence if necessary.

First, assume that (1.8) is valid. Recall that u_k satisfies

$$\begin{cases} d_k \dfrac{1}{\rho}\nabla \cdot (\rho V \nabla u_k) + b(\bar{u}_k - u_k) + g(x) f(u_k) = 0 & \text{in } \Omega, \\ \nu \cdot V \nabla u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the equation for u_k by $\rho(x)(1 - u_k)^{-k_2}$ and integrating, we obtain

$$\int_\Omega \left[-d_k k_2 \frac{\nabla u_k \cdot (V \nabla u_k)}{(1 - u_k)^{k_2+1}} + \frac{b(\bar{u}_k - u_k)}{(1 - u_k)^{k_2}} - \frac{b(\bar{u}_k - u_k)}{(1 - \bar{u}_k)^{k_2}} + g(x) \frac{f(u_k)}{(1 - u_k)^{k_2}} \right] \rho dx = 0,$$

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which gives that

$$\int_\Omega \left[-d_k k_2 \frac{\nabla u_k \cdot (V \nabla u_k)}{(1-u_k)^{k_2+1}} + b \frac{(\bar{u}_k - u_k)\left[(1-\bar{u}_k)^{k_2} - (1-u_k)^{k_2}\right]}{(1-u_k)^{k_2}(1-\bar{u}_k)^{k_2}} \right] \rho dx + \int_\Omega g(x) \frac{f(u_k)}{(1-u_k)^{k_2}} \rho dx = 0.$$

This clearly shows that

$$\int_\Omega g(x) \frac{f(u_k)}{(1-u_k)^{k_2}} \rho dx \geqslant 0.$$

Then by letting $k \to \infty$, one easily sees that, according to condition (1.8), $\bar{g} \geqslant 0$, which is a contradiction.

Secondly, assume that (1.10) is satisfied. Dividing the equation of u_k by $f(u_k)/\rho$ and integrating by parts, we have

$$\int_\Omega \left[d_k \frac{\nabla u_k \cdot (V \nabla u_k)}{f^2(u_k)} f'(u_k) + b \frac{(\bar{u}_k - u_k)^2}{f(u_k) f(\bar{u}_k)} f'(\xi) \right] \rho dx + \bar{g} = 0,$$

where $\xi(x) \in [\min\{u_k(x), \bar{u}_k\}, \max\{u_k(x), \bar{u}_k\}]$. Note that $u_k \to 1$ in $C^2(\bar{\Omega})$, combined with condition (1.10), we get $\bar{g} \geqslant 0$, which is a contradiction.

Therefore, our assertion is established.

Step 2. For any $d > 0$, choose $\Lambda = \Lambda(d) > 0$ sufficiently large such that the mapping

$$u \mapsto \frac{b}{d}(\bar{u} - u) + \frac{1}{d} g(x) f(u) + \Lambda u$$

is increasing for all $x \in \bar{\Omega}$ and $u \in [0, 1]$.

Let $\mathcal{I}$ denote the identity map from $C(\bar{\Omega})$ to $C(\bar{\Omega})$ and $\left(-\frac{1}{\rho}\nabla \cdot (\rho V \nabla) + \Lambda \mathcal{I}\right)^{-1}$ is the inverse of the operator $-\frac{1}{\rho}\nabla \cdot (\rho V \nabla) + \Lambda \mathcal{I}$ with no flux boundary condition. Define $\mathcal{H}(d, u) : \mathbb{R} \times C(\bar{\Omega}) \to C(\bar{\Omega})$ by

$$\mathcal{H}(d, u) = \left(-\frac{1}{\rho}\nabla \cdot (\rho V \nabla) + \Lambda \mathcal{I}\right)^{-1} \left[\frac{b}{d}(\bar{u} - u) + \frac{1}{d} g(x) f(u) + \Lambda u\right].$$

It is standard to check that $\mathcal{H}(d, \cdot)$ is compact, strictly order-preserving and maps $\mathcal{K}$ into itself, where $\mathcal{K}$ is defined in (1.11).

According to the claim in **Step 1**, there exists $\delta_1 = \delta_1(D) > 0$ such that

$$\delta_1 \leqslant u(x) \leqslant 1 - \delta_1 \text{ in } \bar{\Omega}$$

for any non-trivial steady state of (1.6) with $d \geqslant D$. This demonstrates that for any $d \geqslant D$, $\mathcal{H}(d, \cdot)$ has no fixed points on $\partial \mathcal{U}_\epsilon$ where

$$\mathcal{U}_\epsilon := \{u \in C(\bar{\Omega}) : \epsilon < u(x) < 1 - \epsilon, \ \forall \, x \in \bar{\Omega}\},$$

provided that $0 < \epsilon < \delta_1$.

Therefore, due to the homotopy invariance of Leray–Schauder degree, $\deg(\mathcal{I} - \mathcal{H}(d, \cdot), \mathcal{U}_\epsilon, 0)$ is independent of $d \in [D, \infty]$, provided that $0 < \epsilon < \delta_1 = \delta_1(D)$.

Step 3. Suppose that for problem (1.6) with some $d = D$, $u \equiv 0$ is unstable. Then, there exists a minimal non-trivial solution of (3.4), denoted by u_1. Since $u \equiv 0$ is unstable from above and there exists no further fixed point of $\mathcal{H}(d, \cdot)$ between 0 and u_1, we may get strict lower solutions as close to $u \equiv 0$ as we wish.

On the other hand, by Theorem 1.4 (ii), when $\bar{g} < 0$, $u \equiv 1$ is unstable from below. Similarly, there exists a maximal non-trivial solution of (3.4) with $d = D$, denoted by u_2 and we may get strict upper solutions as close to $u \equiv 1$ as we wish.

Now, fix $\epsilon = \delta_1(D)/2$ and choose a strict lower solution $\underline{u}_1$ with $0 < \underline{u}_1 < \epsilon$ and a strict upper solution $\hat{u}_2$ with $1 - \epsilon < \hat{u}_2 < 1$. Then we have $\mathcal{S} \supset \mathcal{U}_\epsilon$, where

$$\mathcal{S} := \{u \in C(\bar{\Omega}) : \underline{u}_1 < u(x) < \hat{u}_2, \ \forall \ x \in \bar{\Omega}\}.$$

It is routine to show that $\mathcal{H}(D, \cdot)$ has no fixed points on $\partial\mathcal{S}$.

Next, take any $u_0 \in \mathcal{S}$ and for $(\sigma, u) \in [0, 1] \times \mathcal{S}$, define

$$h(\sigma, u) := (1 - \sigma)u_0 + \sigma\mathcal{H}(D, u).$$

Thanks to the choice of $\mathcal{S}$, it can be verified that for all $\sigma \in [0, 1]$, $h(\sigma, \cdot)$ has no fixed points on $\partial\mathcal{S}$. We omit the details here. Then, again by the homotopy invariance of Leray–Schauder degree, we see that

$$\deg(\mathcal{I} - \mathcal{H}(D, \cdot), \mathcal{S}, 0) = \deg(\mathcal{I} - u_0, \mathcal{S}, 0) = 1.$$

Moreover, no fixed points of $\mathcal{H}(D, \cdot)$ are allowed in the claim in $\mathcal{S} \setminus \mathcal{U}_\epsilon$ due to **Step 1**, thus it follows that

$$\deg(\mathcal{I} - \mathcal{H}(D, \cdot), \mathcal{U}_\epsilon, 0) = 1. \tag{3.11}$$

However, according to Theorem 1.2, if $\bar{g} < 0$

$$\deg(\mathcal{I} - \mathcal{H}(d, \cdot), \mathcal{U}_\epsilon, 0) = 0 \tag{3.12}$$

when $d > 0$ large enough.

Therefore, based on the conclusion obtained in **Step 2**, (3.11) and (3.12) together give rise to a contradiction. □

We remark that in Theorem 1.3 (i), the assumption $|\{s \in (0, \delta_0) : f'(s) = 0\}| = 0$ in addition to (1.9) cannot be removed due to the following example, which is constructed based on [12, Example 2.4].

Example 3.1. Let $\Omega = (-1, 1)$. For each $k = 1, 2, \ldots$, set $a_k = \left(\frac{1}{2}\right)^{k-1}$ and define

$$f(s)|_{[a_{k+1}, a_k]} = \begin{cases} c_k & \text{if } k \text{ is odd,} \\ f_k & \text{if } k \text{ is even,} \\ 0 & \text{if } k = \infty, \end{cases}$$

where, $c_k = \frac{3a_{k+1}}{2k}$ and f_k is chosen properly as in [12, Example 2.4] to guarantee that f

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satisfies (A3) and (1.9). Then one can check that, for each odd positive integer k,

$$u_k(x) = a_{k+1} + \frac{c_k}{2}\left(-\frac{1}{3}x^3 + x + \frac{2}{3}\right), \quad -1 \leqslant x \leqslant 1$$

is a solution of

$$\begin{cases} \Delta u + (\bar{u} - u) + \left(\frac{3}{2}x - \frac{1}{6}x^3\right) f(u) = 0 & \text{in } (-1,1), \\ 0 \leqslant u \leqslant 1 & \text{in } (-1,1), \\ u'(-1) = u'(1) = 0. \end{cases} \tag{3.13}$$

Moreover, it is routine to verify that

$$\|u_k\|_{L^\infty} = \frac{k+1}{k} a_{k+1} = \frac{k+1}{k}\left(\frac{1}{2}\right)^k \to 0 \text{ as } n \to \infty.$$

Therefore, $u \equiv 0$ is not an isolated solution of (3.13) and $u \equiv 0$ is stable from above.

For the rest of this section, we will focus on the proof of Theorem 1.5, which is mainly based on linearized analysis. Compared with that of Theorem 1.4, the key difference is that $f'(1) < 0$ now.

A useful lemma will be prepared first.

Lemma 3.3 *For the non-local eigenvalue problem*

$$b(\bar{\phi} - \phi) + h(x)\phi = \mu\phi \tag{3.14}$$

where $h(x) \in C(\bar{\Omega})$, $\Omega \subset \mathbb{R}^N$, given any $\epsilon > 0$, there exists $h_\epsilon \in C^N(\bar{\Omega})$ such that $\|h - h_\epsilon\|_{L^\infty} < \epsilon$ and the non-local eigenvalue problem

$$b(\bar{\phi} - \phi) + h_\epsilon(x)\phi = \mu\phi \tag{3.15}$$

admits a principal eigenvalue with strictly positive eigenfunction in $C(\bar{\Omega})$.

This is a standard result for linear operator with non-local diffusion and the proof will be included in Appendix B for the convenience of readers.

Proof of Theorem 1.5

(i) The linearized problem of (1.6) at $u \equiv 1$ is

$$\begin{cases} d\frac{1}{\rho}\nabla \cdot [\rho V \nabla \psi] + b(\bar{\psi} - \psi) + g(x) f'(1)\psi = \lambda\psi & \text{in } \Omega, \\ \nu \cdot V\nabla\psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

and let $\lambda_0(d, b)$ denote the principle eigenvalue, which can be characterized as

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follows:

$$\lambda_0(d,b) = \sup_{0\neq\psi\in H^1(\Omega)} \frac{\int_\Omega \left[-d\nabla\psi\cdot(V\nabla\psi) - b(\psi-\bar{\psi})^2 + g(x)f'(1)\psi^2\right]\rho dx}{\int_\Omega \psi^2\rho dx}. \quad (3.17)$$

This shows that

$$\lambda_0(d,b) > \frac{\int_\Omega g(x)f'(1)\rho dx}{\int_\Omega \rho dx} = f'(1)\bar{g} \geqslant 0,$$

where the above strict inequality is due to the assumption that g is non-constant. Hence, $u \equiv 1$ is linearly unstable for any $d, b > 0$.

(ii) First, we claim that $\mu_0(b)$, *defined in (1.13), admits a unique positive root* b^*. By Lemma 3.3, for any $\epsilon > 0$, there exists $g_\epsilon \in C^N(\bar{\Omega})$ such that $\|g - g_\epsilon\|_{L^\infty} < \epsilon$, the principal eigenvalue of

$$b(\bar{\phi} - \phi) + g_\epsilon(x)f'(1)\phi = \mu_\epsilon\phi \quad \text{in } \Omega \quad (3.18)$$

exists, denoted by $\mu_\epsilon(b)$ and the corresponding normalized eigenfunction $\phi_{\epsilon,b}(x)$, i.e. $\|\phi_{\epsilon,b}\|_{L^2} = \left(\int_\Omega \phi_{\epsilon,b}^2\rho dx\right)^{1/2} = 1$, is strictly positive and continuous in $\bar{\Omega}$.

Noting that

$$\mu_\epsilon(b) = \sup_{\phi\in L^2(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[-b(\phi^2 - \bar{\phi}^2) + g_\epsilon(x)f'(1)\phi^2\right]\rho dx}{\int_\Omega \phi^2\rho dx},$$

we see immediately that

$$f'(1)\bar{g}_\epsilon \leqslant \mu_\epsilon(b) \leqslant \max_{\bar{\Omega}}(f'(1)g_\epsilon(x)) = f'(1)\min_{\bar{\Omega}} g_\epsilon(x). \quad (3.19)$$

Moreover, it follows from (3.18) that

$$\phi_{\epsilon,b}(x) = \frac{b}{b - f'(1)g_\epsilon(x) + \mu_\epsilon(b)}\bar{\phi}_{\epsilon,b}.$$

Letting $b \to \infty$, one sees that $\phi_{\epsilon,b} \to 1$ in L^∞ because of (3.19) and $\|\phi_{\epsilon,b}\|_{L^2} = 1$. Next, integrating (3.18) after multiplying it by ρ, we have

$$\lim_{b\to\infty} \mu_\epsilon(b) = f'(1)\bar{g}_\epsilon < 0.$$

Furthermore, it is routine to verify that

$$\lim_{b\to 0} \mu_\epsilon(b) = f'(1)\min_{\bar{\Omega}} g_\epsilon(x) > 0.$$

According to the definitions of $\mu_0(b)$ and $\mu_\epsilon(b)$, it is obvious that $|\mu_0(b) - \mu_\epsilon(b)| < -f'(1)\epsilon$. Since ϵ is arbitrary, we have

$$\lim_{b\to 0} \mu_0(b) = f'(1)\min_{\bar{\Omega}} g(x) > 0, \quad \lim_{b\to\infty} \mu_0(b) = f'(1)\bar{g} < 0.$$

Moreover, $\mu_0(b)$ is continuous and strictly decreasing in b, thus there exists $b^* > 0$ such that

$$\mu_0(b) > 0 \text{ for } 0 < b < b^*, \;\; \mu_0(b^*) = 0 \text{ and } \mu_0(b) < 0 \text{ for } b > b^*. \tag{3.20}$$

The claim is proved. Then, due to (1.13) and (3.17), it is clear that $\mu_0(b) > \lambda_0(d, b)$ for any $d > 0$. Hence by (3.20), for $d > 0$, $b \geqslant b^*$, $u \equiv 1$ is linearly stable.

Secondly, let us designate d^*. For the eigenvalue problem (1.14), we have

$$\ell_0(d) = \sup_{0 \neq \varphi \in H^1(\Omega)} \frac{\int_\Omega \left[-d\nabla\varphi \cdot (V\nabla\varphi) + g(x)f'(1)\varphi^2\right] \rho dx}{\int_\Omega \varphi^2 \rho dx}.$$

Based on this characterization, it is routine to show that

$$\lim_{d \to 0} \ell_0(d) = f'(1) \min_{\bar{\Omega}} g(x) > 0.$$

On the other hand, similar to (3.19), we have, for all $d > 0$,

$$f'(1)\bar{g} \leqslant \ell_0(d) \leqslant f'(1) \min_{\bar{\Omega}} g(x).$$

then standard arguments show that for any small $\epsilon > 0$, we have $\ell_0(d) \leqslant f'(1)\bar{g} + \epsilon$ for all d large. Thus,

$$\lim_{d \to \infty} \ell_0(d) = f'(1)\bar{g} < 0.$$

Also, one sees that $\ell_0(d)$ is decreasing in d. Therefore, there exists $d^* > 0$ such that

$$\ell_0(d) > 0 \text{ for } 0 < d < d^*, \;\; \ell_0(d^*) = 0 \text{ and } \ell_0(d) < 0 \text{ for } d > d^*.$$

This immediately implies that for $d \geqslant d^*$, $b > 0$, $u \equiv 1$ is linearly stable since $\lambda_0(d, b) < \ell_0(d) \leqslant 0$.

Now, fix $0 < b < b^*$, since $\mu_0(b) > 0$ and $\phi_{\epsilon,b}(x) \in C^N(\bar{\Omega})$, it can be used as a test function to show that when d is small enough, $\lambda_0(d, b) > 0$. Then similar to previous arguments, we obtain that there exists $D = D(b) > 0$ such that

$$\lambda_0(d, b) > 0 \text{ for } 0 < d < D(b), \;\; \lambda_0(D(b), b) = 0 \text{ and } \lambda_0(d, b) < 0 \text{ for } d > D(b).$$

Therefore, $u \equiv 1$ is linearly unstable if $0 < d < D(b)$ and linearly stable if $d > D(b)$. It is also clear that $D(0) \doteq \lim_{b \searrow 0} D(b) = d^*$, $D(b^*) \doteq \lim_{b \searrow b^*} D(b) = 0$ and $D(b)$ is strictly decreasing in b.

Finally, it remains to verify that $D = D(b)$ is concave in $[0, b^*]$. Notice that $D(b)$ satisfies $\lambda_0(D(b), b) = 0$. Then for any $0 \leqslant b_1 < b_2 \leqslant b^*$, it is easy to see that

$$\lambda\left(\frac{D(b_1) + D(b_2)}{2}, \frac{b_1 + b_2}{2}\right) < \frac{1}{2}\left[\lambda_0(D(b_1), b_1) + \lambda_0(D(b_2), b_2)\right] = 0.$$

Thus, $\frac{D(b_1)+D(b_2)}{2} > D(\frac{b_1+b_2}{2})$. The proof is complete. □

Remark 3.1 *One mathematical difficulty in studying non-local diffusion is that linear operator like (1.12) may not have a principal eigenvalue with continuous eigenfunctions; e.g.,* μ_0*,*

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which is defined in (1.13)*, may not be attained by a continuous function. We refer interested readers to [2, 6, 20, 22] and references therein for more information on spectral theory for non-local diffusion operators.*

3.4 Some consequences

In this section, we provide the proofs of Theorems 1.6–1.8, which contain the consequences of Theorems 1.1–1.5.

Proof of Theorem 1.6 In this theorem, $\bar{g} = 0$. Notice that according to the assumptions, both $u \equiv 0$ and $u \equiv 1$ are unstable according to the stability analysis in Theorem 1.3 (i) and Theorem 1.4 (i). Hence, (1.6) admits at least one non-trivial stable steady state for all $d, b > 0$. □

Proof of Theorem 1.7 The key assumptions here are $\bar{g} < 0$ and $f'(0) = 0$. The stability analysis in Theorems 1.3 (iii) and 1.4 (ii) shows that for any $d, b > 0$, $u \equiv 0$ is stable while $u \equiv 1$ is unstable. Together with the results established in Theorems 1.1 (ii) and 1.2 (iii), the conclusion follows. □

Proof of Theorem 1.8

(i) Assume that $\bar{g} > 0$ and $f'(1) = 0$. Indeed, this is the dual version of Theorem 1.7. Thus, it can be proved similarly.

(ii) Assume that $\bar{g} > 0$ and $f'(1) < 0$. By the stability analysis in Theorem 1.3 (ii) and Theorem 1.5, $u \equiv 0$ is always unstable while for any $0 \leqslant b < b^*$ and $0 < d < D(b)$, $u \equiv 1$ is linearly unstable. This yields the existence of non-trivial stable steady state of problem (1.6) provided that $0 \leqslant b < b^*$ and $0 < d < D(b)$. Moreover, note that the condition $f'(1) < 0$ implies condition (1.8) and for either d or b large enough, $u \equiv 1$ is linearly stable. Thus by Theorem 1.2 (i) and (ii), $u \equiv 1$ is globally stable if either d or b is large enough.

□

4 Concluding remarks

No progress on the complete dominance case in the model describing the evolution of genes under the joint action of selection and migration had been made until 2010. Partial panmixia, an additional action which is important in taking into account the action of long-distance migration, has been proposed by T. Nagylaki. Mathematically, this poses a challenging problem involving non-local effects in partial differential equations. In this paper, developing theories to handle the dynamics of non-local equations, we have made substantial progress on this important model in genetics; namely, we have obtained existence, non-existence, and stability properties of steady states. Moreover, the methods we have developed here will be useful for other related problems as well. We plan to pursue the interesting and significant issues – *qualitative as well as quantitative* properties of solutions affected by non-local terms, and the uniqueness of non-trivial steady states (which seems to be a very difficult problem) in a future paper.

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Appendix A Existence of stable global minimizer

In this appendix, we include the following standard arguments for the existence of a stable global minimizer to complete the proof of Theorem 1.1. More precisely, we will show that if

$$\min\{J[u] : u \in H^1(\Omega)\} < \min\{J[0], J[1]\},$$

then there exists at least one stable global minimizer solution, where $J[\cdot]$ is defined in (3.1). Suppose that all the global minimizers are unstable. W.l.o.g., assume that v_0 is a global minimizer which is unstable from above. For clarity, the arguments leading to a contradiction will be divided into two steps.

Step 1. We will show that *there exists a minimal steady state of (1.6) above v_0, denoted by v_1, which is a global minimizer stable from below and satisfies $v_0 < v_1 < 1$.*

For $0 \leqslant v \leqslant 1$, set

$$X^+(v) = \{w : 1 \geqslant w \geqslant v, w \neq v\}, \ X^-(v) = \{w : 0 \leqslant w \leqslant v, w \neq v\},$$
$$S = \{u : u \text{ is a steady state of } (1.6)\}, \ S^+(v) = S \cap X^+(v).$$

Then, define $\Psi(x) = \inf_{\tilde{v} \in S^+(v_0)} \tilde{v}(x), \ x \in \bar{\Omega}$. Note that $S^+(v_0) \neq \emptyset$ since it contains $v \equiv 1$. Hence, Ψ is well defined.

Given any $\epsilon > 0$ and finite number of points $x_1, \dots, x_k \in \bar{\Omega}$, there exist $v_1, \dots, v_k \in S^+(v_0)$ such that

$$\Psi(x_i) \leqslant v_i(x_i) \leqslant \Psi(x_i) + \epsilon, \ i = 1, \dots, k,$$

which gives that

$$\Psi(x_i) \leqslant (v_1 \wedge v_2 \wedge \cdots \wedge v_k)(x_i) \leqslant \Psi(x_i) + \epsilon, \ i = 1, \dots, k.$$

It follows that Ψ is an accumulation point of $S^+(v_0)$ in the topology of point-wise convergence of $\bar{\Omega}$. Furthermore, note that $S^+(v_0)$ is bounded below by v_0 and above by $v \equiv 1$. Thus, $S^+(v_0)$ is relatively compact in $C^1(\bar{\Omega}) \cap C^2(\Omega)$ due to the elliptic regularity. This implies that Ψ is also an accumulation point of $S^+(v_0)$ in the topology of $C^1(\bar{\Omega}) \cap C^2(\Omega)$. Hence, $\Psi \in S^+(v_0) \cup \{v_0\}$.

Now to show $\Psi \in S^+(v_0)$, it suffices to show that v_0 is isolated from $S^+(v_0)$ in the L^∞ norm. Recall that v_0 is unstable from above if there exists $\epsilon_0 > 0$ such that for any $\delta > 0$ small enough, there exists $\psi \in C(\bar{\Omega})$ which satisfies $v_0 \leqslant \psi \leqslant v_0 + \delta \leqslant 1$ in Ω and $u(x_0, t_0; \psi) \geqslant v_0(x_0) + \epsilon_0$ for some $x_0 \in \Omega$, $t_0 > 0$. This indicates that

$$\{w \in C(\bar{\Omega}) : \|w - v_0\|_{L^\infty} < \epsilon_0\} \cap S^+(v_0) = \emptyset.$$

Therefore, there exists a minimal steady state above v_0, denoted by v_1. By maximum principle, it is easy to see that $v_1 > v_0$ in $\bar{\Omega}$. Thus, $\Psi = v_1$.

Next, let us investigate the properties of v_1. For any $\epsilon > 0$, choose $v \in X^+(v_0) \cap X^-(v_1)$ and close to v_0 such that

$$J[v] < J[1], \; J[v] < J[v_0] + \epsilon.$$

Then, either $\omega[v] = \{v_0\}$ or $\omega[v] = \{v_1\}$. The assumption that v_0 is unstable from above implies that $\omega[v] = \{v_1\}$. Thus,

$$J[v_1] < J[v] < J[1], \; J[v_1] < J[v] < J[v_0] + \epsilon.$$

It follows that $v_1 < 1$ and v_1 is also a global minimizer since ϵ is arbitrary. Moreover, it is standard to show that for $v \in X^+(v_0) \cap X^-(v_1)$, $u(\cdot, t; v)$ converges to v_1 in $C(\bar{\Omega}) \cap C^2(\bar{\Omega})$. Hence, v_1 is stable from below.

Step 2. Define

$$S_0 = \{v_0 < u < 1 : \; u \text{ is a global minimizer stable from below.}\}$$

Clearly, **Step 1** tells us $S_0 \neq \emptyset$ since $v_1 \in S_0$. The partial order "$\leqslant$" is defined as

$$u, v \in S, \; u \leqslant v \text{ if } u(x) \leqslant v(x) \text{ in } \bar{\Omega}.$$

By Zorn's Lemma, there exists a maximal well-ordered subset of S_0, denoted by W. Two cases will be handled respectively.

Case 1. Assume that W has a greatest element w. Since $w \in S_0$, then w must be unstable from above. By repeating the arguments in **Step 1**, one obtains a minimal steady state above w, denoted by $\tilde{w}$, such that $\tilde{w} \in S_0$. Hence, $W \cup \{\tilde{w}\}$ is a greater well-ordered subset of S_0 than W. This is a contradiction.

Case 2. Assume that W has no greatest element. Define $\hat{w} = \sup_{u \in W} u(x)$, then $\hat{w} \notin W$. We claim that $\hat{w} \in S_0$. For this purpose, it is enough to find a sequence

$$u_1 \leqslant u_2 \leqslant \cdots \leqslant u_k \leqslant \cdots \quad \text{in } S_0$$

such that $u_k \to \hat{w}$ in $C^2(\bar{\Omega})$. Let $\{x_l \in \Omega : \; l \in \mathbb{N}\}$ be a dense subset of Ω. For every l, there exists a sequence $\{u_k^{(l)}\}_{k \geqslant 1}$ in W such that $u_k^{(l)}(x_l)$ increases to $\hat{w}(x_l)$ as $k \to \infty$.

Note that W is well ordered, so we set

$$u_k = \max\{u_k^{(l)} : \; 1 \leqslant l \leqslant k\}.$$

Then, $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_k \leqslant \cdots$ and $u_k \in W$. It is also clear that

$$\lim_{k \to \infty} u_k(x_l) \geqslant \lim_{k \to \infty} u_k^{(l)}(x_l) = \hat{w}(x_l) \text{ and } \lim_{k \to \infty} u_k(x_l) \leqslant \hat{w}(x_l).$$

Hence,

$$\lim_{k \to \infty} u_k(x_l) = \hat{w}(x_l). \tag{A 1}$$

Furthermore, $\{u_k\} \subset S_0$ yields that $\{u_k\}$ is bounded in $C^{2,\alpha}(\bar{\Omega})$. Hence, by passing to a subsequence if necessary, we have $u_k \to \phi$ in $C^2(\bar{\Omega})$ for some $\phi \in C^2(\bar{\Omega})$.

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We claim that $\phi = \hat{w}$. Assume that this is not true, then on the one hand, there exists $x_0 \in \Omega$ such that

$$\phi(x_0) < \hat{w}(x_0) - \epsilon \quad \text{for some } \epsilon > 0.$$

Since $\phi \in C^2(\bar{\Omega})$, there exists a neighbourhood $\mathcal{O}_1$ of x_0 such that

$$\phi(x) < \hat{w}(x_0) - \epsilon \quad \text{for all } x \in \mathcal{O}_1. \tag{A 2}$$

On the other hand, there exists $u_0 \in W$ such that

$$u_0(x_0) > \hat{w}(x_0) - \epsilon,$$

and similarly, there exists a neighbourhood $\mathcal{O}_2$ of x_0 such that

$$u_0(x) > \hat{w}(x_0) - \epsilon \quad \text{for all } x \in \mathcal{O}_2. \tag{A 3}$$

Then due to (A 1), (A 2) and (A 3), one sees that for any $x_l \in \mathcal{O}_1 \cap \mathcal{O}_2$,

$$\hat{w}(x_0) - \epsilon < u_0(x_l) \leqslant \hat{w}(x_l) = \lim_{k\to\infty} u_k(x_l) = \phi(x_l) < \hat{w}(x_0) - \epsilon.$$

This is impossible and thus $\phi = \hat{w}$.

Moreover, according to the choice of ϕ, it follows immediately that $\phi = \hat{w}$ is a global minimizer stable from below. Hence, $\hat{w} \in S_0$ and $W \cup \{\hat{w}\}$ is a greater well-ordered subset of S_0 than W. This gives the desired contradiction.

Therefore, there exists a stable global minimizer. The proof is complete.

Appendix B Proof of Lemma 3.3

Instead of (3.14), we will study the more general non-local eigenvalue problem

$$\mathcal{L}\phi := d\int_\Omega k(x-y)\phi(y)\rho(y)dy + h(x)\phi(x) = \mu\phi(x), \tag{B 1}$$

with the following assumptions imposed:

(B) $h(x) \in C(\bar{\Omega})$, $k(x) \in C(\mathbb{R}^N)$, $k(x) = k(-x) \geqslant 0$, $k(0) > 0$ and $\int_{\mathbb{R}^N} k(x)dx = 1$.

The following result, which immediately yields Lemma 3.3 with $h(x) = g(x) - b, d = b/\alpha$ and $k(x)$ suitably chosen, will be established.

Proposition B.1 *Assume that (B) is valid. Given any $\epsilon > 0$, there exists $h_\epsilon \in C^N(\bar{\Omega})$ such that $\|h - h_\epsilon\|_{L^\infty}$ and the non-local eigenvalue problem*

$$d\int_\Omega k(x-y)\phi(y)\rho(y)dy + h_\epsilon(x)\phi(x) = \mu\phi(x)$$

admits a principal eigenvalue with strictly positive eigenfunction in $C(\bar{\Omega})$.

Before proving this proposition, we will present some properties related to the eigen-

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values of problem (B 1), and in particular, provide a criterion in determining the existence of principal eigenvalue of (B 1) with strictly positive eigenfunction in $C(\bar{\Omega})$.

Denote

$$\mu_0 = \sup_{\|\phi\|_{L^2}=1} \langle \mathcal{L}\phi, \phi \rangle, \text{ where } \langle \phi, \psi \rangle = \int_\Omega \phi(x)\psi(x)\rho(x)dx,$$

and for $\alpha > \max_{\bar{\Omega}} h(x)$, $\psi \in L^2(\Omega)$, define

$$T_\alpha \psi := \frac{d \int_\Omega k(x-y)\psi(y)\rho(y)dy}{\alpha - h(x)}.$$

Lemma B.1 *Assume that (B) holds, then* $\mu_0 \geqslant \max_{\bar{\Omega}} h(x)$.

Since $h(x) \in C(\bar{\Omega})$, this lemma can be verified easily by choosing a sequence of test functions in $L^2(\Omega)$ which concentrate at the maximum point of $h(x)$. We omit the details.

Lemma B.2 *Assume that (B) holds, then the following statements are equivalent:*

(i) $\mu_0 > \max_{\bar{\Omega}} h(x)$;

(ii) *There exists* $\alpha_0 > \max_{\bar{\Omega}} h(x)$ *such that the spectral radius* $r(T_{\alpha_0}) = 1$;

(iii) *(B 1) admits a principal eigenvalue with a strictly positive eigenfunction in* $C(\bar{\Omega})$.

Furthermore, $\mu_0 = \alpha_0$.

Proof Let $X = C(\bar{\Omega})$ and $X^+ = \{u \in X : u \geqslant 0\}$. First, for any $\alpha > \max_{\bar{\Omega}} h(x)$, it is easy to check that by Arzela–Ascoli Theorem, T_α is a compact operator from X to X. Moreover, since the interior of X^+ is not empty and for $0 \neq u \in X^+$, $T_\alpha u > 0$ in $\bar{\Omega}$, by Krein–Rutman Theorem, we have $r(T_\alpha) > 0$ and $r(T_\alpha)$ is a simple eigenvalue of T_α with a strictly positive eigenfunction in $X = C(\bar{\Omega})$.

Now, assume that (*ii*) holds. Let ψ_0 denote the strictly positive eigenfunction of T_α in $C(\bar{\Omega})$, with $\|\psi_0\|_{L^2} = 1$, corresponding to the simple eigenvalue $r(T_{\alpha_0}) = 1$, i.e.,

$$\frac{d \int_\Omega k(x-y)\psi_0(y)\rho(y)dy}{\alpha_0 - h(x)} = \psi_0,$$

which is equivalent to

$$\mathcal{L}\psi_0 = d\int_\Omega k(x-y)\psi_0(y)\rho(y)dy + h(x)\psi_0 = \alpha_0\psi_0.$$

This implies that $\mu_0 \geqslant \alpha_0$. Thus, (i) is proved.

Next, we will demonstrate that $\mu_0 = \alpha_0$, which automatically yields ψ_0 is the corresponding eigenfunction of (B 1) and thus (iii) follows.

Suppose that this is not true, i.e. $\mu_0 > \alpha_0$. Then, according to the definition of μ_0, there exists a sequence of functions ϕ_n satisfying $\|\phi_n\|_{L^2} = 1$, $\langle \mathcal{L}\phi_n, \phi_n \rangle$ increases to μ_0 as $n \to \infty$ and $\phi_n \rightharpoonup \phi_\infty$ in L^2. Without loss of generality, we may assume that $\phi_n \geqslant 0$, and thus $\phi_\infty \geqslant 0$.

Set $\psi_n = \phi_n - \epsilon\psi_0$, where $\epsilon > 0$ is arbitrary. Clearly,

$$\langle \mathcal{L}\psi_n, \psi_n \rangle \leqslant \mu_0 \langle \psi_n, \psi_n \rangle.$$

Direct computations imply that

$$\langle \mathcal{L}\psi_n, \psi_n \rangle = \langle \mathcal{L}\phi_n - \epsilon\mathcal{L}\psi_0, \phi_n - \epsilon\psi_0 \rangle = \langle \mathcal{L}\phi_n, \phi_n \rangle - 2\epsilon\alpha_0 \langle \phi_n, \psi_0 \rangle + \epsilon^2 \alpha_0,$$

and

$$\mu_0 \langle \psi_n, \psi_n \rangle = \mu_0 \langle \phi_n - \epsilon\psi_0, \phi_n - \epsilon\psi_0 \rangle = \mu_0 \left(1 - 2\epsilon \langle \phi_n, \psi_0 \rangle + \epsilon^2 \right).$$

Thus, it follows that, as $n \to \infty$

$$-2\epsilon\alpha_0 \langle \phi_\infty, \psi_0 \rangle + \epsilon^2 \alpha_0 \leqslant -2\epsilon\mu_0 \langle \phi_\infty, \psi_0 \rangle + \epsilon^2 \mu_0. \tag{B 2}$$

Obviously $\langle \phi_\infty, \psi_0 \rangle \geqslant 0$. If $\langle \phi_\infty, \psi_0 \rangle > 0$, a contradiction arises in (B 2) due to the fact that $\mu_0 > \alpha_0$ and $\epsilon > 0$ is arbitrary. Hence, $\langle \phi_\infty, \psi_0 \rangle = 0$, which yields that $\phi_\infty \equiv 0$ since $\phi_\infty \geqslant 0$ and $\psi_0 > 0$ in $\bar{\Omega}$. Then, using the condition that $\phi_n \rightharpoonup \phi_\infty = 0$ in L^2, it is routine to verify that

$$\begin{aligned} \mu_0 &= \lim_{n\to\infty} \langle \mathcal{L}\phi_n, \phi_n \rangle \\ &= \lim_{n\to\infty} \int_\Omega \left[d \int_\Omega k(x-y)\phi_n(y)\rho(y)dy + h(x)\phi_n(x) \right] \phi_n(x)\rho(x)dx \\ &= \lim_{n\to\infty} \int_\Omega h(x)\phi_n^2(x)\rho(x)dx \leqslant \max_{\bar{\Omega}} h(x). \end{aligned} \tag{B 3}$$

This is a contradiction since $\mu_0 > \alpha_0 > \max_{\bar{\Omega}} h(x)$. Thus, we have established that (ii) implies (iii) and $\mu_0 = \alpha_0$.

Now assume that (iii) holds. Let ϕ_0 denote the strictly positive eigenfunction in $C(\bar{\Omega})$ corresponding to μ_0, i.e.,

$$\mathcal{L}\phi_0 = d \int_\Omega k(x-y)\phi_0(y)\rho(y)dy + h(x)\phi_0(x) = \mu_0\phi_0(x).$$

This is equivalent to

$$(\mu_0 - h(x))\phi_0 = d \int_\Omega k(x-y)\phi_0(y)\rho(y)dy.$$

Clearly, the right-hand side is continuous and strictly positive in $\bar{\Omega}$, therefore $\mu_0 > \max_{\bar{\Omega}} h(x)$, i.e., (i) holds.

It remains to show that (i) implies (ii). Since $\mu_0 > \max_{\bar{\Omega}} h(x)$, similar to the arguments at the beginning of the proof, one sees that $r(T_{\mu_0}) > 0$ and $r(T_{\mu_0})$ is a simple eigenvalue of T_{μ_0} with strictly positive eigenfunction in $C(\bar{\Omega})$, denoted by $\hat{\psi}_0$. We only need demonstrate that $r(T_{\mu_0}) = 1$.

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Suppose that $r(T_{\mu_0}) > 1$. Based on the definition of T_α, it is clear that there exists $\tilde{\alpha} > \mu_0$ such that $r(T_{\tilde{\alpha}}) = 1$. However, according to what we have proved, this yields that $\tilde{\alpha} = \mu_0$, which is impossible. Therefore, $r(T_{\mu_0}) \leqslant 1$.

Now suppose that $r(T_{\mu_0}) < 1$ and a contradiction will be derived. The idea is again similar as before.

According to the definition of μ_0, there exists a sequence of functions $\hat{\phi}_n$ such that $\|\hat{\phi}_n\|_{L^2} = 1$, $\langle \mathcal{L}\hat{\phi}_n, \hat{\phi}_n\rangle$ approaches μ_0 as $n \to \infty$ and $\hat{\phi}_n \rightharpoonup \hat{\phi}_\infty$ in L^2. Without loss of generality, we assume that $\hat{\phi}_n \geqslant 0$, $\hat{\phi}_\infty \geqslant 0$ and $\|\hat{\varphi}_0\|_{L^2} = 1$. Similarly, we set $\hat{\varphi}_n = \hat{\phi}_n - \epsilon\hat{\varphi}_0$, where $\epsilon > 0$ is arbitrary and it holds that

$$\langle \mathcal{L}\hat{\varphi}_n, \hat{\varphi}_n\rangle \leqslant \mu_0 \langle \hat{\varphi}_n, \hat{\varphi}_n\rangle,$$

which can be rewritten as

$$\langle \mathcal{L}\hat{\phi}_n - \epsilon\mathcal{L}\hat{\varphi}_0, \hat{\phi}_n - \epsilon\hat{\varphi}_0\rangle \leqslant \mu_0 \left(1 - 2\epsilon\langle \hat{\phi}_n, \hat{\varphi}_0\rangle + \epsilon^2\right). \tag{B 4}$$

For the left-hand side of (B 4), recall that $\hat{\varphi}_0$ satisfies

$$T_{\mu_0}\hat{\varphi}_0 = \frac{d\int_\Omega k(x-y)\hat{\varphi}_0(y)\rho(y)dy}{\mu_0 - h(x)} = r(T_{\mu_0})\hat{\varphi}_0,$$

which is equivalent to

$$\mathcal{L}\hat{\varphi}_0 = d\int_\Omega k(x-y)\hat{\varphi}_0(y)\rho(y)dy + h(x)\hat{\varphi}_0(x) = \left[r(T_{\mu_0})\mu_0 + (1 - r(T_{\mu_0}))h(x)\right]\hat{\varphi}_0.$$

Then direct computation yields that

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\langle \mathcal{L}\hat{\phi}_n - \epsilon\mathcal{L}\hat{\varphi}_0, \hat{\phi}_n - \epsilon\hat{\varphi}_0\rangle - \mu_0\left(1 - 2\epsilon\langle\hat{\phi}_n, \hat{\varphi}_0\rangle + \epsilon^2\right)\right\} \\
&\quad = \lim_{n\to\infty}\left\{\langle \mathcal{L}\hat{\phi}_n, \hat{\phi}_n\rangle - 2\epsilon\langle\hat{\phi}_n, \mathcal{L}\hat{\varphi}_0\rangle + \epsilon^2\langle\mathcal{L}\hat{\varphi}_0, \hat{\varphi}_0\rangle - \mu_0\left(1 - 2\epsilon\langle\hat{\phi}_n, \hat{\varphi}_0\rangle + \epsilon^2\right)\right\} \\
&\quad = 2\epsilon\left(\mu_0\langle\hat{\phi}_\infty, \hat{\varphi}_0\rangle - \langle\hat{\phi}_\infty, \mathcal{L}\hat{\varphi}_0\rangle\right) + O(\epsilon^2) \\
&\quad = 2\epsilon\left\langle\hat{\phi}_\infty, (1 - r(T_{\mu_0}))(\mu_0 - h(x))\hat{\varphi}_0\right\rangle + O(\epsilon^2) > 0
\end{aligned}$$

for $\epsilon > 0$ small enough if $\hat{\phi}_\infty \not\equiv 0$, since $\mu_0 > \max_{\bar{\Omega}} h(x)$, $r(T_{\mu_0}) < 1$, $\hat{\varphi}_0$ is strictly positive and $\hat{\phi}_\infty$ is non-negative. This contradicts (B 4). Therefore, $\hat{\phi}_\infty \equiv 0$. Then similar to the computation in (B 3), we have $\mu_0 \leqslant \max_{\bar{\Omega}} h(x)$, which contradicts (i).

Therefore, $r(T_{\mu_0}) = 1$ and thus (ii) is established. The proof is complete. □

We also include a simple example to demonstrate that it is possible that $\mu_0 = \max_{\bar{\Omega}} h(x)$ and μ_0 cannot even be achieved by functions in $L^2(\Omega)$. Assume that

- $\Omega = (-1, 1)$;
- $\rho(x) = 1/2$ in Ω;
- $k(x) = 1/k_0$ for $|x| \leqslant 2$, $k(x) = 0$ for $|x| \geqslant 3$, $k(x) \in C(\mathbb{R})$, $\int_{\mathbb{R}} k(x)dx = 1$, $k(x) \geqslant 0$, where $k_0 > 0$ is chosen properly;
- $d = k_0$.

We claim that *set* $h(x) = 2 - c|x|^{2/3}$, *where* $c \geqslant 3$, *then* $\mu_0 = \max_{\bar{\Omega}} h(x) = 2$ *and cannot be achieved in* L^2. First, the non-local eigenvalue problem in (B 1) becomes

$$\mathcal{L}\phi = \frac{1}{2}\int_{-1}^{1} \phi(x)dx + (2 - c|x|^{2/3})\phi(x) = \mu\phi(x)$$

and T_α is defined as follows:

$$T_\alpha \psi = \frac{\frac{1}{2}\int_{-1}^{1} \psi(x)dx}{\alpha - 2 + c|x|^{2/3}}.$$

For any $\alpha > \max_{\bar{\Omega}} h(x) = 2$, according to the proof of Lemma B.2, $r(T_\alpha)$ is a simple eigenvalue of T_α with strictly positive eigenfunction, denoted by ψ_α, in $C(\bar{\Omega})$, i.e.,

$$\frac{\frac{1}{2}\int_{-1}^{1} \psi_\alpha(x)dx}{\alpha - 2 + c|x|^{2/3}} = r(T_\alpha)\psi_\alpha.$$

This implies that

$$r(T_\alpha) = \frac{1}{2}\int_{-1}^{1} \frac{1}{\alpha - 2 + c|x|^{2/3}} dx.$$

Since $r(T_\alpha)$ is decreasing in α, we have

$$r(T_\alpha) < \lim_{\alpha \searrow 2} r(T_\alpha) = \frac{1}{2}\int_{-1}^{1} \frac{1}{2 - 2 + c|x|^{2/3}} dx = \frac{3}{c} \leqslant 1,$$

since $c \geqslant 3$. Thanks to Lemmas B.1, B.2, one sees that $\mu_0 = \max_{\bar{\Omega}} h(x) = 2$. Suppose that $\mu_0 = 2$ is achieved by $0 \neq \phi_0 \in L^2$, then

$$\mathcal{L}\phi_0 = \frac{1}{2}\int_{-1}^{1} \phi_0(x)dx + (2 - c|x|^{2/3})\phi_0(x) = \mu_0\phi_0(x).$$

This gives that

$$\phi_0(x) = \frac{1}{c|x|^{2/3}} \frac{1}{2}\int_{-1}^{1} \phi_0(x)dx,$$

which clearly is not in L^2. The claim is verified.

Now we are ready to verify Proposition B.1.

Proof of Proposition B.1 Since $h(x) \in C(\bar{\Omega})$, there exists $h_\epsilon \in C^N(\bar{\Omega})$ such that $\|h - h_\epsilon\|_{L^\infty} < \epsilon$, $h_\epsilon(x_\epsilon) = \max_{\bar{\Omega}} h_\epsilon(x) = \max_{\bar{\Omega}} h(x)$ for some $x_\epsilon \in \bar{\Omega}$ and all the partial derivatives of h_ϵ at x_ϵ up to order $N - 1$ are 0. Then for

$$T_{\alpha,\epsilon}\psi = \frac{d\int_\Omega k(x-y)\psi(y)\rho(y)dy}{\alpha - h_\epsilon(x)},$$

it is routine to show that

$$\lim_{\alpha \searrow \max_{\bar{\Omega}} h(x)} r(T_{\alpha,\epsilon}) = +\infty \quad \text{and} \quad \lim_{\alpha \to +\infty} r(T_{\alpha,\epsilon}) = 0.$$

Hence, there exists $\alpha_0 > \max_{\bar{\Omega}} h(x)$ such that $r(T_{\alpha_0,\epsilon}) = 1$. The desired conclusion now follows from Lemma B.2 immediately. □

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