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Stability of force-free magnetic fields versus magnetic pitch

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Abstract. Starting from the one-dimensional energy integral and related stability theorems given by Newcomb [*Ann. Phys (NY)* **10**, 232 (1960)] for a linear pinch system, this paper analyses the stability of one-dimensional force-free magnetic fields in cylindrical coordinates (r, θ, z). It is found that the stability of the force-free field is closely related to the radial distribution of the pitch of the field lines: $h(r) = 2\pi r B_z/B_\theta$. The following three types of force-free fields are proved to be unstable: (i) force-free fields with a uniform pitch; (ii) force-free fields with a pitch that increases in magnitude with r in the neighbourhood of $r = 0$ ($d|h|/dr > 0$); and (iii) force-free fields for which $(dh/dr)_{r=0} = 0$, $B_\theta \propto r^m$ in the neighbourhood of $r = 0$, and $(h\, d^2h/dr^2)_{r=0} > -128\pi^2/(2m+4)^2$. On the other hand, the stability does not have a definite relation to the maximum of the force-free factor α defined by $\nabla \times \mathbf{B} = \alpha\mathbf{B}$. Examples will be given to illustrate that force-free fields with an infinite force-free factor at the boundary are stable, whereas those with a force-free factor that is finite and smaller than the lowest eigenvalue of linear force-free field solutions in the domain of interest are unstable. The latter disproves the sufficient criterion for stability of nonlinear force-free magnetic fields given by Krüger [*J. Plasma Phys.* **15**, 15 (1976)] that a nonlinear force-free field is stable if the maximum absolute value of the force-free factor is smaller than the lowest eigenvalue of linear force-free field solutions in the domain of interest.

1. Introduction

In solar active regions, the magnetic pressure is much larger than the gas pressure, so the force-free magnetic field, defined by $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$ and $\nabla \cdot \mathbf{B} = 0$, or, equivalently,

$$\nabla \times \mathbf{B} = \alpha(\mathbf{r})\mathbf{B}, \qquad \mathbf{B} \cdot \nabla\alpha = 0, \tag{1}$$

serves as a good approximation (Low 1982). Here $\alpha(\mathbf{r})$ is the force-free factor, and it is a constant for a linear force-free field and a function of $\mathbf{r}$ for a nonlinear one. A force-free field should be stable when it is to be used to interpret quiet magnetic structures in active regions. On the other hand, instabilities caused by the temporal evolution of force-free fields have often been invoked to explain various explosive phenomena. Therefore, stability analysis is important in the application of force-free fields to both quiet magnetic structures and explosive phenomena in active regions.

A useful approach to the stability problem of magnetostatic equilibria is Bernstein's energy principle (Bernstein et al. 1958). According to this principle, a sufficient and necessary condition for the stability of a system is that its perturbed potential energy, given by a so-called energy integral, is positive for every perturbation displacement ξ satisfying appropriate boundary conditions. This energy principle is also applicable to force-free fields. Based on it, Krüger (1976a) proved that a sufficient and necessary condition for stability of a linear force-free field in a finite domain is that the absolute value of the force-free factor is smaller than the lowest eigenvalue of linear force-free field solutions in the domain. Such a conclusion might lead to the following inference: the magnitude of the force-free factor should also bear on the stability of a nonlinear force-free field. As a matter of fact, Krüger did give a sufficient criterion for stability of nonlinear force-free fields in the same paper: the maximum absolute value of the force-free factor is smaller than the lowest eigenvalue associated with the domain of interest, but the proof was not valid.

For one-dimensional magnetostatic equilibria in cylindrical coordinates, Newcomb (1960) derived a one-dimensional form of Bernstein's energy integral, and presented several stability criteria. Hu (2000) applied Newcomb's energy integral and related stability criteria to the analysis of stability of one-dimensional force-free fields in cylindrical coordinates, and proved that the force-free field with a singular current-density surface given by Low (1993) is stable. The force-free factor of this field approaches infinity at the singular current-density surface. Another force-free field given by Gold and Hoyle (1960) was then proved by Hu (2001) to be unstable, and it has a finite force-free factor everywhere and serves as a disproof to the sufficient criterion of stability for nonlinear force-free fields addressed by Krüger (1976a).

Based on the work by Newcomb (1960) and Hu (2000, 2001), this paper will make a further analysis of the stability of nonlinear force-free fields with emphasis on the relationship between the stability of a nonlinear force-free field and the radial distribution of the pitch of the field lines.

2. The energy integral

For one-dimensional magnetostatic equilibria in cylindrical coordinates (r, θ, z), Newcomb (1960) transformed the energy integral into the following form:

$$W = \frac{\pi}{2} \int_0^R dr \left[f \left(\frac{d\xi}{dr} \right)^2 + g\xi^2 \right], \tag{2}$$

where

$$f = \frac{r(krB_z + mB_\theta)^2}{k^2r^2 + m^2}, \tag{3}$$

$$g = \frac{2k^2r^2}{k^2r^2 + m^2}\frac{dp}{dr} + \frac{k^2r^2 + m^2 - 1}{r(k^2r^2 + m^2)}(krB_z + mB_\theta)^2 + \frac{2k^2r}{(k^2r^2 + m^2)^2}(k^2r^2B_z^2 - m^2B_\theta^2), \tag{4}$$

B_z and B_θ are the magnetic field components, p is the gas pressure, R is the radius of the cylinder domain for the equilibrium, ξ is the amplitude of the r component

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of the perturbation displacement whose phase factor is $\exp(im\theta + ikz)$, m is an integer, and k is a real number. An equilibrium is stable if and only if $W > 0$ for arbitrary m, k, and ξ that satisfies $\xi(r = R) = 0$. Starting from the energy integral, Newcomb proved that the most unstable mode is either $m = 0, k \to 0$ or $m = 1$, $-\infty < k < \infty$. Let us now restate some derivations by Hu (2000) in order to obtain the energy integral. For force-free magnetic fields, $dp/dr = 0$, so that for the mode of $m = 0$, $k \to 0$, one has

$$f = rB_z^2 > 0, \qquad g = \frac{B_z^2}{r} > 0;$$

W is always positive. Therefore, only the mode $m = 1$ $(-\infty < k < \infty)$ needs to be examined. Taking $m = 1$ and $dp/dr = 0$, (2) and (3) become

$$f = \frac{r(krB_z + B_\theta)^2}{1 + k^2r^2}, \tag{5}$$

$$g = \frac{k^2r(krB_z + B_\theta)^2}{1 + k^2r^2} + \frac{2k^2r(k^2r^2B_z^2 - B_\theta^2)}{(1 + k^2r^2)^2}. \tag{6}$$

These two equations were derived by Hu (2000). The restatement is complete.

The pitch of field lines is defined by

$$h(r) = \frac{2\pi rB_z}{B_\theta}, \tag{7}$$

which represents the distance travelled by a field line along the z direction while it revolves one cycle around the cylindrical axis. Inserting (7) into (5) and (6) gives

$$f = \frac{rB_\theta^2(1 + kh/2\pi)^2}{1 + k^2r^2}, \tag{8}$$

$$g = \frac{k^2rB_\theta^2(1 + kh/2\pi)^2}{1 + k^2r^2} + \frac{2k^2rB_\theta^2}{(1 + k^2r^2)^2}\left[\left(\frac{kh}{2\pi}\right)^2 - 1\right]. \tag{9}$$

3. Stability versus the pitch

As can be seen from (8) and (9), f and the first term of g are positive-definite, whereas the sign of the second term of g is indefinite and depends on h, implying the importance of the radial distribution of the pitch on the stability of the force-free field. For a force-free field with uniform pitch, one may consider the most unstable mode of

$$k = \frac{2\pi(\delta - 1)}{h}, \qquad 0 < \delta \ll 1, \tag{10}$$

and (8) and (9) become

$$f = \frac{rB_\theta^2\delta^2}{1 + k^2r^2}, \tag{11}$$

$$g = \frac{k^2rB_\theta^2(3 + k^2r^2)\delta^2}{(1 + k^2r^2)^2} - \frac{4k^2rB_\theta^2\delta}{(1 + k^2r^2)^2}. \tag{12}$$

Only the second term of g is of the first order in δ, being dominant and definitely negative, and thus one can always find a certain positive number $\delta \ll 1$ so as to make the energy integral (2) negative regardless of the value of R. We then have the following theorem.

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Theorem 1. *All force-free fields with uniform pitch are unstable.*

Based on a similar derivation, Hu (2001) proved that the force-free field

$$B_z = \frac{a}{a^2 + r^2}, \qquad B_\theta = \frac{r}{a^2 + r^2} \tag{13}$$

given by Gold and Hoyle (1960) is unstable. This field has a pitch $h = 2\pi a$ and belongs to a force-free one with uniform pitch, so that it is unstable according to Theorem 1.

For the general case where h is a function of r, one may always find a neighbourhood of $r = 0$ in which h varies monotonically with r. In this neighbourhood, one may find $k = k_c$ in such a way that

$$\frac{k_c h(r_c)}{2\pi} + 1 = 0, \qquad \text{or} \qquad k_c = -\frac{2\pi}{h(r_c)}, \tag{14}$$

and r_c serves as the unique regular singular point in the neighbourhood. According to Theorem 3 of Newcomb (1960), a necessary condition for stability of the system is that the energy integral in the subinterval $(0, r_c)$ is non-negative. From (2), (8), (9) and (14), the energy integral in this subinterval reads

$$\begin{aligned} W_c = {} & \frac{\pi}{2}\int_0^{x_c} \frac{xB_\theta^2}{1+x^2}\left(\frac{h}{h_c} - 1\right)^2 \left(\frac{d\xi}{dx}\right)^2 dx \\ & + \frac{\pi}{2}\int_0^{x_c} \left[\frac{xB_\theta^2}{1+x^2}\left(\frac{h}{h_c} - 1\right)^2 + \frac{2xB_\theta^2}{(1+x^2)^2}\left(\frac{h^2}{h_c^2} - 1\right)\right] \xi^2\, dx, \end{aligned} \tag{15}$$

where $x = |k_c| r$, $x_c = |k_c| r_c$, and $h_c = h(r_c)$. Introduce the following perturbation displacement in the subinterval:

$$\xi = \begin{cases} 1 & (0 \leqslant x \leqslant x_c - \epsilon), \\ (x_c - x)/\epsilon & (x_c - \epsilon \leqslant x \leqslant x_c), \end{cases} \tag{16}$$

and thus

$$\frac{d\xi}{dx} = \begin{cases} 0, & (0 \leqslant x \leqslant x_c - \epsilon), \\ -1/\epsilon, & (x_c - \epsilon \leqslant x \leqslant x_c), \end{cases} \tag{17}$$

where $\epsilon \ll x_c$. Considering that in the vicinity of x_c, $h/h_c \approx 1 + h_c'(x - x_c)/h_c$ (where $h_c' = h'(x_c)$ and the prime denotes the derivative of the first order), the first term on the right-hand side of (15) becomes

$$\frac{\pi}{2}\int_{x_c-\epsilon}^{x_c} \frac{xB_\theta^2 h_c'^2 (x - x_c)^2}{(1+x^2) h_c^2 \epsilon^2} dx \approx \frac{\pi x_c B_{\theta c}^2 h_c'^2 \epsilon}{6(1 + x_c^2) h_c^2},$$

which vanishes as $\epsilon \to 0$. As a result, (15) is reduced to

$$W_c = \frac{\pi}{2}\int_0^{x_c} \left[\frac{xB_\theta^2}{1+x^2}\left(\frac{h}{h_c} - 1\right)^2 + \frac{2xB_\theta^2}{(1+x^2)^2}\left(\frac{h^2}{h_c^2} - 1\right)\right] dx. \tag{18}$$

Therefore, if $W_c < 0$, the nonlinear force-free field as a whole is unstable. Since $W_c(x_c = 0) = 0$, one must have $W_c < 0$ when $W_c' < 0$ in the neighbourhood of $x = 0$, and in this case the force-free field is unstable. It may be derived from (18) that

$$W_c'(x_c) = \frac{\pi}{2}\int_0^{x_c} \left[\frac{2xB_\theta^2}{1+x^2}\left(1 - \frac{h}{h_c}\right)\frac{hh_c'}{h_c^2} - \frac{4xB_\theta^2}{(1+x^2)^2}\frac{h^2 h_c'}{h_c^3}\right] dx. \tag{19}$$

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In the following, we assume that x_c is a small quantity and only keep the major term in $W'(x_c)$. The ratio between the first and second terms in the integrand is $(h - h_c)(1 + x^2)/2h$, which is of the order of x_c, so that the second is the major term. Consequently, for sufficiently small x_c, W_c' must have a sign opposite to that of $h_c h_c'$ or simply $|h_c|'$, leading to the following sufficient condition for instability of the force-free field:

$$\frac{d|h(r)|}{dr} > 0 \quad (0 < r \ll 1); \tag{20}$$

namely, all force-free fields with a pitch whose magnitude increases with r in the neighbourhood of $r = 0$ are unstable. The criterion (20) may be cast into a more convenient form for application. When $(d|h(r)|/dr)_{r=0} > 0$, (20) is certainly satisfied. On the other hand, if the derivatives of h with respect to r of up to the $(n-1)$th order vanish at $r = 0$ but the derivative of $|h|$ of the nth order is positive, (20) also holds. Therefore, in place of the criterion (20), we have

Theorem 2. *A sufficient condition for instability of force-free fields is that*

$$\left(\frac{d|h|}{dr}\right)_{r=0} > 0, \tag{21a}$$

or

$$\left(\frac{d^m h}{dr^m}\right)_{r=0} = 0, \quad (m = 1, 2, \ldots, n-1); \quad \left(\frac{d^n|h|}{dr^n}\right)_{r=0} > 0. \tag{21b}$$

In what follows, the application of Theorem 2 will be illustrated by three examples, expressed by

$$B_\theta = \frac{r}{(1+r^2)^\nu} \qquad B_z = \frac{[1 + 2(1-\nu)r^2]^{1/2}}{(2\nu - 1)^{1/2}(1+r^2)^\nu} \qquad (\tfrac{1}{2} < \nu < 1), \tag{22}$$

$$B_\theta = (rV)^{1/2} e^{-r/2}, \qquad B_z = [B_{z0}^2 + V(2-r)e^{-r}]^{1/2} \qquad (V > 0), \tag{23}$$

$$B_\theta = rV^{1/2} e^{-r/2}, \qquad B_z = [B_{z0}^2 + V(2 + 2r - r^2)e^{-r}]^{1/2} \qquad (V > 0). \tag{24}$$

These examples are adapted from the three magnetostatic equilibria given by Cargill et al. (1986). The force-free factor α $(= (1/rB_z)\, d(rB_\theta)/dr)$ and the pitch for each field is

$$\left.\begin{aligned} \alpha &= \frac{2(2\nu-1)^{1/2}[1 + (1-\nu)r^2]}{(1+r^2)[1 + 2(1-\nu)r^2]^{1/2}}, \\ h &= \frac{2\pi[1 + 2(1-\nu)r^2]^{1/2}}{(2\nu-1)^{1/2}}; \end{aligned}\right\} \tag{22a}$$

$$\left.\begin{aligned} \alpha &= \frac{(3-r)(V/r)^{1/2} e^{-r/2}}{2[B_{z0}^2 + V(2-r)e^{-r}]^{1/2}}, \\ h &= 2\pi\left(\frac{r}{V}\right)^{1/2} e^{r/2}[B_{z0}^2 + V(2-r)e^{-r}]^{1/2}; \end{aligned}\right\} \tag{23a}$$

$$\left.\begin{aligned} \alpha &= \frac{(4-r)V^{1/2} e^{-r/2}}{2[B_{z0}^2 + V(2 + 2r - r^2)e^{-r}]^{1/2}}, \\ h &= 2\pi V^{-1/2} e^{r/2}[B_{z0}^2 + V(2 + 2r - r^2)e^{-r}]^{1/2}. \end{aligned}\right\} \tag{24a}$$

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For these fields, we have $h > 0$ and

$$\left(\frac{dh}{dr}\right)_{r=0} = 0, \qquad \left(\frac{d^2h}{dr^2}\right)_{r=0} = \frac{4\pi(1-\nu)}{(2\nu-1)^{1/2}} > 0, \tag{22b}$$

$$\left(\frac{dh}{dr}\right)_{r=0} = \left[\frac{\pi(B_{z0}^2 + 2V)^{1/2}}{(rV)^{1/2}}\right]_{r\to 0} \to +\infty, \tag{23b}$$

$$\left(\frac{dh}{dr}\right)_{r=0} = \frac{\pi(B_{z0}^2 + 2V)^{1/2}}{V^{1/2}} > 0, \tag{24b}$$

respectively. Accordingly, either (21a) or (21b) is satisfied for these fields, and thus they are all unstable based on Theorem 2. Among them, the force-free field (22) has an α decreasing monotonically with r and taking a maximum of $2(2\nu-1)^{1/2}$ at $r = 0$, whereas the force-free field (24) has a finite α everywhere for $B_{z0}^2 > 6Ve^{-4}$. If Krüger's sufficient criterion were correct, the two force-free fields would be judged to be stable in a domain of such a radius R that the maximum face-free factor is smaller than the lowest eigenvalue $3.176/R$ derived by Krüger (1976b). However, they are unstable for any domain according to the analysis made above.

For force-free fields that do not satisfy the criterion (21), it is assumed that

$$\left(\frac{dh}{dr}\right)_{r=0} = 0, \qquad \left(h\frac{d^2h}{dr^2}\right)_{r=0} \neq 0, \qquad (B_\theta)_{r\ll 1} \approx B_0 r^m, \tag{25}$$

where B_0 and m are constants, and m should be positive for physically acceptable force-free field solutions. The first two conditions imply that the pitch takes a non-vanishing extremum at $r = 0$. In this situation, one may find $k = k_0$ so that

$$\frac{k_0 h(0)}{2\pi} + 1 = 0, \qquad \text{or} \qquad k_0 = -\frac{2\pi}{h(0)}. \tag{26}$$

With the use of (8), (9) and (26), the energy integral (2) may be cast into the form

$$W_0 = \frac{\pi}{2}\int_0^{x_0} \frac{xB_\theta^2}{1+x^2}\left(\frac{h}{h_0} - 1\right)^2 \left(\frac{d\xi}{dx}\right)^2 dx$$
$$+\frac{\pi}{2}\int_0^{x_0}\left[\frac{xB_\theta^2}{1+x^2}\left(\frac{h}{h_0}-1\right)^2 + \frac{2xB_\theta^2}{(1+x^2)^2}\left(\frac{h^2}{h_0^2}-1\right)\right]\xi^2\,dx, \tag{27}$$

where $x = |k_0|r$, $x_0 = |k_0|R$, and $h_0 = h(0)$. In the following analysis, we assume $x_0 \ll 1$, namely, our focus is placed on the behaviour of the force-free field in a small neighbourhood of $r = 0$. Within this neighbourhood, we have from (25) that

$$h(r) \approx h_0 + \tfrac{1}{2}h_0'' x^2, \qquad B_\theta \approx B_0\left(\frac{|h_0|x}{2\pi}\right)^m, \tag{28}$$

where $h_0'' = (d^2h/dx^2)_{x=0}$. Inserting (28) into (27) leads to

$$W_0 = \frac{\pi B_0^2|h_0|^{2m}}{2(2\pi)^{2m}}\int_0^{x_0}\frac{x^{2m+5}h_0''^2}{4(1+x^2)h_0^2}\left(\frac{d\xi}{dx}\right)^2 dx$$
$$+\frac{\pi B_0^2|h_0|^{2m}}{2(2\pi)^{2m}}\int_0^{x_0}\left[\frac{x^{2m+5}h_0''^2}{4(1+x^2)h_0^2} + \frac{x^{2m+3}h_0''}{(1+x^2)^2 h_0}\left(2 + \frac{h_0''x^2}{2h_0}\right)\right]\xi^2\,dx,$$

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which, if only the major terms are reserved, is reduced to

$$W_0 = \frac{\pi B_0^2 |h_0|^{2(m-1)} h_0''^2}{8(2\pi)^{2m}} \int_0^{x_0} \left[x^{2m+5} \left(\frac{d\xi}{dx} \right)^2 + \frac{8h_0}{h_0''} x^{2m+3} \xi^2 \right] dx. \tag{29}$$

The Euler–Lagrange equation for the energy integral (29) reads

$$\frac{d}{dx} \left(x^{2m+5} \frac{d\xi}{dx} \right) - \frac{8h_0}{h_0''} x^{2m+3} \xi = 0. \tag{30}$$

The solutions are multiples of x^{-n}, where n is the root of the quadratic algebraic equation

$$n^2 - (2m+4)n - \frac{8h_0}{h_0''} = 0. \tag{31}$$

If

$$(2m+4)^2 + \frac{32h_0}{h_0''} < 0, \tag{32}$$

then n is complex, so that these solutions are oscillatory in the neighbourhood of $r = 0$. According to the analysis made by Newcomb (1960), a certain perturbation displacement ξ can be found so as to make the energy integral (29) negative, and thus the system is unstable. Therefore, (32) serves as a sufficient criterion for instability of force-free fields. Since $r = |h_0|x/2\pi$, so that $h_0'' = h_0^2(d^2h/dr^2)_{r=0}/4\pi^2$, this criterion may be rewritten in the form

$$\left(h \frac{d^2h}{dr^2} \right)_{r=0} > -\frac{128\pi^2}{(2m+4)^2}, \tag{33}$$

and then we come to the following theorem.

Theorem 3. *A force-free field satisfying the condition (25) and the criterion (33) is unstable.*

As the criterion tells us, even if the pitch of a force-free field decreases in magnitude with r in the neighbourhood of $r = 0$ so as to fail to meet the criterion (20), the force-free field remains unstable when the rate of decrease is so small that the criterion (33) is satisfied. In case that the criterion (33) is not satisfied, one has to start with the energy integral (2) and (8) and (9) in general, and to assess stability based on relevant theorems given by Newcomb (1960).

Let us now give three examples to illustrate the application of Theorem 3:

$$B_\theta = J_1(\alpha r), \qquad B_z = J_0(\alpha r), \tag{34}$$

$$B_\theta = \tfrac{1}{4}\lambda^2 r, \qquad B_z = \tfrac{1}{4}\lambda^2[2(r_0^2 - r^2)]^{1/2}, \tag{35}$$

$$B_\theta = re^{-r^2}, \qquad B_z = [a + (0.5 - r^2)e^{-2r^2}]^{1/2} \quad (a > -0.5), \tag{36}$$

where J_0 and J_1 are the Bessel functions of the zeroth and first order. Of these examples, (34) represents a linear force-free field that has been proved to be stable in the domain of $(0, 3.176/\alpha)$ (Krüger 1976b), and (35) is a force-free field with a singular current density surface proved to be stable in its definition domain of $(0, r_0)$ (Hu 2000). The force-free field (36) is taken from Hood and Priest (1980). All these fields satisfy the condition (25), and have $m = 1$, so that the term on the right-hand side of (33) becomes $-\frac{32}{9}\pi^2(\approx -3.56\pi^2)$. From (34), we have

$h_0 = 4\pi/\alpha, (d^2h/dr^2)_{r=0} = -\pi\alpha$, and $h_0(d^2h/dr^2)_{r=0} = -4\pi^2$, whereas the corresponding values are $2^{3/2}\pi r_0, -2^{3/2}\pi/r_0$, and $-8\pi^2$ for (35). Both fields do not meet the criterion (33), which is consistent with the conclusion that they are stable.

Let us turn to the force-free field expressed by (36). It has a force-free factor and pitch given by

$$\alpha = \frac{2(1-r^2)}{[ae^{2r^2} + 0.5 - r^2]^{1/2}}, \qquad h = 2\pi(ae^{2r^2} + 0.5 - r^2)^{1/2}. \tag{36a}$$

The force-free field described by (36) is well defined over the whole space $(0, \infty)$ for $a \geqslant (2e^2)^{-1}$; it is definable in a finite domain of radius r_m otherwise, given by

$$a + (0.5 - r_m^2)e^{-2r_m^2} = 0.$$

r_m is equal to 0 for $a = -0.5$ and to 1 as a approaches $(2e^2)^{-1}$. Besides, this field with $a < (2e^2)^{-1}$ is a force-free field with a singular current-density surface that is located at $r = r_m$. From (36a), one may obtain

$$h_0 = 2\pi(a + 0.5)^{1/2}, \qquad \left(\frac{d^2h}{dr^2}\right)_{r=0} = \frac{2\pi(2a-1)}{(a+0.5)^{1/2}},$$

and then

$$h_0\left(\frac{d^2h}{dr^2}\right)_{r=0} = 4\pi^2(2a-1). \tag{36b}$$

Therefore, the force-free field (36) is judged to be unstable for $a > 0.5$ by Theorem 2, and for $a > \frac{1}{18}$ further by Theorem 3. The field solutions with a ranging from $\frac{1}{18}$ to $\frac{1}{2}$ have a pitch that decreases in the neighbourhood of $r = 0$, and thus their instability can only be judged by Theorem 3. When $a \leqslant \frac{1}{18}$, the criterion (33) is not satisfied, and the stability of the associated force-free field should be judged from the energy integral (2), (8) and (9), and relevant theorems given by Newcomb (1960), as mentioned above. By doing so, the force-free field (36) with a in the range of $(-\frac{1}{2}, \frac{1}{18})$ was found to be stable. Thus, this field is another member of the family of stable force-free fields with singular current-density surfaces. It is interesting to notice that the force-free field (36) with a ranging from $\frac{1}{18} \approx 0.0556$ to $(2e^2)^{-1} \approx 0.0677$ also has a singular current-density surface, but it is unstable.

4. Concluding remarks

On the basis of the hydrodynamic energy principle, we have discussed the stability of one-dimensional, nonlinear force-free magnetic fields in cylindrical coordinates. The radial distribution of the pitch is found to play an important role in the stability of these fields. All force-free fields with either a uniform pitch or a pitch that increases in magnitude with the radial distance in the neighbourhood of the cylindrical axis are definitely unstable. If the first-order derivative of the pitch with respect to r vanishes at $r = 0$ and the product between the pitch and its derivative of the second order at $r = 0$ is excessively small in magnitude (i.e. the decrease of the pitch magnitude in the neighbourhood of $r = 0$ is too slow), then the force-free field is also unstable. These conclusions have been summarized in three theorems, which provide us with useful criteria to assess instability of nonlinear force-free fields. It is also demonstrated that the stability of the nonlinear force-free field does not have a definite relation to the maximum force-free factor. A nonlinear force-free field may be stable even if its force-free factor approaches infinity in local areas, whereas it may be unstable even if its maximum force-free factor is smaller than that of a linear force-free field everywhere in the domain of interest where the linear force-free field is stable. The former agrees with the conclusion reached by Vekshtein (1989) that a large local force-free factor near the boundary is consistent with the magnetohydrodynamic stability of a system. The latter disproves the sufficient criterion for stability of nonlinear force-free magnetic fields given by Krüger (1976a) that a nonlinear force-free field is stable if the maximum absolute value of the force-free factor is smaller than the lowest eigenvalue of linear force-free field solutions in the domain of interest.

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