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DYNAMICS OF A SIMPLE ENDOGENOUS GROWTH MODEL WITH FINANCIAL INTERMEDIATION

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We study an impact of the financial intermediation on economic growth. We assume the simple model of the economic growth in the form of an autonomous dynamical system with a financial sector represented by banks and real sector represented by households and firms. We assume that financial intermediation services are described by financial intermediation technology which is a function depending on the share of labor employed by banks. Investments realized by firms depend not only on savings accumulated by banks but also on financial intermediation technology. We obtain a three-dimensional dynamical system and analyze the existence of a saddle equilibrium in the growth process associated with financial intermediation. Using mathematical methods of dynamical systems, we analyze growth paths, and we study the stationary states of the system and their stability. We found that equilibrium is reached only by trajectories located on two submanifolds. The resulting analysis provides an insight into the saddle solution with a stable incoming separatrix lying on one of the invariant manifolds.

Keywords: Endogenous Growth, Financial Intermediation, Dynamical Systems, Saddle Equilibrium

1. INTRODUCTION

In the study of economic growth, one of the determinants which attracts a serious interest is financial intermediation [McKinnon (1973); Shaw (1973); Bencivenga and Smith (1991); Pagano (1993); Demetriades and Hussein (1996); Greenwood and Smith (1997); Benhabib and Spiegel (2000); Levine et al. (2000); Rioja and Valev (2004); Aghion et al. (2005); Levine (2005)]. Financial intermediation plays an important role in the process of money transformation from savings to investment. In this paper, we build a theoretical model that is able to capture and explain the dynamics of economic growth with the financial intermediation as one of its determinants.

The presented model of economic growth with financial intermediation is a generalization of the model proposed by Eggoh and Villieu (2014). It differs from

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Eggoh and Villieu's model as the former assumes that the increase in the labor force provided by the employees is a positive function of the capital stock change (investment). Our goal is to show the importance of financial intermediation for economic growth and to study the dynamics of the economic growth model with financial intermediation.

Research on the impact of financial sector development on economic growth is the starting point for this analysis. The growth of the financial sector is positively correlated with an increase in the level of economic savings and an increase in the efficiency of capital accumulation, which translates into economic growth. The strength of the presented analysis lies in drawing dynamical connections between financial intermediation and economic growth. The empirical research shows an ambiguous impact of the development of the financial sector on economic growth. The same endogenous factors influence on the level of development of the financial sector and the level of economic growth.

Banks have a profound influence on economic growth and economy as a whole. They are financial channel savings from households to those who need funding, in our case companies. It is reasonable to suppose that the credit financing in the economy is intermediated through the banking system. The analysis of the relation between financial intermediation and economic growth is crucial for understanding the phenomenon of uneven growth. The theory of economic growth and intermediation processes is based on the efficient allocation of resources. To understand the economic behavior of companies and households, it is convenient to think of a subject as being described by a utility function that summarizes its preferences.

McKinnon (1973) and Shaw (1973) emphasized the different aspects of financial intermediation and its impact on economic growth. The literature on economic growth with financial intermediation has shown that the finance sector has a significant effect on economic growth (Levine, 2005). We look in more detail into how one of the most important functions of banks, changes in intermediation technology, relates to the economic growth from only a theoretical perspective. The empirical literature investigates the growth reaction of financial intermediation and changes in intermediation technology has focused only on a number of banking services rather than the quality.

Empirical studies on the influence of financial development on economic growth are not decisive. For example Beck et al. (2000), considering the impact of financial intermediaries on savings and investments in the economy, showed a statistically significant impact on economic growth. Arcand et al. (2015) found that as long as the level of credit to the private sector falls below a threshold of about 80%–100% of aggregate GDP, there is a positive marginal effect of a financial depth on economic growth in economies. However, Ram (1999) found that financial development has a negative impact.

Many studies have relied on the assumption made by Levine (1997, 2005) that economic growth is closely linked to the liquidity function in the banking sector. Therefore, the intermediation process plays a key role as it makes household deposits more liquid, while it invests a part of deposits into companies investments. In their model of economic growth, Bencivenga and Smith (1991) showed that financial institutions allocate a greater part of an economy's savings toward long-term financial instruments or investments. Levine (1997) showed that the industrial revolution may not have occurred without the transformation of liquidity in the financial sector.

Levine (2001) argued that capital flow liberalization accelerates economic growth. International financial flows support liquidity in the real economy, which accelerates productivity growth. Competition from different banks has an effect on becoming more efficient and raise productivity in the real sector. The latest studies have revealed ambiguous results by identifying the existence of a non-linear and non-monotonic relationship between economic growth and financial intermediation [Law (2014); Bucci and Marsiglio (2019); Bucci et al. (2019)].

The main difficulty has been in identifying the growth of investments that is caused by changes in the intermediation process. The aim of this paper is to take a step forward in understanding the mechanics of financial intermediation and changes in financial technology and to obtain a quantitative measure of the effects on economic growth.

In the presented model, we specify the form of the intermediation technology function $\phi(\theta)$. This model depends on the banking employment θ, especially we assume that there is a distinguished employment level in the banking sector. This parameter describing the threshold of employment level, which cannot be overcrossed, is crucial for predicting/controlling the economic growth in the model.

There are two features of the presented model worth highlighting: (i) the endogenous nature and labor employed in the banking sector, (ii) the formulation of the model as the dynamical system which allows analyzing the stability of equilibria and the identification of the type of bifurcations due to change of model parameters. We assume that labor employed by banks cannot be greater than a threshold level. We found that this threshold level of labor is reached at a critical point. The solutions going to the stable equilibrium are located on two two-dimensional submanifolds.

2. THE MODEL

We consider the three-dimensional model in which the economy consists of households, firms representing the productive sector and banks representing the financial sector. In the presented model, financial intermediation is the process performed by banks in which they take in deposits from households and then lends them out to companies. We assume that a direct cash flow between households and companies is not possible. This means that banks are the intermediaries that make it possible to transform the stream of household savings into investment in the productive sector.

For competing firms, we assume that the output produced by the i-th firm $Y_i(t)$ depends on the capital inputs of $K_i(t) \geq 0$ and the units of effective labor $A(t)L_i(t) \geq 0$ in the form of the Cobb–Douglas production function

$$Y_i(t) = K_i(t)^{\alpha}[A(t)L_i(t)]^{1-\alpha},$$

where $\alpha \in (0, 1)$ and $t \in [0, +\infty)$. The amount of knowledge depends on the capital of the whole economy $A(t) = K(t)$. This assumption is the consequence of the knowledge becoming the most important form of capital of enterprises. Thereby, the dependence of traditional resources of capital (technology, land, and money) on knowledge capital has been increasingly deepened on a global scale.

The household maximizes the utility from consumption subject to a standard budget constraint which includes the value of the consumption

$$U = \int_0^{\infty} e^{-\rho t} u(C(t))dt$$

where $C(t) = \frac{C_{\text{total}}(t)}{N} > 0$ represents consumption per capita, C_{total} is the total consumption in the economy, and ρ is the discount rate. The total supply of labor is constant and normalized to unity ($N = 1$).

We assume that the level of household savings accumulated in banks depends on the level of wages $w(t)$, the interest on the bank deposit $r^b B(t)$, dividends received from banks $\Delta(t)$, and the consumption of a single household $C(t)$ (defined as household disposable income, plus a change in net equity in household's bank account, and less consumption)

$$\dot{B}(t) = r^b B(t) + w(t) + \Delta(t) - C(t).$$

with the non-Ponzi game condition $\lim_{t\to\infty} \left[B(t) \exp\left(-\int_0^t r^b(v)\, dv\right)\right] \geq 0$.

The growth rate of consumption depends on savings

$$\frac{\dot{C}(t)}{C(t)} = s\left(r^b - \rho\right).$$

where $r^b - \rho$ is net interest rate spread (a leading determinant of a financial institution's profitability or lack thereof).

Each bank j ($j = 1, \ldots, n$) in the process of financial intermediation transforms household savings into corporate loans using $\theta_j(t)$ labor units

$$\dot{K}_j(t) = \phi(\theta_j(t))\dot{B}_j(t).$$

where $\dot{K}_j(t) = \int_0^1 \dot{K}_i(t)\, di$ is the credit level for i-th company and $\phi(\theta_j(t))$ represents financial intermediation technology (the cost of financial intermediation and different factors driving inefficient intermediation on the banking institution). We assume that $\phi(\theta_j(t)) \leq 1$, $\phi'(\theta_j(t)) > 0$ and $\phi''(\theta_j(t)) < 0$.

Banks and companies want their bottom line to be as profitable as possible. As a competitive firm, the bank must choose prices (interest rates) to maximize profits

$$\max_{(K_i(t),L_i(t))}\{\Pi_i^F(t) = K_i(t)^{\alpha}(L_i(t)K(t))^{1-\alpha} - r_j(t)K_i(t) - w(t)L_i(t)\}$$
$$\max_{(K_j(t),L_j(t))}\{\Pi_j(t)^B = r_j(t)K_j(t) - r^b B_j(t) - w(t)\theta_j(t)\}.$$

As a financial intermediary, a bank must solve the informational problems that exist between companies and households (moral hazard and adverse selection). The bank must also choose prices (interest rates) to maximize profits. The two first-order conditions for the maximization of profit of i-th company are

$$\alpha K_i(t)^{\alpha-1}\,(L_i(t)K(t))^{1-\alpha} = r_j(t)$$
$$(1-\alpha)\,K_i(t)^{\alpha}L_i(t)^{-\alpha}K(t)^{1-\alpha} = w(t).$$

The marginal productivity of capital and labor must be equal to the cost of the credit interest rate and the actual wage level.

We assume that there is no information asymmetry so that in a symmetric equilibrium each bank determines the interest rate according to which the interest-bearing corporate loans are at the same level $r_j(t) = r(t)$, where $j \in \{1, \ldots, n\}$.

Since the interest rates on credit in an equilibrium are the same for every j-th bank, the level of employment in each individual bank behaves similarly. This means that the total financial sector labor is described by $\theta_{\text{total}}(t) = n\theta(t)$. Financial intermediaries are able to transform the risk characteristics of assets because they can resolve an information asymmetry problem. Information asymmetry in financial markets arises because companies know generally more about their investment projects than households do. Banks specialize in collecting information, evaluating projects, and risk sharing. They reduce the cost of channeling money between relatively uninformed households and companies. It leads to a more efficient allocation of investments.

The rate of change in the share of employment in a bank depends on the rate of change in the level of savings (it is a measure of the change in the share of labor employed in a bank)

$$\frac{\dot{\theta}}{\theta} = (a_1 - \theta)\frac{\dot{B}}{B}, \quad a_1 > 0.$$

where $a_1 \in (0, 1)$ is the threshold level of bank sector employment. This threshold level is important in our analysis because we assume that the entire population cannot be employed in one sector. In this case, the employment rate of change in the bank sector is proportional to the rate of change of deposits.

From one of the conditions of the first order of maximizing the profit of the bank, we have $\alpha K_i(t)^{\alpha-1}(L_i(t)K(t))^{1-\alpha} = r_j(t)$, so in the symmetrical equilibrium

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(it is an equilibrium where all banks use the same strategy) we have

$$r(t) = \alpha K(t)^{\alpha-1}(L(t)K(t))^{1-\alpha} = \alpha L(t)^{1-\alpha} = \alpha(1 - n\theta(t))^{1-\alpha}.$$

However, in the market of goods, the investments are equal to savings, in the symmetrical equilibrium

$$\dot{K} = K(t)L(t)^{1-\alpha} - C(t) = K(t)(1 - n\theta(t))^{1-\alpha} - C(t),$$

and

$$\frac{\dot{K}(t)}{K(t)} = (1 - n\theta(t))^{1-\alpha} - \frac{C(t)}{K(t)}.$$

We note that household transfers can be presented as follows:

$$\Delta(t) = \Pi(t)^F + \Pi(t)^B + [1 - \phi(\theta(t))]\dot{B}(t),$$

where

$$\begin{aligned}\Pi(t)^F &= K(t)^\alpha(L(t)K(t))^{1-\alpha} - r(t)K(t) - w(t)L(t)\\ \Pi(t)^B &= r(t)K(t) - r^b(t)B(t) - w(t)\theta(t).\end{aligned}$$

Hence, we get

$$\Delta(t) = Y(t) - [1 - \phi(\theta(t))]\dot{B}(t) - w(t) - r^b B(t).$$

Taking a household budget constraint, we receive

$$Y(t) - C(t) = \phi(\theta(t))\dot{B}(t) = \dot{K}(t).$$

To obtain the final form of the dynamical system, let us define two new variables

$$c = \frac{C}{K}, \quad b = \frac{B}{K}.$$

and keep the third variable θ. In these variables, we obtain the following three-dimensional dynamical system

$$\dot{c}(t) = s\left(r^b - \rho\right)c(t) - (1 - n\theta(t))^{1-\alpha}\, c(t) + c(t)^2 \tag{1a}$$

$$\dot{b}(t) = \left(\frac{1}{\phi(\theta(t))} - b(t)\right)\left[(1 - n\theta(t))^{1-\alpha} - c(t)\right] \tag{1b}$$

$$\dot{\theta}(t) = \left(\frac{a_1\theta(t) - \theta(t)^2}{\phi(\theta(t))b(t)}\right)\left[(1 - n\theta(t))^{1-\alpha} - c(t)\right]. \tag{1c}$$

We analyze this nonlinear dynamical system of economic growth with financial intermediation using the methods of dynamical systems (Perko, 2001).

3. THE LOCAL STABILITY ANALYSIS

First, let us assume that

$$\phi(z(t)) = z(t)^{\beta},$$

and reletter variables

$$x = c, \qquad y = b, \qquad z = \theta.$$

In these variables, we have the following three-dimensional dynamical system

$$\frac{dx}{dt} = \gamma x(t) - (1 - nz(t))^{1-\alpha} x(t) + x^2(t) \tag{2a}$$

$$\frac{dy}{dt} = (z^{-\beta}(t) - y(t))[(1 - nz(t))^{1-\alpha} - x(t)] \tag{2b}$$

$$\frac{dz}{dt} = y^{-1}(t)(a_1 z^{1-\beta}(t) - z^{2-\beta}(t))[(1 - nz(t))^{1-\alpha} - x(t)]. \tag{2c}$$

where $\gamma = s(r^b - \rho)$.

The phase space of the system (2) for the economic meaning of model variables is restricted to

$$E = \{(x, y, z) \colon x > 0, y > 0, z > 0\}.$$

PROPOSITION 1. *In the phase space E, the system (2) has only one critical point p^* which is located at*

$$x^* = (1 - na_1)^{1-\alpha} - \gamma, \qquad y^* = a_1^{-\beta}, \qquad z^* = a_1, \tag{3}$$

where $0 < a_1 < 1/n$. The critical point (3) is of the saddle type.

Let us consider the local stability of the critical point (3). Its stability is characterized by the linearization matrix evaluated at this critical point p^*

$$A = \begin{bmatrix} (1 - na_1)^{1-\alpha} - \gamma & 0 & n(1-\alpha)(1 - na_1)^{-\alpha}[(1 - na_1)^{1-\alpha} - \gamma] \\ 0 & -\gamma & -\beta\gamma a_1^{-\beta-1} \\ 0 & 0 & -a_1\gamma \end{bmatrix}.$$

Then its characteristic equation is given as

$$\det[A - \lambda\mathbb{I}] = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0,$$

and the eigenvalues of the linearization matrix are real and equal to terms on the diagonal

$$\lambda_1 = (1 - na_1)^{1-\alpha} - \gamma, \qquad \lambda_2 = -\gamma, \qquad \lambda_3 = -a_1\gamma.$$

Because we consider the critical point only in the domain E, its coordinates (x^*, y^*, z^*) should be strictly positive. This is guaranteed if the condition $x^* = (1 - na_1)^{1-\alpha} - \gamma$ is satisfied. As a consequence, we obtain that λ_1 should be positive. As γ and a_1 are positive, two other eigenvalues are negative.

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Hence, the critical point (3) has all eigenvalues real, two with negative values and one with a positive value. And the phase space E is a direct sum $E = E_{\text{stable}} \oplus E_{\text{unstable}}$ of the two-dimensional stable submanifold ($\dim E_{\text{stable}} = 2$) and the one-dimensional unstable submanifold ($\dim E_{\text{unstable}} = 1$).

4. INVARIANT SUBMANIFOLDS

Let us rearrange the first and third equations of the system (2)

$$\frac{dx}{dt} = x(t)[\gamma - (1 - nz(t))^{1-\alpha} + x(t)] \quad \text{(4a)}$$

$$\frac{dy}{dt} = (z^{-\beta}(t) - y(t))[(1 - nz(t))^{1-\alpha} - x(t)] \quad \text{(4b)}$$

$$\frac{dz}{dt} = y^{-1}(t)z^{1-\beta}(t)(a_1 - z(t))[(1 - nz(t))^{1-\alpha} - x(t)]. \quad \text{(4c)}$$

We can see that the first equation can be zero and the third equation can be zero which lead to two different two-dimensional submanifolds.

PROPOSITION 2. *The system (4) has a two-dimensional invariant submanifold*

$$\{(x, y, z) \in E\colon x = (1 - nz(t))^{1-\alpha} - \gamma\}. \quad \text{(5)}$$

This submanifold is a plane parallel to the (y, z)-plane of the coordinate system. On this invariant submanifold, we have the two-dimensional dynamical system

$$\frac{dy}{dt} = (z^{-\beta}(t) - y(t))\gamma \quad \text{(6a)}$$

$$\frac{dz}{dt} = y^{-1}(t)z^{1-\beta}(t)(a_1 - z(t))\gamma. \quad \text{(6b)}$$

where $\gamma = s(r^b - \rho)$.

PROPOSITION 3. *The system (6) in the invariant submanifold (5) in the domain $\{(y, z)\colon y > 0, z > 0\}$ has a unique critical point of a stable node which the position is*

$$y^* = a_1^{-\beta}, \qquad z^* = a_1. \quad \text{(7)}$$

The linearization matrix evaluated at the point (7) has the form

$$A = \begin{bmatrix} -\gamma & -\beta\gamma a_1^{-\beta-1} \\ 0 & -\gamma a_1 \end{bmatrix}.$$

As $\gamma > 0$ and $a_1 > 0$, then we have $\operatorname{tr} A = -\gamma(1 + a_1) < 0$ and $\det A = \gamma^2 a_1 > 0$. As the discriminant $\Delta = (\operatorname{tr} A)^2 - 4 \det A = \gamma(1 - a_1)^2 > 0$, the eigenvalues of linearization matrix are real, and the critical point (7) is a stable node.

Let us consider the second submanifold.

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PROPOSITION 4. *The system (4) has a two-dimensional invariant submanifold*

$$\{(x, y, z) \in E \colon z = a_1\}. \tag{8}$$

It represents a plane parallel to the (x, y)-plane of the coordinate system.

The system on this invariant submanifold has the form of two-dimensional dynamical system

$$\frac{dx}{dt} = \delta x(t) + x^2(t) \tag{9a}$$

$$\frac{dy}{dt} = (a_1^{-\beta} - y(t))(\gamma - \delta - x(t)) \tag{9b}$$

where $\delta = \gamma - (1 - na_1)^{1-\alpha}$.

PROPOSITION 5. *The system (9) in the invariant submanifold (8) in the domain* $\{(x, y) \colon x > 0, y > 0\}$ *has a unique critical point of a saddle type which the position is*

$$x^* = -\delta, \qquad y^* = a_1^{-\beta}. \tag{10}$$

Because δ is positive, this critical point belongs to the positive quadrant of the (x, y) coordinate system. Moreover, eigenvalues of its linearization matrix

$$A = \begin{bmatrix} -\delta & 0 \\ 0 & -\gamma \end{bmatrix}.$$

are $\lambda_1 = -\delta, \lambda_2 = -\gamma$. Because $\delta < 0$ and $\gamma > 0$, we have $\lambda_1\lambda_2 < 0$, that is, eigenvalues are real of the different signs. This means that the critical point (10) is of the saddle type.

PROPOSITION 6. *In the enlarged phase space with the boundary* $\{y \colon x = 0\}$*, there exists an additional critical point of the stable node type (located on the y-axis)*

$$x^* = 0, \qquad y^* = a_1^{-\beta}. \tag{11}$$

The linearization matrix at this critical point is

$$A = \begin{bmatrix} \delta & 0 \\ 0 & \delta - \gamma \end{bmatrix}.$$

The eigenvalues of the linearization matrix are $\lambda_1 = \delta < 0$, $\lambda_2 = \delta - \gamma < 0$. Therefore, if we consider the phase space with the adjoining y-axis, we see that the stable node critical point organizes its structure.

For the completeness of the dynamical analysis, it could be convenient to calculate eigenvectors. They define directions of (incoming and outcoming) separatrices of the saddle (11).

For eigenvalue $\lambda_1 = -\delta$ we find that the eigenvector $v_1 = [1, 0]$ and for the eigenvalue $\lambda_2 = -\gamma$ we choose the eigenvector $v_2 = [0, 1]$.

For illustration, the phase portrait of the system (9) is presented in Figure 1.

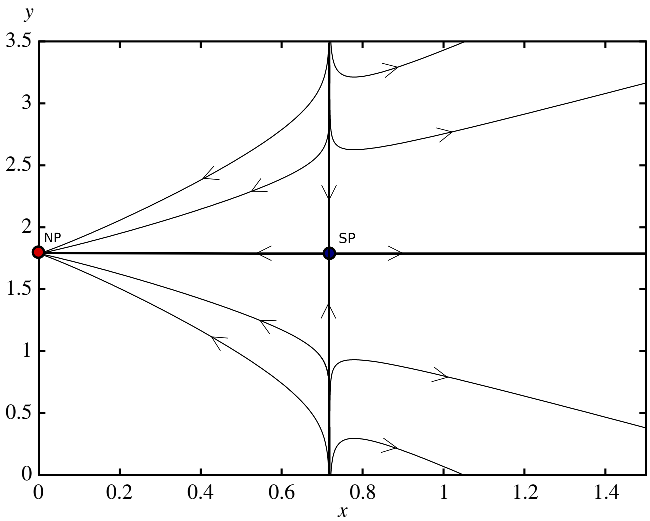

FIGURE 1. The phase portrait of the system (9). The saddle point is denoted as SP and the node is denoted as NP.

5. THE EXACT FORM SOLUTION

From the previous section analysis, we obtain the following corollary.

COROLLARY 1. *Dynamical system (2) admits a straight-line exact solution*

$$x = -\delta, \qquad y = a_1^{-\beta}, \qquad z = a_1.$$

This line defines the location of the saddle point in the three-dimensional phase space and directions of the separatrix.

The dynamical control procedure distinguishes from the economic point of view the vertical incoming separatrix going toward the saddle. On this line $x = \text{const} = -\delta$ and y is varying $y = y(t)$. Analogously, the horizontal outcoming separatrix is going to the stable node as $y = \text{const} = a_1^{-\beta}$ and $x = x(t)$. Of course, both relations $y = y(t)$ and $x = x(t)$ on separatrices as well as an exact solution for all trajectories for all admissible initial conditions can be given.

Let us integrate dynamical system on the invariant submanifold $z = a_1^{-\beta}$. After substituting $z = a_1^{-\beta}$ into (2) and separating variables, we obtain

$$\frac{dx}{x(x+\delta)} = dt,$$

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Integration of both sides gives

$$x(t) = \frac{\delta}{e^{-|\delta(t+C)|} - 1}. \tag{12}$$

To determine the constant of integration, let $x(t = t_0) = x_0$, then $x_0 = \frac{-\delta}{1-e^{-|\delta(t_0+C)|}}$ and after some calculations we obtain

$$e^{-|\delta C|} = \left(1 + \frac{\delta}{x_0}\right) e^{|\delta t_0|}. \tag{13}$$

Putting (13) into (12), $x(t, x_0)$ is given by an exact formula

$$x(t, x_0) = \frac{-\delta}{1 - e^{-|\delta(t-t_0)|}\left(1 + \frac{\delta}{x_0}\right)}. \tag{14}$$

This relation represents the canonical logistic relation.

The separation of the variable y for the system on the invariant submanifold is given

$$\frac{dy}{a_1^{-\beta} - y} = (1 - na_1)^{1-\alpha} + x(t).$$

Integration of both sides gives

$$-\ln |y - a_1^{-\beta}| = (1 - na_1)^{1-\alpha} t + \int^t dt + C.$$

Thus

$$y - a_1^{-\beta} = \pm \left(Ce^{-(1-na_1)^{1-\alpha}t} e^{-\int^t dt}\right).$$

For the incoming separatrix $y \to y^* = a_1^{-\beta}$ as $t \to \infty$. For the outcoming separatrix as $t \to -\infty$ then it reaches $y = y^*$.

The integral $\int x(t)\, dt$ can be calculated after putting (14)

$$y(t) = a_1^{-\beta} \pm C \exp[-(1 - na_1)^{1-\alpha} t] \exp\left[-\int^t \frac{-\delta}{1 - e^{-|\delta(t-t_0)|}(1 + \frac{\delta}{x_0})}\, dt\right].$$

where the constant C is chosen from the initial condition $y(t = t_0) = y_0$. With this initial condition, we have

$$y(t) = y_0 + \exp[-(1 - na_1)^{1-\alpha} t] \exp\left[-\int_{t_0}^t \frac{-\delta}{1 - e^{-|\delta(t-t_0)|}(1 + \frac{\delta}{x_0})}\, dt\right]. \tag{15}$$

Evaluating the integral in (15), we obtain the final solution

$$y(t) = y_0 + e^{-(1-na_1)^{1-\alpha} t}\left[1 + \frac{1}{\left(1 + \frac{\delta}{x_0}\right) e^{\delta t}}\right].$$

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From the economic point of view, there is an interesting case of an incoming separatrix as a solution of dynamical control problem. Therefore, let us consider the stable path reaching the saddle point $x^* = -\delta, y^* = a_1^{-\beta}$.

In equation (9b), we put $z = a_1$ and $x = -\delta$ and along the incoming separatrix we obtain

$$\frac{dy}{dt} = \left(a_1^{-\beta} - y\right)\left[(1 - na_1)^{1-\alpha} + \delta\right],$$

or

$$d(\ln|y - a_1^{-\beta}|) = -\gamma\, dt. \qquad \textbf{(16)}$$

where $\gamma = (1 - na_1)^{1-\alpha} + \delta$.

Integrating equation (16), we obtain

$$|y - a_1^{-\beta}| = Ce^{-\gamma t}.$$

Next, we assume the initial condition $y(t - t_0) = y_0$. And, along the incoming separatrix, we obtain

$$y(t) = a_1^{-\beta} + (y_0 - a_1^{-\beta})e^{-\gamma t}.$$

This incoming separatrix reaches the saddle point at $y \to a_1^{-\beta}$ as $t \to \infty$.

6. SADDLE–NODE BIFURCATION

In this section, we study local bifurcation in the system (1). Let us restrict the analysis to the bifurcation of codimension 1. Assume that we have a three-dimensional dynamical system

$$\dot{x} = f_1(x, y, z)$$
$$\dot{y} = f_2(x, y, z)$$
$$\dot{z} = f_3(x, y, z).$$

The characteristic equation has the form

$$\lambda^3 - (\operatorname{tr} A)\lambda^2 + [(\operatorname{tr} A)^2 - \operatorname{tr} A^2]\lambda - \det A = 0.$$

where $\operatorname{tr} A$ and $\det A$ are a trace and a determinant of the linearization matrix A, respectively. The coefficients of characteristic equation can be expressed in terms of eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$. The quantities are given by

$$\operatorname{tr} A = \lambda_1 + \lambda_2 + \lambda_3$$
$$\operatorname{tr} A^2 - (\operatorname{tr} A)^2 = \lambda_1\lambda_2 + \lambda_1\lambda_2 + \lambda_2\lambda_3$$
$$\det A = \lambda_1\lambda_2\lambda_3.$$

In our case, these quantities assume the following form:

$$\begin{aligned} \operatorname{tr} A &= (1-na_1)^{1-\alpha} - (2+a_1)\gamma \\ \operatorname{tr} A^2 - (\operatorname{tr} A)^2 &= [(1-na_1)^{1-\alpha} - \gamma](1+a_1)\gamma - a_1\gamma^2 \\ \det A &= [(1-na_1)^{1-\alpha} - \gamma]a_1\gamma^2. \end{aligned}$$

In the dynamical system with continuous time, the local bifurcation appears when the value of the real part of eigenvalues $\lambda(p)$ crosses zero as we change a parameter p.

Let us denote p^* a critical value of the bifurcation parameter. It could be useful to distinguish two generic cases (Kuznetsov, 2004). First, when real part of eigenvalues crosses zero: $\lambda(p^*) = 0$, the system undergoes saddle–node bifurcation; second, when real part of complex and conjugate eigenvalue $\lambda(p) = \xi(p) \pm i\omega(p)$ crosses zero then the system undergoes the Hopf bifurcation.

PROPOSITION 7. *The saddle–node bifurcation arises if and only if* $\det A = 0$.

There are three possibilities. The first case is trivial as a bank has no employment $a_1 = 0$. In the second case $r^b = \rho$. The third case is given by the following equation $(1-na_1)^{1-\alpha} = (r^b - \rho)$.

If $\delta = 0$, then the position of the node and saddle coincide is the bifurcation value of a saddle–node bifurcation.

PROPOSITION 8. *The Hopf bifurcation gives rise to the limit cycle either attractive (supercritical) or repulsive (subcritical) if and only if* $\det A = [\operatorname{tr} A^2 - (\operatorname{tr} A)^2](\operatorname{tr} A)$ *and* $\operatorname{tr} A^2 - (\operatorname{tr} A)^2 > 0$.

One can conclude the Hopf bifurcation does not appear as the condition $\det A = [\operatorname{tr} A^2 - (\operatorname{tr} A)^2](\operatorname{tr} A)$ is not satisfied for any model parameters.

7. CONCLUSIONS

In this paper, we considered the model of economic growth with financial intermediation to investigate the relation between the banking sector and the economic growth and look in detail into how financial intermediation relates to the economic growth from a theoretical perspective.

In the model, the households optimize the utility from consumption and their savings are transformed into investment through the bank system. It is assumed that there is some level of employment in the bank system such that there is no change of bank employment with respect to the change of the deposits.

We obtained that the dynamics of the growth model with financial intermediation can be represented as a three-dimensional dynamical system in variables: a ratio of consumption to capital, a ratio of bank deposits to capital, and the level of employment in the bank system. Additional aspects of financial intermediation process such as financial intermediation technology $\phi(\theta)$ and employment in banking system θ should be taken into account in order to make economic growth more predictable.

We showed that the optimal path is localized on the level of constant $\theta = a_1$ which means that the bank system is employing the a_1 units of labor which corresponds to the threshold level of employment in the bank system. We obtained the exact solution for the separatrix incoming to the saddle point. The ratio of consumption to capital stock is constant on this trajectory, and the ratio of deposits to capital stock increases to the value which depends on the threshold employment in the bank sector.

We established the two two-dimensional submanifolds. For any initial conditions located on the first submanifold, trajectories are going to the stable equilibrium. On the second submanifold, there is a saddle path incoming to the equilibrium. For this incoming separatrix, we found an exact form of the solution.

The saddle–node bifurcation was found in the model. Due to this bifurcation, the saddle critical point is created toward which the system evolves along the stable optimal path.

The phase portrait showing the trajectories for all initial conditions to illustrate the dynamics of the invariant submanifold was presented.

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