T. Kaneyama
Nagoya Math. J.
Vol. 57 (1975), 65–86

ON EQUIVARIANT VECTOR BUNDLES ON AN ALMOST HOMOGENEOUS VARIETY

TAMAFUMI KANEYAMA

§ 1. Introduction

Let k be an algebraically closed field of arbitrary characteristic. Let T be an n-dimensional algebraic torus, i.e. $T = G_m \times \cdots \times G_m$ (n-times), where $G_m = \operatorname{Spec}(k[t, t^{-1}])$ is the multiplicative group.

An almost homogeneous variety under an action of T is an algebraic variety X over k endowed with an action of T and which has a dense orbit. Normal effective almost homogeneous varieties under torus action have been classified in [2, 4]. We review the results briefly in § 2.

Let E be a vector bundle on an almost homogeneous variety X with an action of T. For every k-rational point t in T, the action of t on X is denoted by $t: X \to X$ ($x \mapsto tx$). We say that E is equivariant if there exists an isomorphism $\phi_t: t^*E \xrightarrow{\sim} E$ for every k-rational point t of T. Furthermore we say that an equivariant vector bundle E is T-linearized if, for every pair of k-rational points t, t' of $T, \phi_{tt'} = \phi_{t'} \cdot t'^*\phi_t$ holds where

$$\phi_{t'} \cdot t'^*\phi_t : (tt')^*E = t'^*t^*E \xrightarrow{t'^*\phi_t} t'^*E \xrightarrow{\phi_{t'}} E .$$

In this paper we study equivariant vector bundles on a smooth complete almost homogeneous variety. In § 3 we show that an equivariant vector bundle E on X always has a T-linearization. Thus we study T-linearized vector bundles on X. Let $\{U_i\}$ be a covering of X by T-stable affine open sets. We show that the restriction of E to U_i has a semi-invariant base with respect to the action of T, where a section $u \in E(U_i)$ is semi-invariant if for some character ξ

$$\phi_t(u) = e(\xi)(t)u$$

Received July 26, 1974.

holds for every k-rational point t in T. $e(\xi)(t)$ is the value at t of the function $\xi: T \to G_m$. Then in § 4 we describe a T-linearized vector bundle E on X in terms of these semi-invariant, i.e.

Theorem 4.2 Let (X, T) be a smooth complete almost homogeneous variety defined by a cone complex $(\Gamma, \mathscr{C})$. The set of T-linearized vector bundles of rank r up to T-isomorphism corresponds bijectively to the set of (m, P) up to equivalence. A detailed description of (m, P) is in § 4.

Finally, we give examples of equivariant vector bundle of rank 2 on $\boldsymbol{P}^2$ which are indecomposable.

The auther wishes to thank Professor T.Oda for his advise during the preparation of this paper.

§ 2. Almost homogeneous varieties

Let (X, T) be an n-dimensional almost homogeneous variety where T is an n-dimensional algebraic torus. Let $\Xi = \Xi(T) = \mathrm{Hom}_{k-gr}(T, G_m)$ be the additive group of characters of T. Ξ is a free $\boldsymbol{Z}$-module of rank $n = \dim(T)$. Let the exponential map $e: \Xi \to k(T)^*$ be the homomorphism which sends ξ in Ξ to the corresponding rational function $e(\xi)$ on T. Let $\Gamma = \mathrm{Hom}_{k-gr}(G_m, T)$ be the additive group of one-parameter subgroup of T. Γ is a free $\boldsymbol{Z}$-module of rank $n = \dim(T)$ and is the dual $\boldsymbol{Z}$-module of Ξ.

We call a non-empty subset C of $\Gamma_{\boldsymbol{Q}} = \Gamma \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ a strongly convex polyhedral cone with apex at 0, or simply a cone, if $C \cap (-C) = \{0\}$ and if there exists a finite subset $\{\phi_1, \cdots, \phi_m\}$ of Γ such that $C = \boldsymbol{Q}_0\phi_1 + \cdots + \boldsymbol{Q}_0\phi_m$ where $\boldsymbol{Q}_0$ denotes the set of non-negative rational numbers. If $\{\phi_1, \cdots, \phi_m\}$ is irredundant and each ϕ_i is primitive, i.e. is an integral multiple of no element of Γ, we call $\phi_1, \cdots, \phi_m$ verticial elements of the cone C. Let the dimension of C be the dimension of the $\boldsymbol{Q}$-vector space $C + (-C)$. A non-empty subset C' of a cone C is called a facial cone of C if there exists an element ξ of Ξ such that $\phi(\xi) \geqq 0$ for all ϕ in C and that $C' = \{\phi \in C \mid \phi(\xi) = 0\}$.

A cone complex $(\Gamma, \mathscr{C})$, or simyly $\mathscr{C}$, in $\Gamma_{\boldsymbol{Q}}$ is a finite collection of cones of $\Gamma_{\boldsymbol{Q}}$ with properties:

(i) if C' is a facial cone of C in $\mathscr{C}$ then C' is in $\mathscr{C}$,

(ii) if C and C' are in $\mathscr{C}$ then the intersection $C \cap C'$ is a facial cone of C as well as of C'.

THEOREM 2.1. *The category of normal effective almost homogeneous varieties under torus action and equivariant morphisms is equivalent to the category of pairs* $(\Gamma, \mathscr{C})$ *consisting of a free* $\boldsymbol{Z}$*-module* Γ *of finite rank and a cone complex* $\mathscr{C}$ *in* $\Gamma_{\boldsymbol{Q}}$ *and maps which are defined as follows:*

A map from a pair $(\Gamma, \mathscr{C})$ *to another pair* $(\Gamma', \mathscr{C}')$ *is a homomorphism of finite cokernel from* Γ *to* Γ' *whose scalar extension to* $\boldsymbol{Q}$ *sends any cone of* $\mathscr{C}$ *into some cone of* $\mathscr{C}'$.

Moreover, X *is complete if and only if the associated pair satisfies* $\Gamma_{\boldsymbol{Q}} = \bigcup_{C \in \mathscr{C}} C$. X *is smooth if and only if every cone (resp. every maximal cone) in* $\mathscr{C}$ *is regular, i.e. the set of its varticial elements can be extended to a* $\boldsymbol{Z}$*-base of* Γ.

See [4].

PROPOSITION 2.2. *Let* (X, T) *be a smooth almost homogeneous variety and let* x *in* X *be a* T*-fixed closed point. Let* V *be the union of* T*-orbits whose closure contains* x. *Then there exists a base* $\{\alpha_1, \cdots, \alpha_n\}$ *of* Ξ *such that* V *is* T*-equivariantly isomorphic to the affine space* $\operatorname{Spec}(k[e(\alpha_1), \cdots, e(\alpha_n)])$ *which has the canonical diagonal* T*-action* $e(\xi) \mapsto e(\xi) \otimes e(\xi)$.

See [4].

§ 3. Equivariant vector bundles

From now on (X, T) is a complete smooth almost homogeneous variety unless otherwise stated. Hence X is covered by T-stable open sets isomorphic to the n-dimensional affine space with a diagonal action of T by proposition 2.2. For every k-rational point t of T, the action on X of t is denoted by $t: X \to X$ sending x to tx.

DEFINITION 3.1. An equivariant vector bundle E on (X, T) is the vector bundle on X such that, for every k-rational point t in T, there exists an isomorphism $\phi_t: t^*E \xrightarrow{\sim} E$.

DEFINITION 3.2. An equivariant vector bundle (E, ϕ_t) is called T-linearized if, for every pair of k-rational points t, t' of $T, \phi_{tt'} = \phi_{t'} \cdot t'^*\phi_t$ holds where

$$\phi_{t'} \cdot t'^*\phi_t : (tt')^*E = t'^*t^*E \xrightarrow{t'^*\phi_t} t'^*E \xrightarrow{\phi_{t'}} E .$$

https://doi.org/10.1017/S0027763000016561 Published online by Cambridge University Press

Let E be an equivariant vector bundle on (X, T). Let $G(E)$ be the set of pairs (t, ϕ_t) where t is a k-rational point of T and ϕ_t is an isomorphism $\phi_t: t^*E \to E$. $G(E)$ is an algebraic group whose multiplication is given by

$$(t, \phi)(t', \phi') = (tt', \phi' \cdot t'^*\phi) .$$

Then $G(E)$ is an extension of T by the bundle automorphism group Aut (E)

$$0 \longrightarrow \mathrm{Aut}\,(E) \xrightarrow{j} G(E) \xrightarrow{p} T \longrightarrow 0$$

j sends ϕ to $(1, \phi)$ and p sends (t, ϕ) to t. Note that a T-linearization for E corresponds to giving a group section $s: T \to G(E)$.

PROPOSITION 3.3. *The above exact sequence has a group section* $s: T \to G(E)$. *Hence equivariant vector bundle can always be T-linearized.*

Proof. Since T and Aut (E) are linear algebraic group, so is $G(E)$. Let T' be a maximal torus of $G(E)$. Since $p(T') = T$ and a surjective homomorphism from a torus to a torus always has a section (see Borel [1]), there is a section

$$s: T \to T' \subset G(E) . \qquad \text{Q.E.D.}$$

Let E be a T-linearized vector bundle on an X which is defined by a cone complex $(\Gamma, \mathscr{C})$. Let C in $\mathscr{C}^n$ be a maximal cone and let U be the corresponding T-stable affine open set in X, i.e. $U = \mathrm{Spec}\,(A)$ where $A = k[e(D)]$ is a polynomial ring with $D = C^* \cap \Xi$. A is a Ξ-graded ring since there is a T-action on A. Namely

$$A = \bigoplus_{\xi \in \Xi} A_\xi$$

where A_ξ consists of element a of A with

$$a^t = e(\xi)(t)a .$$

a^t is the translation of a induced by the automorphism $t: U \to U$. We note that $A_\xi = ke(\xi)$ if $\xi \in D$ and $A_\xi = 0$ if $\xi \notin D$.

PROPOSITION 3.4. *Let D be a subsemi-group of Ξ such that $A = k[e(D)]$ is a polynomial ring. Let M be an A-module. Then a T-linearization on the sheaf $\tilde{M}$ on* Spec (A) *coincides with a Ξ-graded A-*

module structure on M.

Proof. Let M be a Ξ-graded A-module, i.e.

$$M = \bigoplus_{\xi \in \Xi} M_\xi .$$

Then there is a k-isomorphism

$$t^* : M \to M$$

sending m to $m^t = \sum_{\xi \in \Xi} e(\xi)(t) m_\xi$ if $m = \sum_{\xi \in \Xi} m_\xi$ where $e(\xi)(t)$ is a value at t of a function $e(\xi)$. Since A is a Ξ-graded module there exists a k-isomorphism

$$t^* : A \to A$$

sending a to a^t. We denote $t^*A = A'$. Let A' be an A-module through t^*, $A' \otimes_A M$ is A-module. Furthermore $A' \otimes_{A'} M \simeq M$ by trivial isomorphism $A' \simeq A$. So we have t-semi-linear A-isomorphism

$$t^* : A' \otimes_A M \to A' \otimes_{A'} M$$

sending $a^t \otimes m$ to $a^t \otimes m^t$. Thus we have

$$\phi_t : t^*\tilde{M} \to \tilde{M} .$$

By construction, for every pair $(t, t'), (tt')^* = t'^* \cdot t^*$ means the relation

$$\phi_{tt'} = \phi_{t'} \cdot t'^* \phi_t .$$

So $\tilde{M}$ has a T-linearization. Conversely if $\tilde{M}$ has a T-linearization we can reverse the above order so M is a Ξ-graded module. Q.E.D.

$E|U$ is associated to a projective A-module M. Since $\tilde{M}$ has a T-linearization we see that M is a Ξ-graded A-module. Let D be a sub-semi-group of Ξ generated by a $\boldsymbol{Z}$-base $\{\eta_1, \cdots, \eta_n\}$ of Ξ, i.e. $D = \boldsymbol{Z}_0\eta_1 + \cdots + \boldsymbol{Z}_0\eta_n$, where $\boldsymbol{Z}_0$ is the set of non-negative integers. Let $A = k[e(D)]$, i.e. $A = k[u_1, \cdots, u_n]$ is a polynomial ring of n-variables $u_i = e(\eta_i)$ $(i = 1, \cdots, n)$ with the Ξ-gradation given by $\deg(u_i) = \eta_i$. For $\xi \in \Xi$ we denote by $A(\xi)$ the Ξ-graded A-free module of rank one defined by

$$A(\xi)_\eta = A_{\xi+\eta} .$$

THEOREM 3.5. *Let $A = k[e(D)]$ be as above. If M is a finitely generated Ξ-graded A-projective module of rank r, then there exist $\xi_1, \cdots, \xi_r$ in Ξ such that*

$$M \simeq A(-\xi_1) \oplus \cdots \oplus A(-\xi_r)$$

as Ξ-graded A-module, in particular M is a free A-module.

Proof. Let $\{x_1, \cdots, x_p\}$ be a minimal set of generators of M as an A-module. We may assume $x_1, \cdots, x_p$ are homogeneous with characters $\xi_1, \cdots, \xi_p$ respectively. Let

$$F = \bigoplus_{i=1}^{p} A(-\xi_i)$$

and let e_i be the element of $A(-\xi_i)$ corresponding to 1. We define a degree preserving A-homomorphism

$$f: F \to M$$

by $f(e_i) = x_i$ $(i = 1, \cdots, p)$. f is a surjective homomorphism of Ξ-graded A-modules. Let N be the kernel of f, hence we have an exact sequence

$$0 \longrightarrow N \xrightarrow{i} F \xrightarrow{f} M \longrightarrow 0 .$$

By renumbering if necessary, we may assume that

$$\xi_1 = \cdots = \xi_q , \qquad \xi_j \notin \xi_1 + D \qquad \text{if } j > q .$$

In fact put

$$P = \{\xi_i \mid \xi_i \in \xi_1 + D\} .$$

If $\xi_2 \in P$ and $\xi_2 \neq \xi_1$ then there is non-zero η in D such that $\xi_2 = \xi_1 + \eta$. Put

$$P' = \{\xi_i \mid \xi_i \in \xi_2 + D\} .$$

Since D is a semi-group with zero and with no subgroup we see that $P' \subset P$ and $\xi_1 \notin P'$. Thus we replace ξ_1 by ξ_2. We do the same for this new ξ_1 and keep on doing the same. The process terminates since the original P is finite.

We may assume $q < p$. In fact suppose $\xi_1 = \cdots = \xi_p$ and let

$$\sum_{i=1}^{p} a_i x_i = 0$$

be a homogeneous relation satisfied by $x_i = f(e_i)$ with a_i homogeneous in A. Since $\deg(x_i) = \xi_1$ for all i, there exists an η in D such that

$\deg(a_i) = \eta$ for all i. Hence we can write $a_i = k_i e(\eta)$ where k_i are in k. Therefore our relation becomes

$$e(\eta) \sum_{i=1}^{p} k_i x_i = 0 .$$

Since M is A-flat we have $\sum_{i=1}^{p} k_i x_i = 0$. If there is i such that $k_i \neq 0$, then $x_i = -\sum_{j \neq i} k_i^{-1} k_j x_j$. This is a contradiction to the minimality of the set of generators $\{x_1, \cdots, x_p\}$. If $k_i = 0$ for all i then $a_i = 0$ for all i. Hence $\{x_i, \cdots, x_p\}$ forms an A-free base of M, thus M is free. Therefore we may assume that $q < p$.

We prove proposition 3.4 by induction on $\operatorname{rank}(F) = p$. If $\operatorname{rank}(F) = 1$ we are done. Thus suppose $\operatorname{rank}(F) > 1$ and $q < \operatorname{rank}(F)$. Since M is A-projective there is an A-module section $s: M \to F$ such that $f \cdot s = 1$. This means

$$F = N \oplus s(M)$$

as A-module. We want to show that $F = N \oplus s(M)$ as Ξ-graded A-module by replacing s. Since $\operatorname{Hom}(\tilde{M}, \tilde{F})$ is T-linearized vector bundle $\operatorname{Hom}(M, F)$ is Ξ-graded A-module. So we take s_0 as a section where s_0 is the degree 0 part of s. Since Ξ-graded homomorphism

$$\operatorname{Hom}(M, F) \to \operatorname{Hom}(M, M)$$

is surjective and sending s to identity, so s_0 is not zero. So this s_0 satisfies the assertion.

We continue the proof of proposition 3.4. We may assume that $F = N \oplus s(M)$ as Ξ-graded A-module. Since

$$F_{\xi_1} \otimes_k A = A(-\xi_1) \oplus \cdots \oplus A(-\xi_q)$$

by the choice of ξ_1 and

$$(N \oplus s(M))_{\xi_1} \otimes_k A = (N_{\xi_1} \otimes_k A) \oplus (s(M)_{\xi_1} \otimes_k A)$$

we have

$$\begin{aligned} A(-\xi_{q+1}) \oplus \cdots \oplus A(-\xi_p) &= F/F_{\xi_1} \otimes_k A \\ &= (N \oplus s(M))/\{(N_{\xi_1} \otimes_k A) \oplus (s(M)_{\xi_1} \otimes_k A)\} \\ &= (N/N_{\xi_1} \otimes_k A) \oplus (s(M)/s(M)_{\xi_1} \otimes_k A) . \end{aligned}$$

Since $s(M)/s(M)_{\xi_1} \otimes_k A$ is a direct summand of A-free module

$$A(-\xi_{q+1}) \oplus \cdots \oplus A(-\xi_p)$$

it is A-projective. Thus, by induction assumption, we have

$$s(M)/s(M)_{\xi_1} \otimes_k A = A(-\xi'_1) \oplus \cdots \oplus A(-\xi'_m)$$

is a free A-module for some ξ'_i. Since $s(M)_{\xi_1} \otimes_k A$ is A free we have

$$M \simeq s(M) = A(-\xi'_1) \oplus \cdots \oplus A(-\xi'_m) \oplus (s(M)_{\xi_1} \otimes_k A)$$

is a free module of rank r. Q.E.D.

Let $(\tilde{M}, \phi_t)$ be a T-linearized vector bundle on $U = \operatorname{Spec}(A)$ with $A = k[e(D)]$ for some subsemi-group D of Ξ. We say that $m \in M = \Gamma(U, \tilde{M})$ is semi-invariant if there exists a character $\xi \in \Xi$ such that

$$\phi_t(m) = e(\xi)(t)m$$

is satisfied for every t, i.e. m is homogeneous element of degree ξ of M.

COROLLARY 3.6. *Let $(\tilde{M}, \phi_t)$ be as above. Then $\tilde{M}$ has a semi-invariant base.*

Proof. By the theorem there exist characters $\xi_1, \cdots, \xi_r$ so that

$$M = A(-\xi_1) \oplus \cdots \oplus A(-\xi_r) .$$

Let e_i be the element of $A(-\xi_i)$ corresponding to 1. We denote by the same letter the element of M corresponding to e_i. Then we can easily see that e_i is semi-invariant. Furthermore $\{e_i\}$ is a base of $\tilde{M}$ as $\tilde{A}$-module. Q.E.D.

§ 4. Construction of an equivariant vector bundle

Let (X, T) be a smooth complete almost homogeneous variety defined by the cone complex $(\Gamma, \mathscr{C})$. Let E be a T-linearized vector bundle of rank r on X. We have shown that $E|U$ has a semi-invariant base where U is a T-stable affine open subset of X. Let U and U' be T-stable affine open subsets corresponding to C and C' in $\mathscr{C}^n$ respectively where $\mathscr{C}^n$ is the set of n-dimensional (maximal) cones of $\mathscr{C}$. Let (u_i) and (u'_j) be semi-invariant bases on $E(U)$ and $E(U')$ respectively and let (ξ_i) and (ξ'_j) be corresponding characters respectively. There is a natural pairing $\langle\ ,\ \rangle : \Xi \times \Gamma \to \boldsymbol{Z}$. It can naturally be extended to $\Xi_{\boldsymbol{Q}} \times \Gamma_{\boldsymbol{Q}} \to \boldsymbol{Q}$. We denote $\phi(\xi) = \langle \xi, \phi \rangle$ for $\xi \in \Xi_{\boldsymbol{Q}}$ and $\phi \in \Gamma_{\boldsymbol{Q}}$.

https://doi.org/10.1017/S0027763000016561 Published online by Cambridge University Press

PROPOSITION 4.1. *In the above situation there is a permutation* σ *such that* $\phi(\xi_i) = \phi(\xi'_{\sigma(i)})$ *for every* ϕ *in* $C \cap C'$.

Proof. We note $T \subset X$. Since $U \supset T$ and $U' \supset T$ we have

$$E\,|U|\,T \simeq E\,|U'|\,T\ .$$

We denote

$$\begin{aligned} M &= E(U) = Au_1 \oplus \cdots \oplus Au_r \simeq A(-\xi_1) \oplus \cdots \oplus A(-\xi_r) \\ M' &= E(U') = A'u'_1 \oplus \cdots \oplus A'u'_r \simeq A'(-\xi'_1) \oplus \cdots \oplus A'(-\xi'_r) \end{aligned}$$

where $A = k[e(C^* \cap \Xi)]$ and $A' = k[e(C'^* \cap \Xi)]$ are affine rings corresponding to U and U' respectively. The above isomorphism restricted to T induces the isomorphism

$$k[T] \otimes_A M \simeq k[T] \otimes_{A'} M'$$

as Ξ-graded $k[T]$-modules. Since

$$k[T] \otimes_A M \simeq k[T] \otimes_k \left(\bigoplus_{i=1}^{r} ke(-\xi_i)u_i\right)$$

$$k[T] \otimes_{A'} M' \simeq k[T] \otimes_k \left(\bigoplus_{j=1}^{r} ke(-\xi'_j)u'_j\right)$$

we have k-isomorphism

$$\bigoplus_{i=1}^{r} ke(-\xi_i)u_i \to \bigoplus_{j=1}^{r} ke(-\xi'_j)u'_j$$

i.e. there is a matrix $P = P(C, C')$ in $GL_r(k)$ such that

$$\begin{pmatrix} e(-\xi_1)u_1 \\ \vdots \\ e(-\xi_r)u_r \end{pmatrix} = P \begin{pmatrix} e(-\xi'_1)u'_1 \\ \vdots \\ e(-\xi'_r)u'_r \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} &= \begin{pmatrix} e(-\xi_1) & & 0 \\ & \ddots & \\ 0 & & e(-\xi_r) \end{pmatrix}^{-1} P \begin{pmatrix} e(-\xi'_1) & & 0 \\ & \ddots & \\ 0 & & e(-\xi'_r) \end{pmatrix} \begin{pmatrix} u'_1 \\ \vdots \\ u'_r \end{pmatrix} \\ &= (p_{ij}e(\xi_i - \xi'_j)) \begin{pmatrix} u'_1 \\ \vdots \\ u'_r \end{pmatrix} \end{aligned}$$

where p_{ij} is the (i, j)-entry of P. Since, on $U \cap U'$, $p_{ij}e(\xi_i - \xi'_j)$ are regular functions

$$\phi(\xi_i) \geqq \phi(\xi'_j)$$

for every ϕ in $C \cap C'$ if $p_{ij} \neq 0$. Since

$$\det P = \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) p_{1_{\sigma(1)}} \cdots p_{r_{\sigma(r)}} \neq 0$$

there exists a permutation σ such that $p_{i_{\sigma(i)}} \neq 0$ for every i. Thus

$$\phi(\xi_i) \geqq \phi(\xi'_{\sigma(i)})$$

for every i and every ϕ in $C \cap C'$. Furthermore since

$$\det(p_{ij}e(\xi_i - \xi'_j)) = \det(P)e(\xi_i + \cdots + \xi_r - \xi'_1 - \cdots - \xi'_r)$$

is a unit on $U \cap U'$, we have

$$\phi(\xi_1 + \cdots + \xi_r - \xi'_1 - \cdots - \xi'_r) = 0$$

for every ϕ in $C \cap C'$. Compairing this equality with the above inequality, we have

$$\phi(\xi_i) = \phi(\xi'_{\sigma(i)})$$

for every i and every ϕ in $C \cap C'$. Q.E.D.

By virtue of Proposition 4.1, a T-linearized vector bundle of rank r on X gives rise to the following data:

(i) $m: Sk^1(\mathscr{C}) = \mathscr{C}^1 = \{\phi_1, \cdots, \phi_d\} \to \boldsymbol{Z}^{\oplus r}$ sending ϕ to $m(\phi) = (m(\phi)_1, \cdots, m(\phi)_r)$ where $Sk^1(\mathscr{C})$ is a set of 1-dimensional cones of $\mathscr{C}$, and for every C in $\mathscr{C}^n$

$$m_C: C \cap Sk^1(\mathscr{C}) \to \boldsymbol{Z}^{\oplus r}$$

so that there is a permutation τ such that

$$\begin{aligned} m_C(\phi) &= (m_C(\phi)_1, \cdots, m_C(\phi)_r) \\ &= (m(\phi)_{\tau(1)}, \cdots, m(\phi)_{\tau(r)}) \end{aligned}$$

for every ϕ in $C \cap Sk^1(\mathscr{C})$.

Suppose the data (i) are given, then for C in $\mathscr{C}^n$ we have characters $\xi(C)_i$ by solving equations

$$\phi(\xi) = m_C(\phi)_i$$

for every ϕ in $C \cap Sk^1(\mathscr{C})$. For maximal cones C and C' there exists a permutation τ such that

$$\phi(\xi(C)_i) = \phi(\xi(C')_{\tau(i)})$$

for every i and every ϕ in $C \cap C'$ by the condition on m_C and $m_{C'}$ in (i). Conversely if we have these $\xi(C)_i$ then the data (i) is obtained. So (i) is equivalent to

(i′) $\xi: \mathscr{C}^n \to \Xi^{\oplus r}$
sending C to $\xi(C) = (\xi(C)_1, \cdots, \xi(C)_r)$ such that for every pair of cones C, C' in $\mathscr{C}^n$ there exists a permutation τ so that

$$\phi(\xi(C)_i) = \phi(\xi(C')_{\tau(i)}) \qquad \text{for every } i \text{ and every } \phi \text{ in } C \cap C' .$$

(ii) $P: \mathscr{C}^n \times \mathscr{C}^n \to GL_r(k)$
sending (C, C') to $P(C, C') = (P(C, C')_{ij})$ such that $P(C, C')_{ij} \neq 0$ only if $m_C(\phi)_i \geqq m_{C'}(\phi)_j$ for every ϕ in $(C \cap C') \cap Sk^1(\mathscr{C})$ and such that

$$P(C, C')P(C', C'') = P(C, C'')$$

for every C, C', C'' in $\mathscr{C}^n$.

(iii) For two pairs (m, P) and (m', P'), we say that they are equivalent if there exists a permutation $\sigma = \sigma(C)$ in $\mathfrak{S}_r$ such that

$$(m_C(\phi)_1, \cdots, m_C(\phi)_r) = (m'_C(\phi)_{\sigma(1)}, \cdots, m'_C(\phi)_{\sigma(r)})$$

for every C in $\mathscr{C}^n$ and ϕ in $C \cap Sk^1(\mathscr{C})$ and if there exists

$$\rho: \mathscr{C}^n \to GL_r(k)$$

such that

$$P'(C, C') = \rho(C)^{-1}P(C, C')\rho(C')$$

for every C and C' in $\mathscr{C}^n$.

THEOREM 4.2. *Let (X, T) be a smooth complete almost homogeneous variety defined by a cone complex $(\Gamma, \mathscr{C})$. The set of T-linearized vector bundles of rank r up to T-isomorphism corresponds bijectively to the set of data* (i′) *and* (ii) *up to equivalence* (iii).

Proof. Let (E, ϕ_t) be a T-linearized vector bundle of rank r on X. Let U be an affine open subset of X corresponding to C in $\mathscr{C}^n$. Let (u_i) be semi-invariant base of $E|U$ on U and let (ξ_i) be characters corresponding to (u_i). We define m_C by

$$m_C(\phi)_i = \phi(\xi_i) = \langle \xi_i, \phi \rangle \qquad i = 1, \cdots, r$$

for every ϕ in $C \cap Sk^1(\mathscr{C})$. Then the condition of (i) are satisfied. P is obtained in Proposition 4.1. Let (F, ψ_t) be a T-linearized vector bundle of rank r on X. Let (v_j) be semi-invariant base of $F|U$ on U and let (η_j) be characters corresponding to (v_j). Suppose E and F are T-isomorphic. We denote

$$M = E(U) = Au_1 \oplus \cdots \oplus Au_r$$
$$N = F(U) = Av_1 \oplus \cdots \oplus Av_r$$

where $A = A_C$ is a polynomial ring corresponding to the cone C. Since $E|U \simeq F|U$ we have Ξ-graded A-isomorphism

$$M = A \otimes \left(\bigoplus_{i=1}^{r} ke(-\xi_i)u_i \right) \to N = A \otimes \left(\bigoplus_{j=1}^{r} ke(-\eta_j)v_j \right).$$

So we have k-isomorphism

$$\bigoplus_{i=1}^{r} ke(-\xi_i)u_i \to \bigoplus_{j=1}^{r} ke(-\eta_j)v_j$$

i.e. there exists a matrix $\rho = \rho(C)$ in $GL_r(k)$ such that

$$\begin{pmatrix} e(-\xi_1)u_1 \\ \vdots \\ e(-\xi_r)u_r \end{pmatrix} = \rho \begin{pmatrix} e(-\eta_1)v_1 \\ \vdots \\ e(-\eta_r)v_r \end{pmatrix}.$$

Then

$$\begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} = (\rho_{ij} e(\xi_i - \eta_j)) \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}$$

where ρ_{ij} is the (i, j)-entry of ρ. Since, on $U, \rho_{ij} e(\xi_i - \eta_j)$ are regular functions we have

$$\phi(\xi_i) \geqq \phi(\eta_j)$$

for every ϕ in C if $\rho_{ij} \neq 0$. Since

$$\det(\rho) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \times \rho_{i\sigma(i)} \cdots \rho_{r\sigma(r)} \neq 0$$

there exists a permutation $\sigma = \sigma(C)$ such that $\rho_{i\sigma(i)} \neq 0$ for every i. Thus

$$\phi(\xi_i) \geqq \phi(\eta_{\sigma(i)})$$

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for every i and every ϕ in C. Furthermore since

$$\det(\rho_{ij}e(\xi_i - \eta_j)) = \det(\rho)e(\xi_1 + \cdots + \xi_r - \eta_1 - \cdots - \eta_r)$$

is a unit on U, we have

$$\phi(\xi_1) + \cdots + \phi(\xi_r) = \phi(\eta_{\sigma(1)}) + \cdots + \phi(\eta_{\sigma(r)})$$

for every ϕ in C. So

$$\phi(\xi_i) = \phi(\eta_{\sigma(i)})$$

for every ϕ in C. So this σ satisfies the condition of $\sigma(C)$ of (iii). Let U' be an affine open subset of X corresponding to C' in $\mathscr{C}^n$. Consider the diagram on $U \cap U'$

$$\begin{array}{ccc} E|U & \longrightarrow & F|U \\ \downarrow & \supset & \downarrow \\ E|U' & \longrightarrow & F|U' . \end{array}$$

Let $P = P(C, C')$ be defined in Proposition 4.1 for E and let $Q = Q(C, C')$ for F. Then by virtue of above commutative diagram we have

$$\rho(C)P(C, C') = Q(C, C')\rho(C')$$

where ρ is defined above. This means the second part of (iii).

Conversely if we have the data (i′) and (ii), then the T-linearized vector bundle E of rank r can be constructed as follows. For C in $\mathscr{C}^n$, we write by $U_C = A^n = \mathrm{Spec}(A_C)$ a T-stable affine open subset of X corresponding to C. Suppose m and P are given. Let

$$E(U_C) = A_C(-\xi(C)_1) \oplus \cdots \oplus A_C(-\xi(C)_r)$$

for each C in $\mathscr{C}^n$. The Ξ-graded A_C-module structure on $E(U_C)$ gives rise to a T-linearized vector bundle $E|U_C = \widetilde{E(U_C)}$ on U_C. Let

$$f_{C,C'} : A_{C \cap C'} \otimes_{A_C} E(U_C) \to A_{C' \cap C} \otimes_{A_{C'}} E(U_{C'})$$

be Ξ-graded $A_{C \cap C'}$-module isomorphism defined by

$$\mathrm{diag}\,(e(\xi(C')_1), \cdots, e(\xi(C')_r))P(C, C')^{-1}\,\mathrm{diag}\,(e(-\xi(C)_1), \cdots, e(-\xi(C)_r)) .$$

This $f_{C,C'}$ gives rise to an isomorphism

$$f_{C,C'} : E|U_C|U_C \cap U_{C'} \to E|U_{C'}|U_{C'} \cap U_C$$

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compatible with the action of T. By the condition of (ii) we can patch $E|U_C$ and $E|U_{C'}$ along $U_C \cap U_{C'}$. So we obtain a T-linearized vector bundle $E = E(m, P)$ on X. The data (m, P) is equivalent to the data (m', P') then by the construction $E(m, P)$ and $E(m', P')$ are T-isomorphic T-linearized vector bundles. Q.E.D.

Remark 4.3. Two T-linearized vector bundles $E(m, P)$ of rank r and $E(m', P')$ of rank r' are given. Then the T-linearized vector bundle $E(m, P) \otimes E(m', P')$ is $E(m \otimes m', P \otimes P')$ where

$$(m \otimes m')(\phi) = (\cdots, m(\phi)_i + m'(\phi)_j, \cdots)$$

and $P \otimes P'$ means the Kronecker product. The T-linearized vector bundle $\det E(m, P)$ is $E(\det m, \det P)$ where

$$(\det m)(\phi) = \sum_{i=1}^{r} m(\phi)_i$$

and

$$(\det P)(C, C') = \det (P(C, C')) .$$

Remark 4.4. The case of rank $= 1$. For $\phi_i \in Sk^1(\mathscr{C})$ we denote by m_i the value $m(\phi_i)$. Let D_i be the divisor corresponding to ϕ_i i.e. $D_i = \operatorname{div}(e(\xi_i))$ on T-stable affine open U_k where ξ_i is a character so that $\phi_j(\xi_i) = \delta_{ji}$ for $\phi_j \in Sk^1(\mathscr{C}) \cap C_k$. Then the data $m = (m_i)$ corresponds to the line bundle $O_X(-\sum m_i D_i)$.

§ 5. Examples on P^2

In this section, we consider $X = \boldsymbol{P}^2 = \operatorname{Proj}(k[X_0, X_1, X_2])$ with the standard action of $T = G_m \times G_m$. Let $E(a, b, c)$ be a vector bundle defined by the exact sequence

$$\begin{aligned} 0 \to O_X \to O_X(a) \oplus O_X(b) \oplus O_X(c) \to E(a, b, c)^* \to 0 \\ 1 \to (X_0^a, X_1^b, X_2^c) \qquad\qquad\qquad\qquad \end{aligned}$$

where a, b, c are positive integers. It is easy to see that $E(a, b, c)^*$ is an equivariant vector bundle for positive integers a, b, c.

THEOREM 5.1. *Let T be a 2-dimensional torus. T acts naturally on $\boldsymbol{P}^2$ and it becomes an almost homogeneous variety. An indecomposable equivariant vector bundle of rank 2 on $\boldsymbol{P}^2$ is isomorphic to $E(a, b, c) \otimes O_{P^2}(n)$ for some integer n and some positive integers a, b, c.*

We now compute the data (m, P) for $E(a, b, c)$. Put $S = k[X_0, X_1, X_2]$ and let M be the kernel of

$$\begin{aligned} Se_0 \oplus Se_1 \oplus Se_2 &\to S \\ e_0 &\mapsto X_0^a \\ e_1 &\mapsto X_1^b \\ e_2 &\mapsto X_2^c \,. \end{aligned}$$

The generators of M are

$$X_1^b e_0 - X_0^a e_1 \,, \quad X_1^b e_2 - X_2^c e_1 \,, \quad X_0^a e_2 - X_2^c e_0 \,.$$

Put $x = X_0/X_2, y = X_1/X_2$ and let U_1, U_2, U_3 be affine spaces defined by $X_2 \neq 0, X_0 \neq 0, X_1 \neq 0$ respectively, i.e.,

$$U_1 = \operatorname{Spec}(k[x, y]) \,, \quad U_2 = \operatorname{Spec}\left(k\left[\frac{1}{x}, \frac{y}{x}\right]\right) , \quad U_3 = \operatorname{Spec}\left(k\left[\frac{1}{y}, \frac{x}{y}\right]\right) .$$

Put

$$\begin{aligned} u_1 &= \frac{e_1}{X_2^b} - \left(\frac{X_1}{X_2}\right)^b \frac{e_2}{X_2^c} \,, & v_1 &= \frac{e_0}{X_2^a} - \left(\frac{X_0}{X_2}\right)^a \frac{e_2}{X_2^c} \,, \\ u_2 &= \frac{e_1}{X_0^b} - \left(\frac{X_1}{X_0}\right)^b \frac{e_0}{X_0^a} \,, & v_2 &= -\frac{e_2}{X_0^c} + \left(\frac{X_2}{X_0}\right)^c \frac{e_0}{X_0^a} \,, \\ u_3 &= -\frac{e_0}{X_1^a} + \left(\frac{X_0}{X_1}\right)^a \frac{e_1}{X_1^b} \,, & v_3 &= -\frac{e_2}{X_1^c} + \left(\frac{X_2}{X_1}\right)^c \frac{e_1}{X_1^b} \,. \end{aligned}$$

Let $t = (\lambda, \mu)$ in T be acts by $tX_0 = \lambda X_0, tX_1 = \mu X_1, tX_2 = X_2$. Then

$$te_0 = \lambda^a e_0 \,, \quad te_1 = \mu^b e_1 \,, \quad te_3 = e_3 \,,$$

and

$$tx = \lambda x \,, \qquad ty = \mu y \,.$$

In this case we take semi-invariant basis (u_i, v_i) on U_i and we have

$$\begin{aligned} \begin{pmatrix} x^b y^{-b} & 0 \\ 0 & x^c \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{-b} & 0 \\ 0 & x^{-a} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} , \\ \begin{pmatrix} x^{-a} y^a & 0 \\ 0 & y^c \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x^b y^{-b} & 0 \\ 0 & x^c \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} , \\ \begin{pmatrix} y^{-b} & 0 \\ 0 & x^{-a} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x^{-a} y^a & 0 \\ 0 & y^c \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} . \end{aligned}$$

Let $\phi_1, \phi_2 \in \Gamma$ be such that

$$\phi_1(\lambda) = \phi_2(\mu) = 1\ , \qquad \phi_1(\mu) = \phi_2(\lambda) = 0$$

where λ, μ are characters of T. Then the decomposition of Γ_Q by $(\phi_1, \phi_2, \phi_3 = -\phi_1 - \phi_2)$ defines a cone complex corresponding to $\boldsymbol{P}^2$ (see [4]) i.e. put

$$\begin{aligned} C_1 &= \{p\phi_1 + q\phi_2 \mid p, q \in \boldsymbol{Q}_0\} \\ C_2 &= \{p\phi_2 + q\phi_3 \mid p, q \in \boldsymbol{Q}_0\} \\ C_3 &= \{p\phi_3 + q\phi_1 \mid p, q \in \boldsymbol{Q}_0\} \end{aligned}$$

then $\Gamma_Q = C_1 \cup C_2 \cup C_3$, and C_i corresponds to an affine space U_i. Let ξ_i, η_i be the characters corresponding to u_i, v_i respectively i.e.

$$\begin{aligned} \xi_1(\lambda, \mu) &= \mu^b\ , & \eta_1(\lambda, \mu) &= \lambda^a\ , \\ \xi_2(\lambda, \mu) &= \lambda^{-b}\mu^b\ , & \eta_2(\lambda, \mu) &= \lambda^{-c}\ , \\ \xi_3(\lambda, \mu) &= \lambda^a\mu^{-a}\ , & \eta_3(\lambda, \mu) &= \mu^{-c}\ . \end{aligned}$$

Then

$$\begin{aligned} \phi_1(\xi_1) &= \phi_1(\eta_3) = 0\ , & \phi_1(\eta_1) &= \phi_1(\xi_3) = a\ , \\ \phi_2(\xi_2) &= \phi_2(\xi_1) = b\ , & \phi_2(\eta_2) &= \phi_2(\eta_1) = 0\ , \\ \phi_3(\xi_3) &= \phi_3(\xi_2) = 0\ , & \phi_3(\eta_3) &= \phi_3(\eta_2) = c\ . \end{aligned}$$

These integers mean the data m in § 4, i.e.

$$m(\phi_1) = (0, a)\ , \quad m(\phi_2) = (b, 0)\ , \quad m(\phi_3) = (0, c)\ .$$

To prove Theorem 5.1 we have only to show that the data in § 4 define a vector bundle

$$E(a, b, c) \otimes O_{\boldsymbol{P}^2}(n)$$

for some integer n and some positive integers a, b, c. Let D_i be the divisor corresponding to ϕ_i. By Remarks 4.3 and 4.4, the data $\overline{m}$ for

$$E \otimes O_{\boldsymbol{P}^2}\Big(\sum_{i=1}^{3} m_i D_i\Big)$$

are

$$\begin{aligned} \overline{m}(\phi_1) &= (m(\phi_1)_1 - m_1\ , & m(\phi_1)_2 - m_1) \\ \overline{m}(\phi_2) &= (m(\phi_2)_1 - m_2\ , & m(\phi_2)_2 - m_2) \\ \overline{m}(\phi_3) &= (m(\phi_3)_1 - m_3\ , & m(\phi_3)_2 - m_3) \end{aligned}$$

where $m(\phi_1) = (m(\phi_i)_1, m(\phi_i)_2)$ are the data m for T-linearized vector bundle

E. Thus by tensoring

$$O_{P^2}(m_1 + m_2 + m_3) = O_{P^2}\Big(\sum_{i=1}^{3} m_i D_i\Big)$$

with the T-linearized vector bundle E, if necessary, we may assume that the data m for E are

$$m(\phi_i) = (\alpha_i, 0) \quad \text{or} \quad m(\phi_i) = (0, \alpha_i)$$

for non-negative integers α_i. Furthermore by changing the base if necessary, we may assume that

$$m_{C_1}(\phi_2) = m_{C_2}(\phi_2) = (\alpha_2, 0)$$

and one of

a) $m_{C_2}(\phi_3) = m_{C_3}(\phi_3) = (\alpha_3, 0)$
b) $m_{C_2}(\phi_3) = m_{C_3}(\phi_3) = (0, \alpha_3)$

and one of

1) $m_{C_3}(\phi_1) = m_{C_1}(\phi_1) = (\alpha_1, 0)$
2) $m_{C_3}(\phi_1) = m_{C_1}(\phi_1) = (0, \alpha_1)$
3) $m_{C_3}(\phi_1) = (\alpha_1, 0)$, $m_{C_1}(\phi_1) = (0, \alpha_1)$
4) $m_{C_3}(\phi_1) = (0, \alpha_1)$, $m_{C_1}(\phi_1) = (\alpha_1, 0)$.

We note that if the data P are of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

i.e. $P(C_1, C_2), P(C_2, C_3), P(C_3, C_1)$ are of the above form, then the vector bundle $E(m, P)$ is decomposable.

If one of α_i is zero, we may assume $\alpha_1 = 0$, then

a) $P(C_1, C_2) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $P(C_2, C_3) = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$, $P(C_3, C_1) = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$

or

b) $P(C_1, C_2) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $P(C_2, C_3) = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}$, $P(C_3, C_1) = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$.

In the equivalence data, we take

$$\text{a)}\quad \rho(C_1)=\begin{pmatrix}1 & \dfrac{b}{d}\\ 0 & 1\end{pmatrix},\quad \rho(C_2)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \rho(C_3)=\begin{pmatrix}1 & -\dfrac{b'}{a'}\\ 0 & 1\end{pmatrix}$$

or

$$\text{b)}\quad \rho(C_1)=\begin{pmatrix}1 & -\dfrac{b}{a}\\ 0 & 1\end{pmatrix},\quad \rho(C_2)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \rho(C_3)=\begin{pmatrix}1 & 0\\ \dfrac{c'}{a'} & 1\end{pmatrix}$$

then

$$P'(C_1,C_2)=\rho(C_1)^{-1}P(C_1,C_2)\rho(C_2)=\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}$$

$$P'(C_2,C_3)=\rho(C_2)^{-1}P(C_2,C_3)\rho(C_3)=\begin{pmatrix}a' & 0\\ 0 & d'\end{pmatrix}.$$

Consequently we get

$$P'(C_3,C_1)=\begin{pmatrix}a^{-1}a'^{-1} & 0\\ 0 & d^{-1}d'^{-1}\end{pmatrix}$$

from the relation

$$P'(C_1,C_2)P'(C_2,C_3)P'(C_3,C_1)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

So E is decomposable thus we may assume that α_i are positive integers.

Case a, 1) In the above argument, we may take $c''=0$. So the vector bundle E is decomposable.

Case a, 2) This case means that

$$P(C_1,C_2)=\begin{pmatrix}a & b\\ 0 & d\end{pmatrix},\quad P(C_2,C_3)=\begin{pmatrix}a' & b'\\ 0 & d'\end{pmatrix},\quad P(C_3,C_1)=\begin{pmatrix}a'' & 0\\ c'' & d''\end{pmatrix}.$$

From the relation

$$P(C_1,C_2)P(C_2,C_3)P(C_3,C_1)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

we have

$$c''=0 \quad\text{and}\quad ab'+bd'=0\,.$$

We take

$$\rho(C_1)=\rho(C_3)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \rho(C_2)=\begin{pmatrix}1 & -\dfrac{b}{a}\\ 0 & 1\end{pmatrix}$$

then

$$P'(C_1, C_2)=\rho(C_1)^{-1}P(C_1, C_2)\rho(C_2)=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$$

$$P'(C_2, C_3)=\rho(C_2)^{-1}P(C_2, C_3)\rho(C_3)=\begin{pmatrix}a' & b'+\dfrac{b}{a}d'\\ 0 & b'\end{pmatrix}=\begin{pmatrix}a' & 0\\ 0 & b'\end{pmatrix}$$

$$P'(C_3, C_1)=\rho(C_3)^{-1}P(C_3, C_1)\rho(C_1)=\begin{pmatrix}a'' & 0\\ c'' & d''\end{pmatrix}=\begin{pmatrix}a'' & 0\\ 0 & d''\end{pmatrix}.$$

So E is decomposable.

Case a, 3) In this case

$$P(C_1, C_2)=\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}, \quad P(C_2, C_3)=\begin{pmatrix}a' & b'\\ 0 & d'\end{pmatrix}, \quad P(C_3, C_1)=\begin{pmatrix}a'' & b''\\ c'' & 0\end{pmatrix}.$$

Then

$$P(C_1, C_2)P(C_2, C_3)P(C_3, C_1)=\begin{pmatrix}aa'a''+(ab'+bd')c'' & aa'b''\\ dd'c'' & 0\end{pmatrix}$$

This contradicts the relation

$$P(C_1, C_2)P(C_2, C_3)P(C_3, C_1)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

So this case cannot happen.

Case a, 4) This case cannot happen for the same reason as in the case a, 3).

Case b, 1) This case determine the decomposable vector bundle for the same reason of the case a, 2).

Case b, 2) This case cannot happen for the same reason of the case a, 3).

Case b, 3) In this case

$$P(C_1, C_2)=\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}, \quad P(C_2, C_3)=\begin{pmatrix}a' & 0\\ c' & d'\end{pmatrix}, \quad P(C_3, C_1)=\begin{pmatrix}a'' & b''\\ c'' & 0\end{pmatrix}.$$

By taking

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$$\rho(C_1) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad \rho(C_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(C_3) = \begin{pmatrix} a'^{-1} & 0 \\ 0 & d'^{-1} \end{pmatrix}$$

we may assume that

$$a = d = a' = d' = 1 , \qquad b''c'' = -1 .$$

Since

$$P(C_1, C_2)P(C_2, C_3)P(C_3, C_1) = \begin{pmatrix} (1 + bc')a'' + bc'' & (1 + bc')b'' \\ c'a'' + c'' & c'b'' \end{pmatrix}$$

we have

$$\begin{gathered} (1 + bc')a'' + bc'' = c'b'' = 1 \\ (1 + bc')b'' = c'a'' + c'' = 0 . \end{gathered}$$

Then

$$b'' = -b , \quad c'' = b^{-1} , \quad c' = -b^{-1} , \quad a'' = 1.$$

Furthermore we take

$$\rho(C_1) = \rho(C_2) = \rho(C_3) = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix},$$

then we may assume that

$$P(C_1, C_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P(C_2, C_3) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad P(C_3, C_1) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

We note that transition matrices in the example are

$$\begin{aligned} P(C_2, C_1) &= P(C_1, C_2)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ P(C_3, C_2) &= P(C_2, C_3)^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ P(C_1, C_3) &= P(C_3, C_1)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

So this case determines the vector bundle of the type of the example.

Case b, 4) For the same reason as in the case b, 3), we may assume that

$$P(C_1, C_2) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad P(C_2, C_3) = \begin{pmatrix} 1 & 0 \\ c' & 1 \end{pmatrix},$$

$$P(C_3, C_1) = \begin{pmatrix} 0 & b'' \\ c'' & d'' \end{pmatrix}, \qquad b''c'' = -1 .$$

Then

$$P(C_1, C_2)P(C_2, C_3)P(C_3, C_1) = \begin{pmatrix} bc'' & (1 + bc')b'' + bd'' \\ c'' & cb'' + d'' \end{pmatrix} .$$

So c'' must be zero, then $bc'' = 0$. This is a contradiction. This case cannot happen.

Thus every indecomposable vector bundle of rank 2 on $\boldsymbol{P}^2$ is of the form

$$E(a, b, c) \otimes O_{\boldsymbol{P}^2}(n)$$

for some integer n and some positive integers a, b, c. Q.E.D.

Remark 5.2. If $a = b = c = 1$, then the vector bundle $E(1, 1, 1) \otimes O_{\boldsymbol{P}^2}(n)$ is homogeneous, i.e. equivariant with respect to the standard action of $PGL(2)$ on $\boldsymbol{P}^2$. Conversely every homogeneous indecomposable vector bundle of rank 2 on $\boldsymbol{P}^2$ is necessarily of the form $E(1, 1, 1) \otimes \underline{O}_{\boldsymbol{P}^2}(n)$ (ch $k = 0$). [See 5]

The following problems can be posed about equivariant vector bundles on almost homogeneous varieties.

PROBLEM 5.3. Classification of equivariant vector bundles of rank greater than 2.

PROBLEM 5.4. Is there any indecomposable equivariant vector bundle of $2 \leqq \text{rank} \leqq n - 1$ on P^n $(n \geqq 3)$? We can construct those of rank n as in the case of 2.

PROBLEM 5.5. Classification of equivariant vector bundles on X, when X is an almost homogeneous variety of dimension 2.

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Nagoya University