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SUCCESSIVE ENLARGEMENT OF FILTRATIONS AND APPLICATION TO INSIDER INFORMATION

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Abstract

We model in a dynamic way an insider's private information flow which is successively augmented by a family of initial enlargement of filtrations. According to the *a priori* available information, we propose several density hypotheses which are presented in hierarchical order from the weakest to the strongest. We compare these hypotheses, in particular, with Jacod's one, and deduce conditional expectations under each of them by providing consistent expressions with respect to the common reference filtration. Finally, this framework is applied to a default model with insider information on the default threshold and some numerical illustrations are performed.

Keywords: Successive enlargement; density hypothesis; insider information; credit risk

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1. Introduction

Modelling information is a crucial subject in financial markets. The mathematical tool is based on the theory of initial enlargement of a filtration by a random variable, which was developed by the French school in the 1970s–80s by Jacod [17], [18], Jeulin [19], Jeulin and Yor [21], amongst others ; see also [26, Chapter VI] for an introduction in English. This theory received a new focus in the 1990s for its application in finance, notably for problems occurring in insider modelling. When an insider is present, his/her information is often modelled by the enlargement of the common information filtration by the insider's private information and we investigate problems such as the existence of arbitrage or the value of private information; see, e.g. [2], [12], and [16]. Classically, in these papers the extra information L is revealed at the initial time but does not evolve or become more accurate through time.

In this paper our aim is to generalize previous works and consider an insider who can adjust his/her extra information with time. Let t_i, $i = 1, \dots, n$, be a family of discrete times and L^i be random variables modelling the extra information available at time t_i. The insider's information, which is modelled by the filtration $\mathbb{G}^{\mathrm{I}}$, is given by the successive initial enlargement at time t_i by the random variable L^i. In [1] and [12], Jacod's hypothesis, or the so-called density hypothesis, which assumes the equivalence between the conditional law of L with respect

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to the common reference filtration and the law of L, plays an important role. It implies, in particular, the existence of an equivalent martingale measure and, thus, no free lunch with vanishing risk (NFLVR). Moreover, following Föllmer and Imkeller [10], in [12] an equivalent martingale measure was constructed under which the reference filtration is independent of the random variable L. Our methodology consists of generalizing these properties in the framework of successive initial enlargement. We propose several density hypotheses in a hierarchical order. We show that if a density hypothesis is supposed at each step between the conditional law of L^i with respect to the previous information and the conditional law of L^i with respect to previous information at time t_i, we obtain families of probability measures with favourable properties. Indeed, under this successive density hypothesis, we construct a family of probabilities $\mathbb{P}^i$, $i = 1, \ldots, n$, which decouple, at time t_i, the random variable L^i and $\mathcal{G}^{i-1}_{t_i}$. However, this first family obtained by a natural induction does not preserve at time t_i the law of the next random variables L^k, $i < k \leq n$. To overcome this inconvenience, we propose a second family of probability measures $\mathbb{Q}^i$, $i = 1, \ldots, n$, constructed by a backward change of probability measure. Then we focus on the conditional expectation with successive information. The use of the family $\mathbb{Q}^i$ allows us to obtain an evaluation formula in terms of $\mathbb{F}$-conditional expectations, where $\mathbb{F}$ is the common reference information. Our approach, although less general than the local method solution approach introduced by Song [27], [28], nevertheless provides tractable formulae, in particular, for the computation of conditional expectation, which are useful for financial applications. From this successive density hypothesis, we derive, in addition, stronger formulations where the *a priori* available information concerns the nontrivial or trivial initial σ-algebra, which are more similar to the classical density hypothesis of Jacod in the initial enlargement framework. Moreover, another point of view is to consider a global initial enlargement of the reference filtration $\mathbb{F}$ by the random vector $\boldsymbol{L} = (L^1, \ldots, L^n)$ and a density hypothesis between the conditional law of $\boldsymbol{L}$ and the law of $\boldsymbol{L}$. We investigate the link between the global approach and the successive approach.

The application in finance generalizes the default model in [14] to a dynamic setting. The default time is supposed to be the first time where the value of the firm reaches a random threshold chosen by the manager of the firm and adjusted dynamically. In literature, another 'dynamic' enlargement of filtrations was introduced by Corcuera *et al.* [8], where the private information is affected by an independent noise process vanishing as the revelation time approaches. The authors of [22] and [23] studied progressive filtration expansions with càdlàg processes. Bilina Falafala and Protter proposed in [6] a related model in which the market filtration $\mathbb{F}$ is initially enlarged at an $\mathbb{F}$-stopping time. Assuming Jacod's absolute continuity hypothesis is satisfied on the whole time interval $[0, T]$, they examined conditions for no arbitrage and free lunches on $[0, T]$ and they compared the market trader's and insider's risk in a Föllmer–Schweizer sense; see [11]. With Jacod's absolute continuity hypothesis instead of the equivalent hypothesis, NFLVR may hold only locally and not globally. In this paper we propose an alternative approach in that we do not assume Jacod's hypothesis (neither absolute continuity nor the equivalent density) on the whole interval $[0, T]$ but only on the interval $[t_i, T]$ starting at the disclosure time t_i of the updated information L_i. Besides, our successive density hypothesis on $[t_i, T]$, $i = 1, \ldots, n$, implies a NFLVR setting for the insider. To compare the survival probability for different information, we introduce the standard information available to an investor in credit risk, given by the progressive enlargement, which was studied in, e.g. [4], [5], [20], and [24]. Using our successive enlargement framework, we obtain explicit formulations for the survival probability of the insider and compare the results with those of standard investors by numerical illustrations. Finally, we note a strain of related literature dealing with initial enlargement and the information drift, such as applying Malliavin's calculus [15], [16], or using forward anticipative calculus [3], which provide other perspectives to study the insider information.

The paper is organized as follows. We present the model framework in Section 2. In Section 3 we introduce the successive density hypothesis and propose two constructions of auxiliary probability measures to compute conditional expectations. Then in Section 4 we consider several particular cases of the successive density framework and make comparisons. Finally, in Section 5 we apply this insider information framework to a default model and perform some numerical illustrations.

2. Model framework

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ which satisfies the usual conditions and represents the common information flow on financial markets, where T is a finite-time horizon. The insider has knowledge of extra information which is revealed dynamically with time. Let $\{t_i,\ i = 1,\dots,n\}$ be a family of discrete times. The insider's information is described by a family of random variables $\{L^i,\ i = 1,\dots,n\}$, where L^i is $\mathcal{A}$-measurable and takes values in a Polish space E whose Borel σ-algebra is denoted by $\mathcal{E}$. The insider gets the information on L^i at time t_i, so the total information flow of the insider is described by the filtration $\mathbb{G}^{\mathrm{I}} = (\mathcal{G}^{\mathrm{I}}_t)_{t\ge 0}$, where

$$\mathcal{G}^{\mathrm{I}}_t := \mathcal{F}_t \vee \sigma(L^1) \vee \cdots \vee \sigma(L^i), \qquad t \in [t_i, t_{i+1}). \tag{2.1}$$

We can interpret this information flow in two different but equivalent ways by using the theory of enlargement of filtrations. On the one hand, for any $t \in [0, T]$, define the extra information process as

$$L_t = \sum_{i=1}^{n} L^i\, \mathbf{1}_{[t_i,t_{i+1})}(t) \tag{2.2}$$

then we have $\mathcal{G}^{\mathrm{I}}_t = \mathcal{F}_t \vee \sigma(L_s,\ s \le t)$. The filtration $\mathbb{G}^{\mathrm{I}}$ is the progressive enlargement of the filtration $\mathbb{F}$ by the information process L. On the other hand, let us define a family of filtrations $\mathbb{G}^i = (\mathcal{G}^i_t)_{t\ge 0}$ for all $i = 1,\dots,n$, where

$$\mathcal{G}^i_t := \mathcal{F}_t \vee \sigma(L^1) \vee \cdots \vee \sigma(L^i), \qquad t \in [0, T]. \tag{2.3}$$

By definition, we have $\mathcal{G}^{\mathrm{I}}_t = \mathcal{G}^i_t$ for $t \in [t_i, t_{i+1})$ and $\mathcal{G}^i_t = \mathcal{G}^{i-1}_t \vee \sigma(L^i)$, where we set by convention $\mathcal{G}^0_t = \mathcal{F}_t$. Each filtration $\mathbb{G}^i$ is the initial enlargement of the filtration $\mathbb{G}^{i-1}$ by the random variable L^i. We thus obtain an increasing family of successive initial enlargement of filtrations.

We denote by $\boldsymbol{L}$ the n-dimensional random vector $(L^1,\dots,L^n)$. For any $i = 1,\dots,n$, let $\boldsymbol{L}^{(i)} := (L^1,\dots,L^i)$. Similarly, we use the expression $\boldsymbol{x}$ to denote a vector $(x^1,\dots,x^n)$ in E^n, and let $\boldsymbol{x}^{(i)} := (x^1,\dots,x^i)$. For any $t \in [0, T]$, the σ-algebra $\mathcal{G}^i_t$ is generated by $\mathcal{F}_t$ and $\sigma(\boldsymbol{L}^{(i)})$. Therefore, any $\mathbb{G}^i$-adapted process can be written in the form $(Y_t(\boldsymbol{L}^{(i)}),\ 0 \le t \le T)$, where $Y_t(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{E}^{\otimes i}$-measurable; see [19].

In the classical framework of initial information modelling, the insider obtains the extra information at the initial time $t = 0$ and keeps it until the final time T. In our setting this corresponds to the case where $n = 1$ and $\mathcal{G}^{\mathrm{I}}_t = \mathcal{G}^1_t$ for all $t \in [0, T]$.

In the enlargement of filtration theory, the conditional laws of L^i with respect to different filtrations play an important role. For a random variable X taking values in the Polish space E and a sub-σ-algebra $\mathcal{B}$ of $\mathcal{A}$, we denote by $\mathbb{P}(X \in \cdot \mid \mathcal{B})$ a regular version of the conditional

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probability law of X with respect to $\mathcal{B}$. By definition, it is a map from $\Omega \times \mathcal{B}$ to $[0, 1]$ such that

- for any $\omega \in \Omega$, $\mathbb{P}(X \in \cdot \mid \mathcal{B})(\omega)$ is a probability measure on $(E, \mathcal{E})$;
- for any Borel set S in E, the function $\mathbb{P}(X \in S \mid \mathcal{B})$ on Ω is $\mathcal{B}$-measurable, and is $\mathbb{P}$-almost surely (a.s.) equal to the $\mathcal{B}$-conditional expectation $\mathbb{E}^{\mathbb{P}}[\mathbf{1}_S(X) \mid \mathcal{B}]$.

3. Successive density hypothesis

In order to study the dynamic properties of the filtration $\mathbb{G}^{\mathrm{I}}$, we introduce the following *successive density hypothesis*, which asserts that the terminal conditional law of L^i is equivalent to its $\mathcal{G}_{t_i}^{i-1}$-conditional law. This hypothesis provides a weaker and more flexible form compared to the one for the initial enlargement of filtration in [1] and [12] (see also [18] for comparison) which is widely adopted in the study of asymmetric information. In particular, it allows us to take into account the insider's information in a progressive manner at each time step.

Assumption 3.1. *For any $i \in \{1, \dots, n\}$, the $\mathcal{G}_T^{i-1}$-conditional law of L^i is equivalent to its $\mathcal{G}_{t_i}^{i-1}$-conditional law under the probability $\mathbb{P}$, namely, there exists a positive $\mathcal{G}_T^{i-1} \otimes \mathcal{E}$-measurable function $\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ such that*

$$\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_T^{i-1}) = \alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1}). \tag{3.1}$$

Remark 3.1. (i) In the above assumption, we actually consider the density $\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ as an $(\mathcal{F}_T \otimes \mathcal{E}^{\otimes i-1}) \otimes \mathcal{E}$-measurable function $\alpha_T^{i\,|\,i-1}(\cdot, \cdot)$ evaluated at $\boldsymbol{L}^{(i-1)}$. Note that such a representation need not be unique. More precisely, there may exist another $(\mathcal{F}_T \otimes \mathcal{E}^{\otimes i-1}) \otimes \mathcal{E}$-measurable function $\widetilde{\alpha}_T^{i\,|\,i-1}(\cdot, \cdot)$ such that $\widetilde{\alpha}_T^{i\,|\,i-1}(\boldsymbol{x}^{(i-1)}, x)$ is not identically equal to $\alpha_T^{i\,|\,i-1}(\boldsymbol{x}^{(i-1)}, x)$ for $(\boldsymbol{x}^{(i-1)}, x) \in E^i$ but $\widetilde{\alpha}_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) = \alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)$. We refer the reader to [25] for a general discussion on the stochastic process depending on a parameter.

(ii) In [1] and [12] it was assumed that the $\mathcal{G}_t^{i-1}$-conditional law of L^i is equivalent to its probability law for $t \in \mathbb{R}_+$. The main difference here is that the conditional law $\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1})$, with respect to which we consider the density of $\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_T^{i-1})$, is a random measure instead of a deterministic probability law. Therefore, it is difficult to apply Jacod's method [18, Lemma 1.8] to prove the existence of a martingale version of the density process. Our choice of working with the terminal time T allows us to overcome this difficulty. In fact, Assumption 3.1 implies that, for any $t \in [t_i, T]$, the $\mathcal{G}_t^{i-1}$-conditional law of L^i under $\mathbb{P}$ is equivalent to the $\mathcal{G}_{t_i}^{i-1}$-conditional law of L^i. Moreover, the $\mathcal{G}_t^{i-1} \otimes \mathcal{E}$-measurable function $\mathbb{E}^{\mathbb{P}}[\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) \mid \mathcal{G}_t^{i-1}]$ gives the density of $\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_t^{i-1})$ with respect to $\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1})$, which we denote as $\alpha_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$. We refer the reader to Corollary 3.1 for details.

(iii) Under Assumption 3.1, similar as in [1, Proposition 3.3], the filtration $\mathbb{G}^i$ is right-continuous on $[t_i, T]$, and also is $\mathbb{G}^{\mathrm{I}}$ on $[0, T]$.

3.1. One step enlargement of filtration

The filtration $\mathbb{G}^{\mathrm{I}}$ can be considered as a step-by-step enlargement of $\mathbb{F}$. Also, the successive density hypothesis has an inductive nature. In this subsection we focus on one step of the enlargement and develop tools which will be useful in the inductive study of $\mathbb{G}^{\mathrm{I}}$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathbb{H} = (\mathcal{H}_u)_{u \in [t,T]}$ be a filtration of $\mathcal{A}$, where t is a fixed real number such that $0 \le t < T$. Let X be an $\mathcal{A}$-measurable random variable which takes value in a Polish space $(E, \mathcal{E})$. We assume that there exists a positive $\mathcal{H}_T \otimes E$-measurable function $q_T(\cdot)$ such that

$$\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_T) = q_T(x)\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_t). \tag{3.2}$$

Example 3.1. We give a simple but illustrative example which satisfies hypothesis (3.2) but not the density hypothesis with respect to the probability law of X. Let Y_1 and Y_2 be two independent random variables which both follow the standard normal distribution. Let $X = \max(Y_1, Y_2)$. We consider the filtration $\mathbb{H} = (\mathcal{H}_u)_{u\in[t,T]}$ such that $\mathcal{H}_u = \sigma(Y_1)$ for all $u \in [t, T]$. It is clear that the $\mathcal{H}_T$-conditional law of X has a density with respect to the $\mathcal{H}_t$-conditional law, which equals the constant 1. However, it is not true that this conditional law is absolutely continuous with respect to the probability law of X. In fact, if we denote respectively by Φ and ϕ the probability distribution function and the probability density function of the standard normal distribution, then the probability law of X has the probability density $2\Phi\phi$. However, the $\sigma(Y_1)$-conditional law of X is $\Phi(Y_1)\delta_{Y_1}(\mathrm{d}u) + \mathbf{1}_{[Y_1,+\infty)}\,\phi(u)\,\mathrm{d}u$, which is not absolutely continuous with respect to the Lebesgue measure. This is a typical situation which we cannot handle within the classic framework of density hypothesis.

Remark 3.2. Condition (3.2) is invariant under a change of probability measure. Indeed, if $\mathbb{P}'$ is an equivalent probability measure of $\mathbb{P}$ with $\mathrm{d}\mathbb{P}'/\mathrm{d}\mathbb{P} = Q_T(X)$ on $\mathcal{H}_T \vee \sigma(X)$, where $Q_T(\cdot)$ is a positive $\mathcal{H}_T \otimes \mathcal{E}$-measurable function, then for any nonnegative Borel function f on E,

$$\mathbb{E}^{\mathbb{P}'}[f(X) \mid \mathcal{H}_T] = \frac{\mathbb{E}^{\mathbb{P}}[f(X)Q_T(X) \mid \mathcal{H}_T]}{\mathbb{E}^{\mathbb{P}}[Q_T(X) \mid \mathcal{H}_T]} = \frac{\int_E f(x)Q_T(x)q_T(x)\nu_t(\mathrm{d}x)}{\int_E Q_T(x)q_T(x)\nu_t(\mathrm{d}x)},$$

where $\nu_t(\mathrm{d}x) := \mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_t)$. Moreover, let $Q_t(\cdot)$ be a $\mathcal{H}_t \otimes \mathcal{E}$-measurable function such that $Q_t(X) = \mathbb{E}^{\mathbb{P}}[Q_T(X) \mid \mathcal{H}_t \vee \sigma(X)]$, then $Q_t(X)$ is the Radon–Nikodym density $\mathrm{d}\mathbb{P}'/\mathrm{d}\mathbb{P}$ on $\mathcal{H}_t \vee \sigma(X)$, and, hence,

$$\mathbb{E}^{\mathbb{P}'}[f(X) \mid \mathcal{H}_t] = \frac{\mathbb{E}^{\mathbb{P}}[f(X)Q_t(X) \mid \mathcal{H}_t]}{\mathbb{E}^{\mathbb{P}}[Q_t(X) \mid \mathcal{H}_t]} = \frac{\int_E f(x)Q_t(x)\nu_t(\mathrm{d}x)}{\int_E Q_t(x)\nu_t(\mathrm{d}x)}.$$

Therefore, $\mathbb{P}'(X \in \cdot \mid \mathcal{H}_T)$ is absolutely continuous with respect to $\mathbb{P}'(X \in \cdot \mid \mathcal{H}_t)$, and the corresponding density is given by

$$q'_T(\cdot) = q_T(\cdot)\frac{Q_T(\cdot)}{Q_t(\cdot)}\frac{\int_E Q_t(x)\nu_t(\mathrm{d}x)}{\int_E Q_T(x)q_T(x)\nu_t(\mathrm{d}x)}. \tag{3.3}$$

Note that, if X and $\mathcal{H}_T$ are $\mathbb{P}$-conditionally independent given $\mathcal{H}_t$, then we can choose $Q_t(\cdot)$ to be

$$Q_t(\cdot) := \mathbb{E}^{\mathbb{P}}[Q_T(\cdot) \mid \mathcal{H}_t].$$

In [18], Jacod has proposed a density hypothesis without a (strict) positivity assumption on the density. With the above notation, the $\mathcal{F}_T$-condition law of the random variable X is absolutely continuous (but not necessarily equivalent) to the probability law of X. We can also consider the analogue of this hypothesis in relaxing the positivity condition of the density function. This weakened condition is still invariant under the change of (equivalent) probability measures, and (3.3) still holds. In fact, although $q_T(\cdot)$ is not necessarily positive, the integral $\int_E Q_T(x)q_T(x)\nu_t(\mathrm{d}x)$, which appears in the denominator of the right-hand side of (3.3), is equal to $\mathbb{E}^{\mathbb{P}}[Q_T(X) \mid \mathcal{H}_T]$, which is positive a.s. However, the equivalent hypothesis is essential in Proposition 3.4 in order to obtain the decoupling equivalent probability measure.

Let $\mathbb{G} = (\mathcal{G}_u)_{u\in[t,T]}$ denote the initial enlargement of $\mathbb{H}$ with X, i.e. $\mathcal{G}_u = \mathcal{H}_u \vee \sigma(X)$. By using the conditional density, we can construct a probability measure equivalent to $\mathbb{P}$ under which the random variable X and the filtration $\mathbb{H}$ are *conditionally* independent given $\mathcal{H}_t$.

Proposition 3.1. *Under hypothesis (3.2), there exists an equivalent probability measure $\mathbb{Q}$ to $\mathbb{P}$ such that*

(i) *$\mathbb{Q}$ coincides with $\mathbb{P}$ on $\mathbb{H}$;*

(ii) *X and $\mathbb{H}$ are conditionally independent under $\mathbb{Q}$ given $\mathcal{H}_t$;*

(iii) *X has the same conditional law, given $\mathcal{H}_t$, under $\mathbb{P}$ and $\mathbb{Q}$.*

Moreover, the probability measure $\mathbb{Q}$ is unique on $\mathcal{G}_T$ and given by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}|_{\mathcal{G}_T} = q_T(X)^{-1}$.

We emphasize that, although the result has a form similar as in [10] and [12], under our hypothesis it is, in general, not possible to assume the independence between X and the filtration $\mathbb{H}$ under an equivalent probability measure.

Proof of Proposition 3.1. By taking the expectation of a conditional expectation, we have

$$\mathbb{E}^{\mathbb{P}}[q_T(X)^{-1}] = \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[q_T(X)^{-1} \mid \mathcal{H}_T]] = \mathbb{E}^{\mathbb{P}}\bigg[\int_E q_T(x)^{-1}\nu_T(\mathrm{d}x)\bigg].$$

The hypothesis (3.2) thus leads to

$$\mathbb{E}^{\mathbb{P}}[q_T(X)^{-1}] = \mathbb{E}^{\mathbb{P}}\bigg[\int_E q_T(x)^{-1}q_T(x)\nu_t(\mathrm{d}x)\bigg] = 1.$$

Let $\mathbb{Q}$ be the probability measure on $(\Omega, \mathcal{A})$ defined by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = q_T(X)^{-1}$. If f is a nonnegative Borel function on E, Z_T a nonnegative $\mathcal{H}_T$-measurable random variable, and Y_t a nonnegative $\mathcal{H}_t$-measurable random variable, then a direct computation shows that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[f(X)Z_T Y_t] &= \mathbb{E}^{\mathbb{P}}[f(X)q_T(X)^{-1}Z_T Y_t] \\
&= \mathbb{E}^{\mathbb{P}}\bigg[Z_T Y_t \int_E f(x)q_T(x)^{-1}\nu_T(\mathrm{d}x)\bigg] \\
&= \mathbb{E}^{\mathbb{P}}\bigg[Z_T Y_t \int_E f(x)\nu_t(\mathrm{d}x)\bigg] \\
&= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[Z_T \mid \mathcal{H}_t]Y_t\mathbb{E}^{\mathbb{P}}[f(X) \mid \mathcal{H}_t]].
\end{aligned} \tag{3.4}$$

If we take Z_T to be the constant function 1, it follows that the conditional law of X under $\mathbb{P}$ and $\mathbb{Q}$, given $\mathcal{H}_t$, coincide. If we take f and Y_t to be the constant function 1, it follows that $\mathbb{P}$ and $\mathbb{Q}$ coincide on $\mathcal{H}_T$. Therefore, relation (3.4) implies that

$$\mathbb{E}^{\mathbb{Q}}[f(X)Z_T \mid \mathcal{H}_t] = \mathbb{E}^{\mathbb{Q}}[f(X) \mid \mathcal{H}_t]\mathbb{E}^{\mathbb{Q}}[Z_T \mid \mathcal{H}_t],$$

namely $\sigma(X)$ and $\mathbb{H}$ are conditionally independent given $\mathcal{H}_t$.

For the uniqueness of the probability measure $\mathbb{Q}$ on $\mathcal{G}_T$, it suffices to observe that, for any positive $\mathcal{G}_T$-measurable random variable $Y_T(X)$, we have

$$\mathbb{E}^{\mathbb{Q}}[Y_T(X)] = \mathbb{E}^{\mathbb{Q}}\bigg[\int_E \mathbb{E}^{\mathbb{Q}}[Y_T(x) \mid \mathcal{H}_t]\mathbb{Q}(X \in \mathrm{d}x \mid \mathcal{H}_t)\bigg]$$

by using the conditional independence of $\mathbb{H}$ and $\sigma(X)$ given $\mathcal{H}_t$. Since the probability measures $\mathbb{P}$ and $\mathbb{Q}$ coincide on $\mathbb{H}$ and the conditional probability laws of X given $\mathcal{H}_t$ with respect to $\mathbb{P}$ and $\mathbb{Q}$ coincide, we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[Y_T(X)] &= \mathbb{E}^{\mathbb{P}}\bigg[\int_E \mathbb{E}^{\mathbb{P}}[Y_T(x) \mid \mathcal{H}_t]\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_t)\bigg] \\ &= \mathbb{E}^{\mathbb{P}}\bigg[\int_E Y_T(x)\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_t)\bigg] \\ &= \mathbb{E}^{\mathbb{P}}\bigg[\int_E Y_T(x)q_T(x)^{-1}\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_T)\bigg] \\ &= \mathbb{E}^{\mathbb{P}}[Y_T(X)q_T(X)^{-1}].\end{aligned}$$

Therefore, the Radon–Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$ should be $q_T(X)^{-1}$. □

Corollary 3.1. *For any $u \in [t, T]$, the $\mathcal{H}_u$-conditional law of X is equivalent to the $\mathcal{H}_t$-conditional law of X under the probability $\mathbb{P}$. Moreover, if $q_u(\cdot)$ is a positive $\mathcal{H}_u \otimes \mathcal{E}$-measurable function on $\Omega \times E$ such that $q_u(x) = \mathbb{E}^{\mathbb{P}}[q_T(x) \mid \mathcal{H}_u]$ $\mathbb{P}$-a.s., then we have*

$$\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_u) = q_u(x)\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_t).$$

In particular, the Radon–Nikodym derivative of the probability measure $\mathbb{Q}$ defined in Proposition 3.1 with respect to $\mathbb{P}$ is given by $q_u(X)^{-1}$ on $\mathcal{H}_u$ for $u \in [t, T]$.

Proof. Let $\mathbb{Q}$ be the probability measure on $\mathcal{A}$ defined by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = q_T(X)^{-1}$. By Proposition 3.1, for any $u \in [s, T]$, we obtain

$$\mathbb{Q}(X \in \cdot \mid \mathcal{H}_u) = \mathbb{Q}(X \in \cdot \mid \mathcal{H}_t) = \mathbb{P}(X \in \cdot \mid \mathcal{H}_t). \tag{3.5}$$

Moreover, for any nonnegative Borel function f on E, we have

$$\int_E f(x)\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_u) = \mathbb{E}^{\mathbb{P}}[f(X) \mid \mathcal{H}_u] = \frac{\mathbb{E}^{\mathbb{Q}}[f(X)q_T(X) \mid \mathcal{H}_u]}{\mathbb{E}^{\mathbb{Q}}[q_T(X) \mid \mathcal{H}_u]}. \tag{3.6}$$

Note that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[f(X)q_T(X) \mid \mathcal{H}_u] &= \mathbb{E}^{\mathbb{Q}}\bigg[\int_E f(x)q_T(x)\mathbb{Q}(X \in \mathrm{d}x \mid \mathcal{H}_T) \,\bigg|\, \mathcal{H}_u\bigg] \\ &= \mathbb{E}^{\mathbb{Q}}\bigg[\int_E f(x)q_T(x)\nu_t(\mathrm{d}x) \,\bigg|\, \mathcal{H}_u\bigg] \\ &= \mathbb{E}^{\mathbb{Q}}\bigg[\int_E f(x)\nu_T(\mathrm{d}x) \,\bigg|\, \mathcal{H}_u\bigg],\end{aligned} \tag{3.7}$$

where the second equality comes from (3.5) and we recall that $\nu_t(\mathrm{d}x) = \mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_t)$. In addition, we have, from (3.7),

$$\mathbb{E}^{\mathbb{Q}}[f(X)q_T(X) \mid \mathcal{H}_u] = \int_E f(x)\mathbb{E}^{\mathbb{Q}}[q_T(x) \mid \mathcal{H}_u]\nu_t(\mathrm{d}x) = \int_E f(x)q_u(x)\nu_t(\mathrm{d}x) \tag{3.8}$$

since $\mathbb{Q}$ and $\mathbb{P}$ coincide on $\mathbb{H}$. In particular, when f is the constant function 1, (3.7) yields

$$\mathbb{E}^{\mathbb{Q}}[q_T(X) \mid \mathcal{H}_u] = 1.$$

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Therefore, by (3.6) and (3.8), we obtain

$$\int_E f(x)\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_u) = \int_E f(x)q_u(x)\mathbb{P}(X \in \mathrm{d}x \mid \mathcal{H}_t),$$

namely $q_u(\cdot)$ is the density of $\nu_u(\mathrm{d}x)$ with respect to $\nu_t(\mathrm{d}x)$. □

3.2. Change of probability measures

We now come back to the successive enlargements under Assumption 3.1. In this subsection and the next one, we introduce two different ways to construct equivalent probability measures, which will play an important role in further applications.

We recall that for any $x \in E$ and $t \in [t_i, T]$, $\alpha_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ is defined as the conditional expectation

$$\alpha_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) = \mathbb{E}^{\mathbb{P}}[\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) \mid \mathcal{G}_t^{i-1}].$$

By Corollary 3.1, we have

$$\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_t^{i-1}) = \alpha_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1}).$$

We now introduce a family of probability measures equivalent to $\mathbb{P}$ by using Proposition 3.1 in a recursive manner.

Definition 3.1. Let $\mathbb{P}^0 := \mathbb{P}$, and, for any $i \in \{1, \dots, n\}$, let $\mathbb{P}^i$ be the probability measure on $(\Omega, \mathcal{A})$ such that

$$\frac{\mathrm{d}\mathbb{P}^i}{\mathrm{d}\mathbb{P}^{i-1}} = \frac{1}{\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)})}. \tag{3.9}$$

For any $\boldsymbol{x}^{(i)} \in E^i$, let

$$\psi_t^i(\boldsymbol{x}^{(i)}) := \prod_{k=1}^{i} \frac{1}{\alpha_t^{k\,|\,k-1}(\boldsymbol{x}^{(k)})}, \qquad t \in [t_i, T]. \tag{3.10}$$

We show in Proposition 3.2 below that the probability measures $(\mathbb{P}^i)_{i=1}^n$ are well defined and the Radon–Nikodym density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ is $\psi_t^i(\boldsymbol{L}^{(i)})$ on $\mathcal{G}_t^i$.

Proposition 3.2. *The probability measures $(\mathbb{P}^i)_{i=1}^n$ are well defined and equivalent to $\mathbb{P}$. For any $i \in \{1, \dots, n\}$,*

(i) *the probability measures $\mathbb{P}^i$ and $\mathbb{P}^{i-1}$ coincide on $\mathcal{G}_T^{i-1}$, in particular, all probability measures $(\mathbb{P}^i)_{i=1}^n$ coincide with $\mathbb{P}$ on $\mathcal{F}_T$;*

(ii) *$\boldsymbol{L}^{(i)}$ and $\mathcal{F}_T$ are conditionally independent given $\mathcal{F}_{t_i}$ under $\mathbb{P}^i$;*

(iii) *for any $t \in [t_i, T]$, the Radon–Nikodym density of $\mathbb{P}^i$ with respect to $\mathbb{P}^{i-1}$ is given by $\alpha_t^{i\,|\,i-1}(\boldsymbol{L}^{(i)})^{-1}$ on $\mathcal{G}_t^i$ and, hence, the Radon–Nikodym density of $\mathbb{P}^i$ with respect to $\mathbb{P}$ is given by $\psi_t^i(\boldsymbol{L}^{(i)})$ on $\mathcal{G}_t^i$.*

Proof. We prove the proposition by induction on i. The case when $i = 1$ is true by Proposition 3.1. Suppose that the equivalent probability measures $\mathbb{P}^1, \dots, \mathbb{P}^{i-1}$ are well defined and verify the properties asserted by the proposition. Moreover, Assumption 3.1 holds for the probability measure $\mathbb{P}^{i-1}$ by Remark 3.2. More precisely, the conditional

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law $\mathbb{P}^{i-1}(L^i \in \cdot \mid \mathcal{G}_T^{i-1})$ is absolutely continuous with respect to $\mathbb{P}^{i-1}(L^i \in \cdot \mid \mathcal{G}_{t_i}^{i-1})$, and the corresponding density is

$$\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)\frac{\psi_T^{i-1}(\boldsymbol{L}^{(i-1)})}{\mathbb{E}^{\mathbb{P}}[\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \mid \mathcal{G}_{t_i}^{i-1}]} \times \frac{\int_E \mathbb{E}^{\mathbb{P}}[\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \mid \mathcal{G}_{t_i}^{i-1}]\,\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1})}{\int_E \alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\psi_T^{i-1}(\boldsymbol{L}^{(i-1)})\,\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1})},$$

which is still equal to $\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ since

$$\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) = \int_E \alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\psi_T^{i-1}(\boldsymbol{L}^{(i-1)})\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1})$$

and

$$\mathbb{E}^{\mathbb{P}}[\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \mid \mathcal{G}_{t_i}^{i-1}] = \int_E \mathbb{E}^{\mathbb{P}}[\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \mid \mathcal{G}_{t_i}^{i-1}]\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1}).$$

We now show that (3.9) effectively defines a probability measure $\mathbb{P}^i$. We have

$$\mathbb{E}^{\mathbb{P}^{i-1}}[\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)})^{-1} \mid \mathcal{G}_{t_i}^{i-1}] = \frac{\mathbb{E}^{\mathbb{P}}[\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)})^{-1}\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \mid \mathcal{G}_{t_i}^{i-1}]}{\mathbb{E}^{\mathbb{P}}[\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \mid \mathcal{G}_{t_i}^{i-1}]}.$$

Assumption (3.1) applied to L^i and $\mathbb{G}^{i-1}$ leads to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)})^{-1}\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \mid \mathcal{G}_{t_i}^{i-1}] \\
&\quad = \mathbb{E}^{\mathbb{P}}\bigg[\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \int_E \alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)^{-1}\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_{t_i}^{i-1}) \mid \mathcal{G}_{t_i}^{i-1}\bigg] \\
&\quad = \mathbb{E}^{\mathbb{P}}[\psi_T^{i-1}(\boldsymbol{L}^{(i-1)}) \mid \mathcal{G}_{t_i}^{i-1}].
\end{aligned}$$

Therefore, $\mathbb{E}^{\mathbb{P}^{i-1}}[\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)})^{-1} \mid \mathcal{G}_{t_i}^{i-1}] = 1$ and, hence, $\mathbb{P}^i$ is well defined.

By Proposition 3.1, $\mathbb{P}^i$ and $\mathbb{P}^{i-1}$ coincide on $\mathbb{G}^{i-1}$. In particular, $\mathbb{P}^i$ and $\mathbb{P}$ are the same on $\mathcal{F}_T$, which implies the first assertion. By the induction hypothesis, $\boldsymbol{L}^{(i-1)}$ and $\mathcal{F}_T$ are conditionally independent given $\mathcal{F}_{t_{i-1}}$ under the probability measure $\mathbb{P}^{i-1}$, which implies, since $\mathcal{F}_{t_{i-1}} \subseteq \mathcal{F}_{t_i}$, that $\boldsymbol{L}^{(i-1)}$ and $\mathcal{F}_T$ are conditionally independent given $\mathcal{F}_{t_i}$ under $\mathbb{P}^{i-1}$, and also under $\mathbb{P}^i$ by (i). It then suffices to verify that L^i and $\mathcal{F}_T$ are conditionally independent, given $\mathcal{F}_{t_i}$ under $\mathbb{P}^i$, to prove the second assertion. Note that Proposition 3.1 also shows that L^i and $\mathcal{G}_T^{i-1}$ are conditionally independent given $\mathcal{G}_{t_i}^{i-1}$ under the probability $\mathbb{P}^i$. Let f be a nonnegative Borel function on E, and X is a nonnegative $\mathcal{F}_T$-measurable random variable. By the conditional independence of L^i and $\mathcal{F}_T$ given $\mathcal{G}_{t_i}^{i-1}$ under $\mathbb{P}^i$, we obtain

$$\mathbb{E}^{\mathbb{P}^i}[f(L^i)X \mid \mathcal{F}_{t_i}] = \mathbb{E}^{\mathbb{P}^i}[\mathbb{E}^{\mathbb{P}^i}[f(L^i) \mid \mathcal{G}_{t_i}^{i-1}]\,\mathbb{E}^{\mathbb{P}^i}[X \mid \mathcal{G}_{t_i}^{i-1}] \mid \mathcal{F}_{t_i}].$$

Moreover, since X and $\boldsymbol{L}^{(i-1)}$ are conditionally independent given $\mathcal{F}_{t_i}$ under $\mathbb{P}^i$, we have $\mathbb{E}^{\mathbb{P}^i}[X \mid \mathcal{G}_{t_i}^{i-1}] = \mathbb{E}^{\mathbb{P}^i}[X \mid \mathcal{F}_{t_i}]$. Therefore, we obtain

$$\mathbb{E}^{\mathbb{P}^i}[f(L^i)X \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}^i}[f(L^i) \mid \mathcal{F}_{t_i}]\,\mathbb{E}^{\mathbb{P}^i}[X \mid \mathcal{F}_{t_i}].$$

Finally, the last assertion of the proposition follows from (i) and Corollary 3.1. The proposition is thus proved. □

Remark 3.3. This construction of successive changes of probability measures is natural and uses only the knowledge of $\boldsymbol{L}^{(i)}$ to construct $\mathbb{P}^i$. However, under the probability measure $\mathbb{P}^i$, the law of L^k, $k \in \{i+1, \dots, n\}$, is not identical to the law of L^k under $\mathbb{P}^{i-1}$. We will show in the next subsection that $\mathbb{P}^n$ preserves the $\mathbb{P}$-conditional probability law of L^k given $\mathcal{G}_{t_k}^{k-1}$.

Proposition 3.3. *Let $t, u \in [t_i, T]$, $t \le u$, and $X_u(\boldsymbol{L}^{(i)})$ be a nonnegative $\mathcal{G}_u^i$-measurable random variable. We have*

$$\mathbb{E}^{\mathbb{P}}[X_u(\boldsymbol{L}^{(i)}) \mid \mathcal{G}_t^i] = \frac{\mathbb{E}^{\mathbb{P}}[X_u(\boldsymbol{x}^{(i)})\psi_u^i(\boldsymbol{x}^{(i)})^{-1} \mid \mathcal{F}_t]}{\psi_t^i(\boldsymbol{x}^{(i)})^{-1}}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}.$$

Proof. We use the change of the probability measure to $\mathbb{P}^i$ and obtain

$$\mathbb{E}^{\mathbb{P}}[X_u(\boldsymbol{L}^{(i)}) \mid \mathcal{G}_t^i] = \frac{\mathbb{E}^{\mathbb{P}^i}[X_u(\boldsymbol{L}^{(i)})\psi_u^i(\boldsymbol{L}^{(i)})^{-1} \mid \mathcal{G}_t^i]}{\psi_t^i(\boldsymbol{L}^{(i)})^{-1}}.$$

By Proposition 3.2, $\boldsymbol{L}^{(i)}$ and $\mathcal{F}_T$ are conditionally independent given $\mathcal{F}_t$ under the probability $\mathbb{P}^i$. Therefore,

$$\mathbb{E}^{\mathbb{P}}[X_u(\boldsymbol{L}^{(i)}) \mid \mathcal{G}_t^i] = \frac{\mathbb{E}^{\mathbb{P}^i}[X_u(\boldsymbol{x}^{(i)})\psi_u^i(\boldsymbol{x}^{(i)})^{-1} \mid \mathcal{F}_t]}{\psi_t^i(\boldsymbol{x}^{(i)})^{-1}}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}.$$

Since $\mathbb{P}^i$ and $\mathbb{P}$ coincide on $\mathcal{F}_T$, we obtain the desired result. □

3.3. Backward construction of probability measures

In order to have a family of probability measures under which the conditional law of each L^i remains unchanged, we propose the following construction, using a backward change of probability measures. This method is also crucial in the evaluation of financial claims which we will discuss later.

Definition 3.2. Let $\mathbb{Q}^{n+1} = \mathbb{P}$, and, for $i \in \{1, \dots, n\}$, let $\mathbb{Q}^i$ be a probability measure on $(\Omega, \mathcal{A})$ such that

$$\frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{Q}^{i+1}} := \frac{1}{\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)})}. \tag{3.11}$$

Let

$$\varphi_T^i(\boldsymbol{x}) = \prod_{k=i}^{n} \frac{1}{\alpha_T^{k\,|\,k-1}(\boldsymbol{x}^{(k)})}.$$

Then the Radon–Nikodym derivative of $\mathbb{Q}^i$ with respect to $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{P}} = \varphi_T^i(\boldsymbol{L}). \tag{3.12}$$

Note that $\varphi_T^i(\boldsymbol{L})$ is a $\mathcal{G}_T^n$-measurable random variable.

Proposition 3.4. *The equivalent probability measures $(\mathbb{Q}^i)_{i=1}^n$ are well defined and verify the following properties for any $i \in \{1, \dots, n\}$:*

(i) *$\mathbb{Q}^i$ coincides with $\mathbb{P}$ on $\mathcal{G}_T^{i-1}$;*

(ii) *for any $k \in \{i, \dots, n\}$, L^k and $\mathcal{G}_T^{k-1}$ are conditionally independent given $\mathcal{G}_{t_k}^{k-1}$ under $\mathbb{Q}^i$;*

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(iii) *for any* $k \in \{1, \dots, n\}$*,* L^k *has the same conditional law given* $\mathcal{G}^{k-1}_{t_k}$ *under all* $(\mathbb{Q}^i)_{i=1}^n$ *and* $\mathbb{P}$*.*

Proof. We prove the proposition by a reverse induction on i. The assertion is clearly true when $i = n+1$. Assume that the probability measures $\mathbb{Q}^{i+1}, \dots, \mathbb{Q}^{n+1}$ have been constructed and verify the assertions in the proposition. Since $\mathbb{Q}^{i+1}$ is identical to $\mathbb{P}$ on $\mathcal{G}^i_T$, we have

$$\mathbb{Q}^{i+1}(L^i \in \mathrm{d}x \mid \mathcal{G}^{i-1}_T) = \alpha^{i\,|\,i-1}_T(\boldsymbol{L}^{(i-1)}, x)\mathbb{Q}^{i+1}(L^i \in \mathrm{d}x \mid \mathcal{G}^{i-1}_{t_i}). \tag{3.13}$$

In particular, we have

$$\mathbb{E}^{\mathbb{Q}^{i+1}}\biggl[\frac{1}{\alpha^{i\,|\,i-1}_T(\boldsymbol{L}^{(i)})}\biggm| \mathcal{G}^{i-1}_T\biggr] = 1.$$

Therefore, the probability measure $\mathbb{Q}^i$ equivalent to $\mathbb{Q}^{i+1}$ given by (3.11) is well defined.

By (3.13) and Proposition 3.1, the probability measure $\mathbb{Q}^i$ coincides with $\mathbb{Q}^{i+1}$, and, therefore, with $\mathbb{P}$ on $\mathcal{G}^{i-1}_T$. So assertion (i) is proved, and, hence, for any $k \in \{1, \dots, i-1\}$, L^k has the same conditional law given $\mathcal{G}^{k-1}_{t_k}$ under $\mathbb{Q}^i$ and $\mathbb{P}$. Moreover, L^i is conditionally independent of $\mathcal{G}^{i-1}_T$ given $\mathcal{G}^{i-1}_{t_i}$ under $\mathbb{Q}^i$, and L^i has the same conditional probability law given $\mathcal{G}^{i-1}_{t_i}$ under $\mathbb{Q}^i$ and $\mathbb{Q}^{i+1}$ (and, hence, under $\mathbb{P}$ also). Finally, for $k \in \{i+1, \dots, n\}$, let h be a nonnegative Borel function on E, and Y be a nonnegative $\mathcal{G}^{k-1}_T$-measurable random variable, then

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^i}[h(L^k)Y \mid \mathcal{G}^{k-1}_{t_k}] &= \frac{\mathbb{E}^{\mathbb{Q}^{i+1}}[h(L^k)Y\alpha^{i\,|\,i-1}_T(\boldsymbol{L}^{(i)})^{-1} \mid \mathcal{G}^{k-1}_{t_k}]}{\mathbb{E}^{\mathbb{Q}^{i+1}}[\alpha^{i\,|\,i-1}_T(\boldsymbol{L}^{(i)})^{-1} \mid \mathcal{G}^{k-1}_{t_k}]}\\ &= \frac{\mathbb{E}^{\mathbb{Q}^{i+1}}[h(L^k) \mid \mathcal{G}^{k-1}_{t_k}]\mathbb{E}^{\mathbb{Q}^{i+1}}[Y\alpha^{i\,|\,i-1}_T(\boldsymbol{L}^{(i)})^{-1} \mid \mathcal{G}^{k-1}_{t_k}]}{\mathbb{E}^{\mathbb{Q}^{i+1}}[\alpha^{i\,|\,i-1}_T(\boldsymbol{L}^{(i)})^{-1} \mid \mathcal{G}^{k-1}_{t_k}]}\end{aligned}$$

since, by the induction hypothesis, L^k and $\mathcal{G}^{k-1}_T$ are conditionally independent given $\mathcal{G}^{k-1}_{t_k}$ under $\mathbb{Q}^{i+1}$. Therefore,

$$\mathbb{E}^{\mathbb{Q}^i}[h(L^k)Y \mid \mathcal{G}^{k-1}_{t_k}] = \mathbb{E}^{\mathbb{Q}^{i+1}}[h(L^k) \mid \mathcal{G}^{k-1}_{t_k}]\mathbb{E}^{\mathbb{Q}^i}[Y \mid \mathcal{G}^{k-1}_{t_k}].$$

If we take $Y = 1$ then the $\mathcal{G}^{k-1}_{t_k}$-conditional law of L^k under $\mathbb{Q}^i$ coincides with that under $\mathbb{Q}^{i+1}$, which proves assertion (iii). Moreover, this also shows that

$$\mathbb{E}^{\mathbb{Q}^i}[h(L^k)Y \mid \mathcal{G}^{k-1}_{t_k}] = \mathbb{E}^{\mathbb{Q}^i}[h(L^k) \mid \mathcal{G}^{k-1}_{t_k}]\mathbb{E}^{\mathbb{Q}^i}[Y \mid \mathcal{G}^{k-1}_{t_k}],$$

which gives assertion (ii) and completes the proof. □

3.4. Conditional expectation with successive information

In this subsection we are interested in the computation of conditional expectations with the insider's successive information. The $\mathbb{G}^{\mathrm{I}}$-conditional expectations may represent the dynamic values of a financial claim viewed by the insider. The idea is to make connections with the $\mathbb{F}$-conditional expectations which are easier to deal with in an explicit manner and the result is given in a decomposed form with a regime change at each time t_i when new information is available. We still suppose that Assumption 3.1 holds for the information flow. In particular, we assume that the insider has knowledge on the marginal conditional laws $\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}^{i-1}_{t_i})$, $i \in \{1, \dots, n\}$. We will present the evaluation formula in terms of $\mathbb{F}$-conditional expectations.

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Let $Y_T(\boldsymbol{L})$ be a nonnegative $\mathcal{G}_T^{\mathrm{I}}$-measurable random variable. Our purpose is to determine the conditional expectation of $Y_T(\boldsymbol{L})$ given the insider's information $\mathcal{G}_t^{\mathrm{I}}$ at $t \in [0, T]$. Here we work under the initial probability measure $\mathbb{P}$. Note that the method is valid under an equivalent probability measure since Assumption 3.1 is invariant under an equivalent probability change. By definition (2.1) and (2.3), we have

$$\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] = \sum_{i=1}^{n} \mathbf{1}_{[t_i, t_{i+1})}(t)\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{i}] = \sum_{i=1}^{n} \mathbf{1}_{[t_i, t_{i+1})}(t)\mathbb{E}^{\mathbb{P}}[Y_{t_{i+1}}(\boldsymbol{L}^{(i)}) \mid \mathcal{G}_t^{i}], \tag{3.14}$$

where

$$Y_{t_{i+1}}(\boldsymbol{L}^{(i)}) := \mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_{t_{i+1}}^{i}]. \tag{3.15}$$

It then suffices to determine $Y_{t_{i+1}}(\boldsymbol{L}^{(i)})$ under Assumption 3.1. The result is obtained by using a recursive pricing kernel and we use probability measures constructed in the two previous subsections.

For any $i \in \{1, \dots, n\}$, let J_i be the operator which sends a nonnegative or bounded $\mathcal{G}_T^{i}$-measurable random variable $X_T(\boldsymbol{L}^{(i)})$ to the following integral:

$$\int_E \mathbb{E}^{\mathbb{P}}[X_T(\boldsymbol{L}^{(i-1)}, x^i) \mid \mathcal{G}_{t_i}^{i-1}]\mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_{t_i}^{i-1}), \tag{3.16}$$

which is a $\mathcal{G}_{t_i}^{i-1}$-measurable random variable. Note that, by Proposition 3.3, we have

$$\mathbb{E}^{\mathbb{P}}[X_T(\boldsymbol{L}^{(i-1)}, x^i) \mid \mathcal{G}_{t_i}^{i-1}] = \frac{\mathbb{E}^{\mathbb{P}}[X_T(\boldsymbol{x}^{(i)})\psi_T^{i-1}(\boldsymbol{x}^{(i-1)})^{-1} \mid \mathcal{F}_{t_i}]}{\psi_{t_i}^{i-1}(\boldsymbol{x}^{(i-1)})^{-1}}\bigg|_{\boldsymbol{x}^{(i-1)}=\boldsymbol{L}^{(i-1)}}. \tag{3.17}$$

In other words, the operator J_i can be expressed in terms of an $\mathbb{F}$-conditional expectation and integral with respect to the $\mathcal{G}_{t_i}^{i-1}$-conditional law of L^i.

This operator can be better understood by using the probability measure $\mathbb{Q}^i$ constructed in Subsection 3.3. In fact, by Proposition 3.4, we have

$$\mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_{t_i}^{i-1}) = \mathbb{Q}^i(L^i \in \mathrm{d}x^i \mid \mathcal{G}_{t_i}^{i-1}),$$

and

$$\mathbb{E}^{\mathbb{P}}[X_T(\boldsymbol{L}^{(i-1)}, x^i) \mid \mathcal{G}_{t_i}^{i-1}] = \mathbb{E}^{\mathbb{Q}^i}[X_T(\boldsymbol{L}^{(i-1)}, x^i) \mid \mathcal{G}_{t_i}^{i-1}]$$

since $\mathbb{Q}^i$ and $\mathbb{P}$ coincide on $\mathcal{G}_T^{i-1}$. Therefore, we can write (3.16) as

$$\int_E \mathbb{E}^{\mathbb{Q}^i}[X_T(\boldsymbol{L}^{(i-1)}, x^i) \mid \mathcal{G}_{t_i}^{i-1}]\mathbb{Q}^i(L^i \in \mathrm{d}x^i \mid \mathcal{G}_{t_i}^{i-1}),$$

which implies, since L^i and $\mathcal{G}_T^{(i-1)}$ are conditionally independent given $\mathcal{G}_{t_i}^{i-1}$ under $\mathbb{Q}^i$, that

$$J_i(X_T(\boldsymbol{L}^{(i)})) = \mathbb{E}^{\mathbb{Q}^i}[X_T(\boldsymbol{L}^{(i)}) \mid \mathcal{G}_{t_i}^{i-1}]. \tag{3.18}$$

Therefore, J_i is actually a conditional expectation operator. In particular, it is an $\mathbb{R}$-linear operator which verifies the following equality:

$$J_i(X_T(\boldsymbol{L}^{(i)})Z_{t_i}(\boldsymbol{L}^{(i-1)})) = Z_{t_i}(\boldsymbol{L}^{(i-1)})J_i(X_T(\boldsymbol{L}^{(i)})) \tag{3.19}$$

for any $\mathcal{G}_{t_i}^{i-1}$-measurable random variable $Z_{t_i}(\boldsymbol{L}^{(i-1)})$ such that the left-hand side of the above formula is well defined.

Lemma 3.1. *Let $X_T(\boldsymbol{L})$ be a bounded or nonnegative $\mathcal{G}_T^n$-measurable random variable. We have*

$$\mathbb{E}^{\mathbb{Q}^{i+1}}[X_T(\boldsymbol{L}) \mid \mathcal{G}^i_{t_{i+1}}] = J_{i+1} \circ \cdots \circ J_n(X_T(\boldsymbol{L})U_T^{i+1}(\boldsymbol{L})), \qquad i \in \{0, \ldots, n\}, \tag{3.20}$$

where the operator $J_{i+1} \circ \cdots \circ J_n$ is considered as the identity operator when $i = n$ and

$$U_T^{i+1}(\boldsymbol{L}) := \prod_{k=i+1}^{n} \frac{\alpha_{t_{k+1}}^{k\,|\,k-1}(\boldsymbol{L}^{(k)})}{\alpha_T^{k\,|\,k-1}(\boldsymbol{L}^{(k)})}. \tag{3.21}$$

Proof. We prove the assertion by reverse induction on i. The case when $i = n$ follows from (3.18) since $U_T^n(\boldsymbol{L}) = 1$. In the following, we assume that equality (3.20) is verified for $i+1$ and we now prove it is the case for i.

By the induction hypothesis and the fact that

$$U_T^{i+1}(\boldsymbol{L}) = \frac{\alpha_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})}{\alpha_T^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})} U_T^{i+2}(\boldsymbol{L}),$$

we have

$$\begin{aligned} J_{i+1} \circ \cdots \circ J_n(X_T(\boldsymbol{L})U_T^{i+1}(\boldsymbol{L})) &= J_{i+1}\bigg(\mathbb{E}^{\mathbb{Q}^{i+2}}\bigg[X_T(\boldsymbol{L})\frac{\alpha_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})}{\alpha_T^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})} \;\bigg|\; \mathcal{G}^{i+1}_{t_{i+2}}\bigg]\bigg) \\ &= J_{i+1}(\mathbb{E}^{\mathbb{Q}^{i+1}}[X_T(\boldsymbol{L}) \mid \mathcal{G}^{i+1}_{t_{i+2}}]) \\ &= \mathbb{E}^{\mathbb{Q}^{i+1}}[X_T(\boldsymbol{L}) \mid \mathcal{G}^{i}_{t_{i+1}}], \end{aligned}$$

where the second equality comes from the probability change from $\mathbb{Q}^{i+2}$ to $\mathbb{Q}^{i+1}$, and the last equality follows from (3.18). □

Theorem 3.1. *Let $Y_T(\boldsymbol{L})$ be a bounded or nonnegative $\mathcal{G}_T^{\mathrm{I}}$-measurable random variable. For any $t \in [0, T]$, we have*

$$\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] = \sum_{i=1}^{n} \mathbf{1}_{[t_i, t_{i+1})}(t) \frac{\mathbb{E}^{\mathbb{P}}[Y_{t_{i+1}}(\boldsymbol{x}^{(i)})\psi^i_{t_{i+1}}(\boldsymbol{x}^{(i)})^{-1} \mid \mathcal{F}_t]}{\psi^i_t(\boldsymbol{x}^{(i)})^{-1}}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}, \tag{3.22}$$

where $Y_{t_{i+1}}(\cdot)$ is $\mathcal{F}_{t_{i+1}} \otimes \mathcal{E}^{\otimes i}$-measurable such that $Y_{t_{i+1}}(\boldsymbol{L}^{(i)}) = \mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}^i_{t_{i+1}}]$. Moreover, the sequence of random variables $(Y_{t_{i+1}}(\boldsymbol{L}^{(i)}))_{i=0}^n$ satisfies the following backward recursive relation:

$$Y_{t_{i+1}}(\boldsymbol{L}^{(i)}) = \frac{J_{i+1}(Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)}))}{J_{i+1}(\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)}))}, \qquad i \in \{0, \ldots, n-1\}, \tag{3.23}$$

with the terminal term $Y_{t_{n+1}}(\boldsymbol{L}^{(n)}) = Y_T(\boldsymbol{L})$ and the pricing kernel given by

$$\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)}) := J_{i+2} \circ \cdots \circ J_n(\alpha_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i)}) \cdots \alpha_T^{n\,|\,n-1}(\boldsymbol{L}^{(n)})) \tag{3.24}$$

with convention $\Phi_{t_1} = 1$.

Proof. By (3.14) and Proposition 3.3, we obtain equality (3.22). We now prove relation (3.23) by computing the conditional expectation (3.15) under the change of probability measure to $\mathbb{Q}^{i+1}$ as

$$Y_{t_{i+1}}(\boldsymbol{L}^{(i)}) = \mathbb{E}^{\mathbb{P}}[Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)}) \mid \mathcal{G}^i_{t_{i+1}}] = \frac{\mathbb{E}^{\mathbb{Q}^{i+1}}[Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})\varphi_T^{i+1}(\boldsymbol{L})^{-1} \mid \mathcal{G}^i_{t_{i+1}}]}{\mathbb{E}^{\mathbb{Q}^{i+1}}[\varphi_T^{i+1}(\boldsymbol{L})^{-1} \mid \mathcal{G}^i_{t_{i+1}}]},$$

where $\varphi_T^{i+1}(\boldsymbol{L})$ is the Radon–Nikodym derivative of $\mathbb{Q}^{i+1}$ with respect to $\mathbb{P}$ defined in (3.12). By Lemma 3.1, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^{i+1}}[Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})\varphi_T^{i+1}(\boldsymbol{L})^{-1} \mid \mathcal{G}^i_{t_{i+1}}] \\
&\quad = J_{i+1} \circ \cdots \circ J_n(Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})\varphi_T^{i+1}(\boldsymbol{L})^{-1} U_T^{i+1}(\boldsymbol{L})) \\
&\quad = J_{i+1}(Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})) J_{i+2} \circ \cdots \circ J_n(\varphi_T^{i+1}(\boldsymbol{L})^{-1} U_T^{i+1}(\boldsymbol{L})),
\end{aligned}$$

where the second equality comes from (3.19). Note that, by (3.21), we have

$$\frac{U_T^{i+1}(\boldsymbol{L})}{\varphi_T^{i+1}(\boldsymbol{L})} = \alpha_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i)}) \cdots \alpha_T^{n\,|\,n-1}(\boldsymbol{L}^{(n)}),$$

which implies that

$$\mathbb{E}^{\mathbb{Q}^{i+1}}[Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})\varphi_T^{i+1}(\boldsymbol{L})^{-1} \mid \mathcal{G}^i_{t_{i+1}}] = J_{i+1}(Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)})).$$

In addition, Lemma 3.1 shows that

$$\mathbb{E}^{\mathbb{Q}^{i+1}}[\varphi_T^{i+1}(\boldsymbol{L})^{-1} \mid \mathcal{G}^i_{t_{i+1}}] = J_{i+1}(\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)})),$$

which implies (3.23) and completes the proof. □

Remark 3.4. Since we deal with processes of finite-time horizon, Theorem 3.1 can also be viewed as a characterization of $\mathbb{G}^{\mathrm{I}}$-martingales. In fact, a $\mathbb{G}^{\mathrm{I}}$-adapted process can be written in the form

$$X_t = \sum_{i=1}^{n} \mathbf{1}_{[t_i, t_{i+1})}(t) X_t^{(i)}(\boldsymbol{L}^{(i)}), \qquad t \in [0, T],$$

where, for each $i \in \{1, \dots, n\}$, $X_t^{(i)}(\cdot)$ is an $\mathcal{F}_t \otimes \mathcal{E}^{\otimes i}$-measurable random variable. Theorem 3.1 shows that the above process is an $(\mathbb{G}^{\mathrm{I}}, \mathbb{P})$-martingale if

$$\frac{X_t^{(i)}(\boldsymbol{x}^{(i)})}{\psi_t^i(\boldsymbol{x}^{(i)})} = \mathbb{E}^{\mathbb{P}}\bigg[\frac{X_{t_{i+1}}(\boldsymbol{x}^{(i)})}{\psi_{t_{i+1}}^i(\boldsymbol{x}^{(i)})} \;\bigg|\; \mathcal{F}_t\bigg],$$

where $X_{t_{i+1}}(\cdot)$ is $\mathcal{F}_{t_{i+1}} \otimes \mathcal{E}^{\otimes i}$-measurable such that $X_{t_{i+1}}(\boldsymbol{L}^{(i)}) = \mathbb{E}^{\mathbb{P}}[X_T \mid \mathcal{G}^i_{t_{i+1}}]$, which can be calculated by (3.23). This condition can also be interpreted as below. For any $i \in \{1, \dots, n\}$ and any $\boldsymbol{x}^{(i)} \in E^i$, the process

$$\frac{X_t^{(i)}(\boldsymbol{x}^{(i)})}{\psi_t^i(\boldsymbol{x}^{(i)})}, \qquad t \in [t_i, t_{i+1})$$

is an $(\mathbb{F}, \mathbb{P})$-martingale on $[t_i, t_{i+1})$ which converges to $X_{t_{i+1}}(\boldsymbol{x}^{(i)})/\psi_{t_{i+1}}^i(\boldsymbol{x}^{(i)})$ when t tends to t_{i+1}.

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4. Several stronger density hypotheses

In this section we consider particular cases of our successive density framework by introducing several stronger density hypotheses than Assumption 3.1. We compare these hypotheses and deduce concrete evaluation formulae in each case. For simplicity, we suppose that $\mathcal{F}_0$ is trivial.

4.1. Density hypothesis with different initial σ-algebras

We begin by the case where we consider the conditional law of L^i given the initial σ-algebra of the previous information filtration $\mathcal{G}_0^{i-1} = \sigma(\boldsymbol{L}^{(i-1)})$.

Assumption 4.1. *For any $i \in \{1, \dots, n\}$, the $\mathcal{G}_T^{i-1}$-conditional law of L^i is equivalent to its $\mathcal{G}_0^{i-1}$-conditional law under the probability $\mathbb{P}$, namely there exists a positive $\mathcal{G}_T^{i-1} \otimes \mathcal{E}$-measurable function $\beta_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ such that*

$$\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_T^{i-1}) = \beta_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_0^{i-1}).$$

Similarly to what we have explained in Remark 3.1, we can consider the conditional density $\beta_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ as a positive $(\mathcal{F}_T \otimes \mathcal{E}^{\otimes(i-1)}) \otimes \mathcal{E}$-measurable function $\beta_T^{i\,|\,i-1}(\cdot, \cdot)$ evaluated at $\boldsymbol{L}^{(i-1)}$. For any $t \in [0, T]$, let $\beta_t^{i\,|\,i-1}(\cdot, \cdot)$ be an $(\mathcal{F}_t \otimes \mathcal{E}^{\otimes(i-1)}) \otimes \mathcal{E}$-measurable function such that

$$\beta_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) = \mathbb{E}^{\mathbb{P}}[\beta_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) \mid \mathcal{G}_t^{i-1}].$$

By Corollary 3.1,

$$\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_t^{i-1}) = \beta_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_0^{i-1}). \tag{4.1}$$

Note that $\beta_0^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) = 1$ a.s. for all $x \in E$ and

$$\int_E \beta_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_0^{i-1}) = 1.$$

In particular, if we define for all $t_i \le t \le T$ a function $\alpha_t^{i\,|\,i-1}(\boldsymbol{x})$ on $\Omega \times E^i$ which is $\mathcal{F}_T \otimes \mathcal{E}^i$-measurable such that

$$\alpha_t^{i\,|\,i-1}(\boldsymbol{x}^{(i)}) = \frac{\beta_t^{i\,|\,i-1}(\boldsymbol{x}^{(i)})}{\beta_{t_i}^{i\,|\,i-1}(\boldsymbol{x}^{(i)})}, \tag{4.2}$$

then the random vector $\boldsymbol{L}$ verifies Assumption 3.1 with the conditional density $\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)$ and $\alpha_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) = \mathbb{E}^{\mathbb{P}}[\alpha_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) \mid \mathcal{G}_t^{i-1}]$, $x \in E$.

Let us note that under Assumption 4.1 the filtration $\mathbb{G}^i$ is right-continuous on $[0, T]$, whereas it is *a priori* right-continuous only on $[t_i, T]$ under the weaker Assumption 3.1.

We now apply Theorem 3.1 to compute the conditional expectation under Assumption 4.2 where the recursive operators can be simplified in an explicit manner. As the result can also be obtained in a more straightforward manner using a global approach (see Subsection 4.2), we will give the proof by using the recursive approach in Appendix A.

Proposition 4.1. *We suppose that Assumption 4.1 holds. Let $Y_T(\boldsymbol{L})$ be a nonnegative $\mathcal{G}_T^n$-measurable random variable. Then, for $t \in [0, T]$, we have*

$$\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] = \sum_{i=1}^{n} \mathbf{1}_{[t_i, t_{i+1})}(t) \int_{E^{n-i}} \frac{\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{x}) Z_T^n(\boldsymbol{x}) \mid \mathcal{F}_t]}{Z_t^i(\boldsymbol{x}^{(i)})}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \times \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^i),$$

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where the pricing kernel is defined as

$$Z_t^i(\boldsymbol{x}^{(i)}) := \prod_{k=1}^{i} \beta_t^{k\,|\,k-1}(\boldsymbol{x}^{(k)}). \tag{4.3}$$

We state the following key property of the pricing kernel.

Lemma 4.1. *For* $i \in \{0, \dots, n-1\}$ *and* $t \in [0, T]$,

$$\begin{aligned}
&\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_t^i) \\
&\quad = \frac{Z_t^n(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n)}{Z_t^i(\boldsymbol{L}^{(i)})} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^i)
\end{aligned} \tag{4.4}$$

with convention $Z_t^0 = 1$. *Moreover, we have*

$$Z_t^i(\boldsymbol{L}^{(i)}) = \int_{E^{n-i}} Z_t^n(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n) \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^i). \tag{4.5}$$

Proof. Let I_t^i be the operator sending a nonnegative $\mathcal{G}_t^i$-measurable random variable $Y_t(\boldsymbol{L}^{(i)})$ to

$$\mathbb{E}[Y_t(\boldsymbol{L}^{(i)}) \mid \mathcal{G}_t^{i-1}] = \int_E Y_t(\boldsymbol{L}^{(i-1)}, x^i) \beta_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x^i) \mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_0^{i-1}).$$

On the one hand, by the property of conditional expectation, we have

$$\begin{aligned}
&(I_t^{i+1} \circ \dots \circ I_t^n)(Y_t(\boldsymbol{L})) \\
&\quad = \mathbb{E}^{\mathbb{P}}[Y_t(\boldsymbol{L}) \mid \mathcal{G}_t^i] \\
&\quad = \int_{E^{n-i}} Y_t(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n) \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_t^i).
\end{aligned} \tag{4.6}$$

On the other hand, by the definition of the operators $I_t^{i+1}, \dots, I_t^n$ and the fact that

$$\beta^{i+1\,|\,i}(\boldsymbol{L}^{(i)}, x^{i+1}) \cdots \beta^{n\,|\,n-1}(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n) = \frac{Z_t^n(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n)}{Z_t^i(\boldsymbol{L}^{(i)})},$$

it follows that

$$\begin{aligned}
&(I_t^{i+1} \circ \dots \circ I_t^n)(Y_t(\boldsymbol{L})) \\
&\quad = \int_{E^{n-i}} Y_t(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n) \frac{Z_t^n(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n)}{Z_t^i(\boldsymbol{L}^{(i)})} \\
&\qquad\qquad \times \mathbb{P}(L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^{n-1}) \cdots \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1} \mid \mathcal{G}_0^i) \\
&\quad = \int_{E^{n-i}} Y_t(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n) \frac{Z_t^n(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n)}{Z_t^i(\boldsymbol{L}^{(i)})} \\
&\qquad\qquad \times \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^i).
\end{aligned}$$

Combining with equality (4.6), we deduce the first assertion (4.4) of the lemma, which leads to (4.5) directly. □

Another hypothesis is the Jacod's hypothesis in the successive initial enlargement of filtration setting where the terminal conditional law of each L^i given the previous information filtration $\mathbb{G}^{(i-1)}$ is equivalent to its probability law.

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Assumption 4.2. *For any $i \in \{1, \dots, n\}$, the $\mathcal{G}_T^{i-1}$-conditional law of L^i is equivalent to its conditional law under the probability $\mathbb{P}$, namely there exists a positive $\mathcal{G}_T^{i-1} \otimes \mathcal{E}$-measurable function $p_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ such that*

$$\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_T^{i-1}) = p_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)\mathbb{P}(L^i \in \mathrm{d}x).$$

Note that under the above assumption, for any $t \in [0, T]$, the $\mathcal{G}_t^{i-1}$-conditional law of L^i has the density $p_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) := \mathbb{E}^{\mathbb{P}}[p_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) \mid \mathcal{G}_t^{i-1}]$ with respect to $\mathbb{P}(L^i \in \mathrm{d}x)$. In particular, the family of $(\mathbb{P}, \mathbb{G}^{i-1})$-martingales $p^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, \cdot)$ has the initial value

$$p_0^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) = \frac{\mathbb{P}(L^i \in \mathrm{d}x \mid \mathcal{G}_0^{i-1})}{\mathbb{P}(L^i \in \mathrm{d}x)}, \qquad x \in E. \tag{4.7}$$

Moreover, if $\boldsymbol{L}$ satisfies Assumption 4.2, it also satisfies Assumption 4.1 with $\beta_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)$, where, for all t,

$$\beta_t^{i\,|\,i-1}(\boldsymbol{x}^{(i)}) = \frac{p_t^{i\,|\,i-1}(\boldsymbol{x}^{(i)})}{p_0^{i\,|\,i-1}(\boldsymbol{x}^{(i)})}. \tag{4.8}$$

We give hereafter an example where Assumption 4.2 is satisfied and the density processes $p^{i\,|\,i-1}$ are given explicitly.

Example 4.1. Let (W, W') a two-dimensional Brownian motion and $\mathcal{F}_t = \sigma(W_s,\ s \le t \le T)$. Define the Brownian motion $B = \rho W + (1-\rho)W'$ with $\rho \in [0, 1[$ and let $L^i = B_{t_{i+1}}$ be the endpoint of B at each interval $[t_i, t_{i+1}[$. Then Assumption 4.2 is satisfied and

$$p_t^{i\,|\,i-1}(L^{(i-1)}, x) = \begin{cases} \dfrac{\phi(L^{i-1}, t_{i+1} - t_i, x)}{\phi(0, t_{i+1}, x)}, & t \le t_i, \\[2ex] \dfrac{\phi(L^{i-1} + \rho(W_t - W_{t_i}), \rho^2(t_{i+1} - t) + (1-\rho)^2(t_{i+1} - t_i), x)}{\phi(0, t_{i+1}, x)}, & t_i < t \le t_{i+1}, \\[2ex] \dfrac{\phi(L^{i-1} + \rho(W_{t_{i+1}} - W_{t_i}), (1-\rho)^2(t_{i+1} - t_i), x)}{\phi(0, t_{i+1}, x)}, & t_{i+1} < t \le T, \end{cases}$$

where $\phi(\mu, \sigma^2, x)$ is the probability density function of the normal distribution $N(\mu, \sigma^2)$. We note that $p^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x)$ is a $(\mathbb{P}, \mathbb{G}^{i-1})$-martingale on [0,T].

We deduce from Proposition 4.1 the following result.

Proposition 4.2. *We suppose that Assumption 4.2 holds. Let $Y_T(\boldsymbol{L})$ be a nonnegative $\mathcal{G}_T^n$-measurable random variable. Then, for $t \in [0, T]$, we have*

$$\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] = \sum_{i=1}^n \mathbf{1}_{[t_i, t_{i+1})}(t) \int_{E^{n-i}} \frac{\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{x})\widetilde{Z}_T^n(\boldsymbol{x}) \mid \mathcal{F}_t]}{\widetilde{Z}_t^i(\boldsymbol{x}^{(i)})}\bigg|_{\boldsymbol{x}^{(i)} = \boldsymbol{L}^{(i)}} \times \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}) \cdots \mathbb{P}(L^n \in \mathrm{d}x^n),$$

where $\widetilde{Z}_t^i(\boldsymbol{x}^{(i)}) = \prod_{k=1}^i p_t^{k\,|\,k-1}(\boldsymbol{x}^{(k)})$.

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Proof. We apply Proposition 4.1 under Assumption 4.2. By equality (4.8), we obtain

$$Z_t^i(\boldsymbol{x}^{(i)}) = \prod_{k=1}^{i} \beta_t^{k\,|\,k-1}(\boldsymbol{x}^{(k)}) = \widetilde{Z}_t^i(\boldsymbol{x}^{(i)}) \prod_{k=1}^{i} \frac{1}{p_0^{k\,|\,k-1}(\boldsymbol{x}^{(k)})}.$$

Therefore, Proposition 4.1 leads to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] \\
&= \sum_{i=1}^{n} \mathbf{1}_{[t_i,t_{i+1})}(t) \int_{E^{n-i}} \frac{\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{x})\widetilde{Z}_T^n(\boldsymbol{x}) \mid \mathcal{F}_t]}{\widetilde{Z}_t^i(\boldsymbol{x}^{(i)})}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \\
&\qquad \times \prod_{k=i+1}^{n} \frac{1}{p_0^{k\,|\,k-1}(\boldsymbol{x}^{(k)})} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^i).
\end{aligned}$$

Finally, by (4.7), which implies the following relation:

$$\begin{aligned}
&\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^i) \\
&= \bigg(\prod_{k=i+1}^{n} p_0^{k\,|\,k-1}(\boldsymbol{x}^{(k)})\bigg) \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}) \cdots \mathbb{P}(L^n \in \mathrm{d}x^n),
\end{aligned}$$

we obtain the result of the proposition. □

Remark 4.1. Similarly to Lemma 4.1, for $i \in \{0, \dots, n-1\}$ and $t \in [0, T]$, we have

$$\begin{aligned}
&\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_t^i) \\
&= \frac{\widetilde{Z}_t^n(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n)}{\widetilde{Z}_t^i(\boldsymbol{L}^{(i)})} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}) \cdots \mathbb{P}(L^n \in \mathrm{d}x^n), \qquad (4.9)
\end{aligned}$$

where $\widetilde{Z}_t^0 = 1$ and

$$\widetilde{Z}_t^i(\boldsymbol{L}^{(i)}) = \int_{E^{n-i}} \widetilde{Z}_t^n(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n) \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}) \cdots \mathbb{P}(L^n \in \mathrm{d}x^n).$$

Due to the transitivity of the equivalence relation between probability measures, Assumption 4.2 implies Assumption 4.1 which, in turn, implies Assumption 3.1. We now provide several examples to compare these hypotheses.

Example 4.2. (i) Trivial examples (that lead to no enlargement of filtrations) show that the reciprocal statements are false: e.g. L^i which is a deterministic function of $\boldsymbol{L}^{(i-1)}$ satisfies Assumption 4.1 but not Assumption 4.2; L_i, which is a $\mathcal{G}_{t_i}^{i-1}$-measurable random variable but not $\mathcal{G}_0^{i-1}$-measurable satisfies Assumption 3.1 and not Assumption 4.1.

(ii) More generally, Assumption 4.1 is satisfied but not Assumption 4.2 at step t_i if and only if the distribution of L^i is not equivalent to the conditional distribution of L^i given $\boldsymbol{L}^{(i-1)}$.

(iii) Here is another example, in the context of credit risk and default threshold, in which Assumption 4.1 is satisfied and not Assumption 4.2. Let L^i take two values a or b, $a < b$. At time t_i, the manager has an anticipation of the firm's value $X_{T'+t_i}$ with $T' > T$ and knows if this value will be above or below the constant target c, X being an $\mathbb{F}$-adapted process.

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If $X_{T'+t_{i+1}}$ is above the target and the former threshold L^i was low, then the manager keeps fixing a low level for the threshold L^{i+1}, otherwise he/she will fix a high level for L^{i+1}, i.e.

$$L^{i+1} = a\, \mathbf{1}_{\{X_{T'+t_{i+1}}>c\}}\, \mathbf{1}_{\{L^i=a\}} + b(\mathbf{1}_{\{X_{T'+t_{i+1}}\le c\}} + \mathbf{1}_{\{X_{T'+t_{i+1}}>c\}}\, \mathbf{1}_{\{L^i=b\}}).$$

In this example, the distribution of L^{i+1} has two atoms a and b with positive probability, while the distribution of L^{i+1} given the event $\{L^i = b\}$ is a Dirac measure.

(iv) Similarly, here is an example in which Assumption 3.1 is satisfied but not Assumption 4.1. If $X_{T'+t_{i+1}}$ and the current value $X_{t_{i+1}}$ is above the target c, then the manager keeps fixing a low level for the threshold L^{i+1}, otherwise he/she fixes a high level for L^{i+1}, i.e.

$$L^{i+1} = a\, \mathbf{1}_{\{X_{T'+t_{i+1}}>c\}}\, \mathbf{1}_{\{X_{t_{i+1}}>c\}} + b(\mathbf{1}_{\{X_{T'+t_{i+1}}\le c\}} + \mathbf{1}_{\{X_{T'+t_{i+1}}>c\}}\, \mathbf{1}_{\{X_{t_{i+1}}\le c\}}).$$

In this example, the distribution of L^{i+1} (given $\boldsymbol{L}^{(i)}$) has two atoms a and b with positive probability, while the distribution of L^{i+1}, given the event $\{X_{t_{i+1}} \le c\}$, is a Dirac measure.

As in Proposition 3.4, we can introduce a family of probability measures which satisfy the following properties.

Proposition 4.3. *Under Assumption 4.1 (respectively, Assumption 4.2), there exists a family of equivalent probability measures* $\{\overline{\mathbb{Q}}^i,\ i = 1, \dots, n\}$ *such that*

(i) $\overline{\mathbb{Q}}^i$ *is identical to* $\mathbb{P}$ *on* $\mathcal{G}_T^{i-1}$*;*

(ii) *any* L^k, $k \in \{1, \dots, n\}$, *has the same conditional law given* $\mathcal{G}_0^{k-1}$ *(respectively, the same probability law) under* $\overline{\mathbb{Q}}^i$ *and* $\mathbb{P}$*;*

(iii) *under* $\overline{\mathbb{Q}}^i$*, the vector* $(L^i, \dots, L^n)$ *and* $\mathcal{G}_T^{i-1}$ *are conditionally independent given* $\mathcal{G}_0^{k-1}$ *(respectively, independent).*

Moreover, the Radon–Nikodym derivative is given by

$$\frac{\mathrm{d}\overline{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{G}_T^n} = \prod_{i=k}^{n} \frac{1}{\beta_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)})} = \frac{Z_T^{k-1}(\boldsymbol{L}^{(k-1)})}{Z_T^n(\boldsymbol{L})}$$

(respectively, $\prod_{i=k}^n 1/p_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)}) = \widetilde{Z}_T^{k-1}(\boldsymbol{L}^{(k-1)})/\widetilde{Z}_T^n(\boldsymbol{L})$*).*

4.2. Global enlargement of filtration

In this subsection, instead of assuming the density hypothesis in a successive way for the family of enlarged filtrations, we consider the random variables $L^1, \dots, L^n$ as a vector and treat the Jacod's hypothesis in the following way.

Assumption 4.3. *The* $\mathbb{F}$*-conditional law of* $\boldsymbol{L} = (L^1, \dots, L^n)$ *is equivalent to its probability law, i.e. there exists an* $\mathcal{F}_T \otimes \mathcal{E}^n$*-measurable function* $p_T(\cdot)$ *such that*

$$\mathbb{P}(\boldsymbol{L} \in \mathrm{d}\boldsymbol{x} \mid \mathcal{F}_t) = p_T(\boldsymbol{x})\mathbb{P}(\boldsymbol{L} \in \mathrm{d}\boldsymbol{x}),$$

where $\mathrm{d}\boldsymbol{x} = (\mathrm{d}x^1, \dots, \mathrm{d}x^n)$.

We denote by $(p_t(\boldsymbol{x}),\ t \in [0, T])$ the density process of $\boldsymbol{L}$ given $\mathbb{F}$, which is a $(\mathbb{P}, \mathbb{F})$-martingale for any $\boldsymbol{x} \in E^n$. Define the filtration $\mathbb{G}^{\boldsymbol{L}} = (\mathcal{G}_t^{\boldsymbol{L}})_{t\in[0,T]}$, where $\mathcal{G}_t^{\boldsymbol{L}} := \mathcal{F}_t \vee \sigma(\boldsymbol{L})$

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coincides with $\mathcal{G}_t^n$. Then $\boldsymbol{L}$ and $\mathbb{F}$ are independent under the equivalent probability measure $\mathbb{P}^L$ defined by

$$\frac{\mathrm{d}\mathbb{P}^L}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{G}_t^L} := \frac{1}{p_t(\boldsymbol{L})}.$$

We remark that $L^1, \dots, L^n$ are not mutually independent under $\mathbb{P}^L$. In particular, if $\boldsymbol{L}$ is independent of $\mathcal{F}_T$ then $p_t(\boldsymbol{L}) = 1$.

We make precise the relationship between the global approach and the successive one. In particular, we compare Assumption 4.3 with previous assumptions.

Proposition 4.4. (i) *Assumption 4.3 is equivalent to Assumption 4.1. The conditional densities are given by the following relations. On the one hand,*

$$p_T(\boldsymbol{x}) = \prod_{i=1}^{n} \beta_T^{i\,|\,i-1}(\boldsymbol{x}^{(i)})$$

and, on the other hand,

$$\beta_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x^i) = \frac{\int_{E^{n-i}} p_T(\boldsymbol{L}^{(i-1)}, x^i, \dots, x_n)\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^i)}{\int_{E^{n-i+1}} p_T(\boldsymbol{L}^{(i-1)}, x^i, \dots, x_n)\mathbb{P}(L^i \in \mathrm{d}x^i, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^{i-1})}. \tag{4.10}$$

(ii) *The probability measure $\mathbb{P}^L$ coincides with the probability measure $\overline{\mathbb{Q}}^1$ constructed in Proposition 4.3 under Assumption 4.1.*

Proof. If Assumption 4.1 holds, let $i = 0$ in Lemma 4.1, we obtain

$$\mathbb{P}(\boldsymbol{L} \in \mathrm{d}\boldsymbol{x} \mid \mathcal{F}_t) = Z_t^n(\boldsymbol{x})\mathbb{P}(\boldsymbol{L} \in \mathrm{d}\boldsymbol{x}),$$

which implies Assumption 4.3 with

$$p_t(\boldsymbol{x}) = Z_t^n(\boldsymbol{x}). \tag{4.11}$$

Moreover, by Proposition 4.3, $\mathbb{P}^L = \overline{\mathbb{Q}}^1$, which is the second assertion of the proposition.

Conversely, supposing that Assumption 4.3 holds, and $\mathbb{F}$ and $\boldsymbol{L}$ are independent under $\mathbb{P}^L$; thus, for $i = 1, \dots, n$,

$$\mathbb{P}^L(L^i \in \mathrm{d}x^i \mid \mathcal{F}_T \vee \sigma(\boldsymbol{L}^{(i-1)})) = \mathbb{P}^L(L^i \in \mathrm{d}x^i \mid \boldsymbol{L}^{(i-1)}), \qquad \mathbb{P}\text{-a.s.}$$

and we conclude, using the stability of Assumption 4.1 under an equivalent change of probability measure ($\mathbb{P}^L$ is equivalent to $\mathbb{P}$), that

$$\mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_T^{i-1})(\omega) \sim \mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_0^{i-1}).$$

Moreover, the Radon–Nikodym density $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{P}^L$ on $\mathcal{G}_T^i$ is given by

$$\begin{aligned} Q_T^i(\boldsymbol{L}^{(i)}) :=& \mathbb{E}^{\mathbb{P}^L}[p_T(\boldsymbol{L}) \mid \mathcal{G}_T^i] \\ =& \int_{E^{n-i}} p_T(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n)\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^i) \end{aligned}$$

since $\boldsymbol{L}$ and $\mathbb{F}$ are independent under $\mathbb{P}^L$ and $\mathbb{P}^L$ coincides with $\mathbb{P}$ on $\sigma(\boldsymbol{L})$. Therefore, by Remark 3.2, we obtain

$$\mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_T^{i-1}) = \frac{Q_T^i(\boldsymbol{L}^{(i-1)}, x^i)}{\int_E Q_T^i(\boldsymbol{L}^{(i-1)}, x^i)\mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_0^{i-1})}\mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_0^{i-1}),$$

which leads to (4.10). □

Proposition 4.5. (i) *Assumption 4.3 together with the condition* $\mathbb{P}(\boldsymbol{L} \in \mathrm{d}\boldsymbol{x}) \sim \prod_{i=0}^n \mathbb{P}(L^i \in \mathrm{d}x_i)$ *is equivalent to Assumption 4.2. The conditional densities are given by the following relations. On the one hand,*

$$p_T(\boldsymbol{x}) = \frac{\widetilde{Z}_T^n(\boldsymbol{x})}{\widetilde{Z}_0^n(\boldsymbol{x})} = \prod_{i=1}^n \frac{p_T^{i\,|\,i-1}(\boldsymbol{x}^{(i)})}{p_0^{i\,|\,i-1}(\boldsymbol{x}^{(i)})} \tag{4.12}$$

and, on the other hand,

$$p_T^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x^i) = \frac{\int_{E^{n-i}}(p_T/\zeta)(\boldsymbol{L}^{(i-1)}, x^i, \dots, x^n)\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1})\cdots\mathbb{P}(L^n \in \mathrm{d}x^n)}{\int_{E^{n-i+1}}(p_T/\zeta)(\boldsymbol{L}^{(i-1)}, x^i, \dots, x^n)\mathbb{P}(L^i \in \mathrm{d}x^i)\cdots\mathbb{P}(L^n \in \mathrm{d}x^n)},$$

where $\zeta(\cdot)$ *is the Radon–Nikodym density of* $\prod_{i=1}^n \mathbb{P}(L^i \in \mathrm{d}x^i)$ *with respect to* $\mathbb{P}(\boldsymbol{L} \in \mathrm{d}\boldsymbol{x})$.

(ii) *Under Assumption 4.3 and assuming* $\mathbb{P}(\boldsymbol{L} \in \mathrm{d}\boldsymbol{x}) = \zeta(\boldsymbol{x})^{-1}\prod_{i=1}^n \mathbb{P}(L^i \in \mathrm{d}x^i)$ *with* $\zeta(\cdot)$ *being a positive function on* E^n*, the equivalent probability measure* $\mathbb{Q}^L$ *defined by*

$$\frac{\mathrm{d}\mathbb{Q}^L}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{G}_T^n} = \frac{\zeta(\boldsymbol{L})}{p_T(\boldsymbol{L})} \tag{4.13}$$

satisfies

(a) $\mathbb{F}$ *and the random variables* $L^1, \dots, L^n$ *are mutually independent under* $\mathbb{Q}^L$*;*

(b) *the marginal law of each* $L_1, \dots, L_n$ *under* $\mathbb{Q}^L$ *coincide with the one under* $\mathbb{P}$.

(iii) *The probability measure* $\mathbb{Q}^L$ *coincides with the probability* $\overline{\mathbb{Q}}^1$ *defined in Proposition 4.3 under Assumption 4.2.*

Proof. (i) and (ii) Under Assumption 4.2, by Remark 4.1 and taking $i = 0$ in (4.9), we have

$$\mathbb{P}(\boldsymbol{L} \in \boldsymbol{x} \mid \mathcal{F}_T) = \widetilde{Z}_T^n(\boldsymbol{x})\mathbb{P}(L^1 \in \mathrm{d}x^1)\cdots\mathbb{P}(L^n \in \mathrm{d}x^n)$$

and, in particular,

$$\mathbb{P}(\boldsymbol{L} \in \boldsymbol{x}) = \widetilde{Z}_0^n(\boldsymbol{x})\mathbb{P}(L^1 \in \mathrm{d}x^1)\cdots\mathbb{P}(L^n \in \mathrm{d}x^n). \tag{4.14}$$

Therefore, Assumption 4.3 is true with $p_T(\boldsymbol{x}) = \widetilde{Z}_T^n(\boldsymbol{x})/\widetilde{Z}_0^n(\boldsymbol{x})$.

Conversely, we assume that Assumption 4.3 holds and the condition $\mathbb{P}(\boldsymbol{L} \in \boldsymbol{x}) \sim \prod_{i=1}^n \mathbb{P}(L^i \in \mathrm{d}x^i)$, with

$$\mathbb{P}(\boldsymbol{L} \in \boldsymbol{x}) = \zeta(\boldsymbol{x})^{-1}\prod_{i=1}^n \mathbb{P}(L^i \in \mathrm{d}x^i).$$

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Note that Assumption 4.3 implies the existence of a probability measure $\mathbb{P}^L$ equivalent to $\mathbb{P}$ such that $\boldsymbol{L}$ is independent of $\mathcal{F}_T$ under $\mathbb{P}^L$ and that $\mathbb{P}^L$ coincides with $\mathbb{P}$ on $\mathcal{F}_T$ and on $\sigma(L)$. Therefore,

$$\mathbb{P}^L(\boldsymbol{L}\in \mathrm{d}\boldsymbol{x}) = \mathbb{P}(\boldsymbol{L}\in \mathrm{d}\boldsymbol{x}) = \zeta(\boldsymbol{x})^{-1}\prod_{i=1}^n \mathbb{P}(L^i\in \mathrm{d}x^i) = \zeta(\boldsymbol{x})^{-1}\prod_{i=1}^n \mathbb{P}^L(L^i\in \mathrm{d}x^i),$$

which implies that

$$\mathbb{E}^{\mathbb{P}^L}[\zeta(\boldsymbol{L})] = \int_{E^n}\frac{\zeta(\boldsymbol{x})}{\zeta(\boldsymbol{x})}\prod_{i=1}^n \mathbb{P}^L(L^i\in \mathrm{d}x^i) = 1.$$

We introduce a new probability measure $\mathbb{Q}^L$ on $\mathcal{G}_T^{\boldsymbol{L}}$ such that $\mathrm{d}\mathbb{Q}^L/\mathrm{d}\mathbb{P}^L = \zeta(\boldsymbol{L})$, which is also given by (4.13). We then check (a) and (b) in the second assertion.

- We first prove that $\boldsymbol{L}$ and $\mathcal{F}_T$ are independent under $\mathbb{Q}^L$. Let f be a bounded Borel function on E^n and X be a bounded $\mathcal{F}_T$-measurable random variable. We have

$$\mathbb{E}^{\mathbb{Q}^L}[f(\boldsymbol{L})X] = \mathbb{E}^{\mathbb{P}^L}[\zeta(\boldsymbol{L})f(\boldsymbol{L})X] = \mathbb{E}^{\mathbb{P}^L}[\zeta(\boldsymbol{L})f(\boldsymbol{L})]\mathbb{E}^{\mathbb{P}^L}[X] = \mathbb{E}^{\mathbb{Q}^L}[f(\boldsymbol{L})]\mathbb{E}^{\mathbb{P}^L}[X],$$

  where the second equality comes from the fact that $\boldsymbol{L}$ and $\mathcal{F}_T$ are independent under $\mathbb{P}^L$. Taking $f = 1$ in the last expression leads to

$$\mathbb{E}^{\mathbb{P}^L}[X] = \mathbb{E}^{\mathbb{Q}^L}[X],$$

  therefore $\mathbb{E}^{\mathbb{Q}^L}[f(\boldsymbol{L})X] = \mathbb{E}^{\mathbb{Q}^L}[f(\boldsymbol{L})]\mathbb{E}^{\mathbb{Q}^L}[X]$.

- Moreover, the random variables $L^1,\dots,L^n$ are independent under $\mathbb{Q}^L$. Indeed, if $f_1,\dots,f_n$ are bounded Borel functions on E then

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^L}[f_1(L^1)\cdots f_n(L^n)] &= \mathbb{E}^{\mathbb{P}^L}[\zeta(\boldsymbol{L})f_1(L^1)\cdots f_n(L^n)]\\ &= \int_{E^n}\zeta(\boldsymbol{x})f_1(x^1)\cdots f_n(x^n)\mathbb{P}^L(\boldsymbol{L}\in \mathrm{d}\boldsymbol{x})\\ &= \int_{E^n} f_1(x^1)\cdots f_n(x^n)\prod_{i=1}^n \mathbb{P}^L(L^i\in \mathrm{d}x^i)\\ &= \prod_{i=1}^n \mathbb{E}^{\mathbb{P}^L}[f_i(L^i)].\end{aligned}$$

  Besides, taking $f_j = 1$ for all $j\neq i$ yields

$$\mathbb{E}^{\mathbb{Q}^L}[f_i(L^i)] = \mathbb{E}^{\mathbb{P}^L}[f_i(L^i)] = \mathbb{E}^{\mathbb{P}}[f_i(L^i)].$$

  Therefore,

$$\mathbb{E}^{\mathbb{Q}^L}[f_1(L^1)\cdots f_n(L^n)] = \prod_{i=1}^n \mathbb{E}^{\mathbb{Q}^L}[f_i(L^i)].$$

- The previous two points yield

$$\mathbb{Q}^L(L^i\in \mathrm{d}x^i \mid \mathcal{G}_T^{i-1}) = \mathbb{Q}^L(L^i\in \mathrm{d}x^i).$$

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Moreover, the Radon–Nikodym density $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}^L$ on $\mathcal{G}_T^i$ is given by

$$\mathbb{E}^{\mathbb{Q}^L}\biggl[\frac{p_T(\boldsymbol{L})}{\zeta(\boldsymbol{L})}\biggm| \mathcal{G}_T^i\biggr] = \int_{E^{n-i}} \frac{p_T}{\zeta}(\boldsymbol{L}^{(i)}, x^{i+1}, \dots, x^n)\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}) \cdots \mathbb{P}(L^n \in \mathrm{d}x^n).$$

By Remark 3.2, this implies Assumption 4.2 with

$$p_t^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x) = \frac{\int_{E^{n-i}} (p_T/\zeta)(\boldsymbol{L}^{(i-1)}, x^i, \dots, x^n)\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}) \cdots \mathbb{P}(L^n \in \mathrm{d}x^n)}{\int_{E^{n-i+1}} (p_T/\zeta)(\boldsymbol{L}^{(i-1)}, x^i, \dots, x^n)\mathbb{P}(L^i \in \mathrm{d}x^i) \cdots \mathbb{P}(L^n \in \mathrm{d}x^n)}.$$

Therefore, assertions (i) and (ii) are proved.

(iii) Finally, to prove the third assertion, it suffices to verify that $(\prod_{i=1}^n p_T^{i\,|\,i-1}(\boldsymbol{L}^{(i)}))^{-1}$ is equal to $\zeta(\boldsymbol{L})/p_T(\boldsymbol{L})$. This is a consequence of (4.12) since (4.14) leads to

$$\zeta(\boldsymbol{x}) = \frac{1}{\widetilde{Z}_0^n(\boldsymbol{x})} = \prod_{i=1}^n \frac{1}{p_0^{i\,|\,i-1}(\boldsymbol{x}^{(i)})}.$$

The proposition is thus proved. □

Remark 4.2. In the particular case where the law of $\boldsymbol{L}$ admits a density with respect to the Lebesgue measure on E^n, Assumptions 4.2 and 4.3 are equivalent.

Remark 4.3. The function ζ can be expressed in terms of copulas: $c(u^1, \dots, u^n)$ denotes the density of the copula such that

$$C(u^1, \dots, u^n) = F(F_1^{-1}(u^1), \dots, F_n^{-1}(u^n)) = \int_{-\infty}^{u^1} \cdots \int_{-\infty}^{u^n} c(u^1, \dots, u^n)\,\mathrm{d}u^1 \cdots \mathrm{d}u^n,$$

where $F_1, \dots, F_n$ are marginal distribution functions and F is the joint distribution function, then

$$\zeta(x^1, \dots, x^n) = \frac{1}{c(F_1(x^1), \dots, F_n(x^n))}. \tag{4.15}$$

4.3. Conditional expectation using the global approach

We now apply the global approach to calculate the conditional expectations with respect to the insider's filtration $\mathbb{G}^{\mathrm{I}}$, under the equivalent Assumptions 4.2 and 4.3. The idea is to use the global change of probability measure $\mathbb{P}^L$, which will make the computation easier.

Proposition 4.6. *We suppose that Assumption 4.3 holds. Let $Y_T(\boldsymbol{L})$ be a nonnegative $\mathcal{G}_T^n$-measurable random variable. Then, for $t \in [0, T]$,*

$$\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] = \sum_{i=1}^n \mathbf{1}_{[t_i, t_{i+1})}(t) \times \frac{\int_{E^{n-i}} \mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{x})p_T(\boldsymbol{x}) \mid \mathcal{F}_t]\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \boldsymbol{L}^{(i)})}{\int_{E^{n-i}} p_t(\boldsymbol{x})\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \boldsymbol{L}^{(i)})}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}.$$

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Proof. We use the change of probability measure to $\mathbb{P}^L$ constructed in the global approach of Subsection 4.2. By Bayes' formula, we have

$$\mathbf{1}_{[t_i,t_{i+1})}\,\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] = \mathbf{1}_{[t_i,t_{i+1})}\,\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{i}] = \mathbf{1}_{[t_i,t_{i+1})}\,\frac{\mathbb{E}^{\mathbb{P}^L}[(Y_T p_T)(\boldsymbol{L}) \mid \mathcal{G}_t^{i}]}{\mathbb{E}^{\mathbb{P}^L}[p_T(\boldsymbol{L}) \mid \mathcal{G}_t^{i}]}.$$

Since $\boldsymbol{L}$ and $\mathbb{F}$ are independent under $\mathbb{P}^L$, and $\mathbb{P}^L$ coincides with $\mathbb{P}$ on $\mathbb{F}$ and $\sigma(\boldsymbol{L})$, respectively, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^L}[p_T(L^1,\dots,L^n) \mid \mathcal{G}_t^{i}]\\
&\quad= \bigg(\int_{E^{n-i}} \mathbb{E}^{\mathbb{P}}[p_T(\boldsymbol{x}^{(i)},x^{i+1},\dots,x^n) \mid \mathcal{F}_t]\\
&\qquad\qquad \times \mathbb{P}(L^{i+1}\in \mathrm{d}x^{i+1},\dots,L^n\in \mathrm{d}x^n \mid \boldsymbol{L}^{(i)})\bigg)\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}\\
&\quad= \bigg(\int_{E^{n-i}} p_t(\boldsymbol{x}^{(i)},x^{i+1},\dots,x^n)\mathbb{P}(L^{i+1}\in \mathrm{d}x^{i+1},\dots,L^n\in \mathrm{d}x^n \mid \boldsymbol{L}^{(i)})\bigg)\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}},
\end{aligned}$$

where the second equality results from the martingale property of $(p_t(\boldsymbol{x}))_{t\in[0,T]}$. Moreover,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^L}[(Y_T p_T)(L^1,\dots,L^n) \mid \mathcal{G}_t^{i}]\\
&\quad= \bigg(\int_{E^{n-i}} \mathbb{E}^{\mathbb{P}}[(Y_T p_T)(\boldsymbol{x}^{(i)},x^{i+1},\dots,x^n) \mid \mathcal{F}_t]\\
&\qquad\qquad \times \mathbb{P}(L^{i+1}\in \mathrm{d}x^{i+1},\dots,L^n\in \mathrm{d}x^n \mid \boldsymbol{L}^{(i)})\bigg)\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}},
\end{aligned}$$

which completes the proof. □

Remark 4.4. By equality $p_T(\boldsymbol{x}) = Z_T^n(\boldsymbol{x})$ (see (4.11)) and relation (4.5), we see that Proposition 4.6 yields the same result as in Proposition 4.1 under Assumption 4.1.

Remark 4.5. If Assumption 4.2 is satisfied then

$$\begin{aligned}
&\mathbb{P}(L^{i+1}\in \mathrm{d}x^{i+1},\dots,L^n\in \mathrm{d}x^n \mid \boldsymbol{L}^{(i)})\\
&\quad= \frac{1}{\zeta(\boldsymbol{L}^{(i)},x^{i+1},\dots,x^n)}\mathbb{P}(L^{i+1}\in \mathrm{d}x^{i+1})\cdots\mathbb{P}(L^n\in \mathrm{d}x^n).
\end{aligned}$$

Then as a direct consequence of Proposition 4.6, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] = \sum_{i=1}^n \mathbf{1}_{[t_i,t_{i+1})}(t)\int_{E^{n-i}} &\frac{\mathbb{E}^{\mathbb{P}}[(Y_T(p_T/\zeta))(\boldsymbol{x}^{(i)},x^{i+1},\dots,x^n) \mid \mathcal{F}_t]}{(p_t/\zeta)(\boldsymbol{x}^{(i)},x^{i+1},\dots,x^n)}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}\\
&\times \prod_{k=i+1}^n \mathbb{P}(L^k\in \mathrm{d}x^k).
\end{aligned}\tag{4.16}$$

5. Application and numerical illustration

In this section we apply our framework to a default model with insider information. We are particularly interested in both the default and survival probabilities, and the pricing of defaultable bonds under different information levels.

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We consider the default time of a firm which is supposed to be the first time that a continuous $\mathbb{F}$-adapted process $(X_t,\ t \in [0, T])$ reaches a random threshold, which is determined by the manager of the firm and can be adjusted dynamically. More precisely, let the default threshold $(L_t,\ t \in [0, T])$ be given in the form of (2.2). The default time is defined by

$$\tau := \inf\{t : X_t < L_t\}, \tag{5.1}$$

where the random variables $L^1, \dots, L^n$ represent the private information of the manager on the threshold at times $t_1, \dots, t_n$, which are not available to standard investors. This model extends the one considered in [14]. To make a comparison with a standard investor, we also introduce the information filtration given by $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$, where

$$\mathcal{G}_t = \bigcap_{s\geq t} \mathcal{F}_s \vee \sigma(\tau \wedge s).$$

The filtration $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ by the random time τ and is classically used to model the available information in a default market for a standard investor, in comparison with the filtration $\mathbb{G}^{\mathrm{I}}$ which represents the insider information.

5.1. Conditional survival probability

One of fundamental quantities in the modelling of credit risk is the conditional survival probability given the available information. The following result yields the conditional survival probability given the insider information. For ease of computations, we suppose that Assumption 4.2 holds, but similar computations can be carried out under the other assumptions studied in this paper.

Proposition 5.1. *Let $0 \leq t \leq s \leq T$. We denote by i and j the indexes such that $t_i \leq t < t_{i+1}$ and $t_j \leq s < t_{j+1}$. Then*

$$\mathbb{P}(\tau > s \mid \mathcal{G}^{\mathrm{I}}_t) = \mathbf{1}_{\{\tau>t\}} \frac{\mathbb{E}^{\mathbb{P}}[\chi^i_s(\boldsymbol{x}^{(i)}) \mid \mathcal{F}_t]}{\int_{E^{n-i}} (p_s/\zeta)(\boldsymbol{x}^{(i)}, x^{i+1}, \dots, x^n) \prod_{k=i+1}^n \mathbb{P}(L^k \in \mathrm{d}x^k)}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}, \tag{5.2}$$

where, denoting $X^*_{[t,s[} := \inf_{t\leq u<s} X_u$ and $X^*_t := X^*_{[0,t[} = \inf_{0\leq u<t} X_u$, if $i < j$,

$$\chi^i_s(\boldsymbol{x}^{(i)}) = \int_{E^{n-i}} \frac{p_s}{\zeta}(\boldsymbol{x})\, \mathbf{1}_{\{X^*_{[t,t_{i+1}[}>x^i\}}\, \mathbf{1}_{\{X^*_{[t_{i+1},t_{i+2}[}>x^{i+1}\}} \cdots \mathbf{1}_{\{X^*_{[t_j,s[}>x^j\}} \prod_{k=i+1}^n \mathbb{P}(L^k \in \mathrm{d}x^k),$$

and otherwise, if $i = j$,

$$\chi^i_s(\boldsymbol{x}^{(i)}) = \int_{E^{n-i}} \frac{p_s}{\zeta}(\boldsymbol{x})\, \mathbf{1}_{\{X^*_{[t,s[}>x^i\}} \prod_{k=i+1}^n \mathbb{P}(L^k \in \mathrm{d}x^k).$$

Proof. By definitions (5.1) and (2.2), the survival event can be written as

$$\mathbf{1}_{\{\tau>s\}} = \mathbf{1}_{\{X^*_{[t_1,t_2[}>L^1\}} \cdots \mathbf{1}_{\{X^*_{[t_i,t[}>L^i\}}\, \mathbf{1}_{\{X^*_{[t,t_{i+1}[}>L^i\}} \cdots \mathbf{1}_{\{X^*_{[t_j,s[}>L^j\}}.$$

We apply (4.16) to the random variable

$$Y_T(\boldsymbol{x}) = \mathbf{1}_{\{X^*_{[t_1,t_2[}>x^1\}} \cdots \mathbf{1}_{\{X^*_{[t_i,t[}>x^i\}}\, \mathbf{1}_{\{X^*_{[t,t_{i+1}[}>x^i\}} \cdots \mathbf{1}_{\{X^*_{[t_j,s[}>x^j\}}$$

and obtain the results. □

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We also recall that for the standard information, it is well known (see [4], [9]) that, for $t \le s$,

$$\mathbb{P}(\tau > s \mid \mathcal{G}_t) = \mathbf{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > s \mid \mathcal{F}_t)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}. \tag{5.3}$$

In the following, we will compare the survival probability estimated by these two types of investor in an explicit setting, in order to show the impact of insider information.

5.2. An explicit default model

We consider now a concrete example with three periods $0 = t_1 < t_2 < t_3 = T$, where the value of the firm X follows a geometric Brownian motion (with drift μ and volatility σ). The default threshold information are renewed at t_1 and t_2 respectively as L^1 and L^2 and we suppose that L^1 and L^2 are exponential random variables with intensity λ_1 and λ_2, respectively. In addition, we assume that $\boldsymbol{L} = (L^1, L^2)$ are independent of $\mathcal{F}_T$. We note that the standard investor has the knowledge on the (marginal and joint) laws of $\boldsymbol{L}$, while the insider knows the realization of these thresholds at the renewal times of information. Let the law of $\boldsymbol{L}$ be given by a Gumbel–Barnett copula (see [13]) with parameter $0 \le \theta \le 1$, which is given by

$$C(u_1, u_2) = u_1 + u_2 - 1 + (1 - u_1)(1 - u_2)\mathrm{e}^{-\theta \ln(1-u_1)\ln(1-u_2)}.$$

Then the joint cumulative distribution function of (L^1, L^2) is given by

$$F(x_1, x_2) = 1 - \mathrm{e}^{-\lambda_1 x_1} - \mathrm{e}^{-\lambda_2 x_2} + \mathrm{e}^{-(\lambda_1 x_1 + \lambda_2 x_2 + \theta\lambda_1\lambda_2 x_1 x_2)}.$$

Moreover, by (4.15), we have

$$\frac{1}{\zeta(x_1, x_2)} = \mathrm{e}^{-(\theta\lambda_1\lambda_2 x_1 x_2)}((\theta\lambda_1 x_1 + 1)(\theta\lambda_2 x_2 + 1) - \theta).$$

Let $\nu = \mu - \sigma^2/2$. We recall that for a geometric Brownian motion X with drift μ and volatility σ starting from $X_0 = 1$, the density of the couple (X_t^*, X_t) for $t > 0$ is given by

$$f_t(u, v) = \mathbf{1}_{\{u \le v\}} \mathbf{1}_{\{0 \le u \le 1\}} \frac{2v^{\nu/\sigma^2 - 1}\ln(v/u^2)}{\sigma^3\sqrt{2\pi}t^{(3/2)}u} \exp\bigg(-\frac{\nu^2 t}{2\sigma^2}\bigg) \exp\bigg(-\frac{\ln^2(v/u^2)}{2\sigma^2 t}\bigg)$$

and the density of X_t^* is given by

$$\begin{aligned} f_{X_t^*}(w) = \mathbf{1}_{\{0 < w \le 1\}} \bigg(\frac{1}{\sqrt{2\pi t} w} \bigg(&\exp\bigg(-\frac{(-\ln(w) + \nu t)^2}{2\sigma^2 t}\bigg) \\ &+ w^{2\nu/\sigma^2} \exp\bigg(-\frac{(-\ln(w) - \nu t)^2}{2\sigma^2 t}\bigg)\bigg) \\ &- \frac{\nu}{\sigma^2} w^{2\nu/\sigma^2 - 1} \operatorname{erfc}\bigg(\frac{-\ln(w) - \nu t}{\sigma\sqrt{2t}}\bigg)\bigg), \end{aligned}$$

where $\operatorname{erfc}(x) = (2/\sqrt{\pi}) \int_x^{+\infty} \mathrm{e}^{-v^2}\,\mathrm{d}v$, $x \ge 0$, is the complementary error function.

We now present the explicit formulae for the conditional survival probabilities below.

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5.2.1. *Survival probability for $t \in [t_1, t_2)$. Insider information.* We have

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t^{\mathrm{I}}) = \mathbf{1}_{\{\tau>t\}} \int_0^{+\infty} \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{\{X^*_{[t,t_2[}>x^1\}} \mathbf{1}_{\{X^*_{[t_2,T[}>y\}} \mid \mathcal{F}_t]_{x^1=L^1} \frac{1}{\zeta(L^1, y)} \lambda_2 \mathrm{e}^{-\lambda_2 y}\,\mathrm{d}y$$

since $\int_0^{+\infty}(1/\zeta(L^1, y))\lambda_2\mathrm{e}^{-\lambda_2 y}\,\mathrm{d}y = 1$. To compute more explicitly this quantity, we need the joint law of the running minimum $(X^*_{[t,t_2[}, X^*_{[t_2,s[})$. Using the result of [7], we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[\mathbf{1}_{\{X^*_{[t_2,T[}>y\}} \mid \mathcal{F}_{t_2}] &= \mathbf{1}_{\{y\le X_{t_2}\}}\bigg(1 - \frac{1}{2}\mathrm{erfc}\bigg(\frac{\ln(X_{t_2}/y) + \nu(T-t_2)}{\sigma\sqrt{2(s-t_2)}}\bigg)\bigg).\\ &\quad - \frac{1}{2}\bigg(\frac{y}{X_{t_2}}\bigg)^{2\nu/\sigma^2} \mathrm{erfc}\bigg(\frac{\ln(X_{t_2}/y) - \nu(T-t_2)}{\sigma\sqrt{2(T-t_2)}}\bigg)\\ &=: G(X_{t_2}, y).\end{aligned}$$

Furthermore, using the Markov property and the joint law of $(X^*_{t_2-t}, X_{t_2-t})$, leads to

$$\mathbb{E}^{\mathbb{P}}[\mathbf{1}_{\{X^*_{[t,t_2[}>x^1\}} \mathbf{1}_{\{X^*_{[t_2,T[}>y\}} \mid \mathcal{F}_t] = \iint \mathbf{1}_{\{uX_t>x^1\}} G(vX_t, y) f_{t_2-t}(u, v)\,\mathrm{d}u\,\mathrm{d}v.$$

Standard information. For the progressive information, we use (5.3) where successive conditioning implies that

$$\begin{aligned}&\mathbb{P}(\tau > T \mid \mathcal{F}_t)\\ &\quad = \int_0^1\int_0^{+\infty}\int_0^1 \exp(-\lambda_1 \min(X_t^*, uX_t) - \lambda_2(vwX_t) - \theta\lambda_1\lambda_2 \min(X_t^*, uX_t)(vwX_t))\\ &\qquad\qquad \times f_{X^*_{T-t_2}}(w) f_{t_2-t}(u, v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u.\end{aligned}$$

5.2.2. *Survival probability for $t \in [t_2, T)$.* Straightforward computations imply the following results.

Insider information. We have

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t^{\mathrm{I}}) = \mathbf{1}_{\{\tau>t\}} \int_u^1 f_{X^*_{T-t}}(w)\,\mathrm{d}w|_{u=L^1/X_t}.$$

Standard information. We have

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbf{1}_{\{\tau>t\}} \frac{\int_0^1 F(X^*_{t_2}, \min(X^*_{[t_2,t[}, wX_t)) f_{X^*_{T-t}}(w)\,\mathrm{d}w}{F(X^*_{t_2}, X^*_{[t_2,t[})}.$$

5.3. Numerical results

In this subsection we compare the survival probabilities for the insider and the standard investor by numerical examples. We use the default time model described previously. The value of the parameters are $\mu = 0.05$, $\sigma = 0.8$, $\lambda_1 = 1.5$ and $\lambda_2 = 1$, $t_1 = 0$, $t_2 = 1$, and $t_3 = T = 2$. In particular, we analyse the impact of the correlation between L^1 and L^2 through the parameter θ. The $\theta = 0$ case corresponds to the independence case. We present two examples. In the first one, there is a default event before the maturity, and in the second one, there is no default. In each example, we compare the survival probabilities $\mathbb{P}(\tau > T \mid \mathcal{G}_t^{\mathrm{I}})$ and $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ on a given trajectory of the value of the firm.

In the first example, in Figure 1 we present the realized trajectory of the value of the firm. We suppose that the manager adjusts the threshold level at $t_2 = 1$ from $L^1 = 0.8$ to $L^2 = 1.5$, so there is a high risk of default after time t_2, which is larger than the expected value. We observe

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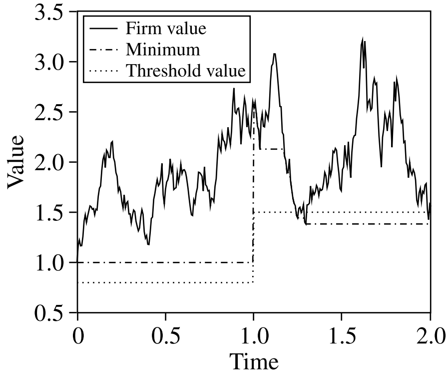

FIGURE 1: First case: default during $[1, 2]$, $L^1 = 0.8$, $L^2 = 1.5$.

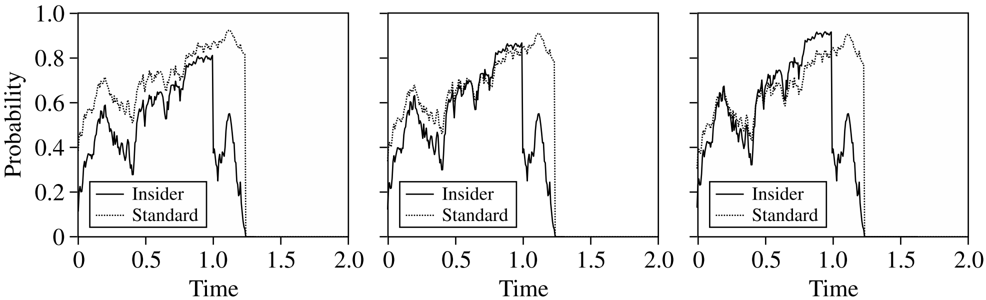

FIGURE 2: Survival probabilities $\mathbb{P}(\tau > T \mid \mathcal{G}_t^{\mathrm{I}})$ and $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ for $\theta = 0, 0.5$ and 1.

from the three graphs in Figure 2 that in all the cases (for different values of θ), the insider will modify immediately the estimations on the survival probability and there is an instantaneous jump at t_2. While the standard investor, who is not accessible to this information, maintains the survival probability at a high level and can adjust the estimation only when the default occurs effectively. Finally, comparing the three graphs where the correlation between L^1 and L^2 varies, we see that when the time approaches t_2, since the value of the firm is at a relatively high level compared to L^1, when there is a strong correlation (with larger θ) between the two thresholds, the insider will have a higher estimation for the survival probability than when there is independence. However, such a difference between the estimations due to different values of θ will be neutralized once the insider obtains the exact information on L^2 at time t_2.

In the second example where the sample path of the value of the firm is given by Figure 3, there is no default before the maturity T. In addition, we suppose that the level of the second threshold $L^2 = 0.6$ is slightly lower than the first one $L^1 = 0.8$ and is close to the expected value. So there is no important readjustment of the insider's estimation at t_2, as shown in all three graphs in Figure 2. However, when the value of the firm descends gradually after time t_2 and approaches the threshold level L^2, the estimations of the survival probability by the insider has dropped significantly; see Figure 4. Only when the value of the firm begins to go back up and when the time approaches the maturity, the insider modifies once again the survival probability to be higher. In contrast, the estimations by the standard investor remain quite stable during all periods in this example. The comparison between the correlation parameter θ is similar to the first example. Since the value of the firm is at a high level during the first

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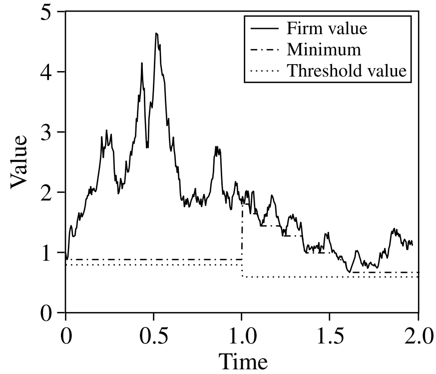

FIGURE 3: Second case: no default , $L^1 = 0.8$, $L^2 = 0.6$.

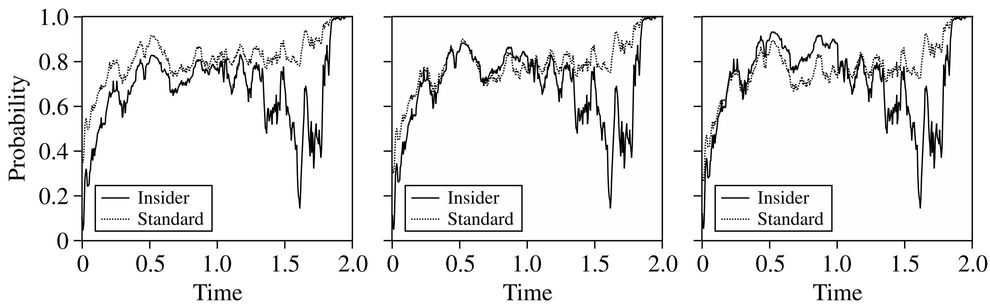

FIGURE 4: Survival probabilities $\mathbb{P}(\tau > T \mid \mathcal{G}_t^{\mathrm{I}})$ and $\mathbb{P}(\tau > T \mid \mathcal{G}_t)$ for $\theta = 0, 0.5$ and 1.

period, if $\theta = 1$, the insider has a higher estimation for the survival probability than in the case if $\theta = 0$. However, such differences are visible only before the second information renewal time.

Appendix A

A.1. Proof of Proposition 4.1

The goal of this subsection is to apply Theorem 3.1 to compute $\mathbb{G}^{\mathrm{I}}$-conditional expectations under Assumption 4.1. We begin by calculating, in several lemmas below, the recursive operators in Theorem 3.1 in an explicit manner and then state the proof of Proposition 4.1. Throughout this subsection Assumption 4.1 holds.

Lemma A.1. *Let $i \in \{1, \dots, n\}$ and $t \in [t_i, T]$. If $X_t(\boldsymbol{L}^{(i)})$ is a nonnegative $\mathcal{G}_t^i$-measurable random variable then*

$$J_i(X_t(\boldsymbol{L}^{(i)}))$$
$$= \int_E \frac{\mathbb{E}^{\mathbb{P}}[X_t(\boldsymbol{x}^{(i)}) Z_t^{i-1}(\boldsymbol{x}^{(i-1)}) \mid \mathcal{F}_{t_i}]}{Z_{t_i}^{i-1}(\boldsymbol{x}^{(i-1)})}\bigg|_{\boldsymbol{x}^{(i-1)}=\boldsymbol{L}^{(i-1)}} \beta_{t_i}^{i\,|\,i-1}(\boldsymbol{L}^{(i-1)}, x^i)\mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_0^{i-1}),$$

where $Z_t^i(\boldsymbol{x}^{(i)})$ is defined by (4.3).

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Proof. We recall the operation J_i defined by (3.16). By (3.17), we have

$$J_i(X_t(\boldsymbol{L}^{(i)})) = \int_E \frac{\mathbb{E}^{\mathbb{P}}[X_t(\boldsymbol{x}^{(i)})\psi_t^{i-1}(\boldsymbol{x}^{(i-1)})^{-1}]}{\psi_{t_i}^{i-1}(\boldsymbol{x}^{(i-1)})^{-1}}\bigg|_{\boldsymbol{x}^{(i-1)}=\boldsymbol{L}^{(i-1)}} \mathbb{P}(L^i \in \mathrm{d}x^i \mid \mathcal{G}_{t_i}^{i-1}).$$

Note that

$$\psi_t^{i-1}(\boldsymbol{x}^{(i-1)})^{-1} = \prod_{k=1}^{i-1} \alpha_t^{k\,|\,k-1}(\boldsymbol{x}^{(k)}) = \prod_{k=1}^{i-1} \frac{\beta_t^{k\,|\,k-1}(\boldsymbol{x}^{(k)})}{\beta_{t_k}^{k\,|\,k-1}(\boldsymbol{x}^{(k)})} = Z_t^{i-1}(\boldsymbol{x}^{(i-1)}) \prod_{k=1}^{i-1} \frac{1}{\beta_{t_k}^{k\,|\,k-1}(\boldsymbol{x}^{(k)})},$$

where the first equality comes from (3.10), the second equality follows from (4.2), and the last equality results from (4.3). Similarly, we have

$$\psi_{t_i}^{i-1}(\boldsymbol{x}^{(i-1)})^{-1} = \prod_{k=1}^{i-1} \frac{\beta_{t_i}^{k\,|\,k-1}(\boldsymbol{x}^{(k)})}{\beta_{t_k}^{k\,|\,k-1}(\boldsymbol{x}^{(k)})} = Z_{t_i}^{i-1}(\boldsymbol{x}^{(i-1)}) \prod_{k=1}^{i-1} \frac{1}{\beta_{t_k}^{k\,|\,k-1}(\boldsymbol{x}^{(k)})}.$$

Therefore,

$$\frac{\mathbb{E}^{\mathbb{P}}[X_t(\boldsymbol{x}^{(i)})\psi_t^{i-1}(\boldsymbol{x}^{(i-1)})^{-1}]}{\psi_{t_i}^{i-1}(\boldsymbol{x}^{(i-1)})^{-1}} = \frac{\mathbb{E}^{\mathbb{P}}[X_t(\boldsymbol{x}^{(i)})Z_t^{i-1}(\boldsymbol{x}^{(i-1)}) \mid \mathcal{F}_{t_i}]}{Z_{t_i}^{i-1}(\boldsymbol{x}^{(i-1)})}.$$

By (4.1), we obtain the announced equality. □

Lemma A.2. *The pricing kernel (3.24) is given, under Assumption 4.1, by*

$$\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)}) = \frac{\beta_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})}{\beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})} \tag{A.1}$$

and

$$J_{i+1}(\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)})) = 1. \tag{A.2}$$

Proof. We have

$$\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)}) = (J_{i+2} \circ \cdots \circ J_n)(\alpha_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i)}) \cdots \alpha_T^{n\,|\,n-1}(\boldsymbol{L}^{(n)})).$$

By Lemma A.1, this can be expressed it as the integral of

$$\mathbb{E}^{\mathbb{P}}\bigg[\alpha_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)}) \prod_{j=i+2}^{n} \bigg(\alpha_{t_{j+1}}^{j\,|\,j-1}(\boldsymbol{x}^{(j)}) \frac{Z_{t_{j+1}}^{j-1}(\boldsymbol{x}^{(j-1)})}{Z_{t_j}^{j-1}(\boldsymbol{x}^{(j-1)})} \beta_{t_j}^{j\,|\,j-1}(\boldsymbol{x}^{(j)})\bigg) \;\bigg|\; \mathcal{F}_{t_{i+2}}\bigg]_{\boldsymbol{x}^{(i+1)}=\boldsymbol{L}^{(i+1)}}$$
$$= \frac{\beta_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})}{\beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})} \frac{\mathbb{E}[Z_T^n(\boldsymbol{x}) \mid \mathcal{F}_{t_{i+2}}]_{\boldsymbol{x}^{(i+1)}=\boldsymbol{L}^{(i+1)}}}{Z_{t_{i+2}}^{i+1}(\boldsymbol{L}^{(i+1)}, x^{i+2})}$$
$$= \frac{\beta_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})}{\beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})} \frac{Z_{t_{i+2}}^n(\boldsymbol{L}^{(i+1)}, x^{i+2}, \ldots, x^n)}{Z_{t_{i+2}}^{i+1}(\boldsymbol{x}^{(i+2)})}$$

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with respect to $\mathbb{P}(L^{i+2} \in \mathrm{d}x^{i+2}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^{i+1})$. By (4.5) we obtain the first equality. We then apply Lemma A.1 to write $J_{i+1}(\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)}))$ as

$$\begin{aligned}
&\int_E \mathbb{E}^{\mathbb{P}}\bigg[\frac{\beta_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)})}{\beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)})}\frac{Z_{t_{i+2}}^{i}(\boldsymbol{x}^{(i)})}{Z_{t_{i+1}}^{i}(\boldsymbol{x}^{(i)})}\beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)}) \;\bigg|\; \mathcal{F}_{t_{i+1}}\bigg]_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1} \mid \mathcal{G}_0^{i}) \\
&\quad = \int_E \mathbb{E}^{\mathbb{P}}\bigg[\frac{Z_{t_{i+2}}^{i}(\boldsymbol{x}^{(i)})}{Z_{t_{i+1}}^{i}(\boldsymbol{x}^{(i)})}\beta_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)}) \;\bigg|\; \mathcal{F}_{t_{i+1}}\bigg]_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1} \mid \mathcal{G}_0^{i}) \\
&\quad = \int_E \mathbb{E}^{\mathbb{P}}\bigg[\frac{Z_{t_{i+2}}^{i+1}(\boldsymbol{x}^{(i+1)})}{Z_{t_{i+1}}^{i}(\boldsymbol{x}^{(i)})} \;\bigg|\; \mathcal{F}_{t_{i+1}}\bigg]_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1} \mid \mathcal{G}_0^{i}).
\end{aligned}$$

Note that, by Lemma 4.1, we have

$$\mathbb{P}(L^1 \in \mathrm{d}x^1, \dots, L^i \in \mathrm{d}x^i \mid \mathcal{F}_t) = Z_t^i(\boldsymbol{x}^{(i)})\mathbb{P}(L^1 \in \mathrm{d}x^1, \dots, L^i \in \mathrm{d}x^i \mid \mathcal{F}_0).$$

Therefore, $(Z_t^{i+1}(\boldsymbol{x}^{(i+1)}))_{t\in[0,T]}$ is an $(\mathbb{F}, \mathbb{P})$-martingale, so we obtain

$$J_{i+1}(\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)})) = \int_E \beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{L}^{(i)}, x^{i+1})\mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1} \mid \mathcal{G}_0^{i}) = 1. \qquad \square$$

Proof of Proposition 4.1. Let $Y_T(\boldsymbol{L})$ be a nonnegative $\mathcal{G}_T^n$-measurable random variable. Then, for $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{L}) \mid \mathcal{G}_t^{\mathrm{I}}] = \sum_{i=1}^{n} \mathbf{1}_{[t_i,t_{i+1})}(t) \int_{E^{n-i}} &\frac{\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{x})Z_T^n(\boldsymbol{x}) \mid \mathcal{F}_t]}{Z_t^i(\boldsymbol{x}^{(i)})}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \\
&\times \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}_0^{i}). \qquad \square
\end{aligned}$$

Proof. Apply Theorem 3.1 and compute the sequence of random variables $(Y_{t_{i+1}}(\boldsymbol{L}^{(i)}))_{i=0}^n$ under Assumption 4.1. By the backward recursive relation (3.23) and equalities (A.1) and (A.2), we have

$$Y_{t_{i+1}}(\boldsymbol{L}^{(i)}) = \frac{J_{i+1}(Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)}))}{J_{i+1}(\Phi_{t_{i+2}}(\boldsymbol{L}^{(i+1)}))} = J_{i+1}\bigg(Y_{t_{i+2}}(\boldsymbol{L}^{(i+1)})\frac{\beta_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})}{\beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{L}^{(i+1)})}\bigg),$$

where the second equality comes from (3.19). By Lemma A.1, we can write it as

$$\begin{aligned}
&\int_E \mathbb{E}^{\mathbb{P}}\bigg[Y_{t_{i+2}}(\boldsymbol{x}^{(i+1)})\frac{\beta_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)})}{\beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)})}\frac{Z_{t_{i+2}}^{i}(\boldsymbol{x}^{(i)})}{Z_{t_{i+1}}^{i}(\boldsymbol{x}^{(i)})}\beta_{t_{i+1}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)}) \;\bigg|\; \mathcal{F}_{t_{i+1}}\bigg]_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \\
&\quad \times \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1} \mid \mathcal{G}_0^{i}) \\
&\qquad = \int_E \mathbb{E}^{\mathbb{P}}\bigg[Y_{t_{i+2}}(\boldsymbol{x}^{(i+1)})\frac{Z_{t_{i+2}}^{i}(\boldsymbol{x}^{(i)})}{Z_{t_{i+1}}^{i}(\boldsymbol{x}^{(i)})}\beta_{t_{i+2}}^{i+1\,|\,i}(\boldsymbol{x}^{(i+1)}) \;\bigg|\; \mathcal{F}_{t_{i+1}}\bigg]_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1} \mid \mathcal{G}_0^{i}).
\end{aligned}$$

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Therefore, it follows that $Y_{t_{i+1}}(\boldsymbol{L}^{(i)})$ is the integral

$$\int_{E^{n-i}} \mathbb{E}^{\mathbb{P}}\biggl[Y_T(\boldsymbol{x}) \prod_{j=i+1}^{n} \frac{Z^{j-1}_{t_{j+1}}(\boldsymbol{x}^{(j-1)})}{Z^{j-1}_{t_j}(\boldsymbol{x}^{(j-1)})} \beta^{j\,|\,j-1}_{t_{j+1}}(\boldsymbol{x}^{(j)}) \;\Big|\; \mathcal{F}_{t_{i+1}}\biggr]_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}$$
$$\times\, \mathbb{P}(L^n \in \mathrm{d}x^n \mid \mathcal{G}^{n-1}_0) \cdots \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1} \mid \mathcal{G}^i_0)$$
$$= \int_{E^{n-i}} \frac{\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{x}) Z^n_T(\boldsymbol{x}) \mid \mathcal{F}_{t_{i+1}}]}{Z^i_{t_{i+1}}(\boldsymbol{x}^{(i)})}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}^i_0).$$

We deduce that, for $t \in [t_i, t_{i+1})$,

$$\frac{\mathbb{E}^{\mathbb{P}}[Y_{t_{i+1}}(\boldsymbol{x}^{(i)}) \psi_{t_{i+1}}(\boldsymbol{x}^{(i)})^{-1} \mid \mathcal{F}_t]}{\psi^i_t(\boldsymbol{x}^{(i)})^{-1}}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}}$$
$$= \int_{E^{n-i}} \frac{\mathbb{E}^{\mathbb{P}}[Y_T(\boldsymbol{x}) Z^n_T(\boldsymbol{x}) \mid \mathcal{F}_t]}{Z^i_t(\boldsymbol{x}^{(i)})}\bigg|_{\boldsymbol{x}^{(i)}=\boldsymbol{L}^{(i)}} \mathbb{P}(L^{i+1} \in \mathrm{d}x^{i+1}, \dots, L^n \in \mathrm{d}x^n \mid \mathcal{G}^i_0).$$

The proposition is thus proved. □

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