

ON REALS WITH Δ_2^0 -BOUNDED COMPLEXITY AND COMPRESSIVE POWER

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Abstract. The (prefix-free) Kolmogorov complexity of a finite binary string is the length of the shortest description of the string. This gives rise to some ‘standard’ lowness notions for reals: A is K -trivial if its initial segments have the lowest possible complexity and A is low for K if using A as an oracle does not decrease the complexity of strings by more than a constant factor. We weaken these notions by requiring the defining inequalities to hold only up to all Δ_2^0 orders, and call the new notions Δ_2^0 -bounded K -trivial and Δ_2^0 -bounded low for K . Several of the ‘nice’ properties of K -triviality are lost with this weakening. For instance, the new weaker definitions both give uncountable set of reals. In this paper we show that the weaker definitions are no longer equivalent, and that the Δ_2^0 -bounded K -trivials are cofinal in the Turing degrees. We then compare them to other previously studied weakenings, namely *infinitely-often K -triviality* and *weak lowness for K* (in each, the defining inequality must hold up to a constant, but only for infinitely many inputs). We show that Δ_2^0 -bounded K -trivial implies infinitely-often K -trivial, but no implication holds between Δ_2^0 -bounded low for K and weakly low for K .

§1. Introduction. The prefix-free Kolmogorov complexity, $K(\sigma)$, of a binary string σ is the length of the shortest self-delimiting program (in a given language) whose output is σ . We can extend this to a notion on reals by examining the complexities of all finite initial segments of their binary expansions. We say a real is *Martin-Löf random* if the complexities of its initial segments are as high as possible, i.e., up to an additive constant c we have for all n , $K(A \upharpoonright_n) \geq n - c$, where $A \upharpoonright_n$ is the initial segment of A of length n . In this way we capture a notion of randomness that coincides with being difficult to describe. The Martin-Löf random reals are one of the cornerstones of the field of Algorithmic Randomness. On the other end of the spectrum, we have reals whose initial segment complexity is as low as possible. A string of length n can always be used as a description of the number n , so the lowest complexity we can achieve is $K(n)$. We say real A is *K -trivial* if up to an additive constant c we have for all n , $K(A \upharpoonright_n) \leq K(n) + c$ (for an $n \in \omega$, we use $K(n)$ to mean the complexity of a string of n zeros). The K -trivials are another set of reals that are well-studied and of great interest in this field.

Another way of comparing reals using Kolmogorov complexity is to examine their compressive power. By allowing programs to have oracle access to reals, we get a notion of relativized Kolmogorov complexity; the length of the shortest description of σ that can use A as an oracle. We can then compare the plain complexities of strings with their complexities relative to a given real to get some idea of the

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additional power that the real is providing to compression. Some reals, for example Martin-Löf randoms, have high compressive power, since used as an oracle they can give very short descriptions of their own initial segments, which are impossible to compress by an oracle-free program. However, we also get a concept of ‘lowness’ for reals for this measure. We say a real is *low for K* if up to an additive constant c , for all finite binary strings σ , $K(\sigma) \leq K^A(\sigma) + c$, that is, A provides no more than a constant amount of additional compression to any string. It is a remarkable fact due to Nies [11] that lowness for K coincides exactly with K -triviality; having minimal complexity is the same as having minimal compressive power. However, intensionally the definitions are quite different and as we weaken the definitions slightly the notions come apart. We formalize some notation to be used throughout this paper.

We use ω to denote the least countable ordinal, identified with the set of natural numbers. We use $2^{<\omega}$ to denote the set of finite binary strings and 2^ω for the set of infinite binary sequences, identified with the binary expansions of reals. We use the symbol ‘ \frown ’ to denote the operation of concatenation on $2^{<\omega}$, omitting it where there will be no confusion, and the symbol ‘ \prec ’ to denote the initial segment relation on $2^{<\omega} \times 2^{<\omega}$ and $2^{<\omega} \times 2^\omega$. We denote the restriction of an element $A \in 2^\omega$ to its finite initial segment of length n by $A \upharpoonright_n$. In contexts that involve both finite binary strings and natural numbers, we will use $\langle \sigma \rangle$ to denote the *string* σ as opposed to the natural number with decimal expansion σ (i.e., $\langle 10 \rangle$ is the binary string of 1 followed by 0, while 10 is the natural number ‘ten’), unless this can be omitted without confusion. By a *tree* we mean a subset of $2^{<\omega}$ that is closed downwards under \prec . For such a tree T , we use $[T]$ to denote the set of infinite paths through T , i.e., $[T] = \{A \in 2^\omega : \forall n A \upharpoonright_n \in T\}$. As stated above, we use n to denote the string consisting of n zeros. For computations or processes that may or may not converge, we use \downarrow to denote convergence and \uparrow to denote divergence. We use the symbol ‘ \leq^+ ’ to denote that an inequality holds up to an additive constant. We will use standard terminology and definitions for recursion theoretic concepts as given in for example [14].

By a *machine* we mean a partial recursive function $\mathcal{M} : 2^{<\omega} \rightarrow 2^{<\omega}$. A machine \mathcal{M} is *prefix-free* if for any $\sigma \prec \tau$ in $2^{<\omega}$, if $\mathcal{M}(\sigma) \downarrow$ then $\mathcal{M}(\tau) \uparrow$. For a prefix-free machine \mathcal{M} , the *prefix-free Kolmogorov complexity relative to \mathcal{M}* of a string σ is $\min\{|\tau| : \mathcal{M}(\tau) = \sigma\}$ and is denoted $K_{\mathcal{M}}(\sigma)$. Solomonoff, Kolmogorov, and Chaitin each independently showed the existence of universal prefix-free machines, that is, machines \mathcal{M}_U such that for any other prefix-free machine \mathcal{M} , for all $\sigma \in 2^{<\omega}$, $K_{\mathcal{M}_U}(\sigma) \leq^+ K_{\mathcal{M}}(\sigma)$. We fix some such universal prefix-free machine and denote it \mathbb{U} and the associated Kolmogorov complexity simply K . For many of our proofs we will need to construct our own machines, and we will need the following result.

A *Kraft-Chaitin set* is a recursively enumerable subset W of $2^{<\omega} \times \omega$ such that $\sum_{(\sigma,n) \in W} 2^{-n} < 1$. The Kraft-Chaitin Theorem, which appeared independently in

work of Levin, states that for any such set W there exists a prefix-free machine \mathcal{M} such that for any pair $(\sigma, n) \in W$ there is a $\tau \in 2^{<\omega}$ such that $|\tau| = n$ and $\mathcal{M}(\tau) = \sigma$. Note that as long as this sum is finite there is some constant c such that the sum $\sum_{(\sigma,n+c) \in W} 2^{-n-c}$ is less than 1, so we will be able to find a machine that performs as

requested, up to an additive constant. We call the sum $\sum_{(\sigma,n) \in W} 2^{-n}$ the *mass* of the Kraft-Chaitin set W .

In this paper the main objects of study are the weakenings of the standard lowness notions discussed above derived from replacing the constants with slow-growing functions.

DEFINITION 1.1.

- For a function $f : 2^{<\omega} \rightarrow \omega$, a real A is *low for K up to f* if for all $\sigma \in 2^{<\omega}$

$$K(\sigma) \leq^+ K^A(\sigma) + f(\sigma).$$

- For a function $g : \omega \rightarrow \omega$, a real A is *K -trivial up to g* if for all $n \in \omega$

$$K(A \upharpoonright_n) \leq^+ K(n) + g(n).$$

We write $\mathcal{LK}(f)$ for the set of reals that are low for K up to f , and $\mathcal{KT}(g)$ for the set of reals that are K -trivial up to g . In this notation $\mathcal{KT}(0)$ is the set of standard K -trivials and $\mathcal{LK}(0)$ is the set of standard lows for K (so $\mathcal{KT}(0) = \mathcal{LK}(0)$).

The question now arises as to which functions it will be fruitful to consider for these f and g . Obviously some functions grow quickly enough that $\mathcal{LK}(f)$ or $\mathcal{KT}(g)$ is all of $2^{<\omega}$. On the other hand, many functions (any with a finite lim sup) will just give us $\mathcal{KT}(0)$ or $\mathcal{LK}(0)$ again. As these functions represent the rates of growth of some quantities, it is natural to consider *orders*, that is, functions that are unbounded and nondecreasing (some sources additionally require that orders be recursive, but we make no such restriction). In principle we can consider orders of arbitrarily high arithmetic complexity. However, Csima and Montalbán showed that there is a Δ_4^0 order f such that $\mathcal{KT}(f) = \mathcal{KT}(0)$, that is, A is K -trivial if and only if $K(A \upharpoonright_n) \leq^+ K(n) + f(n)$ [5]. Later Baartse and Barmpalias improved this by constructing a Δ_3^0 order with this property [1], and showed that no such so-called ‘gap function’ could be Δ_2^0 . Thus, the Δ_2^0 order case is where these weakenings can be interesting and can be handled in a general way. It will often be more convenient in the proofs to work with a slightly more general notion than being a Δ_2^0 order, which we define below.

For a total function $f : \omega \rightarrow \omega$, a *recursive approximation* is a uniformly recursive series of functions (f_s) such that for all x , $\lim_{s \rightarrow \infty} f_s(x) = f(x)$. By the Schoenfield Limit Lemma and Post’s Theorem (both in, for example [14]) a function has a recursive approximation if and only if it is Δ_2^0 . We use some effective listing of all partial recursive approximations and write $\phi_{e,s}$ for the s th stage of the e th approximation.

DEFINITION 1.2. A Δ_2^0 function $f : \omega \rightarrow \omega$ is *finite-to-one approximable* if it is total and has a recursive approximation $f_s \rightarrow f$ such that for any $n \in \omega$, for all but finitely many $m \in \omega$, for all s , $f_s(m) > n$. Such an approximation is called a *finite-to-one approximation*.

We note that this is a more restrictive notion than having an approximation that is finite-to-one at each stage (any Δ_2^0 function will have such an approximation). We require rather that for a given output there are only finitely many inputs that are

ever in its preimage (so the function is finite-to-one over the whole approximation). With a simple diagonalization one can even construct a finite-to-one Δ_2^0 function that fails to have a finite-to-one approximation in the above sense, which necessitates the complication of terminology.

Finite-to-one approximability may seem like an odd condition to impose, but in some sense it is a generalization of being an order. Any Δ_2^0 order is finite-to-one approximable, by taking any recursive approximation and selecting only the stages where it looks like an order on initial segments of increasing length (and replacing the tail with the identity, if necessary). Moreover, any finite-to-one approximable function pointwise dominates some Δ_2^0 order. Since each n will only ever appear in the output for finitely many inputs, each time it does so we can drop the value on all smaller inputs to n to maintain monotonicity. Eventually we reach a point where n never appears again, so our new function will have \liminf greater than n .

A central concept of this paper will be those reals that are K -trivial or low for K up to every Δ_2^0 order. We use

$$\mathcal{KT}(\Delta_2^0) = \bigcap_{f \text{ a } \Delta_2^0 \text{ order}} \mathcal{KT}(f),$$

to denote the set of reals that are K -trivial up to every Δ_2^0 order and

$$\mathcal{LK}(\Delta_2^0) = \bigcap_{f \text{ a } \Delta_2^0 \text{ order}} \mathcal{LK}(f).$$

to denote the set of reals that are low for K up to every Δ_2^0 order. By the discussion above it should be clear that these coincide with the reals that are K -trivial or low for K up to every finite-to-one approximable function.

Since no Martin-Löf random real can be K -trivial or low for K up to even $\log(n)$, it is clear that both of these sets have measure 0. The reals in $\mathcal{KT}(\Delta_2^0)$ are within every Δ_2^0 order of being K -trivial. Another way to think of these reals is that, while they may not be K -trivial, there is no Δ_2^0 witness to their non- K -triviality. Any function $f: \omega \rightarrow \omega$ such that $f(c)$ gives an n with $K(A \upharpoonright_n) > K(n) + c$ cannot be Δ_2^0 , and the analogous statement holds for $\mathcal{LK}(\Delta_2^0)$. To see this, note that from such a function f , we can define the function $g(n) =$ the smallest c with $f(2c) \geq n$, which will be a Δ_2^0 order such that the inequality $K(A \upharpoonright_n) \leq K(n) + g(n) + c$ will fail to hold for any c (as witnessed by $n = f(2c)$). We use this observation to show that among the Δ_2^0 reals the only reals in $\mathcal{KT}(\Delta_2^0)$ or $\mathcal{LK}(\Delta_2^0)$ must be the K -trivials.

PROPOSITION 1.3. *If A is Δ_2^0 and not K -trivial, then $A \notin \mathcal{KT}(\Delta_2^0)$ and $A \notin \mathcal{LK}(\Delta_2^0)$.*

PROOF. First, we note that the function $K(n)$ is itself Δ_2^0 . To approximate $K(n)$ we can at stage s run s steps of the computation of the universal machine \mathbb{U} on the length-lexicographically first s -many strings and output the length of the shortest string that we find working as a description of n so far. Eventually s will be large enough that we find the actual shortest length, so this approximation converges. Now, if A is also Δ_2^0 , then so is the function that sends n to $K(A \upharpoonright_n)$, since we can approximate the complexity of our current approximation to $A \upharpoonright_n$ and eventually these will both converge.

Since A is not K -trivial, for any c there will be an n such that $K(A \upharpoonright_n) > K(n) + c$, and since these quantities have recursive approximations, we can recursively approximate the smallest such n that works for a given c . This gives us a Δ_2^0 f that witnesses A 's non- K -triviality, as mentioned above, so A is not in $\mathcal{KT}(\Delta_2^0)$.

To see that A is also not in $\mathcal{LK}(\Delta_2^0)$ a similar argument will work, or we can use the following proposition, which gives us that $\mathcal{LK}(\Delta_2^0)$ is in fact a subclass of $\mathcal{KT}(\Delta_2^0)$. \dashv

PROPOSITION 1.4. *If $A \in \mathcal{LK}(\Delta_2^0)$, then $A \in \mathcal{KT}(\Delta_2^0)$.*

PROOF. For any $f : \omega \rightarrow \omega$, let $\hat{f} : 2^{<\omega} \rightarrow \omega$ be given by $\hat{f}(\sigma) = f(|\sigma|)$. Clearly if f is a Δ_2^0 order then so is \hat{f} . If $A \in \mathcal{LK}(\Delta_2^0)$, then for any Δ_2^0 order f we have $A \in \mathcal{LK}(\hat{f})$, i.e., for some c , for all $\sigma \in 2^{<\omega}$, we have $K(\sigma) \leq K^A(\sigma) + \hat{f}(\sigma) + c$. In particular we get for all n , $K(A \upharpoonright_n) \leq K^A(A \upharpoonright_n) + f(n) + c$. Now, relative to A there is a very short description of $A \upharpoonright_n$: read off the first n bits of the oracle. All this machine requires to produce this initial segment is a description of the number n , so $K^A(A \upharpoonright_n) \leq^+ K^A(n)$, and this can be no larger than $K(n)$. Thus, for all n , $K(A \upharpoonright_n) \leq^+ K(n) + f(n)$, so $A \in \mathcal{KT}(f)$. Since this holds for any Δ_2^0 order f , A must be an element of $\mathcal{KT}(\Delta_2^0)$. \dashv

In this paper we examine some properties of $\mathcal{KT}(\Delta_2^0)$ and $\mathcal{LK}(\Delta_2^0)$. In particular, we are interested in how these sets compare to each other, to the standard notions of lowness for K and K -triviality, and to some other weakenings of these notions. We first review some results about the classic case and what is known about certain weakenings.

In the classic case, as noted above it is a result of Nies that $\mathcal{LK}(0) = \mathcal{KT}(0)$ [11]. $\mathcal{LK}(0)$ (and thus $\mathcal{KT}(0)$) is clearly closed downwards under \leq_T , since a real can simulate the compression done by any real it can compute. Chaitin [4] showed that $\mathcal{KT}(0)$ was countable and all its members are Δ_2^0 . Downey, Hirshfeldt, Nies, and Stephan have shown that $\mathcal{KT}(0)$ is closed under effective join [7].

In contrast Baartse and Barmपालias [1] constructed for any Δ_2^0 order g a perfect set of reals in $\mathcal{KT}(g)$, so this set is uncountable and has non- Δ_2^0 elements. Hirschfeldt and Weber, though they did not use this terminology, first showed that for any finite-to-one approximable f , $\mathcal{LK}(f)$ contains an r.e. set that is not in $\mathcal{LK}(0)$ [9]. In an earlier paper [8], the author has shown that there is a perfect set of reals in $\mathcal{LK}(f)$ for any finite-to-one approximable f , and in fact in $\mathcal{LK}(\Delta_2^0)$. Additionally, the perfect set constructed in that paper has the property that for any real A there are two elements B_1 and B_2 of the set such that $B_1 \oplus B_2 \geq_T A$, so in general $\mathcal{LK}(f)$ and $\mathcal{LK}(\Delta_2^0)$ are not closed under effective join (i.e., except for the trivial case when $\mathcal{LK}(f) = 2^\omega$).

In Section 2 we discuss the downwards closure of $\mathcal{KT}(\Delta_2^0)$ under \leq_T and implication between $\mathcal{KT}(\Delta_2^0)$ and $\mathcal{LK}(\Delta_2^0)$. In Section 3 we give some positive closure results for $\mathcal{KT}(\Delta_2^0)$, and in Section 4 we compare these notions with other lowness notions related to Kolmogorov complexity and closures under weaker reducibilities. We give further directions for study in Section 5.

§2. Downwards closure. It is easy to see that lowness for K as a property of reals is closed downwards under \leq_T , since a real can simulate the compression algorithms of any real it can compute. From this and the Nies's Theorem it follows

that K -triviality is also closed downwards. The same argument shows that $\mathcal{KK}(f)$ is closed downwards for any f , but we do not have an analog of Nies's Theorem in the weaker case, so it does not necessarily follow that $\mathcal{KT}(g)$ is closed downwards in general. In fact, this is not the case. The downwards closure under \leq_T fails in a very strong sense, with $\mathcal{KT}(\Delta_2^0)$ (and hence each $\mathcal{KT}(g)$ for any Δ_2^0 order g) actually being cofinal in the Turing degrees.

THEOREM 2.1. *For any real B there is an $A \geq_T B$ such that $A \in \mathcal{KT}(\Delta_2^0)$.*

PROOF. We wish to build an A that is Turing-above B and that is K -trivial up to every finite-to-one approximation. We do not know *a priori* which $\phi_{e,s}$ are finite-to-one approximations, and, since we need Kraft-Chaitin sets to be recursively enumerable, we will have to build a tree T and use the branching nodes to mark guesses as to the behaviors of the $\phi_{e,s}$'s. The tree we build will be independent of B and will contain a witness A for every real B .

Placing the branching nodes in our tree will be a delicate operation. We will use a system of markers $\gamma(\alpha)$ to keep track of the values where corresponding $\phi_{e,s}$'s are large enough to place another branching node, which we will use to guess the behavior of the next $\phi_{e+1,s}$. Additionally, we will use these markers to mark 'coding locations' where the bits of a given real can be stored, which will make the behavior of the tree around these nodes slightly more complicated. We will have to introduce another kind of branching node which will keep track of which value is in the i th bit of B . Once we put a marker $\gamma(\alpha)_s$ at a node to make it a guessing node, both successors of that node, $\gamma(\alpha)_s \hat{\ } 1$ and $\gamma(\alpha)_s \hat{\ } 0$, will also both be in T_s , and a path taking one or the other of these nodes will correspond to guessing whether $\phi_{|\alpha|}$ is or is not a finite-to-one approximation. After this branching we will immediately introduce another branching, so that $\gamma(\alpha)_s \hat{\ } 11$, $\gamma(\alpha)_s \hat{\ } 10$, $\gamma(\alpha)_s \hat{\ } 01$, and $\gamma(\alpha)_s \hat{\ } 00$ will all be in T_s . The value of a path through $\gamma(\alpha)$ at $|\gamma(\alpha)| + 2$ will correspond to the $|\alpha|$ th bit of B . We distinguish between the two kinds of branching nodes as either *guessing nodes* or *coding nodes*. To *kill* a node is to make a commitment never to add nodes to T above it, and a node that has not yet been killed is *living*.

We make some definitions for ease of bookkeeping. We say a path ρ through T follows a string α through the branching nodes of T if at least $|\alpha|$ -many branching nodes are initial segments of ρ and for each $i < |\alpha|$, $\eta_i \hat{\ } \alpha(i) \preceq \rho$, where η_i is the i th branching node with $\eta_i \prec \rho$ (note that here we count the branching nodes starting with a 0th one). A path that follows α through the first $|\alpha|$ -many branching nodes is only making guesses about the first $|\alpha|/2$ -many $\phi_{e,s}$'s, since only the even-numbered branching nodes correspond to guesses. We say that the number e is a *guessing member* of the string α if and only if $\alpha(e) = 1$ and e is even. We denote this $e \in_g \alpha$. For each $\alpha \in 2^{<\omega}$, we use $\psi_{\alpha,s}(n) = \min\{\phi_{e,s}(n) : e \in_g \alpha\}$ to denote the function that makes all of α 's guesses. Note that ψ_α considers up to $|\alpha|/2$ many $\phi_{e,s}$'s. We will need to ensure we branch at a rate that forces these values to be large enough that we can afford to pay for multiple initial segments of the same length into our Kraft-Chaitin sets. We will want to ensure that each $\psi_{\alpha,s}$ takes values at least as large as $2|\alpha|$ before we add another branching.

We build a separate Kraft-Chaitin set M_e to witness A 's K -triviality up to ϕ_e . M_e will only take requests to describe initial segments that are on the tree above nodes that guess that $\phi_{e,s}$ is a total finite-to-one approximation, so for a correct path

A through T these M_e together will witness that $A \in \mathcal{KT}(\Delta_2^0)$. Recall that the *mass* of the set M_e is the sum $\sum_{(\sigma,n) \in M_e} 2^{-n}$.

Ensuring that there is such an A that is Turing-above B will be handled after the construction. Essentially, we will just show that the path through the ‘true subtree’ that contains the bits of B in its coding locations can find these locations recursively.

The requirements we are trying to meet are

B_α : The path through T that follows α through the branching nodes branches twice more at a level n where $\psi_\alpha(n) \geq 2|\alpha|$

for all $\alpha \in 2^{<\omega}$ with $|\alpha|$ even, and

R_i^e : For all n with $2i \leq \phi_e(n) < 2i + 2$, $K(A \upharpoonright_n) \leq^+ K(n) + 2i$

for all $i, e \in \omega$ with $e \leq i$.

We order these requirements $B_{\langle \rangle}, R_0^0, B_{\langle 00 \rangle}, B_{\langle 01 \rangle}, B_{\langle 10 \rangle}, B_{\langle 11 \rangle}, R_1^0, R_1^1, B_{\langle 0000 \rangle}, \dots$. The construction will be an injury construction, and we give the strategies for meeting each of the requirements. Because some of the $\phi_{e,s}$ will not actually be finite-to-one approximations, we may fail to satisfy some of these requirements in the construction. We will need to organize the construction so that injuries caused by the ill-behaved $\phi_{e,s}$ do not interfere with the satisfaction of requirements for the well-behaved $\phi_{e,s}$.

A B_α requirement will *require attention* at a stage s if there is not a living branching node τ above the path that follows α through T_s with $\psi_{\alpha,s}(|\tau|) \geq 2|\alpha|$. The strategy for satisfying B_α is

1. Search for an n such that $\psi_{\alpha,s}(n) \geq 2|\alpha|$.
2. Extend the longest path that follows α with a string of 0s to a length $n' + 1$ where $n' > n$ has not been used yet in the construction. Put the marker $\gamma(\alpha)_{s+1}$ at the node on this branch of length $n' + 1$. Put both extensions of length $n' + 2$ and all four extensions of this node of length $n' + 3$ into T_{s+1} .

An R_i^e requirement will *require attention* at a stage s if there is an $n \leq s$ with $2i \leq \phi_{e,s}(n) < 2i + 2$ and there is a living path τ through T_s of length at least n and there is an α such that e is a guessing member of α and $\gamma(\alpha)_s \preceq \tau$ such that there is not a request in $M_{e,s}$ for a description of $\tau \upharpoonright_n$ of length less than $K_s(n) + 2i$. The strategy for satisfying R_i^e is

1. For all such n and τ , for the longest α such that $\gamma(\alpha)_s \preceq \tau$ and e is a guessing member of α , put the request $(\tau \upharpoonright_n, K_s(n) + 2|\alpha|)$ into $M_{e,s+1}$.

We now give the construction, which will call these subroutines as necessary.

STAGE 0: $T_0 = \emptyset, M_0 = \emptyset, \gamma(\langle \rangle)_0 = \langle \rangle$ and $\gamma(\alpha)_0$ undefined for all other α .

STAGE $s + 1$:

1. Compute $\phi_{e,s+1}(n)$ for $e, n \leq s + 1$.
2. If there are an n and an α such that $|\gamma(\alpha)_s| < n$ and $\psi_{\alpha,s+1}(n) < 2|\alpha|$, then $\gamma(\alpha)_s$ is no longer marking a point after which ψ_α is greater than $2|\alpha|$, so for the length-lexicographically first α
 - (a) Kill all branches of the tree above $\gamma(\alpha)_s$.

- (b) Let $\gamma(\alpha)_s^-$ be the initial segment of $\gamma(\alpha)_s$ of length $|\gamma(\alpha)_s| - 1$ and put $\gamma(\alpha)_s^- \frown 1$ into T_{s+1} as a living node. Note that again by construction $\gamma(\alpha)_s$ always ends in a 0 and is always a node of length at least 2 longer than any number seen earlier in the construction, hence $\gamma(\alpha)_s^- \frown 1$ will not have been used before this point.
 - (c) Repeat 2a) and 2b) for all other such α with $\gamma(\alpha)_s$ still living, in length-lexicographic order.
3. For the highest priority requirement that requires attention of the first $s + 1$ many requirements, run $s + 1$ -many steps of its strategy.
 4. Repeat 3.) for any of the first $s + 1$ -many requirements that still require attention, in order of decreasing priority.

This completes the construction. We let $T = \bigcup_s T_s$, $M_e = \bigcup_s M_{e,s}$, and $\gamma(\alpha) = \lim_s \gamma(\alpha)_s$. Unfortunately, in this construction, as in life, not all requirements can be satisfied. We call a string α *correct* if for all $n \leq |\alpha|$, if $n = 2m$ is even then $\alpha(n) = 1$ if and only if $\phi_{m,s}$ is a finite-to-one approximation. We need to show that the requirements that are relevant to building correct paths through T are all satisfied, and that the mass we put into M_e to satisfy the R_i^e requirements is bounded.

LEMMA 2.2. *The requirements B_α for correct α and R_i^e for e such that $\phi_{e,s}$ is a finite-to-one approximation are all eventually satisfied.*

PROOF. First we argue that each of these requirements can be subject to at most finitely many injuries. An injury to any B_α requirement only occurs when for some $e \in_g \alpha$ the value of $\phi_{e,s}(n)$ drops below $2|\alpha|$ for some $n > |\gamma(\alpha)_s|$. When this happens, we respond by moving $\gamma(\alpha)_{s+1}$ to a higher level. Now, since there are only finitely many $e \in_g \alpha$, if there were infinitely many such injuries then at least one $e \in_g \alpha$ would be responsible for infinitely many. There would then have to be infinitely many n such that for some t , $\phi_{e,t}(n) < 2|\alpha|$, so $\phi_{e,s}$ would not be a finite-to-one approximation. Then $e \in_g \alpha$ is a contradiction to α 's correctness.

For R_i^e requirements, since $\phi_{e,s}$ is a total finite-to-one approximation we will eventually reach some stage s where $\phi_{e,s}$ has converged on all n such that $\phi_e(n) < 2i + 2$. At this stage all the n that R_i^e will ever be concerned about have been found. We let the largest of these n be n' . Then injuries to R_i^e can only occur either when K_s changes for one of these n , but this happens only finitely often, or when there is a change in T below n' on some branch that is guessing that $\phi_{e,s}$ is a total finite-to-one approximation. Each of these changes moves some marker to a point larger than n' , and, since there were only finitely many markers at positions lower than n' at stage s , this too can only happen finitely often.

Now we need to show that once these requirements are no longer injured they will be able to act to satisfy themselves. For B_α requirements, the strategy waits until it finds an n such that $\psi_{\alpha,s}(n) \geq 2|\alpha|$, and then extends a path in the tree to this height and branches twice. Since every $e \in_g \alpha$ is in fact a total finite-to-one approximation, there will exist an n for which $\psi_\alpha(m) \geq 2|\alpha|$ for all $m > n$, and so eventually B_α will find such an m and act and be satisfied. For R_i^e requirements, the strategy puts requests into M_e . After it is no longer injured, it needs to act at most once for each n with $\phi_e(n) < 2i + 2$ and there are only finitely many of these so it is eventually satisfied. ⊖

The R_i^e requirements for $\phi_{e,s}$ that are finite-to-one approximations will act until satisfied, but this happens just by placing the relevant requests into M_e . We now need to find an upper bound on the mass put into M_e in this way, to ensure we get a machine that serves our purpose.

LEMMA 2.3. *For all e , $\mu(\text{dom}(M_e)) \leq 8$.*

PROOF. Let us start by fixing an e . We will bound the amount paid into M_e for an arbitrary n using some description σ of n from \mathbb{U} . For any path τ of length n in T , while τ is alive there is some maximal α such that $\gamma(\alpha)_s \preceq \tau$. We recall that $\gamma(\alpha)$'s are placed based on the behavior of the e that are guessing members of α , so for a given n the collection of strings that are maximal for some nodes of length n at some stage s can include strings of different lengths. Now, a given α can be maximal for at most 4 such τ 's since T_s will branch twice immediately after $\gamma(\alpha)_s$ and then no more until $\gamma(\alpha')$ for some $\alpha' \succ \alpha$, at which point α' would be maximal. Changes in T_s below level n will kill off some of the nodes at that level and then later new nodes will be added. Rather than keep track of the different τ 's of length n that are living at different points in the construction, it will be easier to keep track of the α 's that are maximal in this sense for some nodes of length n as we go through the construction.

An injury in the construction kills the tree above some $\gamma(\alpha)_s$, and when B_α acts again at a later stage t , the new $\gamma(\alpha)_t$ will be placed at a node of length n' that is larger than any number seen before in the construction, so clearly any $\beta \succeq \alpha$ that was maximal for nodes of length n will no longer be after the change (although it will be maximal for some nodes of much longer lengths). In this way any change in T_s below level n will make some new α' maximal for some new nodes of length n , where α' will be an initial segment of some α that was maximal before the change. Thus, a given α can only ever be maximal for 4 nodes of length n over the entire course of the construction. Recall that $\gamma(\alpha)$ are only placed for α of even length.

For each α of even length, we know that $\gamma(\alpha)_s$ is placed at a node such that $\psi_{\alpha,s}(|\gamma(\alpha)_s|) \geq 2^{|\alpha|}$, and so we know the rate at which we will pay into M_e for all τ 's for which α is maximal is $2^{-2^{|\alpha|}}$. Here we use the fact that we only pay for paths with $e \in_g \alpha$. Now all that remains is to add up the total mass that could be paid into M_e when any α is maximal such that $\gamma(\alpha) < n$. This gives us the sum

$$\sum_{\substack{|\alpha| \text{ even} \\ 0 \leq |\alpha| < \infty}} 4 \cdot 2^{-|\sigma|} 2^{-2^{|\alpha|}}.$$

Since there are 2^{2i} -many α of length $2i$ this can be rewritten as

$$4 \cdot \sum_{i=0}^{\infty} 2^{2i} \cdot 2^{-|\sigma|-4i}.$$

This sum reduces to $4 \cdot 2^{-|\sigma|} \cdot \sum_{i=0}^{\infty} 2^{-2i}$, and the sum here is bounded by 2, so we can bound the mass paid into M_e on behalf of n using σ by $8 \cdot 2^{-|\sigma|}$. We now sum over all σ in the domain of the universal machine to find a bound of the mass paid into M_e for any n using any σ . We get that this mass is bounded by $\sum_{\sigma \in \text{dom}(\mathbb{U})} 8 \cdot 2^{-|\sigma|} \leq 8$, and we are done. ⊖

This lemma gives us that a path through T that guesses correctly that $\phi_{e,s}$ is a finite-to-one approximation will be K -trivial up to ϕ_e . Thus, paths through T that guess correctly about the behavior of every $\phi_{e,s}$ (i.e., that are *correct*) will be K -trivial up to every finite-to-one approximation. Now all that remains is to show that given a real B we can find a path through this true subtree of T that computes B .

LEMMA 2.4. *For any B , there is an $A \in [T]$ such that $A \geq_T B$ and A is correct.*

PROOF. Suppose we are given B . We define a sequence of strings α_i . Let α_i be such that $|\alpha_i| = 2i$ and for all $n < 2i$, if $n = 2m + 1$ then $\alpha_i(2m + 1) = B(m)$ and if $n = 2m$ then $\alpha_i(n) = 1$ if and only if $\phi_{m,s}$ is a finite-to-one approximation. Since α_i is correct and $|\alpha_i|$ is even, B_{α_i} is a requirement in our construction that is eventually satisfied. This means that we will eventually place the marker $\gamma(\alpha_i)$ on some node that follows α_i through the first $2i$ many branching nodes, where it will remain for the rest of the construction. Now $\alpha_i < \alpha_{i+1}$ for all i , so $\gamma(\alpha_i)$ must necessarily be an initial segment of $\gamma(\alpha_{i+1})$ for all i . Then we can let $A = \bigcup_i \gamma(\alpha_i)$ and this is well-defined. Now, A follows α_i through T for every i , and each $\gamma(\alpha_i)$ is correct about its guesses, so A must also be correct about its guesses. From this and the previous lemma, we know that $A \in \mathcal{KT}(\Delta_2^0)$. All that remains is to show that $A \geq_T B$.

First, we know that the bits of B are encoded somewhere in A , since they are the odd bits of the α_i 's. A follows α_i through the branching nodes of T , so $A(|\gamma(\alpha_i)| + 2) = B(i)$. A can simulate the construction of T and the approximation to $\gamma(\alpha)$ for each α . When we move a $\gamma(\alpha)$ we first kill the tree above $\gamma(\alpha)_s$ and then start building above $\gamma(\alpha)_s^- \frown 1$. Thus, A can tell when it finds a $\gamma(\alpha_i)_s$ whether this will be the location of $\gamma(\alpha_i)$ at the end of the construction. When it reaches a stage such that $\gamma(\alpha_i)_s < A$ it knows this is the final location of $\gamma(\alpha_i)$, and so it can retrieve the i th bit of B . ⊢

This was the final step in the proof of Theorem 2.1. ⊢

We end this section with a few remarks on the proof. First, we note that the paths through T that we construct to compute B may have much higher Turing degree than is necessary. By the same process that A uses to compute the bits of B , A can deduce which $\phi_{e,s}$ are total finite-to-one approximations and the index set $\{e : \phi_{e,s} \text{ is a total finite-to-one approximation}\}$ is Π_3^0 -complete. Of course, for certain B there may be much less complicated A (for example, if B is recursive then $A = B$ is in $\mathcal{KT}(\Delta_2^0)$).

Turning to analogues of Nies's Theorem (that K -triviality and lowness for K coincide) with these weaker notions, we saw in Proposition 1.4 that one direction still holds, i.e., that lowness for K up to Δ_2^0 orders implies K -triviality up to Δ_2^0 orders. In the other direction, however, Theorem 2.1 gives a strong negative result. While $\mathcal{LK}(\Delta_2^0)$ must be closed downwards in the Turing degrees, $\mathcal{KT}(\Delta_2^0)$ is cofinal in this structure. This separates $\mathcal{KT}(\Delta_2^0)$ from $\mathcal{LK}(f)$ for any f , not just Δ_2^0 orders.

COROLLARY 2.5. *For any function $f : 2^{<\omega} \rightarrow \omega$ either $\mathcal{LK}(f) = 2^\omega$ or $\mathcal{KT}(\Delta_2^0) \not\subseteq \mathcal{LK}(f)$.*

PROOF. If there is an B such that $B \notin \mathcal{LK}(f)$, then by Theorem 2.1 there is an $A \geq_T B$ such that $A \in \mathcal{KT}(\Delta_2^0)$. Now, since $\mathcal{LK}(f)$ is closed downwards under \leq_T

and $B \notin \mathcal{LK}(f)$, A cannot be in $\mathcal{LK}(f)$. Thus, there is an A that is in $\mathcal{KT}(\Delta_2^0)$ but not in $\mathcal{LK}(f)$. ⊣

We know there are reals that are not in $\mathcal{KT}(\Delta_2^0)$, so in particular $\mathcal{KT}(\Delta_2^0) \neq 2^\omega$. This gives us the following corollary, which demonstrates that it is impossible to capture this notion of bounded initial segment complexity with any notion of bounded compressive power.

COROLLARY 2.6. *There is no collection F of functions $2^{<\omega} \rightarrow \omega$ such that $\mathcal{LK}(F) = \mathcal{KT}(\Delta_2^0)$, where $\mathcal{LK}(F) = \bigcap_{f \in F} \mathcal{LK}(f)$.*

In particular, since we know $\mathcal{LK}(\Delta_2^0) \neq 2^\omega$, we have that $\mathcal{KT}(\Delta_2^0) \not\subseteq \mathcal{LK}(\Delta_2^0)$.

§3. Other closures. Now, by Theorem 2.1 we know that in general $\mathcal{KT}(g)$ and $\mathcal{KT}(\Delta_2^0)$ are not closed downwards under \leq_T , so traditionally they would not be considered ‘lowness’ notions. We examine some other reducibility notions under which these sets are closed downwards. First we show that for a stronger computational reducibility $\mathcal{KT}(\Delta_2^0)$ is closed downwards. A is *weak truth-table reducible* to B (denoted $A \leq_{wtt} B$) if there is a Turing functional Φ and a recursive function f such that $\Phi^B = A$ and for any n , the use of the computation of the n th bit of A from B (the largest bit of B that is queried in the computation) is no more than $f(n)$. That is, not only can we use B to compute A , but we have a recursive bound on how much of B is needed to compute a given amount of A . It follows easily from the definition of K -triviality that $KT(0)$ is closed downwards under \leq_{wtt} and we show that this closure is preserved under the weakening to $\mathcal{KT}(\Delta_2^0)$.

THEOREM 3.1. *If $A \leq_{wtt} B$ and $B \in \mathcal{KT}(\Delta_2^0)$, then $A \in \mathcal{KT}(\Delta_2^0)$.*

PROOF. Suppose B can compute A via Turing functional Φ , with use ϕ^B . For any recursive f , we can find a recursive function that majorizes f and that is monotone increasing, so without loss of generality we can assume the we have an increasing recursive bound f on the use of Φ^B . Now, given $B \upharpoonright_{f(n)}$ we can find n , since f is recursive and injective, and then we can run Φ on this initial segment of B to get the initial segment of A of length n . Thus, to describe $A \upharpoonright_n$ all we need is $B \upharpoonright_{f(n)}$ and some constant that is a code for the functional Φ , so we have $K(A \upharpoonright_n) \leq^+ K(B \upharpoonright_{f(n)})$. We wish to show that for an arbitrary finite-to-one approximable function g , $K(A \upharpoonright_n) \leq^+ K(n) + g(n)$. Given such a g , we define a new function h , by $h(n) = g(m)$, where m is the greatest number such that $f(m) \leq n$. Finding this m can be done recursively, so h is also finite-to-one approximable. Thus, since B is K -trivial up to h , $K(B \upharpoonright_{f(n)}) \leq^+ K(f(n)) + h(f(n))$. Now f is recursive, so $K(n) =^+ K(f(n))$, and by definition $h(f(n)) = g(n)$, so finally we get $K(A \upharpoonright_n) \leq^+ K(n) + g(n)$, as desired. ⊣

Another closure property we get for $\mathcal{KT}(\Delta_2^0)$, in contrast to $\mathcal{LK}(\Delta_2^0)$ as in [8], is that $\mathcal{KT}(\Delta_2^0)$ is closed under effective join (the effective join of reals A and B , denoted $A \oplus B$, is the real whose binary expansion is given by $A \oplus B(2n) = A(n)$ and $A \oplus B(2n+1) = B(n)$). The proof follows closely the proof that $\mathcal{KT}(0)$ is closed under effective join that was given by Downey, Hirschfeldt, Nies, and Stephan [7].

THEOREM 3.2. *For any reals A and B , $A, B \in \mathcal{KT}(\Delta_2^0)$ if and only if $A \oplus B \in \mathcal{KT}(\Delta_2^0)$.*

PROOF. Given an A and B in $\mathcal{KT}(\Delta_2^0)$, we take an arbitrary finite-to-one approximable function f and show that $K(A \oplus B \upharpoonright_n) \leq^+ K(n) + f(n)$. Without loss of generality, we can assume that $f(n)$ is monotonic. We note that it suffices to show this inequality holds for just the even n , since for any σ , $K(\sigma) =^+ K(\sigma \frown 1) =^+ K(\sigma \frown 0)$ and for any n , $K(n+1) =^+ K(n)$.

We define a new finite-to-one approximable function $g(n) = \lfloor f(2n)/3 \rfloor$. Since A and B are in $\mathcal{KT}(\Delta_2^0)$, in particular they are K -trivial up to this new g , so for some constants b_A and b_B , for every n , we have $K(A \upharpoonright_n) \leq K(n) + g(n) + b_A$ and $K(B \upharpoonright_n) \leq K(n) + g(n) + b_B$. We let $b = \max\{b_A, b_B\}$. It is a theorem of Downey et al. (Theorem 5.5 in [7]) that there is a constant c such that for any k the cardinality of the set $S_{n,k} = \{\sigma : |\sigma| = n \ \& \ K(\sigma) \leq K(n) + k\}$ is no more than $2^c 2^k$. Importantly, it does not depend on n . Thus, we know that $A \upharpoonright_n$ and $B \upharpoonright_n$ are both elements of the set $S_{n,g(n)+b}$, which is relatively small. Moreover, the set $S_{n,g(n)+b}$ is uniformly recursively enumerable in $K(n)$, $g(n)$, and b , so we can describe $A \upharpoonright_n$ and $B \upharpoonright_n$ relatively easily by giving their positions in the enumeration of this smallish set.

Formally, we define a prefix-free machine M that works as follows. On a string $\tau = 0^k \frown 1 \frown \sigma \frown \alpha \frown \beta$, where α and β are both strings of length $k + c$, M runs the universal machine \mathbb{U} on σ until it converges and then defines $n = \mathbb{U}(\sigma)$. M interprets α and β as binary representations of numbers between 1 and 2^{c+k} (using $\langle 0^{c+k} \rangle$ for the number 1 and $\langle 1^{c+k} \rangle$ for the number 2^{c+k}), and waits for the α th and β th strings of length n to receive descriptions from the universal machine of lengths less than $|\sigma| + k$. When it finds these two strings, it outputs their effective join.

M is clearly partial recursive and its prefix-freeness follows from the prefix-freeness of \mathbb{U} and the call to \mathbb{U} on σ . If we take a $\tau = 0^{g(n)+b} \frown 1 \frown \sigma \frown \alpha \frown \beta$ where σ is a shortest description of n , and $A \upharpoonright_n$ and $B \upharpoonright_n$ are the α th and β th strings of length n to receive descriptions shorter than length $n + g(n) + b$, then $M(\tau) = A \upharpoonright_n \oplus B \upharpoonright_n = A \oplus B \upharpoonright_{2n}$. This string τ has length $g(n) + b + 1 + K(n) + 2(c + g(n) + b)$. Thus, by the universality of \mathbb{U} , we get that $K(A \oplus B \upharpoonright_{2n}) \leq^+ K(n) + 3g(n) + 2c + 3b + 1$. Since c and b are constants that do not depend on n , and $K(n) =^+ K(2n)$, we can rewrite this as $K(A \oplus B \upharpoonright_{2n}) \leq^+ K(2n) + 3g(n) \leq^+ K(2n) + f(2n)$, by the definition of g . This suffices to show that $A \oplus B$ is in $\mathcal{KT}(f)$.

For the other direction, for any A and B , $K(A \upharpoonright_n) \leq^+ K(A \oplus B \upharpoonright_{2n})$ and for all n , $K(n) =^+ K(2n)$. Thus, if $A \oplus B \in \mathcal{KT}(\Delta_2^0)$, then $K(A \upharpoonright_n) \leq^+ K(n) + f(2n)$ for any finite-to-one approximable f . To show that $A \in \mathcal{KT}(g)$ for a given finite-to-one approximable g , we simply take an f such that $f(2n) = f(2n + 1) = g(n)$ for all n . This is clearly finite-to-one approximable if g is. By a symmetrical argument, B must also be in $\mathcal{KT}(\Delta_2^0)$. ⊣

By Theorems 3.1 and 3.2, we get that $\mathcal{KT}(\Delta_2^0)$ is an uncountable ideal in the *wtt*-degrees, so it is not too far removed from being a legitimate ‘lowness notion.’ One rather interesting side note is that $\mathcal{KT}(\Delta_2^0)$ and $\mathcal{LK}(\Delta_2^0)$ exhibit exactly the opposite behavior in terms of being Turing ideals. $\mathcal{KT}(\Delta_2^0)$ is closed under \oplus but for any real A there is a real $B \in \mathcal{KT}(\Delta_2^0)$ with $B \geq_T A$, while the set $\mathcal{LK}(\Delta_2^0)$ is closed downwards under \leq_T but for any A there are B and C in $\mathcal{LK}(\Delta_2^0)$ with $B \oplus C \geq_T A$ (so $\mathcal{LK}(\Delta_2^0)$ is cofinal in the T -degrees under \oplus). This goes some way to suggest how each definition contributes to the various closure properties of $\mathcal{KT}(0) = \mathcal{LK}(0)$ and demonstrates how important Nies’s Theorem is to our understanding of these properties.

§4. Weak reducibilities. We now consider these sets of reals under other, weaker reducibility notions. A natural reducibility notion under which to consider the reals in $\mathcal{KT}(\Delta_2^0)$ is K -reducibility, introduced by Downey, Hirschfeldt, and LaForte in [6]. We say A is K -reducible to B ($A \leq_K B$) if for all n , $K(A \upharpoonright_n) \leq^+ K(B \upharpoonright_n)$. Equivalently, we often say that B is K -above A or that A is K -below B . From the definition it is immediate that $\mathcal{KT}(g)$ for any g and $\mathcal{KT}(\Delta_2^0)$ are closed downwards under \leq_K . Because a K -reduction does not need to have a concrete object as a witness (the way a Turing-reduction needs a Turing functional), it is not necessarily the case that the set of reals reducible to a given real will always be countable. In fact this is often not the case, as every Martin-Löf random real has an uncountable lower cone in the K -degrees[15]. The following theorem shows that for reals in $\mathcal{KT}(\Delta_2^0)$ at least, this is the case. A precise characterization of those reals with countable lower \leq_K -cones is at this time still an open problem.

THEOREM 4.1 (with F. Stephan). *If A is in $\mathcal{KT}(\Delta_2^0)$, then A has a countable lower \leq_K -cone.*

PROOF. We show that if $A \in \mathcal{KT}(\Delta_2^0)$ then A is infinitely often K -trivial (i.e., for some constant c there are infinitely many n satisfying $K(A \upharpoonright_n) \leq K(n) + c$). Infinitely often K -trivial reals have been studied by Barmpalias and Vlek [3], and in particular they have shown that if A is infinitely often K -trivial, then any $B \leq_K A$ is Δ_2^0 in A , and so A 's lower \leq_K -cone must be countable.

In fact, $A \in \mathcal{KT}(\log \log n)$ suffices to ensure that A is infinitely often K -trivial. To show this, we assume $A \in \mathcal{KT}(\log \log n)$ and find infinitely many n where $K(A \upharpoonright_n) \leq^+ K(n)$.

Since $A \in \mathcal{KT}(\log \log n)$, for any m , $K(A \upharpoonright_{2^m}) \leq^+ K(2^m) + \log \log 2^m =^+ K(m) + \log m$. Now, for any number m , $K(m)$ is always up to a constant less than $\log m + 2 \log \log m$ (see, for example Chapter 2 of [12]), so we get that $K(A \upharpoonright_{2^m}) \leq^+ 2 \log m + 2 \log \log m$, and for large enough m this quantity is less than m . Thus, there is some $\sigma \in 2^{<\omega}$ with $|\sigma| < m$ such that $\mathbb{U}(\sigma) = A \upharpoonright_{2^m}$. Because $|\sigma| < m$, σ can be interpreted as the binary representation of a natural number, $\text{num}(\sigma)$, with value less than 2^m . Now for this σ , $K(A \upharpoonright_{\text{num}(\sigma)}) \leq^+ K(\sigma)$, since from a description for σ one can run the universal machine on σ to get $A \upharpoonright_{2^m}$ and then compute $\text{num}(\sigma)$ and truncate this string to $A \upharpoonright_{\text{num}(\sigma)}$. It is a recursive process to go from a binary string σ to the natural number it is a binary representation of, so $K(\sigma) =^+ K(\text{num}(\sigma))$, and so $K(A \upharpoonright_{\text{num}(\sigma)}) \leq^+ K(\text{num}(\sigma))$. For every sufficiently large m , such a σ exists, and they are necessarily distinct for distinct m , so A is infinitely often K -trivial. Thus, by the result of Barmpalias and Vlek, the set $\{B : B \leq_K A\}$ is countable. \dashv

Just as K -reducibility is a way to preorder reals by their relative initial segment complexities, there is an analogous reducibility notion for relative compressive power. We say a real A is LK -reducible to B ($A \leq_{LK} B$) if for all $\sigma \in 2^{<\omega}$, $K^B(\sigma) \leq^+ K^A(\sigma)$, that is, A compresses strings at most as well as B does. It is clear that the sets $\mathcal{LK}(f)$ and $\mathcal{LK}(\Delta_2^0)$ are closed downwards under \leq_{LK} . Analogously to infinitely often K -trivial reals, we have reals that are *weakly low for K* .

DEFINITION 4.2. A real A is *weakly low for K* if there is a $c \in \omega$ such that for infinitely many $\sigma \in 2^{<\omega}$, $K(\sigma) \leq K^A(\sigma) + c$.

Miller [10] showed that these reals correspond to the low for Ω reals defined by Nies, Stephan, and Terwijn in [13]. A real A is called *low for Ω* if for there is a $c \in \omega$ such that for all $n \in \omega$, $K^A(\Omega \upharpoonright_n) \geq n - c$, where Ω is the measure of the domain of the universal machine \mathbb{U} . In the same paper Miller showed that these weakly low for K reals had countable lower cones in the LK -degrees, and conjectured that these were in fact the only reals that do. Barmpalias and Lewis [2] settled this question in the affirmative. Following Theorem 4.1 we could hope to show that the reals in $\mathcal{LK}(\Delta_2^0)$ all have countable lower LK -cones, but unfortunately this is not the case. The rest of this section comprises a proof of the incomparability of $\mathcal{LK}(\Delta_2^0)$ and weak lowness for K . One direction follows easily from the existence of Martin-Löf random reals that are low for Ω , since none of these can be in $\mathcal{LK}(\Delta_2^0)$. The other direction is more complicated.

THEOREM 4.3. *There are reals in $\mathcal{LK}(\Delta_2^0)$ that are not low for Ω , and so not weakly low for K .*

PROOF. The construction will be similar to the one in the proof of Theorem 2.1. The idea is to build a perfect binary branching tree whose branching nodes will represent guesses as to which of the $\phi_{e,s}$ are finite-to-one approximations and alongside this to build a Kraft-Chaitin set to witness that each path is low for K up to those $\phi_{e,s}$ which it guesses are finite-to-one approximations. To ensure that the path is not low for Ω , we enumerate an oracle Kraft-Chaitin set using potential paths through the tree as oracles and giving short descriptions to initial segments of Ω relative to these paths. Ω is Δ_2^0 , so we can approximate its initial segments recursively. Tension arises between trying to put short descriptions of initial segments of Ω onto paths through our tree while also trying to match descriptions relative to paths through the tree with almost-the-same-length descriptions by our oracle-free Kraft-Chaitin sets. The construction will generate a lot of waste mass into all the Kraft-Chaitin sets since we have at best an approximation to Ω and to the finite-to-one approximable $\phi_{e,s}$'s. This part will be more difficult than the proof of Theorem 2.1, since here we are trying to use an oracle-free Kraft-Chaitin set to pay for descriptions of strings matching those relative to an oracle. As we change the tree the oracles will change, so our opponent will get mass back with which to challenge us while the mass we spent will have been wasted.

Branchings in the tree T will alternate between *guessing nodes*, which are associated with guesses as to the behavior of some ϕ_e , and *compression intervals*, collections of i -many branchings (so 2^i -many top-level nodes) on which, for some n , we place descriptions of possible $\Omega \upharpoonright_n$ of length $n - i$. We distribute the descriptions as evenly as possible among the top-level nodes of a compression interval, to ensure that the measure of the domain of the machine we construct is finite with respect to each of the paths as an oracle.

In the proof of Theorem 2.1 we waited till the various $\phi_{e,s}$ took values greater than $2i$; here we will use the sequence $c_i = 80(i + 1)^4$ for the same purpose. We use similar terminology to the previous proof.

For a given $\alpha \in 2^{<\omega}$, we will say e is a *member* of α if $\alpha(e) = 1$ and denote this $e \in \alpha$. For each $\alpha \in 2^{<\omega}$ we define a function that guesses that α is correct: $\psi_{\alpha,s}(\sigma) = \min\{\phi_{e,s}(\sigma) : e \in \alpha\}$.

We will say a path ρ though T follows a string α through the guessing nodes of T if for each $i < |\alpha|$, $\eta_i \frown \alpha(i) \preceq \rho$, where η_i is the i th guessing node that is an

initial segment of ρ . Such a path is *minimal* if it is minimal under the \leq relation (i.e., $|\rho| = |\eta|_{|\alpha|} + 1$).

The requirements that we will try to meet are:

R_α : For all paths through T that follow α at the *guessing* nodes, there is a level where they all branch $(2|\alpha|) + 1$ -many more times

for all $\alpha \in 2^{<\omega}$,

S_i^e : For all σ with $c_i \leq \phi_e(\sigma) < c_{i+1}$, for all $A \in [T]$, $K(\sigma) \leq^+ K^A(\sigma) + c_i$

for all $i, e \in \omega$ with $i \geq e$, and

N_α : For any minimal path ρ that follows α through the guessing nodes of T there is an extension, ρ' to an $(|\alpha| + 1)$ st guessing node and an m such that $K^{\rho'}(\Omega \upharpoonright_m) \leq^+ m - |\alpha|$

for all $\alpha \in 2^{<\omega}$ with $|\alpha| \geq 1$.

Note that we only have S_i^e requirements for $i \geq e$. This prevents ϕ_e from injuring the tree below the guessing node for e .

We order the requirements $R_{\langle \rangle}, S_0^0, R_{\langle 0 \rangle}, R_{\langle 1 \rangle}, N_{\langle 0 \rangle}, N_{\langle 1 \rangle}, S_1^0, S_1^1, R_{\langle 00 \rangle}, R_{\langle 01 \rangle}, R_{\langle 10 \rangle}, R_{\langle 11 \rangle}, N_{\langle 00 \rangle}, N_{\langle 01 \rangle}, N_{\langle 10 \rangle}, N_{\langle 11 \rangle}, S_2^0, S_2^1, S_2^2, \dots$.

The various S_i^e requirements will be concerned with different σ throughout the construction as the approximations to the $\phi_{e,s}$ settle. We will say S_i^e is *e-responsible* for σ at stage s if at stage s we have $c_i \leq \phi_{e,s}(\sigma) < c_{i+1}$ and σ is one of the length-lexicographically first s elements of $2^{<\omega}$. Note that for each e for a given string σ and stage s at most one S_i^e is *e-responsible* for σ at s . As in the last proof, we will want to keep track of our guessing levels and to this end we will use a collection of markers $n_{\alpha,s}$. Each $n_{\alpha,s}$ will mark the the end of the compression interval where the paths that follow α through the first $|\alpha|$ -many guessing nodes of T_s have descriptions of $\Omega \upharpoonright_m$ that are shorter by $|\alpha|$, and will be the next guessing level. For clarity, we note that we start the count of branching nodes and guessing nodes with 0, so that the e th guessing node up the tree guesses about the behavior of $\phi_{e,s}$; the first time we branch is the 0th guessing node. As before, to *kill* a node is to make a commitment to never add nodes above it into T . Nodes in T_s that have not been killed are *living*. A node is a *leaf node* at stage s if neither of its successors is in T_s .

In the construction we will build various Kraft-Chaitin sets to witness the inequalities we are trying to achieve. First, for each e we will be enumerating a Kraft-Chaitin set L_e in order to satisfy the S_i^e requirements. When a new description of some σ that S_i^e is *e-responsible* for at s converges on some living path through the tree T_s , we will want to respond by putting a request into L_e for a description of σ that is at most c_i longer than the one we see using the path as an oracle, as long as doing so will not put us at risk of adding too much mass to L_e (we give a definition of this case, and how we respond to it, below). In addition to all these L_e , we will also be enumerating an *oracle Kraft-Chaitin set*, M , as we try to show that some path through T will compress initial segments of Ω . Oracle Kraft-Chaitin sets allow us to build machines that query an oracle, and so we need to enumerate triples (σ, n, α)

that request a description of the string σ of length n , relative to an oracle with α as an initial segment. A relativized version of the Kraft-Chaitin theorem will give us an oracle machine that behaves as we request, as long as we ensure that for any A , the sum $\sum_{(\sigma,n,\alpha) \in M, \alpha \prec A}$ is less than 1 (it suffices to show this is bounded, and our machine will perform as we request up to an additive constant). For our M , N_α will be enumerating requests for descriptions of initial segments of our approximation to Ω that are shorter by $|\alpha|$, relative to oracles that are paths through our compression intervals.

We now give the strategies for satisfying each of our requirements.

An R_α requirement *requires attention* at a stage s if the guessing level $n_{\alpha,s}$ is not defined. The strategy for meeting R_α is

1. Let n be some number larger than any seen before in the construction.
2. For every living leaf node, η , of T_s that follows α through the first $|\alpha|$ -many guessing nodes, add the path $\eta \frown \beta \frown \gamma$ to T_s to get T_{s+1} , where $|\eta| + |\beta| = n$, $\beta(i) = 0$ for all i where it is defined, for every $\gamma \in 2^{2^{|\alpha|+1}}$.
3. Let $n_{\alpha,s+1} = n + 2|\alpha|$. This is now a guessing level.

An N_α requirement *requires attention* at a stage s if one of the minimal nodes ρ that follows α through the guessing nodes of T_s does not have an extension $\rho' \in T_s$ to an $|\alpha| + 1$ st guessing node for which the request $(\Omega_s \upharpoonright_{m_{\alpha,s}}, m_{\alpha,s} - |\alpha|, \rho')$ has been put into M , for the current $m_{\alpha,s}$. The strategy for meeting this requirement is

1. If $m_{\alpha,s}$ is undefined, pick some $m > 2|\alpha| + 1$ that is also larger than anything seen so far in the construction and let $m_{\alpha,s} = m$.
2. For the current $m_{\alpha,s}$, for the $|\alpha| + 1$ st guessing node, ρ' in T_s that follows α through the first $|\alpha|$ -many guessing nodes and is the leftmost that has had the least amount of mass put into M so far, put the request $(\Omega_s \upharpoonright_{m_{\alpha,s}}, m_{\alpha,s} - |\alpha|, \rho')$ into M_{s+1} .

An S_i^e requirement *requires attention* at a stage s if there is a σ that it is e -responsible for and there is a living partial path γ in T_s that, if it goes through at least e -many guessing nodes then it takes the ‘1’ branch after the e th one, and we have $K_s^?(\sigma) + \phi_{e,s}(\sigma)$ is less than the shortest description of σ in $L_{e,s}$. This means that the shorter description of σ is on a path that either has not reached a guessing node for $\phi_{e,s}$ or is guessing that it is a finite-to-one approximation, so we will need to act. The strategy for meeting this requirement is

1. Find the length-lexicographically least σ and for this σ the length-lexicographically least γ that are causing S_i^e to require attention. By the choice of these as length-lexicographically least, we must have that the use of the computation $\mathbb{U}_s^?(\tau) \downarrow = \sigma$ that is causing S_i^e to require attention is $|\gamma|$, that is, no initial segment of γ yields the same computation.
2. For this γ , let α be maximal such that γ follows α through the guessing nodes of T_s . In other words, α is the collection of guesses that are being made on the path γ .
3. If $|\alpha| \leq e$, then γ has not guessed about the behavior of $\phi_{e,s}$, but we know $i \geq e$, so we can afford to pay for a description of σ on this part of the tree anyway. Put a request $(\sigma, K_s^?(\sigma) + c_i)$ into $L_{e,s}$ to get $L_{e,s+1}$.

4. Otherwise, $|\alpha| > e$, and since S_i^e requires attention, we must have $\alpha(e) = 1$. We now consider whether $|\alpha| \leq i + 1$, in order to check whether γ has too many guessing nodes and will cause us to injure the tree. If $|\alpha| \leq i + 1$, then we have not yet branched $(i + 1)^2 - 1$ times (where the $(i + 1)$ st guessing node would be) so we can pay. Put a request $(\sigma, K_s^\gamma(\sigma) + c_i)$ into $L_{e,s}$ to get $L_{e,s+1}$.
5. Otherwise, $|\alpha| > i + 1$, so γ is longer than the $(i + 1)$ st guessing level. Since $\phi_{e,s}(\sigma) < c_{i+1}$, putting a request into L_e for a description of σ up to $\phi_{e,s}(\sigma)$ longer could potentially cost us too much (not necessarily just for this σ , but we have allowed so many branchings that the same description could be reused on many different oracles and we would have to meet all of them with our oracle-free L_e), so we
 - (a) Injure $R_{\alpha \upharpoonright_{i+1}}$ and run the Injury Subroutine for it.
 - (b) Let $T_{s+1} = T_s$, $L_{e,s+1} = L_{e,s}$.

The Injury Subroutine for an R_α strategy at stage s is

1. For every minimal path η that follows α through the guessing nodes of T_s , find the living leaf node $\eta' \succ \eta$ such that

$$\sum_{\tau: \mathbb{U}_s^{\eta'}(\tau) \downarrow, \mathbb{U}_s^\eta(\tau) \uparrow} 2^{-|\tau|}$$

is maximal. If there is more than one, take η' to be the leftmost.

2. For every pair (η, η') found above, keep η' alive in T_s and kill all other extensions of η .
3. For every $\beta \succeq \alpha$, set R_β to requiring attention (i.e., set $n_{\beta,s+1}$ to be undefined) and set $m_{\beta,s+1} = m_{\beta,s} + 1$. This injures all these R_β and N_β .

The skeleton of the construction is

STAGE 0: Set $T_0 = \emptyset$, $M_0 = \emptyset$, $L_{e,0} = \emptyset$ and for every e , and $n_{\alpha,0}$ undefined for all α .

STAGE $s + 1$:

1. Compute $\phi_{e,s+1}(\sigma)$ and $K_{s+1}^\eta(\sigma)$ for all living branches η in T_s , the first $s + 1$ -many σ 's, and $e \leq s + 1$.
2. In order of priority, run the strategy for each of the first $s + 1$ -many requirements that require attention, including executing the Injury Subroutine as necessary.
3. For any $n_{\alpha,s}$ or $m_{\alpha,s}$ that were not affected, set $n_{\alpha,s+1} = n_{\alpha,s}$ and $m_{\alpha,s+1} = m_{\alpha,s}$.

Now let $T = \bigcup_s T_s$, $L_e = \bigcup_s L_{e,s}$, $n_\alpha = \lim_s n_{\alpha,s}$, $M = \bigcup_s M_s$, $m_\alpha = \lim_s m_{\alpha,s}$. This completes the construction. The verification follows.

As in the proofs of Theorem 2.1, we will not be able to ensure that all requirements are satisfied, but only those that are correct about their guesses. We would like to show that every path through the subtree of T generated by all the correct guesses about the $\phi_{e,s}$ is in $\mathcal{LK}(\Delta_2^0)$ and at least one of them is not low for Ω . We call α *correct* if for every $e \leq |\alpha|$, $e \in \alpha$ if and only if $\phi_{e,s}$ is a finite-to-one approximation.

LEMMA 4.4. *For all correct α , R_α and N_α are injured only finitely often.*

PROOF. By construction, for any α , the requirements R_α and N_α can only be injured by S_i^e requirements with $e \in \alpha$ and $i < |\alpha|$, of which there are only

finitely many. Assuming α is correct, the only ϕ_e that can cause these injuries are then indeed total finite-to-one approximations. Since R_α and N_α are always injured together, it will suffice to show that R_α is injured only finitely often.

To derive a contradiction, first let us assume there is some R_α for a correct α that is injured infinitely often, and take α to be a minimal such string. Each of the S_i^e 's that can injure R_α only ever has e -responsibility for finitely many σ since ϕ_e is a finite-to-one approximation, so there must be at least one σ that is the cause of infinitely many injuries to R_α . Let us take the length-lexicographically least such σ and fix an e such that S_i^e injures R_α infinitely often on this σ 's account.

Let us assume we are at a stage s such that $\phi_{e,s}(\sigma)$ and $K_s(\sigma)$ have settled and such that no R_β for $\beta \prec \alpha$ will ever be injured again. Since σ causes infinitely many more injuries to R_α , it must be the case that S_i^e has e -responsibility for σ for all stages $t \geq s$, for $i = |\alpha| - 1$.

Now, each run of the Injury Subroutine for R_α at some stage t will, for each minimal η that follows α through the guessing nodes of T_t , keep at least the most massive branch (the branch on which a maximal amount of mass has converged, as specified in step 1.) of the Injury Subroutine) above η alive and kill all other branches above η . There are always $1 \cdot 4 \cdot 16 \cdots 2^{2|\alpha|}$ many living nodes at height $n_{\alpha,t}$ that follow α through the guessing nodes, and a run of the Injury Subroutine for R_α extends each of these to a leaf node in the way that maximizes the mass placed along it. Since, by assumption, no earlier R_β will ever be injured again, the tree below these paths never change so this mass is never lost. We are at a stage t such that $K_t(\sigma)$ has already converged, so each injury caused by σ must be caused by our finding a description of σ of length less than $K(\sigma)$ along one of these paths. That is, at least $2^{-K(\sigma)}$ much mass must converge on one of the paths at height $n_{\alpha,t}$ that follow α through the guessing nodes. This is a fixed amount of mass that is added infinitely often to a finite number of oracles, so the measure of the domain of \mathbb{U} relative to one of these oracles is infinite. This is a contradiction.

Thus, R_α (and so N_α) can only be injured finitely often. ⊥

LEMMA 4.5. *For all correct α and all e such that $\phi_{e,s}$ is a total finite-to-one approximation, the requirements R_α , N_α , and S_i^e are all eventually satisfied.*

PROOF. Some requirements may cause infinitely many injuries to requirements above them in the ordering or require attention infinitely often. However, at any stage s of the construction we allow any of the first s requirements to act, and our actions affect different parts of the tree T_s (the paths that follow different α 's through the guessing nodes), so the poorly behaved requirements will not interfere with our actions in satisfying the correct ones.

By the above lemma, for correct α , the requirements R_α and N_α are only injured finitely often. After the last injury, R_α will need to act once more before it is satisfied, while N_α may need to act several times as it waits for $\Omega \upharpoonright_{p_\alpha}$ to converge (p_α only changes when N_α gets injured). Eventually this happens and after that stage we will put a description of $\Omega \upharpoonright_{p_\alpha}$ onto one of the relevant paths and satisfy N_α permanently.

If $\phi_{e,s}$ is a finite-to-one approximation, then, also by the proof above, the requirement S_i^e can only cause finitely many injuries. Its actions that do not cause injuries are just those in steps 3. and 4. of its strategy and these consist of putting a request $(\sigma, K^\gamma(\sigma) + c_i)$ into L_e , for some σ it is e -responsible for and some partial path γ

through T_s . Since $\phi_{e,s}$ is a finite-to-one approximation, there are only finitely many σ for which S_i^e is ever e -responsible. For each of these, there are only finitely many requests to put into L_e , since we only need to put new ones in if the new $K^\gamma(\sigma)$ is less than all previous ones. Thus, S_i^e will need to act only finitely often. \dashv

Lemmas 4.4 and 4.5 give us that the strategies relevant to the construction of the true subtree will eventually stop acting, but it remains to be shown that the Kraft-Chaitin sets enumerated by these strategies have bounded mass (and so produce the required machines). We start with the machine \mathcal{M} whose existence will be witnessed by M .

LEMMA 4.6. *For every real A , the sum $\sum_{(\sigma,p,\eta) \in M, \eta \prec A} 2^{-p} < 2$.*

PROOF. To prove this lemma we consider the amount of mass a given N_α requirement can contribute to M for an oracle A . It is clear that for different α of the same length, the sets of oracles the N_α 's use will be disjoint, since they will be paths through T that take different directions at at least one of the guessing nodes. Thus, it suffices to show that a given N_α will add at most $2^{-|\alpha|+1}$ to any oracle, and so a path that receives mass from many N_α 's will receive no more than 2 total mass.

Let us first fix an α of length at least 1 and let $d = |\alpha|$. For any stage s in the construction where N_α is active, we have some $m_{\alpha,s}$ which is always larger than $2|\alpha| + 1$. Now, there are naïvely $2^{m_{\alpha,s}}$ many possible strings that could be $\Omega \upharpoonright_{m_{\alpha,s}}$. When N_α requires attention, it puts a request $(\Omega \upharpoonright_{m_{\alpha,s}}, m_{\alpha,s} - d, \eta)$ into M_s for a node η in T_s at level $n_{\alpha,s}$ that follows α through the guessing nodes which has had the minimum amount of mass already placed on it by N_α . This contributes $2^{-m_{\alpha,s}+d}$ much mass to that path. Without any injuries, this strategy would distribute $2^{m_{\alpha,s}} \cdot 2^{-m_{\alpha,s}+d} = 2^d$ much mass evenly among the 2^{2d} many branches in this compression interval, giving 2^{-d} to each. Of course, injuries will complicate matters. A run of the Injury Subroutine will fix an amount of this mass onto each of the paths that follow α through the guessing nodes of T_{s+1} , which will be initial segments of the new nodes at $n_{\alpha,s+1}$ (after R_α acts again). In principle, this is fine. We now have 2^{2d} many nodes at the new $n_{\alpha,s+1}$ and N_α has put some amount p of mass onto an initial segment of all of them and will continue by sharing its remaining mass equally over them. In the ideal case this situation is no different from continuing to share the mass over the nodes before the injury. They each had (almost) p much mass already placed on them by N_α , and the fact that N_α is continuing to act means whatever potential initial segments of Ω they paid for have since been rejected.

The trouble, of course, is in the 'almost'. If some node at $n_{\alpha,s}$ had received more mass than the others before the injury occurred, it is possible for the Injury Subroutine to have picked a path through that node to keep, in which case slightly more mass than has now converged on the initial segment of the new nodes at $n_{\alpha,s+1}$. In the worst case, N_α may act only once between injuries, each of which keeps alive the only path that N_α has added mass to. This will concentrate all the mass that N_α has to distribute onto a single path. For this reason we increment $m_{\alpha,s}$ by 1 every time N_α is injured. We note that this does not affect the amount of mass that N_α has left to distribute; once we see that $\gamma \neq \Omega \upharpoonright_p$ we know that neither extension of γ is $\Omega \upharpoonright_{p+1}$. The difference in mass that N_α can have added to paths before an injury

is always at most $2^{-m_{\alpha,s}+d}$, since it always seeks to distribute the mass evenly, and it does so in quanta of $2^{-m_{\alpha,s}+d}$. Thus, even if injuries are selecting the paths with slightly higher than average mass, no more than $\sum_s 2^{-m_{\alpha,s}+d}$ can accumulate on a path above the average of 2^{-d} . This sum is bounded by $2^d \cdot 2^{-m_{\alpha,s_0}+1}$, where s_0 is the stage at which N_α is initialized. Since we always choose our first m_{α,s_0} to be larger than $2d + 1$, this term is less than 2^{-d} . Thus, the total amount that N_α can have contributed to a path is 2^{-d+1} , and so the total mass in M for any oracle is bounded by $\sum_d 2^{-d+1} = 2$. -1

Since M is a legitimate oracle Kraft-Chaitin set and the N_α requirements are satisfied for all correct α , there will be an infinite path A through T that is not low for Ω . A will be the path that always guesses correctly as to the behavior of $\phi_{e,s}$ at the e th guessing node, and between guessing nodes follows the path through the compression interval for which M has a short description of an actual initial segment of Ω .

All that remains to be shown is that this path is actually in $\mathcal{LK}(\Delta_2^0)$, that is, that the sets L_e that we construct actually witness the existence of machines ensuring that $K(\sigma) \leq^+ K^A(\sigma) + \phi_e(\sigma)$ for all σ , for all paths through T that guess that ϕ_e is finite-to-one approximable. This is the most complicated part of the proof, since the mass paid into L_e can be wasted by injuries to the construction.

LEMMA 4.7. *For every e , $\sum_{(\sigma,n) \in L_e} 2^{-n} < 1$.*

PROOF. We consider separately mass that is put into L_e by the actions of each S_i^e requirement. First, we fix an e and i in ω with $i \geq e$. We note that we do not require $\phi_{e,s}$ to be a finite-to-one approximation; in the event that it is not, actions of S_i^e strategies may cause infinitely many injuries to the part of the tree that guesses that it is, and then L_e will witness that the finitely many infinite branches on this part of the tree are all low for K .

Now, S_i^e requires attention if for some σ for which it has e -responsibility, it sees a new shorter description of σ converge on some path γ through T_s^e (let us take γ minimal to cause this). However, S_i^e only puts a request into L_e if γ goes through no more than $i + 1$ guessing nodes, and otherwise it causes an injury. Therefore, when examining the mass contributed by S_i^e it suffices to consider the finite initial segment of T_s given by

$$T_{i,s} = \{\sigma \in T_s : \sigma \text{ goes through no more than } i + 1\text{-many guessing nodes}\}.$$

We only put requests into L_e in response to computations converging on parts of the tree that are either below the e th guessing level, or follow the ‘1’ path at the e th guessing node itself, but for getting a rough upper bound on the mass of L_e , we can ignore this and consider the full subtree $T_{i,s}$.

The subtree $T_{i,s}$ contains the guessing nodes at $n_{\alpha,s}$ for every $\alpha \in 2^{<\omega}$ with $|\alpha| \leq i + 1$, and all the compression intervals between these nodes. For a given α with $|\alpha| \leq i + 1$, for each minimal η that follows α through $T_{i,s}$, there are $2^{2^{|\alpha|}}$ branches in the compression interval above η that reach level $n_{\alpha,s}$.

We will consider these branches as ‘reservoirs’ of mass, and descriptions converging using one of these branches as an oracle as mass getting added to the corresponding reservoir. If the use of a computation is exactly one of these minimal η ’s, we can consider that much mass being added to *each* of η ’s reservoirs. An injury

will cause many of these branches to be killed, so the mass will be spilled out, but the most massive branch will be kept alive and, after enough R requirements act again, the mass from that branch will end up in a lower reservoir for a shorter α' . It is important to note that for any α the reservoirs corresponding to α are all end extensions of reservoirs for α^- , the immediate predecessor of α , and so the total mass in any collection of nested reservoirs must be no more than 1.

The number of reservoirs associated with an α is fixed throughout the construction, although we may have to wait for R requirements to act to replace reservoirs that were emptied. For α with $|\alpha| = 1$, there are 4 reservoirs above α , since the compression interval has length 2, so there are 8 total reservoirs at this level. Above each of these reservoirs the tree branches once immediately, and then reaches a compression interval of length 4, so each of the level 1 reservoirs has $2 \cdot 2^4 = 32$ reservoirs above it at level 2, for 2^8 level 2 reservoirs total. In general there will be

$$\prod_{j=1}^i 2^{2^{j+1}} = 2^{\sum_{j=1}^i 2^{j+1}} = 2^{(i+1)^2 - 1}$$

reservoirs at level i . Note that there is a single reservoir at level 0, the stem of the tree before $n_{\langle \rangle}$, but since $R_{\langle \rangle}$ is never injured, no mass is ever poured down into this reservoir, so we need not worry about it in our computations.

When considering the contributions of S_i^e to L_e it suffices to consider only the reservoirs at level $i+1$ (for instance, S_0^0 puts mass into the level 1 reservoirs, after the guessing node for $\phi_{0,s}$). For any descriptions that converge lower in the tree $T_{i,s}$ for σ that S_i^e has e -responsibility for, we can instead put the corresponding amount of mass into each of the $i+1$ -level reservoirs above the actual use of the computation, since this has the same effect on the subsequent amount of mass that can be put into these reservoirs. Then to attain a rough bound on the amount of mass that S_i^e puts into L_e , we can make the simplifying assumption that S_i^e will pay for all the mass that passes through the reservoirs at level $i+1$, at a rate of 2^{-c_i} (the largest this can be without causing an injury). Now all that remains is to find a bound for the amount of mass that can pass through these reservoirs.

As we said above, any injury to a relevant R_α will spill the mass from all but the most massive reservoir, and pour this saved mass into a reservoir below. Thus, in the worst, impossible, case, S_i^e could have to pay for all the $i+1$ -level reservoirs being filled with 1 total mass each, then an injury could empty all but one of these, and pass that 1 down to the i -level ones, and this could repeat till all the i -level reservoirs are full. Then an injury could pour out all but one of these and fill one of the $i-1$ -level reservoirs, and this larger process could repeat till all the $i-1$ -level reservoirs are filled. Continuing like this, S_i^e could be forced to pay for the mass that is used to fill *all* the reservoirs at all levels up to and including the $i+1$ st one, while spilling as much as possible at every step. For each 1st level reservoir, we would have to fill every 2nd level one, and for each of these we would have to fill every 3rd level one, continuing until level $i+1$. This gives us that the mass required, just to fill all the 1st level reservoirs is

$$\prod_{j=1}^{i+1} 2^{(j+1)^2 - 1},$$

and thus to fill all the reservoirs up to level $i + 1$ would require

$$\sum_{k=1}^{i+1} \prod_{j=k}^{i+1} 2^{(j+1)^2-1}$$

much mass to pass through the $i + 1$ -level reservoirs.

We approximate a very rough upper bound for this:

$$\begin{aligned} \sum_{k=1}^{i+1} \prod_{j=k}^{i+1} 2^{(j+1)^2-1} &\leq \sum_{k=1}^{i+1} \prod_{j=1}^{i+1} 2^{(j+1)^2-1} \leq i + 1 \cdot 2^{\sum_{j=1}^{i+1} (j+1)^2-1} \\ &\leq 2^{(i+1)} \cdot 2^{[(2i^3+15i^2+37i+24)/6]-i-1} \\ &\leq 2^{2i^3+15i^2+37i+24}. \end{aligned}$$

Thus, the amount of mass the S_i^e puts into L_e is bounded by $2^{2i^3+15i^2+37i+24} \cdot 2^{-c_i}$. Since we have taken $c_i = 80(i + 1)^4$, this is the same as $2^{2i^3+15i^2+37i+24-80(i+1)^4}$, which is always no more than 2^{-i-1} . Since this is the contribution for each S_i^e , the total mass of requests that go into L_e for all i is less than $\sum_i 2^{-i-1} = 1$. -|

This completes the proof of Theorem 4.3 -|

§5. Further questions. Theorem 4.3 and the result of Barmpalias and Lewis [2] show that not every Δ_2^0 -bounded low for K real has a countable lower \leq_{LK} -cone, but the question of which of these reals do have a countable lower \leq_{LK} -cone remains open. At present, the only reals in the intersection are the K -trivials themselves.

QUESTION 5.1. *Can we characterize the reals that are both Δ_2^0 -bounded low for K and weakly low for K , i.e., that have countable lower \leq_{LK} -cones? Are there any that are not K -trivial?*

The analysis carried out in this paper was entirely in terms of prefix-free Kolmogorov complexity, but there are analogous notions in terms of plain Kolmogorov complexity (where the domains of decoding machines are not required to be prefix-free) that can also be weakened to Δ_2^0 -bounded versions. In the case of C -triviality and lowness for C , we know by results of Chaitin [4] that these notions coincide with each other and contain only the recursive sets. So far we know nothing about the Δ_2^0 -bounded versions.

QUESTION 5.2. *What can we say about Δ_2^0 -bounded lowness for C or C -triviality?*

Finally, we have considered here reals with initial segment complexity or compressive power bounded by all Δ_2^0 orders. It may be interesting to consider the internal structure of the various bounded notions, i.e., $\mathcal{LK}(f)$ and $\mathcal{KT}(g)$ for various f and g . Many of the results for the Δ_2^0 -bounded notions carry over trivially, for instance, the cofinality in the Turing degrees of $\mathcal{KT}(g)$ for any Δ_2^0 order g , but the theorems in Section 3 do not necessarily carry over, as the proofs depended on applying bounded initial segment complexity for different bounds. Clearly for some choices of f we have $\mathcal{LK}(f) = 2^\omega$ and for others it is much smaller (and similarly with $\mathcal{KT}(g)$), but it is open whether there is a strict cutoff between the two cases.

QUESTION 5.3. *What can we say about the sets $\mathcal{LK}(f)$ and $\mathcal{KT}(g)$ for single Δ_2^0 orders f and g ? What is the structure of these sets under \leq_K or \leq_{LK} ?*

REFERENCES

- [1] M. BAARTSE, and G. BARMPALIAS, *On the gap between trivial and nontrivial initial segment prefix-free complexity*. *Theory of Computing Systems*, vol. 52 (2013), no. 1, pp. 28–47.
- [2] G. BARMPALIAS, and A. E. M. LEWIS, *Chaitin's halting probability and the compression of strings using oracles*. *Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences*, vol. 467 (2011), no. 2134, pp. 2912–2926.
- [3] G. BARMPALIAS, and C. S. VLEK, *Kolmogorov complexity of initial segments of sequences and arithmetical definability*. *Theoretical Computer Science*, vol. 412 (2011), no. 41, pp. 5656–5667.
- [4] G. J. CHAITIN, *Information-theoretic characterizations of recursive infinite strings*. *Theoretical Computer Science*, vol. 2 (1976), no. 1, pp. 45–48.
- [5] B. F. CSIMA, and A. MONTALBÁN, *A minimal pair of K-degrees*. *Proceedings of the American Mathematical Society*, vol. 134 (2006), no. 5, pp. 1499–1502.
- [6] R. G. DOWNEY, D. R. HIRSCHFELDT, and G. LAFORTE, *Randomness and reducibility*. *Journal of Computer and System Sciences*, vol. 68 (2004), no. 1, pp. 96–114.
- [7] R. G. DOWNEY, D. R. HIRSCHFELDT, A. NIES, and F. STEPHAN, *Trivial Reals*, Proceedings of the 7th and 8th Asian Logic Conferences (Singapore, 2003), Singapore University Press, pp. 103–131.
- [8] I. HERBERT, *A perfect set of reals with finite self-information*, this JOURNAL, vol. 78 (2013), no. 4, pp. 1229–1246.
- [9] D. R. HIRSCHFELDT, and R. WEBER, *Finite self-information*. *Computability* vol. 1 (2012), no. 1, pp. 85–97.
- [10] J. S. MILLER, *The K-degrees, low for K-degrees, and weakly low for K sets*. *Notre Dame Journal of Formal Logic*, vol. 50 (2009–10), no. 4, pp. 381–391.
- [11] A. NIES, *Lowness properties and randomness*. *Advances in Mathematics*, vol. 197 (2005), no. 1, pp. 274–305.
- [12] ———, *Computability and Randomness*, Oxford Logic Guides, vol. 51, Oxford University Press, Oxford, 2009.
- [13] A. NIES, F. STEPHAN, and S. A. TERWIJN, *Randomness, relativization and Turing degrees*, this JOURNAL, vol. 70 (2005), no. 2, pp. 515–535.
- [14] R. I. SOARE, *Recursively Enumerable Sets and Degrees*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987.
- [15] L. YU, D. DING, and R. DOWNEY, *The Kolmogorov complexity of random reals*. *Annals of Pure and Applied Logic*, vol. 129 (2004), no. 1–3, pp. 163–180.

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