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Minimum Number of k-Cliques in Graphs with Bounded Independence Number

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Erdős asked in 1962 about the value of $f(n,k,l)$, the minimum number of k-cliques in a graph with order n and independence number less than l. The case $(k,l) = (3,3)$ was solved by Lorden. Here we solve the problem (for all large n) for $(3,l)$ with $4 \leqslant l \leqslant 7$ and $(k,3)$ with $4 \leqslant k \leqslant 7$. Independently, Das, Huang, Ma, Naves and Sudakov resolved the cases $(k,l) = (3,4)$ and $(4,3)$.

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1. Introduction

Let us give some definitions first. As usual, a *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is the *vertex set* and the *edge set* $E(G)$ consists of unordered pairs of vertices. An *isomorphism* between graphs G and H is a bijection $f : V(G) \to V(H)$ that preserves edges and non-edges. For a graph G, let $\overline{G} = (V(G), \binom{V(G)}{2} \setminus E(G))$ denote its *complement* and let $v(G) = |V(G)|$ denote its *order*. For graphs F and G with $v(F) \leqslant v(G)$, let $P(F,G)$ be the number of $v(F)$-subsets of $V(G)$ that induce in G a subgraph isomorphic to F; further, define the *density of F in G* to be

$$p(F,G) = P(F,G)\binom{v(G)}{v(F)}^{-1}. \tag{1.1}$$

Let K_k denote the complete graph on k vertices. Let $\alpha(G) = \max\{l : P(\overline{K}_l, G) > 0\}$ be the *independence number* of G, that is, the maximum size of an edge-free set of vertices.

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Given a graph F on $[m] = \{1, \dots, m\}$ and a sequence of disjoint sets $V_1, \dots, V_m$, let the *expansion* $F(\!(V_1, \dots, V_m)\!)$ be the graph on $V_1 \cup \cdots \cup V_m$ obtained by putting the complete graph on each V_i and putting, for each edge $\{i, j\} \in E(F)$, the complete bipartite graph between V_i and V_j. An expansion is *uniform* if $||V_i| - |V_j|| \leqslant 1$ for any $i, j \in [m]$. If we consider expansion in terms of complements, then it amounts to blowing up each vertex i of $\overline{F}$ by factor $|V_i|$ (and taking the complement of the obtained graph). Clearly, expansions cannot increase the independence number.

We consider the following extremal function:

$$f(n,k,l) = \min\{P(K_k, G) : v(G) = n,\ \alpha(G) < l\},$$

that is, the minimum number of k-cliques in a graph with n vertices that does not contain $\overline{K}_l$. This function (in its full generality) was first defined by Erdős [6] in 1962.

Earlier, Goodman [10] determined $f(2n, 3, 3)$; his bounds also give the asymptotic value of $f(2n+1, 3, 3)$. Lorden [14] determined $f(n, 3, 3)$ and showed that the complement of $T_2(n)$ is the unique extremal graph when $n \geqslant 12$, where the *Turán graph* $T_m(n)$ is the complete m-partite graph on $[n]$ with parts being nearly equal. (In other words, $T_m(n)$ is the complement of the uniform expansion of $\overline{K}_m$.)

Erdős [6] asked if perhaps

$$f(n,k,l) = P(K_k, \overline{T}_{l-1}(n)), \tag{1.2}$$

that is, if the uniform expansion of $\overline{K}_{l-1}$ gives the value of $f(n,k,l)$ and, specifically, if

$$f(3n, 3, 4) = 3\binom{n}{3}. \tag{1.3}$$

Nikiforov [15] showed that the limit

$$c_{k,l} = \lim_{n\to\infty} \frac{f(n,k,l)}{\binom{n}{k}} \tag{1.4}$$

exists for every pair (k, l) and that the upper bound $c_{k,l} \leqslant (l-1)^{1-k}$ given by the graphs $\overline{T}_{l-1}(n)$ as $n \to \infty$ can be sharp only for finitely many pairs (k, l). Thus, it was too optimistic to expect that (1.2) holds.

The main motivation of the papers [6, 10] came from Ramsey's theorem [19], which implies that $f(n,k,l) > 0$ when $n \geqslant n_0(k,l)$ is sufficiently large. Both papers also considered the related problem of minimizing $p(K_k, G) + p(\overline{K}_k, G)$ over an (arbitrary) order-n graph G. The last question, known as the *Ramsey multiplicity problem*, attracted a lot of attention and led to many important developments.

On the other hand, the problem of determining $f(n,k,l)$ was rather neglected although it was mentioned in the book by Bollobás [3, Problem 11 on page 361] and the survey by Thomason [22, Section 5.5]. One possible reason is that determining $c_{k,l}$, even for some small k and l, might require keeping track of too many different subgraph densities than is practically feasible when doing calculations 'by hand'.

Razborov [20] introduced a powerful formal system for deriving inequalities between subgraph densities, where a computer can be used to do routine book-keeping. One aspect of his theory (discussed in [21]) allows us to minimize linear combinations of subgraph densities by setting up and solving a semi-definite program. In some cases, the numerical solution thus obtained can be converted into a rigorous mathematical proof. Baber and Talbot [2] and Vaughan [23] (see [8, 9]) wrote openly available software for doing such calculations.

By using *Flagmatic* [23], we can solve the problem (for all large n) when $k = 3$ with $4 \leqslant l \leqslant 7$ or $l = 3$ with $4 \leqslant k \leqslant 7$. Independently, Das, Huang, Ma, Naves and Sudakov [5] solved the problem when n is large and $(k, l) = (3, 4)$ or $(4, 3)$, also by using flag algebras.

We state our results as three separate theorems.

Theorem 1.1 (Asymptotic Result).

$$c_{3,l} = (l-1)^{-2}, \qquad 4 \leqslant l \leqslant 7, \tag{1.5}$$
$$c_{4,3} = 3/25, \tag{1.6}$$
$$c_{5,3} = 31/5^4 = 31/625, \tag{1.7}$$
$$c_{6,3} = 19211/2^{20} = 19211/1048576, \tag{1.8}$$
$$c_{7,3} = 98491/2^{24} = 98491/16777216. \tag{1.9}$$

Furthermore, we have in each of these cases that

$$f(n,k,l) = c_{k,l}\binom{n}{k} + O(n^{k-1}). \tag{1.10}$$

The upper bounds in (1.5), (1.6), and (1.7) are obtained by taking a uniform expansion of F, where F is respectively $\overline{K}_{l-1}$, the 5-cycle C_5, and (again) C_5. Easy calculations show that the density of k-cliques in these graphs is as required. These upper bounds on $c_{4,3}$ and $c_{5,3}$ come from Nikiforov [15]. In a subsequent paper [16], he also showed that an order-n graph G with $\alpha(G) < 3$ satisfies $P(K_4, G) \geqslant (\frac{3}{25} + o(1))\binom{n}{4}$ under the additional assumption that G is close to being regular.

The upper bounds in (1.8) and (1.9) come from a more complicated construction. The *Clebsch graph* L has binary 5-sequences of even weight (*i.e.*, with an even number of entries equal to 1) for vertices, with two vertices being adjacent if the term-wise sum modulo 2 of the corresponding sequences has weight 4. For example, the neighbours of $00011 \in V(L)$ are 01100, 10100, 11000, 11101, and 11110. It easily follows from this description that the Clebsch graph is triangle-free and vertex-transitive. For example, an automorphism that maps 00000 to 11000 is to flip the first two bits.

The complement $F = \overline{L}$ of the Clebsch graph is a 10-regular graph on 16 vertices. Take a uniform expansion F' of F of large order n. The limit of $p(K_k, F')$ as $n \to \infty$ is equal to the probability that, if we independently sample uniformly distributed vertices $x_1, \ldots, x_k \in V(L)$, they do not induce any edge in L. By the vertex-transitivity of L, we can fix $x_1 = 00000$. The Clebsch graph has the following maximal independent sets containing 00000: the sequences that we add to 00000 must have weight 2, with the corresponding pairs of indices forming either $K_{1,4}$ (the star with 4 edges) or K_3 (the triangle). There are 5 of the former sets and 10 of the latter sets, of sizes 5 and 4 respectively. A straightforward

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inclusion–exclusion counting shows that the above probability is

$$\frac{5\cdot 5^{k-1}+10\cdot 4^{k-1}-30\cdot 3^{k-1}+20\cdot 2^{k-1}-4}{16^{k-1}}.$$

By plugging in $k=6$ and 7, we get the upper bounds on $c_{k,3}$ stated in (1.8) and (1.9).

The upper bound in (1.10) follows by observing that if we pick a random injection $\phi : [k] \to V(F')$, where F' is a uniform expansion of F of order n, and condition on the restriction of ϕ to $[i]$ for $i<k$, then the probability that $\phi(i+1)$ belongs to a particular part of F' is $1/v(F)+O(1/n)$. Thus $p(K_k,F')$ is within additive term $O(1/n)$ from its limit as $n\to\infty$.

The lower bounds of Theorem 1.1 are proved in Section 3 by using flag algebras.

We say that two graphs G and H of the same order are at *edit distance at most m* (or are *m-close*) if G can be made isomorphic to H by changing (adding or deleting) at most m edges. By inspecting the proof certificate returned by a flag algebra computation, one can sometimes describe the structure of all almost extremal graphs up to a small edit distance (see, for example, [4, 11, 17]). This also works here, and we can establish the following results that apply when (k,l) is one of the pairs $(3,l)$ with $3\leqslant l\leqslant 7$, $(k,3)$ with $4\leqslant k\leqslant 5$, and $(k,3)$ with $6\leqslant k\leqslant 7$, while F is respectively $\overline{K}_{l-1}$, C_5, and $\overline{L}$.

Theorem 1.2 (Stability Property). *Let k,l,F be as above. Then, for every $\varepsilon>0$ there exist $\delta>0$ and n_0 such that every graph G of order $n\geqslant n_0$ with $\alpha(G)<l$ and $P(K_k,G)\leqslant (c_{k,l}+\delta)\binom{n}{k}$ is $\varepsilon\binom{n}{2}$-close to a uniform expansion of F.*

We see that, in each case above, almost extremal graphs on $[n]$ have the same structure up to the edit distance of $o(n^2)$. Such extremal problems are called *stable*. The stability property, besides being of interest on its own, is often very helpful in establishing the exact result for all large n. Here, we also use stability to prove the following theorem.

Theorem 1.3 (Exact Result). *Let k,l,F be as above. Then there is n_0 such that every graph G of order $n\geqslant n_0$ with $\alpha(G)<l$ and the minimum number of K_k-subgraphs contains an expansion $F'=F(\!(V_1,\ldots,V_m)\!)$ as a* spanning subgraph *(that is, $V_1\cup\cdots\cup V_m=V(G)$ and $E(F')\subseteq E(G)$).*

Let n be sufficiently large. Since G in Theorem 1.3 is extremal and F' is $\overline{K}_{l-1}$-free, we have that $P(K_k,G)=P(K_k,F')$, that is, the value of $f(n,k,l)$ is attained by some expansion of F. Furthermore, if $l=3$ and $4\leqslant k\leqslant 7$, then G is necessarily equal to F' because the addition of any extra edge to F' creates at least one copy of K_k. Next, consider the four remaining cases, that is, $k=3$ and $4\leqslant l\leqslant 7$. It is easy to show that $\overline{T}_{l-1}(n)$ has the smallest number of triangles among all order-n expansions of $\overline{K}_{l-1}$. Thus Theorem 1.3 proves Erdős's conjecture (1.3) for all large n. However, note that there are other extremal constructions for $f(n,3,l)$ with $4\leqslant l\leqslant 7$ that can be obtained from $\overline{T}_{l-1}(n)$ by adding edges so that no new triangles are created.

As asked in [5], it would be interesting to determine those l for which $c_{3,l}=(l-1)^{-2}$. We know now that this is the case for all $2\leqslant l\leqslant 7$. Nikiforov [15] showed that this equality can hold for only finitely many l. Das, Huang, Ma, Naves and Sudakov [5] proved that no $l \geqslant 2074$ satisfies it.

Although our proofs rely on extensive computer calculations, new mathematical ideas are also introduced (such as, for example, Theorem 5.1, which deals with all studied cases in a unified manner). Hopefully, these ideas and results will be useful for other problems. For example, the concept of a *phantom edge* introduced here in Section 3.4 has been successfully applied to another extremal problem [7].

2. Notation

Here we collect some graph theory notation that we use.

The cycle (resp. path) with k vertices is denoted by C_k (resp. P_k).

Let G and H be graphs. We write $H \subseteq G$ and say that H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H \subseteq G$ is called *spanning* if $V(H) = V(G)$. It is called *induced* if $H = G[V(H)]$, where we denote $G[X] = (X, \{\{x,y\} \in E(G) : x, y \in X\})$ for $X \subseteq V(G)$. A *strong homomorphism* from H to G is a map $\phi : V(H) \to V(G)$ that preserves both edges and non-edges. For example, H admits a strong homomorphism to K_2 if and only if H is a complete bipartite graph. An *embedding* is a strong homomorphism which is injective; in other words, it is an isomorphism from H to an induced subgraph of G.

An *automorphism* of G is a map $V(G) \to V(G)$ that preserves both edges and non-edges (*i.e.*, an isomorphism of G to itself). A graph G is *vertex-transitive* if for every two vertices there is an automorphism of G mapping one to the other. The *neighbourhood* of a vertex $x \in V(G)$ is

$$\Gamma_G(x) = \{y \in V(G) : \{x, y\} \in E(G)\}.$$

The *closed neighbourhood* of x is $\hat{\Gamma}_G(x) = \Gamma_G(x) \cup \{x\}$.

The *Ramsey number* $R(k,l)$ is the minimum n such that every order-n graph has a k-clique or an independent set of size l. Thus $f(n,k,l) > 0$ if and only if $R(k,l) \geqslant n$.

3. Lower bounds in Theorem 1.1

3.1. Proof certificates

As we have already mentioned, our lower bounds are proved with the help of a computer by using flag algebras and semi-definite programming; see Razborov [20, 21]. This method is described in a number of research publications ([2, 8, 9, 12, 20, 21]), so we will be brief.

We used *Flagmatic* (Version 2.0) [23] for the computations. For each proof that we present, we provide a certificate that contains the information needed for others to be able to verify all claims. The script `inspect_certificate.py` that comes with *Flagmatic* can be used for investigating the certificates and performing some level of verification. The certificates are in a documented format [23] and it is hoped that others will be able to independently verify them.

Also, we include the code that generated each certificate as well as the transcript of each session, to aid the reader in repeating our calculations. This may be helpful if the reader would like to experiment with the software by changing parameters (or to apply *Flagmatic* to some related problems).

These materials are available from *Flagmatic*'s website at

`http://flagmatic.org/examples/Fkl.tgz`

Each solved case (k,l) is supported by the following data: the complete code, the transcript of the session, and all generated certificates. For example, the corresponding files for the case $(k,l) = (7,3)$ are `73.sage`, `73.txt`, and two certificates `73.js` and `73a.js`.

Alternatively, the ancillary folder of [18] contains all files except some certificates whose sizes are larger than arXiv's allowance. The reader should be able to generate these certificates by running the appropriate scripts with *Flagmatic 2.0*.

Also, the cases $(3,4)$ and $(k,3)$ with $4 \leqslant k \leqslant 7$ were previously solved with Version 1.5 of *Flagmatic*; see [18] (Version 3) for all details. This is reassuring as *Flagmatic 2.0* was re-written essentially from scratch (when it was decided to do everything inside *sage* for greater functionality).

Our presentation is different from that of Das, Huang, Ma, Naves and Sudakov [5], who worked hard at making their paper self-contained and the proof as human-readable as possible. This has many advantages (such as giving more insight into the problem) but makes the paper rather long. Our objective is to present formal rigorous proofs of all claimed results. We do so by describing the information that is contained in the certificates and by showing how it implies the stated results. While the certificates are not very suitable for direct inspection (some of them are very large and contain integers with hundreds of digits), the reader may verify all stated properties by using *Flagmatic* or by writing an independent script.

Let us give some definitions that are needed to describe the certificates. Fix one of the pairs (k,l) as above.

Let us call a graph *admissible* if its independence number is less than l. A *type* is a pair (H,ϕ) where H is an admissible graph and $\phi : [v] \to V(H)$ is a bijection, where $v = v(H)$. Given a type $\tau = (H,\phi)$ as above, a *τ-flag* is a pair (G,ψ) where G is an admissible graph and $\psi : [v] \to V(G)$ is an injection such that $\psi \circ \phi^{-1} : V(H) \to V(G)$ is an embedding (that is, an injection that preserves both edges and non-edges). Informally, a type is a vertex-labelled graph and a τ-flag is a partially labelled graph such that the labelled vertices induce τ. The *order* $v((G,\psi))$ of a type or a flag is $v(G)$, the number of vertices in it.

For two τ-flags (G_1,ψ_1) and (G_2,ψ_2) with $n_1 \leqslant n_2$ vertices, let $P((G_1,\psi_1),(G_2,\psi_2))$ be the number of n_1-subsets $X \subseteq V(G_2)$ such that $X \supseteq \psi_2([v])$ (*i.e.*, X contains all labelled vertices) and the τ-flags (G_1,ψ_1) and $(G_2[X],\psi_2)$ are *isomorphic*, meaning that there is a graph isomorphism that preserves the labels. Also, define the *density*

$$p((G_1,\psi_1),(G_2,\psi_2)) = \frac{P((G_1,\psi_1),(G_2,\psi_2))}{\binom{n_2-v}{n_1-v}}$$

to be the probability that a uniformly drawn random n_1-subset X of $V(G_2)$ with $X \supseteq \phi_2([v])$ induces a copy of the τ-flag (G_1,ψ_1) in (G_2,ψ_2).

Now, we can present the information that is contained in each certificate (a file with extension js) and is needed in the proof.

First, the certificate lists all (up to an isomorphism) admissible N-vertex graphs for some integer N. Let us denote these graphs by $G_1, \ldots, G_g$. Then the certificate describes some types $\tau_1, \ldots, \tau_t$ such that their graph components are pairwise non-isomorphic (as unlabelled graphs) and $N - v(\tau_i)$ is a positive even number for each $i \in [t]$.

The certificate contains, for each $i \in [t]$, the list $(F_1^{\tau_i}, \ldots, F_{g_i}^{\tau_i})$ of all τ_i-flags (up to isomorphism of τ_i-flags) with exactly $(N + v(\tau_i))/2$ vertices.

Also, for each $i \in [t]$, the certificate (indirectly) contains a symmetric positive semi-definite $g_i \times g_i$-matrix Q^{τ_i}. More precisely, the matrix Q^{τ_i} is represented in the following manner: we have a diagonal matrix Q' all whose diagonal entries are positive rational numbers and a rational matrix R such that

$$Q^{\tau_i} = RQ'R^T. \tag{3.1}$$

This decomposition automatically implies that the matrix Q^{τ_i} is positive semi-definite.

Now, let G be an admissible graph of large order n. Initially, let $a = 0$. Let us do the following for each v such that $N - v$ is a positive even integer. Enumerate all $n(n-1)\ldots(n-v+1)$ injections $\psi : [v] \to V(G)$. If the *induced type* $G[\psi] = (G[\psi([v])], \psi)$ is isomorphic to some τ_i (as vertex-labelled graphs), then we add $\mathbf{x}_\psi Q^{\tau_i} \mathbf{x}_\psi^T$ to a, where

$$\mathbf{x}_\psi = \left(P\left(F_1^{\tau_i}, (G, \psi)\right), \ldots, P\left(F_{g_i}^{\tau_i}, (G, \psi)\right)\right). \tag{3.2}$$

Since each Q^{τ_i} is positive semi-definite, we have that $\mathbf{x}_\psi Q^{\tau_i} \mathbf{x}_\psi^T \geqslant 0$ and that the final a is non-negative.

Let us take some type τ of order v and two τ-flags F_1 and F_2 with respectively ℓ_1 and ℓ_2 vertices. Let $\ell = \ell_1 + \ell_2 - v$. Consider the sum

$$\sum_{\psi\,:\,G[\psi]\cong\tau} P(F_1, (G, \psi))\, P(F_2, (G, \psi)), \tag{3.3}$$

taken over all injections $\psi : [v] \to V(G)$ such that the induced type $G[\psi]$ is isomorphic to τ. Each term $P(F_i, (G, \psi))$ in (3.3) can be expanded as the sum over ℓ_i-sets X_i with $\psi([v]) \subseteq X_i \subseteq V(G)$ of the indicator function that $(G[X_i], \psi)$ is a τ-flag isomorphic to F_i. Ignoring the choices when X_1 and X_2 intersect outside $\psi([v])$, the remaining terms can be generated by choosing an ℓ-set $X = X_1 \cup X_2$ first, then an injective map $\psi : [v] \to X$, and finally X_1 and X_2. Clearly, the terms that we ignore contribute at most $O(n^{\ell-1})$ in total. Also, the contribution of each ℓ-set X to (3.3) depends only on the isomorphism class H of $G[X]$. Thus the sum in (3.3) can be written (modulo an additive error term $O(n^{\ell-1})$) as an explicit linear combination of the subgraph counts $P(H, G)$, where H runs over unlabelled graphs with ℓ vertices; see, *e.g.*, [20, Lemma 2.3].

By the above discussion, if we expand each quadratic form $\mathbf{x}_\psi Q^{\tau_i} \mathbf{x}_\psi^T$ in the definition of a and take the sum over all injections ψ, then we will get a representation

$$0 \leqslant a = \sum_{i=1}^{g} \alpha_i P(G_i, G) + O(n^{N-1}), \tag{3.4}$$

where each α_i is a rational number that does not depend on n and can be computed given the above information (types, flags, and matrices). An explicit formula for α_i is rather messy, so we do not state it.

The crucial property that our certificates possess is that

$$\alpha_i \leqslant p(K_k, G_i) - c'_{k,l}, \quad \text{for every } i \in [g], \tag{3.5}$$

where $c'_{k,l}$ is the right-hand side of the appropriate statement (1.5)–(1.9), *i.e.*, $c'_{k,l}$ is the lower bound on $c_{k,l}$ that we want to prove. This property (involving rational numbers) can be verified by the stand-alone script `inspect_certificate.py`, which uses exact arithmetic.

If we assume that (3.5) holds, then we have, by Bayes' formula, that

$$p(K_k, G) - c'_{k,l} = \sum_{i=1}^{g} (p(K_k, G_i) - c'_{k,l}) p(G_i, G) \geqslant \sum_{i=1}^{g} \alpha_i p(G_i, G) \geqslant -O(1/n). \tag{3.6}$$

Thus we derived not only $c_{k,l} \geqslant c'_{k,l}$ but also the claimed lower bound in (1.10).

At this point, we may stop and assume that Theorem 1.1 has been proved (modulo verifying all the claims above with the help of a computer). However, it may be useful to say a few words about how these certificates were obtained. Finding matrices $Q^{\tau_1}, \ldots, Q^{\tau_t}$ amounts to solving a semi-definite program. The program is usually quite large. So it is generated by a computer as well; *Flagmatic* provides a highly customizable way of doing this. Then the obtained program is fed into an SDP-solver which returns floating-point matrices. It is a good idea to start with N as small as possible and keep increasing it until the obtained (floating-point) bound seems to be equal to the conjectured value. We found it beneficial, at this stage, to use the double-precision `spda_dd` solver, which usually returns the correct values of around 20 first decimal digits.

In fact, this was how the extremal configuration for $c_{6,3}$ was discovered. The solver seemed to give the same bound $c_{6,3} \geqslant 19211/2^{20}$ for both $N = 7$ and 8. Here, the denominator is a high power of 2. This suggested that an extremal configuration might be a uniform expansion of a graph with 16 vertices, which made us look at such graphs.

This process of converting the obtained floating-point matrices into those that satisfy (3.5) exactly also uses a computer. It is fairly automated in *Flagmatic*, although it sometimes requires adjustment of various parameters and options. Of course, once we have found suitable rational matrices that provide a rigorous proof, we can ignore their floating-point lineage altogether.

One strategy to simplify the proof certificates, once N has been fixed, is to reduce the number of types as much as possible by re-running the SDP-solver and checking that we still get the same bound. Note that $\tau_1, \ldots, \tau_t$ need not enumerate all types. The removal of some type τ effectively means that we make the corresponding matrix Q^τ to be identically 0. (Likewise, $F_1^{\tau_i}, \ldots, F_{g_i}^{\tau_i}$ need not enumerate all τ_i-flags but this observation does not seem to be very useful.)

Another useful trick comes from the following lemma.

Lemma 3.1. *Suppose that we have a flag algebra proof, as specified above, that the value of $c_{k,l}$ is given by uniform expansions of a $\overline{K}_l$-free graph F. Fix $i \in [t]$. Let the ith type τ_i*

be (H,ϕ) *and let* $v=v(\tau_i)$. *Let* n *be large and let* G *be a uniform expansion of* F *of order* n. *Let* $\psi:[v]\to V(G)$ *be an injection such that* $\psi\circ\phi^{-1}$ *is an embedding of* H *into* G. *Then* $\mathbf{x}_\psi Q^{\tau_i}\mathbf{x}_\psi^T=O(n^{N-v-1})$, *where* $\mathbf{x}_\psi$ *is defined by* (3.2).

Proof. Since each part V_i of G is homogeneous, any modification of the injection ψ such that its values stay in the same parts is an embedding. These new injections give the same vector $\mathbf{x}_\psi$. Thus, with $m=v(F)$,

$$0\leqslant\left(\frac{n}{m}+O(1)\right)^v\mathbf{x}_\psi Q^{\tau_i}\mathbf{x}_\psi^T\leqslant a. \tag{3.7}$$

Let us run our flag algebra proof on G. It shows in fact that $p(K_k,G)\geqslant c_{k,l}+a/\binom{n}{N}+O(1/n)$. Also, as we have previously remarked, $p(K_k,G)$ deviates from $c_{k,l}$ by at most $O(1/n)$. We conclude that $a=O(n^{N-1})$, implying the lemma by (3.7). □

Thus, when we let $n\to\infty$ and scale $\mathbf{x}_\psi$ to have the ℓ_1-norm equal to 1, we obtain a zero eigenvector of Q^{τ_i} in the limit. (Note that $\mathbf{x}Q\mathbf{x}^T=0$ for $Q\succeq 0$ implies that $Q\mathbf{x}^T=\mathbf{0}$.) We call such a zero eigenvector *forced*. By inspecting the graph F that gives the upper bound in Theorem 1.1, we can identify forced zero eigenvectors. It is crucial to know all forced zero eigenvectors during the rounding step because a small but uncontrolled perturbation of Q^{τ_i} may result in negative eigenvalues. *Flagmatic 2.0* takes care of this by ensuring that the column space of the matrix R in (3.1) is orthogonal to all forced zero eigenvectors of Q^{τ_i} (when an extremal construction is supplied using the function `set_extremal_construction`).

Lemma 3.1 can be generalized to many other problems. This idea was first used by Razborov [21].

There are further relations that have to hold in a flag algebra proof. For $i\in[g]$, call the graph G_i *sharp* if (3.5) becomes an equality, that is, $\alpha_i=p(K_k,G_i)-c_{k,l}$. (We know by now that $c_{k,l}=c'_{k,l}$.)

Lemma 3.2. *Suppose that we have a flag algebra proof, as specified above, that the value of* $c_{k,l}$ *is given by uniform expansions of a* $\overline{K}_l$*-free graph* F. *Let* n *be large and let* G *be a uniform expansion of* F *of order* n. *Let* $i\in[g]$ *be such that* G_i *embeds into* G. *Then* G_i *is sharp.*

Proof. Let $m=v(F)$. Note that $P(G_i,G)\geqslant(n/m+O(1))^N/N!$: if we take an embedding f of G_i into $F(\!(U_1,\dots,U_m)\!)$, then any injection $f':V(G_i)\to V(G)$ with $f(x)$ and $f'(x)$ belonging to the same part U_j is also an embedding. We have by (3.4) and (3.5) that

$$\begin{aligned}p(K_k,G)-c_{k,l}&\geqslant\sum_{j\in[g]}(p(K_k,G_j)-c_{k,l}-\alpha_j)p(G_j,G)+O(1/n)\\&\geqslant(p(K_k,G_i)-c_{k,l}-\alpha_i)p(G_i,G)+O(1/n)\\&\geqslant\frac{p(K_k,G_i)-c_{k,l}-\alpha_i}{m^N}+O(1/n).\end{aligned} \tag{3.8}$$

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Since $p(K_k, G) - c_{k,l} = o(1)$ by our assumption, we conclude (by using (3.5) again) that G_i is sharp, as required. □

Flagmatic also uses the restrictions given by Lemma 3.2 for rounding (if a construction is provided). In some cases, the large amount of data and/or the presence of tiny but non-zero coefficients required us to reduce the number of types as much as possible (essentially by trial and error) and to use the double-precision SDP-solver `sdpa_dd`. Below we mention briefly how this process went in each solved case and what further actions (if any) were needed.

3.2. Cases $(k, l) = (4, 3)$ or $(5, 3)$

The rounding procedure worked without any issues for these two cases. In both cases, we used the 6-vertex universe that contains 38 graphs with independence number at most 2.

3.3. Cases $(k, l) = (6, 3)$ or $(7, 3)$

In these cases, we found it more convenient to work with the complements: namely, we forbid K_3 and minimize the density of $\overline{K}_k$ for $k = 6, 7$. These cases went through without any problems. While $c_{6,3}$ could be computed by using graphs with at most 7 vertices, it seems that the determination of $c_{7,3}$ by this method requires 8-vertex graphs.

3.4. Cases $k = 3$ and $4 \leqslant l \leqslant 7$

One difficulty that we had to overcome is that there are some further relations that a flag algebra proof of $c_{3,l} \geqslant (l-1)^{-2}$ has to satisfy, in addition to those given by Lemmas 3.1 and 3.2.

Lemma 3.3. *Suppose that we have a flag algebra proof that $c_{3,l} \geqslant (l-1)^{-2}$ as above. Let n be large and let $T = \overline{T}_{l-1}(n) = \overline{K}_{l-1}(\!(V_1, \ldots, V_{l-1})\!)$. Let T' be obtained from T by adding one extra edge $\{x_1, x_2\}$ between V_1 and V_2. If some G_i admits an embedding f into T', then it is sharp.*

Proof. Let $\varepsilon > 0$ be a small constant and let $n \to \infty$. Let the graph G be obtained from T by adding all edges between U_1 and U_2, where $U_i \subseteq V_i$ is a set of size $\lfloor \varepsilon n \rfloor$. We have $\alpha(G) < l$ and

$$P(K_3, G) - P(K_3, T) \leqslant \binom{2\varepsilon n}{3} = O(\varepsilon^3 n^3), \tag{3.9}$$

as each triangle in G but not in T has to lie inside $U_1 \cup U_2$. Let us plug this G into (3.6). As we have just observed, the left-hand side of (3.6) is $O(\varepsilon^3)$. Since G_i embeds into T', we have that $p(G_i, G) \geqslant \Omega(\varepsilon^2)$. (Indeed, if we take any $f' : V(G_i) \to V(G)$ so that $f'(x)$ and $f(x)$ always belong to the same part of $\overline{T}_{l-1}(n)$ while $f'(x) \in U_j$ if and only if $f(x) = x_j$, then we obtain at least $(1 - o(1)) \times (\varepsilon n)^2 \times (\frac{n}{l-1} - \varepsilon n)^{N-2}$ different embeddings f'.) As ε can be arbitrarily small, it follows that G_i is sharp by a version of (3.8). □

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Lemma 3.3 shows that some further graphs are necessarily sharp in addition to those that embed into $\overline{T}_{l-1}(n)$. Likewise, by unfolding the last inequality in (3.6) for the graph G from the proof of Lemma 3.3 and using (3.9), we conclude that $a = O(\varepsilon^3 n^N)$. Each of the t summands in

$$a = \sum_{i=1}^{t} \sum_{\psi\,:\,G[\psi]\cong\tau_i} \mathbf{x}_\psi Q^{\tau_i} \mathbf{x}_\psi^T \tag{3.10}$$

is non-negative and is therefore at most $O(\varepsilon^3 n^N)$. Thus all terms in the right-hand side of (3.10) that can have magnitude $\Omega(\varepsilon^2 n^N)$ have to disappear. In particular, for every type τ_i that embeds into T' but not into T, there are some further zero eigenvectors of Q^{τ_i} (that are not caught by the direct application of Lemma 3.1).

Once we understood 'phantom' edges, the rounding problem went through without any problems. The option `phantom_edge` (see the scripts) instructs *Flagmatic* to take into account all such extra sharp graphs and zero eigenvectors.

A similar phenomenon was encountered in the maximum codegree problem for 3-graphs with independent neighbourhoods (see [7]), and a version of Lemma 3.3 was crucial for rounding the numerical solution there.

4. Proving the Stability Property

Here we prove Theorem 1.2. Our proof is similar in spirit to the proof of Theorem 2 in [17]. Let (k,l) and F be as in the theorem. Let $N = N(k,l)$ be the number of vertices that was used in the flag algebra proof of Section 3; thus $N(3,4) = 5$, $N(3,5) = N(4,3) = N(5,3) = 6$, $N(3,6) = N(6,3) = 7$, and $N(3,7) = N(7,3) = 8$.

Suppose on the contrary that there is $\varepsilon > 0$ such that for infinitely many $n \to \infty$ there is a graph G of order n such that $\alpha(G) < l$ and $p(K_k, G) = c_{k,l} + o(1)$ but G is $\varepsilon\binom{n}{2}$-far from a uniform expansion of F. Let $V = V(G)$.

Recall that G_i is *sharp* if we have equality in (3.5). Call an admissible graph G_i *singular* if G_i is not contained as an induced subgraph in any expansion of F. Note that these definitions apply only to the order-N graphs $G_1,\dots,G_g$. The following observation is well known (compare it with Lemma 3.2).

Lemma 4.1. *Let $i \in [g]$. If G_i is not sharp, then $p(G_i, G) = o(1)$.*

Proof. Note that we have already established that $c'_{k,l} = c_{k,l}$. Let us run our flag algebra proof on G. Similarly to (3.8), we obtain that

$$p(K_k, G) - c_{k,l} \geqslant (p(K_k, G_i) - c_{k,l} - \alpha_i)p(G_i, G) + O(1/n).$$

Since G is almost extremal, we have that $p(K_k, G) - c_{k,l} = o(1)$. The lemma follows from (3.5). □

4.1. Cases $(k,l) = (4,3)$ or $(5,3)$

Let $l = 3$ and $k = 4$ or 5. Here F is the 5-cycle C_5 and $N = 6$.

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The scripts verify that the number of graphs of order 6 that occur with positive density in a large expansion of F is the same as the number of sharp graphs (namely, there are 17 graphs in each list). Thus these two lists coincide by Lemma 3.2. (In other words, each G_i is either sharp or singular.)

By Lemma 4.1, we conclude that $p(G_i, G) = o(1)$ for every singular G_i. The Induced Removal Lemma of Alon, Fischer, Krivelevich and Szegedy [1] implies that we can change $o(n^2)$ edges in G and destroy all singular graphs and, additionally, preserve the property $p(\overline{K}_3, G) = 0$. Since changing $o(n^2)$ edges affects each $p(H,G)$ by $o(1)$, we can assume that G itself does not contain any singular induced subgraph. This means the following.

Claim 4.2. *For any subset $U \subseteq V(G)$ with at most 6 vertices there is a partition $U = U_0 \cup \cdots \cup U_4$ such that $G[U] = C_5(\!(U_0, \ldots, U_4)\!)$.* □

By the Induced Removal Lemma we can additionally assume that either the density of C_5 in G is $\Omega(1)$ or G does not have a single induced 5-cycle. In fact, the first alternative necessarily holds.

Claim 4.3. $p(C_5, G) = \Omega(1)$.

Proof of Claim. Suppose on the contrary that G does not contain an induced pentagon. Take a longest induced path $(u_1, \ldots, u_s)$. By Claim 4.2, we have $s \leqslant 4$. Also, $s \geqslant 3$ for otherwise G is the union of disjoint cliques, of which there can be at most two because the independence number is at most 2; but then the K_k-density is at least $1/2^{k-1} + o(1)$, contradicting the extremality of G. Take any vertex $x \in V(G)$. The set $X = \{u_1, \ldots, u_s, x\}$ induces some expansion of C_5 by Claim 4.2. Since we do not have an induced pentagon and s is maximal, X in fact induces an expansion of the s-vertex path P_s. Let $\{x, u_i\}$ be the part of this expansion that contains x. We assign this vertex x to the ith part, thus obtaining a partition $V(G) = U_1 \cup \cdots \cup U_s$.

We have in fact $G = P_s(\!(U_1, \ldots, U_s)\!)$. Indeed, if we take any two vertices x, y and apply Claim 4.2 to $\{u_1, \ldots, u_s, x, y\}$, we see that the adjacency relation between x and y in G is exactly as dictated by the expansion.

Thus we can make G into the union of two disjoint cliques by removing some edges and without creating $\overline{K}_3$. This cannot increase the density of K_k and, as we have just seen, leads to a contradiction. □

Claim 4.4. *Let $u_0, \ldots, u_4 \in V(G)$ induce a pentagon in G with $\{u_i, u_{i+1}\} \in E(G)$ for $i \in \mathbb{Z}_5$, where $\mathbb{Z}_5$ denotes the residues modulo 5. Let $U = \{u_0, \ldots, u_4\}$. Then, for every $u \in V(G) \setminus U$, there is $j \in \mathbb{Z}_5$ such that $\{u, u_i\} \in E(G)$ if and only if $i \in \{j-1, j, j+1\}$.*

Proof of Claim. Take the partition $U \cup \{u\} = U_0 \cup \cdots \cup U_4$ given by Claim 4.2. For every distinct $i, j \in \mathbb{Z}_5$, the vertices u_i and u_j have different neighbourhoods in $U \setminus \{u_i, u_j\}$, so they belong to different parts. Without loss of generality assume that $u_i \in U_i$ for each i. If the vertex u belongs to U_j, then the neighbours of u are u_{j-1}, u_j, u_{j+1}, as required. □

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Fix some $u_0,\ldots,u_4 \in V(G)$ that induce C_5 with $\{u_i,u_{i+1}\} \in E(G)$ for $i \in \mathbb{Z}_5$; such vertices exist by Claim 4.3. Let $U = \{u_0,\ldots,u_4\}$. Claim 4.4 gives a partition of $V(G)$ into 5 parts $U_0,\ldots,U_4$ where we classify vertices according to their neighbourhoods in U:

$$U_i = \{u_i\} \cup \{u \in V(G) \setminus U : \Gamma_G(u) \cap U = \{u_{i-1},u_i,u_{i+1}\}\}. \tag{4.1}$$

Claim 4.5. *For every $i \in \mathbb{Z}_5$ the induced subgraph $G[U_i]$ is complete.*

Proof of Claim. By symmetry, let $i = 0$. Take any distinct $u,v \in U_0$. By the definition of U_0, we have that $v,u_1,\ldots,u_4$ induce a 5-cycle. Also, u is adjacent to u_4 and u_1. By Claim 4.4 we conclude that $\{u,v\} \in E(G)$. □

Claim 4.6. *Let $i,j \in \mathbb{Z}_5$ be distinct and let $v_i \in U_i$ and $v_j \in U_j$ be arbitrary. Then v_i and v_j are adjacent if and only if $i = j \pm 1$.*

Proof of Claim. Assume that $v_i \neq u_i$ and $v_j \neq u_j$, for otherwise we are done by (4.1). First, let $i = 0$ and $j = 1$. The vertex $v_1 \in U_1$ is adjacent to the vertices u_1 and u_2 but not to u_3 of the induced 5-cycle on $v_0,u_1,\ldots,u_4$. By Claim 4.4, v_0 and v_1 are adjacent. Next, let $i = 0$ and $j = 2$. The vertex $v_2 \in U_2$ is adjacent to the vertices u_1, u_2 and u_3 of the induced 5-cycle on $v_0,u_1,\ldots,u_4$. By Claim 4.4, v_0 and v_2 are not adjacent. This covers all the cases of Claim 4.6 up to symmetry. □

Thus we see that G is exactly an expansion of C_5 with parts $U_0,\ldots,U_4$. Choose an arbitrary subsequence of n such that each $|U_i|/n$ approaches some limit α_i. It remains to show that each $\alpha_i = \frac{1}{5}$. One approach to showing this would be to argue that an explicit degree-k polynomial, that approximates $p(K_k,G)$, has the unique minimizer $(\frac{1}{5},\ldots,\frac{1}{5})$. This approach seems rather messy.

However, there is another way to get the desired conclusion: namely, by applying Lemma 3.1. Let us consider type τ_6 which is obtained by labelling the vertices of the 3-edge path by $3,1,2,4$ as we go along the path. (It is `4:121324` in *Flagmatic*'s notation.) There are exactly 8 non-isomorphic τ_6-flags on 5 vertices, which we denote by $F_1^{\tau_6},\ldots,F_8^{\tau_6}$. Three of these flags, labelled by *Flagmatic* as $F_6^{\tau_6},F_7^{\tau_6},F_8^{\tau_6}$, do not embed into any expansion of C_5 when we view them as unlabelled graphs. Thus, by Claim 4.2, we have that $p(F_i^{\tau_6},(G,\phi)) = 0$ for every ϕ and $i = 6,7,8$. Every embedding ψ of τ_6 into $G = C_5(\!(U_0,\ldots,U_4)\!)$ uses four different parts. Note that each part has size $\Omega(n)$ by Claim 4.3. When we form the vector $\mathbf{x}_\psi$ as in (3.2), we have to count the number of τ_6-flags on 5 vertices that we obtain over all $n-4$ choices of an unlabelled vertex $u \in V(G) \setminus \psi([4])$. Up to symmetry, there are only 5 different choices of u depending on which part U_i contains u. Each i contributes either $|U_i|$ or $|U_i|-1$ to some coordinate of $\mathbf{x}_\psi$ and different i contribute to different coordinates. Thus, up to a permutation of coordinates, $\mathbf{x}_\psi$ is equal to $(\alpha_1 n + o(n),\ldots,\alpha_5 n + o(n),0,0,0)$. It follows from a version of Lemma 3.1 that some permutation of $(\alpha_1,\ldots,\alpha_5,0,0,0)$ is a zero eigenvector of Q^{τ_6}. On the other hand, Lemma 3.1 implies that $(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},0,0,0)$ is a forced zero eigenvector of Q^{τ_6} (that comes from analysing our flag algebra proof on the uniform expansion of C_5).

Moreover, the scripts verify that the rank of the rational 8×8-matrix Q^{τ_6} is exactly 7 (so its null-space has dimension 1). Since $\alpha_1+\cdots+\alpha_5=1$, we conclude that each $\alpha_i=\frac{1}{5}$. This proves the desired stability property (that is, it contradicts our assumption that each graph G is $\varepsilon\binom{n}{2}$-far from a uniform expansion of C_5).

4.2. Cases $k=3$ and $4\leqslant l\leqslant 7$

The scripts verify that the number of sharp graphs and the number of those order-N graphs that embed into $\overline{T}_{l-1}(n)$ with one edge added are the same: namely, 10, 20, 33, and 55 graphs when (l,N) is respectively $(4,5)$, $(5,6)$, $(6,7)$, and $(7,8)$. Thus these lists coincide by Lemma 3.3. By applying the Induced Removal Lemma, we can assume that G does not contain any non-sharp N-vertex graph. In other words, the following holds.

Claim 4.7. *Every subset $U\subseteq G$ with at most N vertices admits a partition $U=U_1\cup\cdots\cup U_{l-1}$ such that $G[U]$ is equal to $\overline{K}_{l-1}(\!(U_1,\dots,U_{l-1})\!)$ with at most one added edge.* □

Define an equivalence relation $\sim$ on vertices of G, where $x\sim y$ if and only if $x=y$ or there is a chain of intersecting triangles in G that connects x to y. Each equivalence class is a clique by Claim 4.7 as $N\geqslant 5$. Let U_0 be the union of equivalence classes of size 1, that is, U_0 consists of those vertices that are not contained in a triangle. Since G does not contain $\overline{K}_l$, we have that $|U_0|+1$ is at most the Ramsey number $R(3,l)$. Remove U_0 from $V(G)$ as this will not affect the stability property.

Let $U_1,\dots,U_s$ be the remaining $\sim$-equivalence classes. Each U_i spans a clique and has at least three vertices.

Let us derive a contradiction by assuming that some U_i sends at least two edges to $V(G)\setminus U_i$, say $\{w,x\}$ and $\{y,z\}$ with $w,y\in U_i$. Take some 5-set $X\supseteq\{w,x,y,z\}$ with $|U_i\cap X|=3$. Then $G[X]$ is a subgraph that contains at least one triangle (on $X\cap U_i$) plus at least two extra edges incident to it. By Claim 4.7, $\{w,x,y,z\}$ spans a clique, which contradicts the fact that $x,z\notin U_i$.

Thus by removing at most one vertex from each U_i, we can eliminate all edges across the parts. As U_i is still non-empty, we have that $s<l$ by the $\overline{K}_l$-freeness of G.

A simple optimization shows that, in fact, $s=l-1$ and each U_i has $(\frac{1}{l-1}+o(1))n$ vertices. This proves the stability property for $f(n,3,l)$ with $4\leqslant l\leqslant 7$.

4.3. Cases $(k,l)=(6,3)$ or $(7,3)$

Here $N=7$ if $k=6$ and $N=8$ if $k=7$. Let G be a K_3-free graph of large order n with $p(\overline{K}_k,G)=c_{k,l}+o(1)$. Recall that, for notational convenience, we prefer to work with the graph complements in these cases. Also note that an expansion corresponds to a blow-up of a graph when we look at the complements.

The scripts verify that the numbers of the sharp graphs and of those N-vertex graphs that appear in a blow-up of the Clebsch graph are the same (namely, 86 graphs for $(k,N)=(6,7)$ and 232 graphs for $(k,N)=(7,8)$). So these lists coincide by Lemma 3.2. As before, by applying the Induced Removal Lemma we can additionally assume that G has the following property.

Claim 4.8. *No singular graph is an induced subgraph of G, that is, every induced N-vertex subgraph of G is a blow-up of the Clebsch graph L.* □

We need some further definitions before we can proceed with the proof.

Let $X \subseteq V(H)$ be a subset of vertices in some graph H. Two vertices $x, y \in V(H)$ are *X-equivalent*, denoted as $x \sim_X y$, if $\Gamma_H(x) \cap X = \Gamma_H(y) \cap X$, that is, if they are adjacent to the same vertices of X. Note that we allow x or y to belong to X and it is possible that some $x \in X$ and $y \notin X$ are X-equivalent. Clearly, $\sim_X$ is an equivalence relation. Let $[x]_X = \{y \in V(H) : y \sim_X x\}$ denote the equivalence class of x.

Let C_5' be obtained from the 5-cycle on $x_1, \dots, x_5$ by adding an extra isolated vertex x_0. Let ϕ be a strong homomorphism from C_5' to the Clebsch graph L that maps the isolated vertex to 00000 and maps the remaining vertices to the cyclic shifts of 00011. This ϕ is injective and its image is

$$X = \{00000, 00011, 01100, 10001, 00110, 11000\}. \tag{4.2}$$

Claim 4.9. *Let ϕ and X be as above. Then the following claims hold.*

(1) *Let H be obtained from C_5' by removing at most one vertex. Then, for every strong homomorphism ψ from H to L, there is an automorphism σ of L such that $\psi = \sigma \circ \phi|_{V(H)}$. (In particular, ψ is injective.)*

(2) *The X-equivalence relation is trivial on $V(L)$, that is, $x \sim_X y$ if and only if $x = y$.*

(3) *For every two distinct vertices $x, y \in V(L)$ there is $z \in X$ such that, for $Z = X \setminus \{z\}$, we have $x \not\sim_Z y$ and the bipartite subgraph of L induced by $[x]_Z$ and $[y]_Z$ is either complete or empty. Also, for every $x \in V(L)$ there is $z \in X$ such that $[x]_{X\setminus\{z\}} = \{x\}$.*

Proof of Claim. First, let $H = C_5'$ or let H be obtained from C_5' by removing a vertex of degree 2, say $H = C_5' - x_5$. Up to an automorphism of L, each strong homomorphism ψ from H to L is as follows. By the vertex-transitivity of L, we can assume that $\psi(x_0) = 00000$. Thus every other vertex of H has to be mapped to a sequence of weight 2. (No other vertex can be mapped to 00000 because x_0 is the unique isolated vertex of H.) By permuting indices $1, \dots, 5$ (which gives an automorphism of L), we can assume that $\psi(x_2) = 00011$. Next, up to a permutation of indices $1, 2, 3$, we can assume that $\psi(x_3) = 01100$ and $\psi(x_1) = 11000$. (Note that $\psi(x_3) \neq \psi(x_1)$ because of $x_4 \in \Gamma_H(x_3) \setminus \Gamma_H(x_1)$.) Up to a transposition of 4 and 5, we can also assume that $\psi(x_4) = 10001$. Also, if $H = C_5'$, then $\psi(x_5) = 00110$ is uniquely determined. Thus $\psi = \phi|_{V(H)}$ up an automorphism of L. The remaining case $H \cong C_5$ can be done by a similar analysis, finishing part (1) of the claim.

Every 5-sequence of weight 0, 4 and 2 sends respectively 0, 3, and 1–2 edges to X, so X distinguishes vertices of different weight. An easy case analysis for each possible weight shows part (2) of the claim. For example, 00011 is identified among all weight-2 sequences already by the set $\{01100, 11000\} \subseteq X$.

In order to establish part (3), we use the fact that any cyclic permutation or the reversal of the indices preserves X. Up to these symmetries, there are 12 different unordered pairs x, y to check. The following table lists a vertex z that establishes the claim and the

Z-equivalence classes of x and y, where $Z = X \setminus \{z\}$:

x	y	z	$[x]_Z$	$[y]_Z$
00000	00011	10001	$\{00000, 01010\}$	$\{00011\}$
00000	00101	00011	$\{00000, 10100\}$	$\{00101\}$
00000	01111	00011	$\{00000, 10100\}$	$\{01111\}$
00011	01100	00110	$\{00011\}$	$\{01100\}$
00011	00110	00011	$\{00011\}$	$\{00110\}$
00011	00101	00011	$\{00011\}$	$\{00101\}$
00011	01010	00110	$\{00011\}$	$\{01010\}$
00011	10100	10001	$\{00011\}$	$\{01100, 10100\}$
00101	01010	00110	$\{00101\}$	$\{01010\}$
00101	01001	00110	$\{00101\}$	$\{00000, 01001\}$
01111	10111	00011	$\{01111\}$	$\{10111\}$
01111	11011	00011	$\{01111\}$	$\{11011\}$

Alternatively, the *Mathematica* notebook `Clebsch.nb`, available from the ancillary folder of [18], verifies the existence of z by the brute-force enumeration of all cases. The remaining statement of part (3) is easy to verify: for example, we can take z to be 00000, 00011, 00011, and 00011 if x is respectively 00000, 00011, 00101, and 01111. This finishes the proof of the claim. □

Claim 4.10. $P(C'_5, G) = \Omega(n^6)$.

Proof of Claim. Suppose on the contrary that $p(C'_5, G) = o(1)$. By the Induced Removal Lemma, we can additionally assume that $P(C'_5, G) = 0$. We let *Flagmatic* prove some lower bound on the density of $\overline{K}_k$ given that both K_3 and C'_5 are forbidden. The obtained bound (with the certificates `63a.js` and `73a.js`) is strictly larger than $c_{k,3}$. This contradicts $p(\overline{K}_k, G) = c_{k,3} + o(1)$ for all large n, proving the claim. □

Fix one embedding ψ of C'_5 into G. Let us view C'_5 as the subgraph of L induced by $X \subseteq V(L)$, where $X = V(C'_5)$ is defined by (4.2). Thus $\psi : X \to V(G)$. Let $Y = \psi(X)$.

Claim 4.11. *For every $y \in V(G)$ there is a (unique) vertex $x \in V(L)$ whose adjacencies to X match those of y to Y, that is, $\psi(\Gamma_L(x) \cap X) = \Gamma_G(y) \cap Y$.*

Proof of Claim. The subgraph $H = G[Y \cup \{y\}]$, which has at most $7 \leqslant N$ vertices, admits an embedding into a blow-up of the Clebsch graph by Claim 4.8. This implies that there is a strong homomorphism ξ from H into L. By part (1) of Claim 4.9, we can assume that the composition $\xi \circ \psi$ is the identity map $\mathrm{Id}_X : X \to X$. Now, $x = \xi(y)$ satisfies the claim. The uniqueness of x follows from part (2) of Claim 4.9. □

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Thus each $y \in V(G)$ falls into one of at most sixteen Y-equivalence classes that are naturally labelled as U_x for $x \in V(L)$, where $x = x(y)$ is given by Claim 4.11. In particular, for each $x \in X$, the part containing $\psi(x)$ is labelled by U_x.

Claim 4.12. *For every adjacent $x, y \in V(L)$, the induced bipartite subgraph $G[U_x, U_y]$ is complete. For non-adjacent $x, y \in V(L)$ the induced bipartite subgraph $G[U_x, U_y]$ is empty. (In particular, each part U_x forms an independent set.)*

Proof of Claim. Let $x, y \in V(L)$ be adjacent. Let $x' \in U_x$ and $y' \in U_y$ be arbitrary.

Pick $z \in X$ given by part (3) of Claim 4.9 and let $Z = X \setminus \{z\}$. The induced subgraph $H = G[\psi(Z) \cup \{x', y'\}]$ has at most $7 \leqslant N$ vertices. By Claim 4.8, H admits a strong homomorphism ξ to L. By part (1) of Claim 4.9, we can assume that $\xi \circ \psi|_Z$ is the identity on Z. Then $\xi(x') \in [x]_Z$ and $\xi(y') \in [y]_Z$. However, the bipartite subgraph induced by $[x]_Z$ and $[y]_Z$ in L is complete by the choice of z (since $\{x, y\} \in E(L)$). Thus x' and y' are adjacent. The second part of the claim follows in a similar manner. □

Thus we know that G is a blow-up of L with parts $U_{00000}, \dots, U_{11110}$. It remains to argue that each part U_x has $(\frac{1}{16} + o(1))n$ vertices.

Let $k = 7$. We proceed very similarly as we did at the end of Section 4.1, so we will be rather brief. We consider the type τ_{37}, which is a labelling of C_5'. It is `6:1213243545` in *Flagmatic*'s notation. There are 22 τ_{37}-flags on 7 vertices. By Claim 4.10, there are $\Omega(n^6)$ embeddings ψ of τ_{37} into G. By parts (1)–(2) of Claim 4.9, each obtained vector $\mathbf{x}_\psi$ consists of sixteen entries $|U_x| + O(1)$, one for each $x \in V(L)$, and six zeros. On the other hand, the script `73.sage` verifies that the 22×22-matrix $Q^{\tau_{37}}$ from our flag algebra proof has rank 21. Moreover, by Lemma 3.1, the matrix $Q^{\tau_{37}}$ has one forced zero eigenvector consisting of sixteen entries equal to $1/16$ and six entries equal to 0. It follows in the same way as in Section 4.1 that each U_x has size $(\frac{1}{16} + o(1))n$.

Let $k = 6$. We consider the type τ_{11} that consists of the 3-edge path plus an isolated vertex (it is `5:121324` in *Flagmatic*'s notation). Since C_5' contains τ_{11} as a subgraph, Claim 4.10 implies that there are $\Omega(n^5)$ embeddings ξ of τ_{11} into G. Fix an embedding ξ such that its image avoids all parts U_x of size $o(n)$. (A typical ξ has this property.) By part (1) of Claim 4.9, we can relabel the parts U_x so that the image Y of ξ has exactly one vertex in each of the parts $U_{00000}, U_{00011}, U_{01100}, U_{10001}, U_{00110}$. The Y-equivalence relation on G makes each part U_x into a separate equivalence class except for the following three Y-equivalence classes:

$$U_{00000} \cup U_{00101}, \quad U_{00011} \cup U_{10010}, \quad U_{00110} \cup U_{01010}. \tag{4.3}$$

On the other hand, the 16×16-matrix $Q^{\tau_{11}}$ of our solution has rank 15. Moreover, it has one forced zero eigenvector that has ten entries equal to $1/16$, three entries equal to $2/16$, and three entries equal to 0 by Lemma 3.1. (This follows from (4.3) when applied to the uniform blow-up of L.) This implies that each of the ten parts that do not appear in (4.3) has size $(\frac{1}{16} + o(1))n$ while each of the three sets in (4.3) has $(\frac{2}{16} + o(1))n$ vertices.

The graph G has other copies of τ_{11}, for example via

$$U_{10100}, \quad U_{01111}, \quad U_{11000}, \quad U_{10111}, \quad U_{11101}.$$

The adjacency pattern to these $(\frac{n}{16}+o(n))^5$ copies τ_{11} uniquely identifies parts U_{00000}, U_{00011}, and U_{01010}. As before, we conclude that that each of these parts has size $(\frac{1}{16}+o(1))n$. This is enough to determine the sizes of all six parts that appear in (4.3). Thus G is $o(n^2)$-close to a uniform blow-up of L. The stability property has been established.

Remark. By running everything with $N = 8$ (see the script `63.sage` and the certificate `63b.sage`), it is possible to shorten the 'human' part of the proof of Theorem 1.2 for $(k,l) = (6,3)$. (For example, part (3) of Claim 4.9 and the argument around (4.3) become redundant.) However, we believe that the ability to solve this case within the universe of 7-vertex graphs justifies the extra work, as the ideas introduced for this task may be useful for other problems.

5. Exact result

First, we present a rather general Theorem 5.1 and then verify in Section 5.2 that it implies Theorem 1.3. Theorem 5.1 could in principle be strengthened in various ways but we state only the current version as it suffices for all the cases that we need.

5.1. A general result

We need to give some definitions first, given an arbitrary pair (k,l) and any admissible graph F with vertex set $[m]$.

We say that F is a *stability graph* for (k,l) if for every $\varepsilon > 0$ there are n_0 and $\delta > 0$ such that the following holds. Let G be an arbitrary graph such that $n = v(G) \geqslant n_0$, $\alpha(G) < l$, and $p(K_k, G) \leqslant c_{k,l} + \delta$. Then there is a partition $V(G) = V_1 \cup \cdots \cup V_m$ such that the part sizes differ at most by 1 and

$$|E(F(\!(V_1,\ldots,V_m)\!)) \bigtriangleup E(G)| \leqslant \varepsilon \binom{n}{2}.$$

In other words, F is a stability graph for (k,l) if every large almost extremal graph for the $f(n,k,l)$-problem is $o(n^2)$-close in the edit distance to a uniform expansion of F. Clearly, this property is preserved if we replace F by an isomorphic graph or by $F(\!(U_1,\ldots,U_m)\!)$ with $|U_1| = \cdots = |U_m| > 0$.

We give some further definitions related to the graph F, which will be illustrated in the next paragraph. Let us call a set of vertices $X \subseteq [m]$ *legal* if $F - X$ does not contain $\overline{K}_{l-1}$. Let the *gradient* $\mathrm{grad}(X)$ of X be the probability, when we independently pick $k-1$ uniformly distributed vertices $x_1,\ldots,x_{k-1} \in [m]$, that all belong to X and for every $i,j \in [k-1]$ the vertices x_i and x_j are adjacent or equal. Let us call a stability graph F *strict* if $\mathrm{grad}(X) > c_{k,l}$ for every legal X for which there is no $i \in [m]$ with $X = \hat{\Gamma}_F(i)$. Recall that

$$\hat{\Gamma}_F(i) = \{i\} \cup \{j \in V(F) : \{i,j\} \in E(F)\}$$

is the closed neighbourhood of i.

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The above definitions are motivated by the addition of a new vertex x to $F' = F(\!(V_1,\dots,V_m)\!)$ with $|V_1| = \cdots = |V_m| = n/m$ so that x is adjacent to precisely $\cup_{i\in X} V_i$. The new graph is still $\overline{K}_l$-free if and only if X is legal. Also, the number of k-cliques that contain x is $\mathrm{grad}(X)\binom{n}{k-1} + O(n^{k-2})$. If $X = \hat{\Gamma}_F(i)$, then adding x is the same as enlarging the part V_i by one vertex and, if F is a stability graph, then the number of k-cliques increases by $(c_{k,l}+o(1))\binom{n}{k-1}$; see Claim 5.4 below. Thus F is strict if the number of the new k-cliques is by $\Omega(n^{k-1})$ larger for every other legal X.

Theorem 5.1. *Let a pair (k,l) admit a stability graph F which is strict. Then there is n_0 such that every graph G with $n = v(G) \geqslant n_0$, $\alpha(G) < l$, and $P(K_k, G) = f(n,k,l)$ contains an expansion of F as a spanning subgraph.*

Proof. Let $V(F) = [m]$. Choose positive constants

$$\varepsilon_2 \gg \varepsilon_1 \gg \varepsilon_0 \gg 1/n_0 > 0, \tag{5.1}$$

each being sufficiently small, depending on the previous ones. We show that n_0 satisfies the conclusion of the theorem.

Since there are finitely many different subsets $X \subseteq [m]$, we can assume that

$$\mathrm{grad}(X) \geqslant c_{k,l} + 2km\varepsilon_2 \tag{5.2}$$

for every legal X that is not the closed neighbourhood of some vertex. Also, we may assume that for every $n \geqslant n_0$ we have

$$f(n,k,l) \geqslant (c_{k,l} - \varepsilon_0)\binom{n}{k}, \tag{5.3}$$

Let G be an arbitrary $f(n,k,l)$-extremal graph with $n \geqslant n_0$ vertices. Let $V = V(G)$. Since $f(n,k,l) = (c_{k,l}+o(1))\binom{n}{k}$ by (1.4) and F is a stability graph, we have that

$$|E(G) \triangle E(F')| \leqslant \varepsilon_0 \binom{n}{2} \tag{5.4}$$

for some uniform expansion $F' = F(\!(V_1,\dots,V_m)\!)$ on V.

We are going to modify the partition $V = V_1 \cup \cdots \cup V_m$. Given a current partition, let $B = E(F') \setminus E(G)$ and $S = E(G) \setminus E(F')$. We call the pairs in B *bad* and those in S *superfluous.*

Iteratively repeat the following operation as long as possible (updating $V_1,\dots,V_m$, F', B and S as we proceed): if we can move some vertex x of F' to another part and decrease the number of bad pairs by least $\varepsilon_1 n$, then we perform this move.

Since we had initially at most $\varepsilon_0\binom{n}{2}$ bad pairs, we perform at most $\varepsilon_0\binom{n}{2}/\varepsilon_1 n < \varepsilon_1 n/4$ moves. Let $V_1,\dots,V_m,F',B,S$ refer to the final configuration. What we have achieved is that for every vertex $x \in V_j$ and every $i \in [m]$

$$|\Gamma_{\overline{G}}(x) \cap \cup_{h\in\hat{\Gamma}_F(i)} V_h| > |\Gamma_{\overline{G}}(x) \cap \cup_{h\in\hat{\Gamma}_F(j)} V_h| - \varepsilon_1 n. \tag{5.5}$$

Also, the current expansion F' is not far from being uniform:

$$\left| |V_i| - \frac{n}{m} \right| \leqslant \varepsilon_1 n, \quad \text{for all } i \in [m]. \tag{5.6}$$

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In addition, we have

$$|E(G) \triangle E(F')| \leqslant \varepsilon_0 \binom{n}{2} + \frac{\varepsilon_1 n}{4} n < \varepsilon_1 \binom{n}{2}. \tag{5.7}$$

Claim 5.2. *The removal of any edge $\{x, y\}$ from F' creates $\overline{K}_l$.*

Proof of Claim. First, suppose that x and y belong the same part V_i. Partition $V_i = X \cup Y$ into two almost equal parts so that $x \in X$ and $y \in Y$. Let F'' be obtained from F' by removing all edges between X and Y. By (5.7) we have rather roughly that

$$\begin{aligned} P(K_k, F'') &\leqslant P(K_k, F') - \frac{1}{2}\binom{|V_i|}{k} \\ &\leqslant P(K_k, G) + \varepsilon_1 \binom{n}{2}\binom{n-2}{k-2} - \frac{(n/m)^k}{4\,k!} \; < \; P(K_k, G). \end{aligned}$$

By the extremality of G, we conclude that F'' contains an independent set I of size l. Clearly, I has exactly one vertex in each X and Y. Since any permutation of the vertices of X (and of Y) is an automorphism of F'', we can assume that $x, y \in I$, as required.

If x, y come from different parts V_i and V_j, then a similar argument works where we remove all edges of F' between V_i and V_j. □

Claim 5.3. *For every bad pair $\{x_1, x_2\} \in B$ we have $d_S(x_1) + d_S(x_2) \geqslant n/(3m^{l-2})$.*

Proof of Claim. Let $x_1 \in V_{i_1}$ and $x_2 \in V_{i_2}$. By Claim 5.2, $F' - \{x_1, x_2\}$ has $\overline{K}_l$ as a subgraph. This means that we can find distinct $i_3, \ldots, i_l \in [m] \setminus \{i_1, i_2\}$ such that no pair of vertices $i_1, \ldots, i_l$ except $\{i_1, i_2\}$ is adjacent in F.

For every choice of $\mathbf{x} = (x_3, \ldots, x_l)$ such that $x_j \in V_{i_j}$, at least one pair $\{x_j, x_h\}$ with $1 \leqslant j < h \leqslant l$ is superfluous (for otherwise we get an independent set of size l in G). It is impossible that both j and h are at least 3 for at least half of the choices of $\mathbf{x}$: otherwise, as each superfluous pair is overcounted at most n^{l-4} times, we would have that

$$|S| \geqslant \frac{1}{2}\left(\left(\frac{1}{m} - \varepsilon_1\right) n\right)^{l-2} \frac{1}{n^{l-4}} > \varepsilon_1 \binom{n}{2},$$

which contradicts (5.7). Thus, for at least half of the choices of $\mathbf{x}$ there is a superfluous pair intersecting $\{x_1, x_2\}$. Since each such pair is over-counted at most n^{l-3} times, we obtain that

$$d_S(x_1) + d_S(x_2) \geqslant \frac{1}{2}\left(\left(\frac{1}{m} - \varepsilon_1\right) n\right)^{l-2} \times \frac{1}{n^{l-3}},$$

which implies the claim provided that $\varepsilon_1 = \varepsilon_1(m, l)$ is sufficiently small. □

Let K_k^1 be the flag obtained from K_k by labelling one vertex. Thus $P(K_k^1, (H, x))$ is the number of k-cliques in a graph H that contain $x \in V(H)$.

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Claim 5.4. *For any two vertices $x, y \in V$, we have*

$$|P(K_k^1,(G,x)) - P(K_k^1,(G,y))| \leqslant \binom{n-2}{k-2}.$$

Proof of Claim. If we delete x but add a clone y' of y (putting an edge between y and y'), then we do not create a copy of $\overline{K}_l$ while the number of k-cliques changes by at most $P(K_k^1,(G,y)) - P(K_k^1,(G,x)) + \binom{n-2}{k-2}$. Since G is extremal, this has to be non-negative. By swapping the roles of x and y, we derive the claim. □

Claim 5.4 and the extremality of G imply that for every $x \in V(G)$ we have

$$P(K_k^1,(G,x)) \leqslant \frac{k\,f(n,k,l)}{n} + \binom{n-2}{k-2}, \tag{5.8}$$

for otherwise

$$P(K_k,G) = \frac{1}{k}\sum_{y\in V(G)} P(K_k^1,(G,y)) > \frac{n}{k}\left(P(K_k^1,(G,x)) - \binom{n-2}{k-2}\right)$$

is too large.

Suppose that B is not empty, for otherwise we are done: G contains F' as a spanning subgraph.

By Claim 5.3, there is a vertex x whose S-degree is at least $n/(6m^{l-2})$. Define

$$X = \{i \in [m] : |V_i \setminus \Gamma_G(x)| \leqslant \varepsilon_2 n\}.$$

Claim 5.5. *X is legal.*

Proof of Claim. Suppose that this is false. Then there are distinct $i_1,\ldots,i_{l-1} \in [m]\setminus X$ that span $\overline{K}_{l-1}$ in F. Let $x_l = x$. For every choice of $(x_1,\ldots,x_{l-1})$ with $x_j \in \Gamma_{\overline{G}}(x)\cap V_{i_j}$, the $(l-1)$-set $\{x_1,\ldots,x_{l-1}\}$ has to span at least one edge in G (otherwise together with x it induces $\overline{K}_l$). This edge is necessarily in S. On the other hand, any pair in S is over-counted at most n^{l-3} times. Thus $|S| \geqslant (\varepsilon_2 n)^{l-1}/n^{l-3}$, contradicting (5.7). □

Claim 5.6. *There is $i \in [m]$ such that $X = \hat{\Gamma}_F(i)$.*

Proof of Claim. Suppose that the claim is false. As F is strict, we have that (5.2) holds. Let F'' be obtained from F' by changing edges at x so that the new neighbourhood of x is exactly $Y = (\cup_{j\in X} V_j)\setminus\{x\}$. The number of K_k-subgraphs in F'' via x is

$$P(K_k^1,(F'',x)) \geqslant (c_{k,l} + 2km\varepsilon_2)\binom{n-1}{k-1} - \varepsilon_1 mn\binom{n-2}{k-2} + O(1/n). \tag{5.9}$$

(Here, the middle term corresponds to the fact that, by (5.6), we can make F' into a uniform expansion by moving at most $\varepsilon_1 mn$ vertices between parts.) On the other hand, G and F' differ in at most $\varepsilon_1\binom{n}{2}$ edges by (5.7) while at most $\varepsilon_2 mn$ edges between x and

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Y can be missing in G by the definition of X. Thus, rather roughly,

$$P(K_k^1,(G,x)) \geqslant P(K_k^1,(F'',x)) - \varepsilon_1 \binom{n}{2}\binom{n-3}{k-3} - \varepsilon_2 mn\binom{n-2}{k-2}.$$

However, this inequality contradicts (5.3), (5.8) and (5.9) by our choice of the constants in (5.1). □

Fix the i that is returned by Claim 5.6.

Claim 5.7. $d_B(x) < 2\varepsilon_1 n$.

Proof of Claim. Suppose on the contrary that $d_B(x) \geqslant 2\varepsilon_1 n$.

Consider moving x to V_i. (The following statements are also true if x is already in V_i.) By (5.5), the new number of bad pairs at x would be at least $d_B(x) - \varepsilon_1 n > \varepsilon_2 mn$ and each one would connect x to $\cup_{h\in\hat{\Gamma}_F(i)} V_h$.

Hence, in the graph G, x has more than $\varepsilon_2 n$ non-neighbours in some V_h with $h \in \hat{\Gamma}_F(i)$, meaning that $X \neq \hat{\Gamma}_F(i)$ and contradicting Claim 5.6. □

Let $x \in V_j$ (where possibly $j = i$). Fix a vertex $y \in V_j$ that has at most the average number of superfluous edges over the vertices of V_j. We have

$$d_S(y) \leqslant \frac{|E(G) \triangle E(F')|}{|V_j|} \leqslant \frac{\varepsilon_1\binom{n}{2}}{(1/m - \varepsilon_1)n} \leqslant \varepsilon_1 mn.$$

This and Claim 5.7 imply that

$$|\Gamma_G(y) \setminus \Gamma_G(x)| \leqslant d_S(y) + d_B(x) \leqslant \varepsilon_1(m+2)n.$$

On the other hand, x sends at least $d_S(x)/m \geqslant n/(6m^{l-1})$ superfluous edges to some part V_h. By (5.7), all but at most $\varepsilon_1\binom{n}{2}$ pairs of V_h are edges of G. Thus the superfluous edges at x create at least

$$\binom{n/(6m^{l-1})}{k-1} - \varepsilon_1\binom{n}{2}\binom{|V_h|-2}{k-3} > (2m+5)\varepsilon_1 n\binom{n-2}{k-2}$$

copies of K_k through x. We conclude that

$$\begin{aligned} P(K_k^1,(G,x)) - P(K_k^1,(G,y)) &> (2m+5)\varepsilon_1 n\binom{n-2}{k-2} - 2\varepsilon_1(m+2)n\binom{n-2}{k-2} \\ &> \binom{n-2}{k-2}, \end{aligned}$$

contradicting Claim 5.4. This final contradiction to $B \neq \emptyset$ proves Theorem 5.1. □

5.2. Verifying Theorem 1.3

Theorems 1.2 and 5.1 imply Theorem 1.3 provided we can verify that the appropriately defined F is strict. The cases $F = \overline{K}_{l-1}$ or C_5 are straightforward to verify. Namely, every legal set X that is not a closed neighbourhood of a vertex has at least 2 vertices for

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$\overline{K}_{l-1}$ and at least 4 vertices for C_5; any such X contains some closed neighbourhood as a proper subset and has a strictly larger gradient.

Let $(k,l) = (6,3)$ or $(7,3)$. Let us check that $\overline{L}$ satisfies Theorem 5.1. We already know by Theorem 1.2 that $\overline{L}$ is a stability graph for (k,l). Let $X \subseteq V(L)$ be any legal set, meaning that $Y = V(L) \setminus X$ spans no edge in L. By the vertex-transitivity of L, we can assume that $00000 \in Y$. Thus all other sequences in Y have weight 2 and, furthermore, no two such sequences can have 1s in disjoint positions. If $|Y| = 5$, then up to a symmetry the only possibility is $Y = \{00000, 00011, 00101, 01001, 10001\}$ but then X is precisely the closed neighbourhood of 11110 in $\overline{L}$. If $|Y| = 4$ and X does not contain a closed neighbourhood, then, up to an automorphism of L, we have $Y = \{00000, 00011, 00101, 00110\}$. The script `Clebsch.nb` shows that, if $k = 6$, then $\mathrm{grad}(X) = 1437/2^{16} > c_{6,3}$ and if $k = 7$, then $\mathrm{grad}(X) = 14503/2^{21} > c_{7,3}$. Every other Y is a subset of one of the sets that we have already considered and the gradient of $X = V(L) \setminus Y$ is strictly larger than we had before. Thus $\overline{L}$ is strict. This finishes the remaining cases of Theorem 1.3.

6. Concluding remarks

Let us call a graph G *extremal* (s,t)*-Ramsey* if G has neither K_s nor $\overline{K}_t$ as an induced subgraph while the order of G is $R(s,t) - 1$, that is, the maximum possible. Das, Huang, Ma, Naves and Sudakov [5, page 365] asked whether, for every (k,l) and large n, the value of $f(n,k,l)$ is attained by an expansion of some extremal Ramsey graph. The cases $(k,l) = (6,3)$ and $(7,3)$ that we solved here show that the answer is in the negative. Interestingly, $\overline{L}$ is nonetheless related to Ramsey numbers, but to 3-colour ones: Kalbfleisch and Stanton [13] showed that there are two different 3-edge-colourings of K_{16} without a monochromatic triangle but each colour class (in either colouring) is isomorphic to the Clebsch graph (and thus the union of any two colour classes is isomorphic to $\overline{L}$).

Das, Huang, Ma, Naves and Sudakov [5, page 365] mention that they ran the SDP-solver for the cases $(k,l) = (5,3)$, $(3,5)$ and $(3,6)$ and the obtained floating-point bound suggested that $c_{5,3} = 31/625$, $c_{3,5} = 1/16$, and $c_{3,6} = 1/25$, with extremal configurations being an expansion of respectively C_5, $\overline{K}_4$ and $\overline{K}_5$. Since their paper was already quite long they did not try to convert it into a rigorous proof. The current paper makes these statements rigorous.

It would be interesting to identify further pairs (k,l) amenable to this approach. One promising case is $f(n,4,4)$, where we make the following conjecture.

Conjecture 6.1.

$$c_{4,4} = \frac{14 \cdot 2^{1/3} - 11}{192}. \tag{6.1}$$

The upper bound in (6.1) comes from taking expansions of the (unique) (3,4)-Ramsey graph F with 8 vertices and 10 edges. More specifically, let F be obtained from the 8-cycle on $1,\dots,8$ by adding the two 'diameters' $\{1,5\}$ and $\{2,6\}$ as edges. Take an expansion $F' = F(\!(U_1,\dots,U_8)\!)$ with parts U_1, U_2, U_5, and U_6 (those corresponding to degree-3

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vertices of F) having size $(\alpha + o(1))n$ and the other four parts having size $(\frac{1}{4} - \alpha + o(1))n$, where $\alpha = \frac{1}{12}(1 - 2^{1/3} + 2^{2/3})$. Routine calculations show that the density of K_4 approaches the right-hand side of (6.1) as $n \to \infty$. On the other hand, *Flagmatic* suggests that this construction is asymptotically optimal and, perhaps, a flag algebra proof exists within the 8-vertex universe (*i.e.*, taking $N = 8$). Unfortunately, we have not been able to round the floating-point solution.

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