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FINITE GROUPS WITH LARGE CHERMAK–DELGADO LATTICES

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Abstract

Given a finite group G, we denote by $L(G)$ the subgroup lattice of G and by $\mathcal{CD}(G)$ the Chermak–Delgado lattice of G. In this note, we determine the finite groups G such that $|\mathcal{CD}(G)| = |L(G)| - k$, for $k = 1, 2$.

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1. Introduction

Let G be a finite group and $L(G)$ be the subgroup lattice of G. The *Chermak–Delgado measure* of a subgroup H of G is defined by

$$m_G(H) = |H||C_G(H)|.$$

Let

$$m^*(G) = \max\{m_G(H) \mid H \le G\} \quad \text{and} \quad \mathcal{CD}(G) = \{H \le G \mid m_G(H) = m^*(G)\}.$$

Then the set $\mathcal{CD}(G)$ forms a modular, self-dual sublattice of $L(G)$, which is called the *Chermak–Delgado lattice* of G. It was first introduced by Chermak and Delgado [4] and revisited by Isaacs [5]. In the last few years, there has been a growing interest in understanding this lattice (see [1–3, 6–8, 11–14]). We recall several important properties of the Chermak–Delgado measure:

- if $H \le G$, then $m_G(H) \le m_G(C_G(H))$, and if the measures are equal, then $C_G(C_G(H)) = H$;
- if $H \in \mathcal{CD}(G)$, then $C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$;
- the maximal member M of $\mathcal{CD}(G)$ is characteristic and $\mathcal{CD}(M) = \mathcal{CD}(G)$;
- the minimal member $M(G)$ of $\mathcal{CD}(G)$ (called the *Chermak–Delgado subgroup* of G) is characteristic, abelian and contains $Z(G)$.

In [12], the Chermak–Delgado measure of G has been seen as a function:

$$m_G : L(G) \longrightarrow \mathbb{N}^*, \quad H \mapsto m_G(H) \quad \text{for all } H \in L(G).$$

If G is nontrivial, then m_G has at least two distinct values, or equivalently $\mathcal{CD}(G) \neq L(G)$ (see [11, Corollary 3]). This leads to the following natural question.

QUESTION 1.1. How large can the lattice $\mathcal{CD}(G)$ be?

The dual problem of finding finite groups with small Chermak–Delgado lattices has been studied in [6, 7].

Our main result is stated as follows.

THEOREM 1.2. *Let G be a finite group. Then:*

(a) $|\mathcal{CD}(G)| = |L(G)| - 1$ *if and only if* $G \cong \mathbb{Z}_p$ *or* $G \cong Q_8$*;*
(b) $|\mathcal{CD}(G)| = |L(G)| - 2$ *if and only if* $G \cong \mathbb{Z}_{p^2}$*.*

For the proof of the above theorem, we need the following well-known result (see, for example, [10, Volume II, (4.4)]).

THEOREM 1.3. *A finite p-group has a unique subgroup of order p if and only if it is either cyclic or a generalised quaternion 2-group.*

We recall that a *generalised quaternion 2-group* is a group of order 2^n for some positive integer $n \geq 3$, defined by

$$Q_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2, a^{2^{n-1}} = 1, b^{-1}ab = a^{-1}\rangle.$$

We also need the following theorem taken from [12].

THEOREM 1.4. *Let G be a finite group. For each prime p dividing the order of G and $P \in \mathrm{Syl}_p(G)$, let $|Z(P)| = p^{n_p}$. Then*

$$|\mathrm{Im}(m_G)| \geq 1 + \sum_p n_p. \tag{1.1}$$

Finally, we indicate a natural open problem concerning the above study.

OPEN PROBLEM. Determine the finite groups G such that $|\mathcal{CD}(G)| = |L(G)| - k$, where $k \geq 3$.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [5]. For subgroup lattice concepts, we refer to [9].

2. Proof of the main result

First of all, we solve the problem for generalised quaternion 2-groups.

LEMMA 2.1. *With the above notation:*

(a) $|\mathcal{CD}(Q_{2^n})| = |L(Q_{2^n})| - 1$ *if and only if* $n = 3$*, that is,* $G \cong Q_8$*;*
(b) $|\mathcal{CD}(Q_{2^n})| \neq |L(Q_{2^n})| - 2$ *for all* $n \geq 3$*.*

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PROOF. We easily obtain

$$m^*(Q_{2^n}) = 2^{2n-2} \quad \text{for all } n \geq 3,$$

and

$$\mathcal{CD}(Q_{2^n}) = \begin{cases} \{Q_8, \langle a\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2\rangle\} & \text{if } n = 3 \\ \{\langle a\rangle\} & \text{if } n \geq 4. \end{cases}$$

These lead immediately to the desired conclusions. □

We are now able to prove our main result.

PROOF OF THEOREM 1.2. We divide the proof into two parts corresponding to the two parts of the theorem.

PART (a). Since $|\mathcal{CD}(G)| = |L(G)| - 1$, we have $\mathcal{CD}(G) = L(G) \setminus \{H_0\}$, where $H_0 \leq G$. We infer that $|\text{Im}(m_G)| = 2$ and so G is a p-group with $|Z(G)| = p$ by Theorem 1.4. Then $m_G(1) < m_G(Z(G))$, implying that

$$H_0 = 1 \quad \text{and} \quad m^*(G) = m_G(Z(G)) = p^{n+1},$$

where $|G| = p^n$.

Assume that there exists $H \leq G$ with $|H| = p$ and $H \neq Z(G)$. Then $H \notin \mathcal{CD}(G)$, which shows that $H = 1$ and this contradicts the hypothesis. Thus, G has a unique subgroup of order p and Theorem 1.3 leads to

$$G \cong \mathbb{Z}_{p^n} \quad \text{or} \quad G \cong Q_{2^n} \quad \text{for some } n \geq 3. \tag{2.1}$$

In the first case, we easily get $n = 1$, that is, $G \cong \mathbb{Z}_p$, while in the second one, we get $G \cong Q_8$ by Lemma 2.1(a).

PART (b). The condition $|\mathcal{CD}(G)| = |L(G)| - 2$ means $\mathcal{CD}(G) = L(G) \setminus \{H_1, H_2\}$, where $H_1, H_2 \leq G$. Then $|\text{Im}(m_G)| \leq 3$. Recall that we cannot have $|\text{Im}(m_G)| = 1$.

If $|\text{Im}(m_G)| = 2$, then

$$m_G(H_1) = m_G(H_2) \neq m^*(G)$$

and again G is a p-group with $|Z(G)| = p$. It is clear that one of the two subgroups H_1 and H_2 must be trivial, say $H_1 = 1$. Then G has at most two subgroups of order p, namely $Z(G)$ and possibly H_2. This implies that it has exactly one subgroup of order p because the number of subgroups of order p in a finite p-group is congruent to $1 \pmod p$. Consequently, one obtains again (2.1). For $G \cong \mathbb{Z}_{p^n}$, we easily get $n = 2$, that is, $G \cong \mathbb{Z}_{p^2}$, while for $G \cong Q_{2^n}$, we get no solution by Lemma 2.1(b).

If $|\text{Im}(m_G)| = 3$, then $m_G(H_1)$, $m_G(H_2)$ and $m^*(G)$ are distinct. Also, (1.1) becomes

$$3 \geq 1 + \sum_p n_p.$$

Since $n_p \geq 1$ for all p, we have the two possibilities described in Cases 1 and 2.

Case 1: $|G| = p^n$ and $|Z(G)| \in \{p, p^2\}$.

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Obviously, if G is abelian, we get $G \cong \mathbb{Z}_{p^2}$. Assume that G is not abelian. Since $m_G(1) < m_G(Z(G)) = m_G(G)$, we infer that one of the two subgroups H_1 and H_2 is trivial and that

$$m^*(G) = m_G(Z(G)) = m_G(G).$$

If $|Z(G)| = p$, then G has a unique subgroup of order p and so it is a generalised quaternion 2-group, contradicting Lemma 2.1(b). The same can also be said when $|Z(G)| = p^2$ because all subgroups of order p of G are outside of $\mathcal{CD}(G)$.

Case 2: $|G| = p^n q^m$ and the Sylow p-subgroups and q-subgroups of G have centres of orders p and q, respectively.

Let P be a Sylow p-subgroup and Q be a Sylow q-subgroup of G. Since $P \subseteq C_G(Z(P))$, we have

$$m_G(Z(P)) = p\,|C_G(Z(P))| = p^{n+1}q^x \quad \text{for some } x \text{ with } 0 \le x \le m,$$

and similarly,

$$m_G(Z(Q)) = p^y q^{m+1} \quad \text{for some } y \text{ with } 0 \le y \le n.$$

Also,

$$m_G(1) = p^n q^m \quad \text{and} \quad m_G(G) = p^n q^m |Z(G)|.$$

We observe that the measures $m_G(Z(P))$, $m_G(Z(Q))$ and $m_G(1)$ are distinct and consequently they are all possible measures of the subgroups of G. We distinguish two subcases.

Subcase 2.1: $Z(G) = 1$.

Then $m^*(G) = m_G(1) = m_G(G)$. Indeed, if $m^*(G) = m_G(Z(P))$, then 1, G and $Z(Q)$ will be outside of $\mathcal{CD}(G)$, and this contradicts the hypothesis. In the same way, we cannot have $m^*(G) = m_G(Z(Q))$. Since $m_G(P)$ is divisible by p^{n+1} and $m_G(Q)$ is divisible by q^{m+1}, we infer that $m_G(P) = m_G(Z(P))$ and $m_G(Q) = m_G(Z(Q))$. Thus, $P, Z(P), Q, Z(Q) \notin \mathcal{CD}(G)$ and our hypothesis implies that $P = Z(P)$ and $Q = Z(Q)$, that is, G is a nonabelian group of order pq. Assume that $p < q$. Then $\mathcal{CD}(G)$ consists of the unique subgroup of order q of G and therefore we obtain $|L(G)| = 3$, and this contradicts the hypothesis.

Subcase 2.2: $Z(G) \neq 1$.

Then $m_G(1) < m_G(G)$, which shows that $m_G(G)$ equals either $m_G(Z(P))$ or $m_G(Z(Q))$. Assume that $m_G(G) = m_G(Z(P))$. Then $x = m$ and $|Z(G)| = p$, implying that

$$Z(G) = Z(P). \tag{2.2}$$

Note that we cannot have $m^*(G) = m_G(Z(Q))$ because in this case, 1, $Z(G)$ and G will be outside of $\mathcal{CD}(G)$, and this contradicts the hypothesis. Consequently,

$$m^*(G) = m_G(Z(P)) = m_G(P) = m_G(Z(G)) = m_G(G).$$

It follows that 1, $Z(Q)$ and Q are not contained in $\mathcal{CD}(G)$, which leads to $Q = Z(Q)$. In other words, $\mathcal{CD}(G) = L(G) \setminus \{1, Q\}$. Thus, $\mathcal{CD}(G)$ is the lattice interval

$$[G/Z(G)] = \{H \in L(G) \mid Z(G) \leq H \leq G\}$$

and [11, Corollary 2] shows that G is nilpotent. Then $G = P \times Q$ and it follows that $Z(G) = Z(P) \times Q$, which contradicts (2.2).

This completes the proof. □

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