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Bi-orders do not arise from total orders

Samuel M. Corson

Abstract. We present a Zermelo–Fraenkel (**ZF**) consistency result regarding bi-orderability of groups. A classical consequence of the ultrafilter lemma is that a group is bi-orderable if and only if it is locally bi-orderable. We show that there exists a model of **ZF** plus dependent choice in which there is a group which is locally free (ergo locally bi-orderable) and not bi-orderable, and the group can be given a total order. The model also includes a torsion-free abelian group which is not bi-orderable but can be given a total order.

1 Introduction

The goal of this note is to explore the set theoretic strength of bi-orderability in the setting of Zermelo–Fraenkel (**ZF**) set theory. Let **ZF** denote Zermelo–Fraenkel set theory minus **AC**, the axiom of choice. Recall that a *total order* on a set X is a binary relation $<$ for which exactly one of $x < y$ or $y < x$ holds for distinct $x, y \in X$, $x < x$ is false for all $x \in X$, and $x < y$ and $y < z$ imply $x < z$.

If G is a group, we say that a total order $<$ on G is a *left-order* (respectively *right-order*) provided for all $g, h, k \in G$ we have that $g < h$ implies $kg < kh$ (resp. $gk < hk$). We say G is *left-orderable* provided there exists a left order on G. One could similarly define *right-orderable* but since a left-order explicitly defines a right-order and vice-versa, questions of left- or right-orderability of a group are equivalent. Left-orderable groups are torsion-free. A group order is a *bi-order* if it is both a left- and right-order and a group is *bi-orderable* provided such an order exists.

The ultrafilter lemma (every filter on a set extends to an ultrafilter) implies the classically known local-to-global bi-orderability result (see [4, Proposition 1.4]):

A group G is bi-orderable if and only if every finitely generated subgroup is bi-orderable.

In a nice setting, one can have explicit bi-orders without having recourse to this local-to-global theorem. Given a total order on a set X, one immediately obtains a bi-order on the free abelian group $F_{ab}(X)$ generated by X by considering the lexicographic order, and a bi-order on a free abelian group restricts to a total order on the free set of generators. Importantly, the assertion that every set can be given a total order cannot be proved from **ZF**, so a total order on an arbitrary set X does not

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exist a priori. Thus in **ZF**, a free abelian group is bi-orderable if and only if a free set of generators can be given a total order. By a more elaborate argument, in **ZF** a free group is bi-orderable if and only if a free generating set has a total order [2]. It seems natural to ask whether in **ZF** a total order on a locally free, or a torsion-free abelian, group implies bi-orderbility (by a total order on a group we mean, of course, a total order on the group's underlying set). We show that this is not even the case in the presence of dependent choices.

Recall that the *principle of dependent choices* is the assertion that if R is a binary relation on a nonempty set X for which $(\forall x \in X)(\exists y \in X)[xRy]$, then there exists a sequence $\{x_n\}_{n\in\omega}$ for which $x_n R x_{n+1}$. This principle, which is a consequence of the axiom of choice, implies many of the standard results in analysis and also implies the axiom of countable choices.

Theorem 1.1 *If **ZF** is consistent then there exists a model of **ZF** in which the following hold:*

(1) *There exists a group $\mathcal{G}$ which is locally free and can be given a total order, but $\mathcal{G}$ is not bi-orderable.*
(2) *There exists a torsion-free abelian group $\mathcal{A}$ which can be given a total order, but $\mathcal{A}$ is not bi-orderable.*
(3) *The principle of dependent choices.*

The overall strategy in this independence proof is to work in a permutation model of set theory, constructing the claimed groups via presentations, and using the permutations of the model to eliminate any possibility of a bi-order.

We leave some remaining questions regarding bi-orderability. We have mentioned that the local-to global bi-orderability theorem, which we will denote **LG**, follows from the ultrafilter lemma. Also, **LG** implies the ordering principle (every set can be totally ordered) by considering the free abelian group on a set, which is locally free abelian and therefore locally bi-orderable. Thus we ask:

Question *Is **LG** strictly weaker than the ultrafilter lemma?*

Question *Is **LG** strictly stronger than the ordering principle?*

Since the ultrafilter lemma is strictly stronger than the ordering principle [3], the answer to at least one of the two above questions is "yes."

2 The proof

We will work in a modification of the model of van Douwen (see [8] or [1, Model $\mathcal{N}2\,(LO)$]). We let $\mathcal{M}$ be a model of **ZFA** + **AC** with set A of atoms such that $|A| = \aleph_1$. Write A as a disjoint union $A = \bigcup_{\alpha<\aleph_1} A_\alpha$ with each A_α being countably infinite and endowed with a total order $<_\alpha$ which makes A_α order isomorphic to $\mathbb{Z}$. Let Γ be the set of bijections τ on A for which $\tau \restriction A_\alpha \in \mathrm{Aut}(A_\alpha, <_\alpha)$ for all $\alpha < \aleph_1$. Let $\mathcal{F}$ be the normal filter on Γ given by the ideal of countable subsets of A. Let $\mathcal{N} \subseteq \mathcal{M}$ denote

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the permutation model of hereditarily $\mathcal{F}$-symmetric objects in $\mathcal{M}$. For each $B \subseteq A$ we let $\mathrm{fix}(B) = \{\tau \in \Gamma \mid (\forall a \in B)\tau(a) = a\}$ and for an object $x \in \mathcal{M}$ we let $\mathrm{stab}(x) = \{\tau \in \Gamma \mid \tau(x) = x\}$. For each $a \in A$ we let $s(a)$ denote the next largest element under $<_\alpha$ in A_α, where $a \in A_\alpha$.

That the model $\mathcal{N}$ satisfies the principle of dependent choices follows from the fact that the ideal defining the filter $\mathcal{F}$ is closed under taking countable unions (see [1, Note 144]). It is not difficult to see that the existence of the claimed groups in Theorem 1.1 is boundable in the sense of Pincus [5]. Thus, our main result will follow from the transfer principle [6, Theorem 4] (or see [1, p. 286]) provided that we can establish the existence of the claimed groups in the model $\mathcal{N}$.

Let $J = A \times \{0,1\}$ and $\mathbb{F}(J) = (W_J, \circ_J, ^{-1}, 1_{W_J})$ denote the free group on the set J, with W_J denoting the set of reduced words over the alphabet $J^{\pm 1}$, $\circ_J$ and $^{-1}$ denoting the group multiplication and group inversion operations, and 1_{W_J} denoting the trivial element. This group, which we have defined in $\mathcal{M}$, is clearly in $\mathcal{N}$ as well; moreover, $\mathrm{stab}(W_J) = \mathrm{stab}(\circ_J) = \mathrm{stab}(^{-1}) = \Gamma$. Notice that the subset $X_J = \{(s(a),0)(a,1)(s(a),0)^{-1}(s(a),1)\}_{a\in A} \subseteq W_J$ is also in $\mathcal{N}$ and also supported by $\varnothing \subseteq A$. Therefore, the normal subgroup $N_J = \langle\langle X_J\rangle\rangle \trianglelefteq \mathbb{F}(J)$ is in $\mathcal{N}$ and supported by $\varnothing$, and the similar claims hold for the quotient $\mathcal{G} = \mathbb{F}(J)/N_J$. We emphasize that the identity element N_J of $\mathcal{G}$, which we will denote $1_{\mathcal{G}}$, is supported by $\varnothing$.

$\mathcal{G}$ is locally free. For each $\alpha < \aleph_1$, let $J_\alpha = (\bigcup_{\beta\le\alpha} A_\beta) \times \{0,1\}$. Similarly, define the free group $\mathbb{F}(J_\alpha)$ and notice that $\mathbb{F}(J_\alpha)$ is in $\mathcal{N}$ and the set of reduced words in $J_\alpha^{\pm 1}$, the group multiplication operation and the inverse operation are supported by $\varnothing$. Let $r_\alpha : \mathbb{F}(J) \to \mathbb{F}(J_\alpha)$ denote the retraction map given by deleting all letters in $J^{\pm 1} \smallsetminus J_\alpha^{\pm 1}$ and freely reducing, and notice that r_α is in $\mathcal{N}$ and supported by $\varnothing$. Letting $\mathcal{G}_\alpha = F(J_\alpha)N_J$, we see that $\mathcal{G}_\alpha$ is also in $\mathcal{N}$ and supported by $\varnothing$. Also, $r_\alpha(X_J) \subseteq X_J \cup \{1_{W_J}\}$ and so $r_\alpha(N_J) \subseteq N_J$. Then the retraction homomorphism $\mathcal{G} \to \mathcal{G}_n$ given by taking a coset K of N_J to $r_n(K)N_J$ is in $\mathcal{N}$ and is similarly invariant under Γ.

We will show that each $\mathcal{G}_\alpha$ is locally free and this is sufficient since any finitely generated subgroup of $\mathcal{G}$ includes into some $\mathcal{G}_\alpha$. Fix $\alpha < \aleph_1$. Let T_α denote the group

$$\mathbb{F}(J_\alpha)/\langle\langle\{(s(a),0)(a,1)(s(a),0)^{-1}(s(a),1)\}_{a\in\bigcup_{\beta\le\alpha} A_\beta}\rangle\rangle.$$

It is easy to see that T_α is in $\mathcal{N}$ and that the identity map on the generators induces an isomorphism with $\mathcal{G}_\alpha$ (and this isomorphism is also in $\mathcal{N}$). We establish that T_α is locally free.

By selecting $a_\beta \in A_\beta$ for each $\beta \le \alpha$, we have $\mathrm{fix}(\{a_\beta\}_{\beta\le\alpha}) = \mathrm{fix}(\bigcup_{\beta\le\alpha} A_\beta)$. Since the object T_α is hereditarily supported by $\mathrm{fix}(\{a_\beta\}_{\beta\le\alpha})$ and $\mathcal{M}$ is a model of **ZFA** + **AC**, we may use **AC** in arguing that T_α is locally free. It is clear that T_α is the free product of $|\alpha| + 1$ copies of the group H given by presentation.

(1) $$\langle\{x_m\}_{m\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}} \mid \{y_n = x_{n+1}^{-1} y_{n+1}^{-1} x_{n+1}\}_{n\in\mathbb{Z}}\rangle.$$

Since the class of locally free groups is closed under taking free products, we now need to show that H is locally free.

Lemma 2.1. *The group H is locally free and all generators $\{x_m\}_{m\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$ are nontrivial elements in H.*

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Proof Notice that for a fixed $N \in \mathbb{Z}$, the presentation defining H does not require the relators $\{y_n = x_{n+1}^{-1} y_{n+1}^{-1} x_{n+1}\}_{n<N}$ and the generators $\{y_n\}_{n<N}$ since the relators $\{y_n = x_{n+1}^{-1} y_{n+1}^{-1} x_{n+1}\}_{n<N}$ are only used in giving names to the elements $\{y_n\}_{n<N}$. This is because for any positive $k \in \omega$ we can write

$$y_{N-k} = x_{N-k+1}^{-1} x_{N-k+2}^{-1} \cdots x_N^{-1} y_N^{(-1)^k} x_N \cdots x_{N-k+2} x_{N-k+1}.$$

In particular, for any fixed $N \in \mathbb{Z}$, we know that H is isomorphic to the group H_N with presentation

$$\langle \{x_m\}_{m \in \mathbb{Z}} \cup \{y_n\}_{n \geq N} \mid \{y_n = x_{n+1}^{-1} y_{n+1}^{-1} x_{n+1}\}_{n \geq N} \rangle \tag{2}$$

via the map ρ_N determined by

$$\begin{aligned} x_m &\mapsto x_m \text{ for all } n \in \mathbb{Z} \\ y_n &\mapsto y_n \text{ for } n \geq N \\ y_{N-k} &\mapsto x_{N-k+1}^{-1} x_{N-k+2}^{-1} \cdots x_N^{-1} y_N^{(-1)^k} x_N \cdots x_{N-k+2} x_{N-k+1} \text{ for } k \geq 1. \end{aligned}$$

Consider the normal subgroup $K = \langle\langle \{x_n\}_{n>N} \rangle\rangle \trianglelefteq H_N$. The quotient H_N/K has presentation

$$\langle \{x_m\}_{m \leq N} \cup \{y_n\}_{n \geq N} \mid \{y_n = y_{n+1}^{-1}\}_{n \geq N} \rangle$$

and this group is simply the free group in the generators $\{x_m\}_{m \leq N} \cup \{y_N\}$. This implies that for each $N \in \mathbb{Z}$ the set $\{x_m\}_{m \leq N} \cup \{y_N\}$ freely generates a subgroup of H_N.

For any finite set of words $\{w_0, \ldots, w_r\}$ in the letters $\{x_m\}_{m \in \mathbb{Z}}^{\pm 1} \cup \{y_n\}_{n \in \mathbb{Z}}^{\pm 1}$, there exists some N for which each of the words $w_0, \ldots, w_r$ is written in the letters $\{x_m\}_{m \leq N}^{\pm 1} \cup \{y_n\}_{n \leq N}^{\pm 1}$. Then applying ρ_N to the group elements represented by $w_0, \ldots, w_r$ places this set within the subgroup $\langle \{x_m\}_{m \leq N} \cup \{y_N\} \rangle \leq H_N$, and since this subgroup is free, we have that H is locally free. The second claim follows immediately from our proof since we showed that for each $N \in \mathbb{Z}$ the set $\{x_m\}_{m \leq N} \cup \{y_N\}$ freely generates a subgroup of H. ■

$\mathcal{G}$ can be given a total order. Toward producing a total order on $\mathcal{G}$, we produce, in $\mathcal{M}$, a normal form for $\mathcal{G}$. Since **AC** holds in $\mathcal{M}$, we shall freely use choices in this construction, and the fact that the normal form is also in $\mathcal{N}$ will become apparent. Recall that a word rewriting system on a free monoid $\mathrm{Mon}(X)$ on set X is a set of rules $\mathcal{R}$ whose inputs and outputs are words in the monoid (see [7, Section 1.7]). We define binary relation $\rightarrow_{\mathcal{R}}$ on $\mathrm{Mon}(X)$ by letting $w_0 \rightarrow_{\mathcal{R}} w_1$ if there exist $v_0, v_1, v_1', v_2 \in \mathrm{Mon}(X)$ with $w_0 \equiv v_0 v_1 v_2$ and $w_1 \equiv v_0 v_1' v_2$ and $(v_1, v_1') \in \mathcal{R}$. Let $\rightarrow_{\mathcal{R}}^*$ be the smallest transitive binary relation including $\rightarrow_{\mathcal{R}}$ and let $\leftrightarrow_{\mathcal{R}}^*$ denote the smallest equivalence class including $\rightarrow_{\mathcal{R}}^*$. Rewriting system $\mathcal{R}$ is *confluent* if whenever $w_0 \rightarrow_{\mathcal{R}}^* w_1$ and $w_0 \rightarrow_{\mathcal{R}}^* w_2$, there exists w_3 for which $w_1 \rightarrow_{\mathcal{R}}^* w_3$ and $w_2 \rightarrow_{\mathcal{R}}^* w_3$. It is *locally confluent* if whenever $w_0 \rightarrow_{\mathcal{R}} w_1$ and $w_0 \rightarrow_{\mathcal{R}} w_2$, there exists w_3 for which $w_1 \rightarrow_{\mathcal{R}}^* w_3$ and $w_2 \rightarrow_{\mathcal{R}}^* w_3$.

Rewriting system $\mathcal{R}$ is *terminating* if each sequence $w_0 \rightarrow_{\mathcal{R}} w_1 \rightarrow_{\mathcal{R}} w_2 \cdots \rightarrow_{\mathcal{R}} w_n$ must eventually stabilize. A word w is a *terminus* of $\mathcal{R}$ if $w \rightarrow_{\mathcal{R}} v$ implies $w \equiv v$. If $\mathcal{R}$ is terminating and locally confluent, then it is confluent, and if $\mathcal{R}$ is terminating

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and confluent then each equivalence class in $\leftrightarrow^*_{\mathcal{R}}$ contains a unique terminus (see [7, Section 1.7]).

We let $\mathrm{Mon}(J^{\pm 1})$ denote the free monoid on the set $\{(a,0)\}_{a\in A} \cup \{(a,1)\}_{a\in A} \cup \{(a,0)^{-1}\}_{a\in A} \cup \{(a,1)^{-1}\}_{a\in A}$, and let e denote the empty word. Consider the rewriting system $\mathcal{R}$ under which for all $a \in A$ we have rules

(1) $(a,0)(a,0)^{-1} \mapsto e$
(2) $(a,0)^{-1}(a,0) \mapsto e$
(3) $(a,1)(a,1)^{-1} \mapsto e$
(4) $(a,1)^{-1}(a,1) \mapsto e$
(5) $(s(a),0)(a,1) \mapsto (s(a),1)^{-1}(s(a),0)$
(6) $(s(a),0)(a,1)^{-1} \mapsto (s(a),1)(s(a),0)$
(7) $(s(a),0)^{-1}(s(a),1) \mapsto (a,1)^{-1}(s(a),0)^{-1}$
(8) $(s(a),0)^{-1}(s(a),1)^{-1} \mapsto (a,1)(s(a),0)^{-1}$

The idea of this system is to both freely reduce and to move the $(a,0)^{\pm 1}$ letters to the right.

Lemma 2.2. *The rewriting system $\mathcal{R}$ is locally confluent.*

Proof We will argue in cases. It is easy to see that if rules are applied independently to nonoverlapping subwords then the order of application makes no difference. More explicitly if we have a word $w \equiv u_0u_1$ and consider an application of a rule to u_0 to obtain $w \to_{\mathcal{R}} u_0'u_1$, and consider the application of a possibly different rule to u_1 to obtain $w \to_{\mathcal{R}} u_0u_1'$ then clearly $u_0'u_1 \to_{\mathcal{R}} u_0'u_1'$ and $u_0u_1' \to_{\mathcal{R}} u_0'u_1'$ and so there can be no obstruction to local confluence in this setting. Thus, it will only be necessary to consider cases where rule applications are to overlapping subwords.

If $w \to_{\mathcal{R}} w_0$ and $w \to_{\mathcal{R}} w_1$ are each obtained by an application of a free reduction rule (i.e., each is obtained by one of (i)–(iv)) then by applying free reductions to each of w_0 and w_1, we obtain a unique freely reduced word w_2, so that $w_0 \to^*_{\mathcal{R}} w_2$ and $w_1 \to^*_{\mathcal{R}} w_2$. Next, if $w \to_{\mathcal{R}} w_0$ and $w \to_{\mathcal{R}} w_1$ and each of these was given by an application of possibly different rules among (v)–(viii) then either $w_0 \equiv w_1$ or these rules were applied on distinct nonoverlapping subwords, and this latter case was considered above.

Next, we suppose that $w \equiv v_0(s(a),0)(a,1)(a,1)^{-1}v_1$. By applying (iii), one has $w \to_{\mathcal{R}} v_0(s(a),0)v_1$. By instead applying (v) to w we see that

$$w \to_{\mathcal{R}} v_0(s(a),1)^{-1}(s(a),0)(a,1)^{-1}v_1$$

and by applying (vi) and then (iv), we see that

$$v_0(s(a),1)^{-1}(s(a),0)(a,1)^{-1}v_1 \to_{\mathcal{R}} v_0(s(a),1)^{-1}(s(a),1)(s(a),0)v_1 \to_{\mathcal{R}} v_0(s(a),0)v_1.$$

The cases where w is of form

$$w \equiv v_0(s(a),0)(a,1)^{-1}(a,1)v_1;$$
$$w \equiv v_0(s(a),0)^{-1}(s(a),1)(s(a),1)^{-1}v_1; \text{ or}$$
$$w \equiv v_0(s(a),0)^{-1}(s(a),1)^{-1}(s(a),1)v_1$$

are each handled similarly.

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Suppose that we have a word $w \equiv v_0(s(a),0)^{-1}(s(a),0)(a,1)v_1$. If one applies (ii), then one has $w \to_{\mathcal{R}} v_0(a,1)v_1$. If we instead apply (v), then we get $w \to_{\mathcal{R}} v_0(s(a),0)^{-1}(s(a),1)^{-1}(s(a),0)v_1$, and applying rule (viii) and then (ii), we get

$$v_0(s(a),0)^{-1}(s(a),1)^{-1}(s(a),0)v_1 \to_{\mathcal{R}} v_0(a,1)(s(a),0)^{-1}(s(a),0)v_1 \to_{\mathcal{R}} v_0(a,1)v_1.$$

The check in case w is of form

$$w \equiv v_0(s(a),0)^{-1}(s(a),0)(a,1)^{-1}v_1;$$
$$w \equiv v_0(s(a),0)(s(a),0)^{-1}(s(a),1)v_1; \text{ or}$$
$$w \equiv v_0(s(a),0)(s(a),0)^{-1}(s(a),1)^{-1}v_1$$

is similar. Thus local confluence holds. ∎

We note also that the rewriting system is terminating. To see this, given a word w we consider the function

$$j(w) = \sum_{0 \le i < \mathrm{Len}(w), w(i) \in \{(a,0)^{\pm 1}\}_{a \in A}} |\{i < k < \mathrm{Len}(w) \mid w(k) \in \{(a',1)^{\pm 1}\}_{a' \in A}\}|$$

which counts the total number of times that a letter of form $(a',1)^{\pm 1}$ appears in the word somewhere to the right of a letter of form $(a,0)^{\pm 1}$. Each application of a rule will lower the value of the function $\mathrm{Len}(w) + j(w)$ (where $\mathrm{Len}(\cdot)$ denotes the length of the word) and so the fact that the system is terminating follows. Thus, each equivalence class under $\leftrightarrow^*_{\mathcal{R}}$ contains a unique terminus.

All elements of the set R of words which are the terminus of a word in $\mathrm{Mon}(J^{\pm 1})$ under $\mathcal{R}$ are freely reduced. The set R is also obviously in $\mathcal{N}$ (notice that the rules are themselves invariant under the action of Γ) and supported by $\varnothing$. Furthermore, it is straightforward to see that each element in R is a unique representative of an element in $\mathcal{G}$. We give an order $<^l$ to the letters in $J^{\pm 1}$ as follows:

$$(a,0)^{-1} <^l (a,0) <^l (a,1)^{-1} <^l (a,1) <^l (a',0)^{-1} <^l (a',0) <^l (a',1)^{-1} <^l (a',1),$$

where either $a, a' \in A_\alpha$ with $a <_\alpha a'$ or $a \in A_\alpha$ and $a' \in A_{\alpha'}$ with $\alpha < \alpha'$. Endow the elements of R with the shortlex order $<^o$: $w_0 <^o w_1$ if either $\mathrm{Len}(w_0) < \mathrm{Len}(w_1)$, or $\mathrm{Len}(w_0) = \mathrm{Len}(w_1)$ and for the least $0 \le i < \mathrm{Len}(w_0)$ at which $w_0(i) \ne w_1(i)$ we have $w_0(i) <^l w_1(i)$. It is clear that both $<^l$ and $<^o$ are in $\mathcal{N}$, and more particularly they are supported by $\varnothing$.

$\mathcal{G}$ **is not bi-orderable.** To see that $\mathcal{G}$ is not bi-orderable, we suppose for contradiction that $<_{\mathcal{G}}$ is a bi-order on $\mathcal{G}$ in $\mathcal{N}$. Select countable $B \subseteq A$ for which $\mathrm{fix}(B) \le \mathrm{stab}(<_{\mathcal{G}})$. Select $\alpha < \aleph_1$ such that $A_\alpha \cap B = \varnothing$. Let $\tau \in \Gamma$ be given by

$$\tau(a) = \begin{cases} a \text{ if } a \notin A_\alpha \\ s(a) \text{ if } a \in A_\alpha. \end{cases}$$

Let $a \in A_\alpha$ be given. By Lemma 2.1, we know that $(a,1)N_J$ is nontrivial. If $1_{\mathcal{G}} <_{\mathcal{G}} (a,1)N_J$ then $1_{\mathcal{G}} <_{\mathcal{G}} (s(a),0)(a,1)(s(a),0)^{-1}N_J = (s(a),1)^{-1}N_J$, from which we see

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that $(s(a),1)N_J <_{\mathcal{G}} 1_{\mathcal{G}}$, but on the other hand

$$1_{\mathcal{G}} = \tau(1_{\mathcal{G}}) <_{\mathcal{G}} \tau((a,1))N_J = (s(a),1)N_J$$

which is a contradiction. The proof in case $(a,1)N_J <_{\mathcal{G}} 1_{\mathcal{G}}$ is symmetric.

The group $\mathcal{A}$ and its properties. We construct the group $\mathcal{A}$ which is claimed to be in the model $\mathcal{N}$. In checking its various properties, we will simply sketch over the aspects of the proofs which are nearly identical to those in the case of $\mathcal{G}$. We take $\mathbb{F}(A)$ to be the free group on the set A of atoms. Consider the subset $X_A = \{[a,a']\}_{a,a'\in A} \cup \{a(s(a))^2\}_{a\in A}$, where $[a,a']$ denotes the commutator $aa'a^{-1}(a')^{-1}$. This set is in $\mathcal{N}$ and supported by $\varnothing$, and similarly for the normal subgroup $N_A = \langle\langle X_A\rangle\rangle$ and relevant group operations and underlying set of $\mathcal{A} = \mathbb{F}(A)/N_A$. Letting $0_{\mathcal{A}}$ denote the identity element, we emphasize that $0_{\mathcal{A}}$ is supported by $\varnothing$. Let $B_\alpha = \bigcup_{\beta\le\alpha} A_\beta$ and $r_\alpha : \mathbb{F}(A) \to \mathbb{F}(B_\alpha)$ be the retraction. Let $\mathcal{A}_\alpha = \mathbb{F}(B_\alpha)N_A$. Since $r_\alpha(Y) \subseteq Y \cup \{1\}$ we have $r_n(N_A) \subseteq N_A$. Thus, we have a retraction map $\mathcal{A} \to \mathcal{A}_\alpha$ given by $K \mapsto r_\alpha(K)N_A$ which is in $\mathcal{N}$ and supported by $\varnothing$.

Let L_α denote the group $\mathbb{F}(B_\alpha)/\langle\langle\{[a,a']\}_{a,a'\in B_\alpha} \cup \{a(s(a))^2\}_{a\in B_\alpha}\rangle\rangle$. Notice that $L_n \simeq \mathcal{A}_n$ via the identity map on the generators (and this isomorphism is in $\mathcal{N}$). Taking $a_\beta \in A_\beta$ for each $\beta \le \alpha$ we have again that $\mathrm{fix}(\{a_\beta\}_{\beta\le\alpha}) = \mathrm{fix}(B_\alpha)$. Thus, we may utilize **AC**, which holds in $\mathcal{M}$, in analyzing L_α. It is easy to see that L_α is isomorphic to a direct sum of $|\alpha|+1$ copies of the additive group $\mathbb{Z}[\frac{1}{2}]$. Thus $\mathcal{A}_\alpha$, and therefore all of $\mathcal{A}$, is torsion-free abelian and for each $a \in A$ we have aN_A nontrivial in $\mathcal{A}$.

A normal form on $\mathcal{A}$ is given by words of the form

$$a_0^{z_0} a_1^{z_1} \cdots a_m^{z_m},$$

where for each $0 \le i \le m$ we have $z_i \in \mathbb{Z} \setminus 2\mathbb{Z}$ and $a_i \in A_{j_i}$ with $j_0 < j_1 < \cdots < j_m$. The set of all such words is in $\mathcal{N}$ and supported by $\varnothing$. Order the letters $A^{\pm 1}$ by order $<^l$ given by

$$a^{-1} <^l a <^l (a')^{-1} <^l a',$$

where $a, a' \in A_\alpha$ for some $\alpha < \aleph_1$ and $a <_\alpha a'$, or $a \in A_\alpha$ and $a' \in A_{\alpha'}$ with $\alpha < \alpha'$. This order $<^l$ is invariant under Γ. Order $\mathcal{A}$ using shortlex on the normal form.

Now suppose that $<_{\mathcal{A}}$ is a bi-order on $\mathcal{A}$. Let $B \subseteq A$ be countable with $\mathrm{fix}(B) \le \mathrm{stab}(<_{\mathcal{A}})$. Select $\alpha \in \omega$ such that $A_\alpha \cap B = \varnothing$. Let $\tau \in \Gamma$ be given by

$$\tau(a) = \begin{cases} a \text{ if } a \in A \setminus A_\alpha, \\ s(a) \text{ if } a \in A_\alpha. \end{cases}$$

Let $a \in A_\alpha$. Suppose that $0_{\mathcal{A}} <_{\mathcal{A}} aN_A$. On one hand, we have that $0_{\mathcal{A}} = \tau(0_{\mathcal{A}}) <_{\mathcal{A}} \tau(aN_A) = s(a)N_A$, but on the other hand, we have $s(a)^2 N_A = a^{-1}N_A <_{\mathcal{A}} 0_{\mathcal{A}}$, a contradiction. The proof in case $aN_A <_{\mathcal{A}} 0_{\mathcal{A}}$ is symmetric.

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School of Mathematics, University of Bristol, Fry Building, Woodland Road, Bristol, BS8 1UG, United Kingdom
e-mail: sammyc973@gmail.com

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