Navigational Stimuli in the Development of Mathematical Science

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This paper looks at the stimulus given by the practice and theory of navigation to certain problems in mathematical astronomy. The need for more accurate techniques of finding latitude and longitude, brought about largely by the great voyages of discovery and exploration as well as an increase in sea trade, gave rise to navigational instrument makers who produced devices of increasing accuracy. These craftsmen also made better measuring devices for the new observatories. Improvements in measurements led not only to new discoveries, but also made greater demands of theories underlying the practice of navigation. This gave impetus to the search for solutions to related problems in mathematical astronomy. Methods and special functions developed in this context were eventually to find application in a much wider range of problems in theoretical physics and engineering.

1. INTRODUCTION. Mathematical astronomy was a major stimulus to the development of certain areas of mathematics: mathematical astronomy itself was sharpened by the need to solve the longitude problem. In many large enterprises the required scientific understanding often develops side by side with the technology necessary to put the science to practical purposes. Two recent examples are the development of the atomic bomb, and the commitment to land a man on the Moon. More than three hundred years ago, the need to find longitude at sea was just such an undertaking. Although many possible solutions were proposed, it was the astro-navigational approach that was to prove the most promising. The longitude problem provides an interesting case of how the progress of abstract sciences can be driven by practical considerations.

The major part of this paper will be concerned with the mathematical spin-off that resulted from theoretical demands of the need to produce almanacs which could be used by mariners to find longitude. However, there was an earlier requirement for improved angular measuring devices and methods of calculating necessary nautical tables to find latitude. There were some very early links between astronomy, navigation and lunar almanacs.

2. VOYAGING STARS AND THE ORIGINS OF THE CONSTELLATIONS. The constellations provided the first framework against which movements of the Sun, Moon and planets could be analysed. It is not known when constellations were first drawn up in the form we know them today, but some speculations on the origins of the constellations link them with navigational practices of the Minoans who sailed about the Aegean sometime between 1600 and 1400 B.C.

Professor Alexander Gurshtein, a Russian historian of science, has recently discussed the origin of the zodiac constellations, and speculated that they arose from changes in the constellations in which the equinoxes and solstices occurred, brought about by the precession of the equinoxes over a period of five thousand

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years.[1] Thus he claims that in 5000 B.C. the winter solstice, the spring equinox, the summer solstice and the autumn equinox occurred in Pisces, Sagittarius, Virgo and Gemini respectively. By 2700 B.C. these points had shifted to Aquarius, Scorpio, Leo and Taurus, as a result of precession, and to Capricorn, Libra, Cancer and Aries by 1200 B.C. He does not discuss the origin of the constellations that do not lie along the ecliptic. However, other astronomers have written on the origins of all the constellations.

In 1965 Ovenden, then from Glasgow University, analysed the poem *The Phaenomena* by Aratus (315–250 B.C.) in his attempts to find out when the constellations were first put down in systematic form.[2] Aratus was a poet and not an astronomer, but his poem was a celebration of the astronomical work of Eudoxus (409–356 B.C.), whose book on astronomy, by the same name, did not survive. Eudoxus was what we today would call a theoretical astronomer. He made very few observations of his own but tried to formulate theories explaining the observations of others. Aratus's *The Phaenomena* contains a description of the sky and from this we can work out the latitude of the location and the date to which the description applies. Ovenden, and later Roy,[3] also from Glasgow University, concluded that the sky was consistent with what would have been visible from the Cyclades and Crete sometime between 1600 and 1400 B.C.

Roy speculates that the constellations were drawn up to act as navigation aids by Minoan navigators as they sailed from one island to the next. It is well known that a set of stars with nearly the same angular distance from the celestial equator can be used at rising and setting to mark compass directions. This method of navigation is still used by the inhabitants of the Caroline Islands in the Pacific.[4]

3. THE ORIGINS OF LUNAR ALMANACS. The method of lunar distances for finding longitude naturally makes use of lunar almanacs, and the drawing up of such tables provided a strong spur to the general improvement of mathematical methods which could be used to calculate ephemerides of the Moon and the planets. The need to produce such sets of tables goes back to the time of ancient Babylon, and even at this earlier stage in the history of science we find that lunar almanacs also provided the motivation for developing a very basic form of mathematical astronomy.

At first the Babylonians started their lunar month on the day when the thin crescent of the Moon was sighted in the west just after sunset. The purpose of their theoretical astronomy was to free themselves of the necessity of making the actual observation, and to allow them to calculate when, in principle, such a sighting would be possible. In order to do this they had to find a way to describe in numerical terms the varying angular speeds of Sun and Moon against the zodiac constellations, the angles which their respective paths made with each other and the angle between the line joining their centres and the horizon.

They used the arithmetical equivalent of our graphical zig-zag and step functions. This was later extended to predict when eclipses were likely to occur and to calculate relative positions of the planets with respect to each other and to the stellar background. It should be stressed that their approach was not based on any geometrical model of how the planets, Earth, Sun and Moon were arranged in space.

Professor Aaboe,[5] nevertheless, considered their methods important to subsequent work in theoretical predictions:

> Thus the astronomical tradition in the West is linked to Babylonian astronomy. Mathematical astronomy was, however, not only the principal carrier and generator of certain mathematical techniques, but it became the model for the new exact sciences which learned from it their principal goal: to give a mathematical description of a particular class of natural phenomena capable of yielding numerical predictions that can be tested against observations.

4. LATITUDE, ANGULAR MEASURING DEVICES AND NAUTICAL TABLES. The need to find latitude more accurately made greater demands on those who made angular measuring devices than requirements of terrestrial surveying had done. So this was an important stimulus to the growth of the instrument-making industry. Although observatories were beginning to be established, they were far less numerous than ships and each ship required instruments for navigational purposes. Greater accuracy in measurements had to be matched by improvements in the precision of nautical tables, in particular Pole Star tables and declination tables for the Sun. The Portuguese, inspired by the great voyages of discovery, made important contributions in this direction. However, it was the longitude problem that was to tax the ingenuity of the craftsmen, instrument makers and the astronomers.

5. FOUNDING OF OBSERVATORIES AT PARIS AND GREENWICH. Two possible methods of finding longitude had been proposed before the founding of the two great observatories at Paris and Greenwich. Soon after his discovery of Jupiter's moons, Galileo suggested they could be used as a celestial clock for the purposes of finding longitude. Part of the early work of the Paris Observatory, founded in 1667, was to use the moons of Jupiter to determine the longitude of places along the coastline and boundaries of France. However, this method obviously could not be used at sea because of the difficulties of using a telescope to find the positions of these moons, with respect to the planet, on board a ship. Nevertheless, the method required a detailed study of the movements of the moons and this led Olaus Roemer to discover the fact that light had a finite and measurable speed.

The Greenwich Observatory was founded in 1675 to seek a way of finding longitude using our own Moon. This was made very clear in the Royal Warrant issued by Charles II (June 22, 1675):

> Whereas, in order to the finding out of the longitude of places for perfecting navigation and astronomy, we have resolved to build a small observatory within our park at Greenwich...

An earlier warrant, issued on March 4, 1675, concerned the appointment of the first Astronomer Royal, the Rev. John Flamsteed:

> Whereas, we have appointed our trusty and well-beloved John Flamsteed,..., our astronomical observator, forthwith to apply himself with the most care and diligence to rectifying the tables of motions of the heavens, and the places of the fixed stars, so as to find out the so much-desired longitude of places for perfecting the art of navigation.[6]

The wording of the Royal Warrants was most probably the work of Christopher Wren. It was this careful wording that focused the attention of astronomers at Greenwich, and thus Greenwich was to become one of the most important observatories in the world, for its work on positional astronomy. Two astronomical problems needed to be solved to make the method of lunar distances at sea practicable. The first problem was to determine positions of stars with a greater accuracy than that found in any previous catalogues, because the stellar background would provide the reference system against which navigators could measure the Moon's position. The second was to study movements of the Moon and to develop a theory of the Moon's motion which could be used to produce necessary tables of the Moon's position on the celestial sphere.

6. THE MATHEMATICAL PROBLEMS POSED BY THE MOON'S MOTION. The motion of the Moon is really a special example of the gravitational three-body problem, because the Moon is moving in the combined gravitational fields of the Earth and the Sun, and the distances between these bodies are continually varying. Newton tried to solve this problem but his solution was not accurate enough for calculating tables which could be used to find longitude at sea *via* lunar distances.[7] Several outstanding European mathematicians were to address this problem, and they made important contributions to the methods of mathematical astronomy and physics.

(a) *Leonhard Euler (1707–83)*. The Swiss mathematician Leonard Euler was the first to produce an approximate solution to the Moon's motion with sufficient accuracy to be used to calculate the lunar tables for the nautical almanac. For this purpose he invented the method of perturbation theory. This method, as applied to the Moon, starts by ignoring the gravitational force of the Sun on the Moon, and simply works out the orbit of the Moon around the Earth. It then considers the effects that the Sun has on this orbit, and uses this to calculate a new orbit. The whole process can then be repeated. This method of successive approximation can be continued until the desired precision is achieved. It seems that this was the first time that the method of perturbation theory was applied to a physical or astronomical problem, although it is now used in many different applications in physics, astronomy and engineering.

Euler first addressed the problem of the lunar perigee. It was well known at the time that observations did not match available theory. Some astronomers and mathematicians, including Euler, thought the problem lay with Newton's law of gravitation. Clairaut, in 1749, found a mistake in the calculation methods that most researchers had been using, and realised that by including the second approximation term, the size of the discrepancy between theory and observation could be considerably reduced.[8] This stimulated Euler to do more work on the problem.

In 1753 he published an outstanding treatise on the motion of the Moon called *Theoria motus lunae exhibens omnes eius inequalitates*,[9] which included the method of the variation of the elements, now accepted as a very powerful tool in treating perturbations in physics and astronomy. Perhaps the most important consequence of this work was that it showed it to be possible to account for the major irregularities of the Moon's movement entirely within the framework of Newtonian celestial mechanics.

At this time another problem concerning the Moon's motion had began to manifest itself. By comparing observed times of ancient lunar eclipses and the calculation of such times working backward, using the best methods available in 1749, Dunthorne was able to show that the average motion of the Moon was actually changing at the rate of about 10 seconds per century. (This change is referred to as the secular acceleration of the Moon, although it is a decrease in the orbital velocity of the Moon.) Euler wrote another essay on this problem in 1772, proposing that there must be an ethereal fluid in space which offered resistance to the motions of Earth and Moon.[10] It was an unsatisfactory explanation, but this particular problem had to await further developments in mathematics before it could be solved.

(b) *Tobias Mayer (1723–62)*. Despite the fact that there were still discrepancies between theory and observations of the Moon's motion, the theory was sufficiently well developed by 1755 for Mayer, a German mathematical astronomer, to work out reasonably accurate solar and lunar tables, which he published in that year. For the lunar tables he used Euler's method of perturbations.

For the method of lunar distances to work at sea, it was necessary to know the position of the Moon to an accuracy of about 1 minute. Such an error would allow geographical longitude accuracy of approximately 27 minutes. Acting on instructions from the Admirality, James Bradley, 3rd Astronomer Royal, compared Mayer's tables with the Greenwich observations of the Moon and found that their accuracy was of this order of magnitude. In 1770 these tables were published by the Admiralty, together with methods and instructions which had been prepared by Mayer, as an important aid to navigation. By then Mayer had died, but the British Government recognised his contribution by awarding a grant of £3000 to Mayer's widow.[11,12]

By this time other astronomers and mathematicians were beginning to consider related problems in planetary and cometary science. These researches also gave a further stimulus to certain mathematical techniques and the development of some special functions.

(c) *Lagrange (1736–1813)*. Lagrange was another mathematician to address some important astronomical problems. He is best known for his more general work on the analytical approach to mechanics which brought to a climax the move away from the Newtonian geometrical approach. The methods he developed opened up the way to dealing with more complicated problems in classical mechanics: they also provided, in the twentieth century, the basis for the formal development of quantum mechanics from the earlier quantum theory. Although there is no evidence that this work was undertaken specifically to deal with astronomical problems, he nevertheless used analytical techniques to make notable contributions to cometary perturbations and to the three-body problem.

In 1780 Lagrange again won a prize from the French Academy of Sciences for his work on the perturbations of comets by the planets. In 1772 he had shared this prize with Euler, for some of his work on the three-body problem, and in

1774 he had won it for his work concerning the effects of the shapes of the Earth and Moon on the orbit of the Moon. However, his attempts to find an explanation for the secular acceleration of the Moon ended in failure, and he concluded that the historical evidence for this change in the Moon's motion was in doubt.

In astronomy Lagrange is best known for work on a restricted version of the three-body problem, for which he found an exact analytical solution. In this problem we have two bodies with large gravitational fields and a third, much less massive body, all in synchronous orbit with respect to each other. Lagrange showed that there were five stable points at which the smaller body could orbit with respect to the other two. Two of these points, called the Lagrangian points, would form two equilateral triangles, with the two larger bodies at two of the apexes and the smaller body at the third. These conditions are very nearly fulfilled by the Sun-Jupiter system and a group of minor bodies, called the Trojan asteroids. These are the same distance from the Sun as Jupiter, and their heliocentric longitudes differ from that of Jupiter by sixty degrees on either side of the planet.[13]

(d) *Laplace (1749–1827)*. Laplace was an outstanding mathematician who made contributions to a wide variety of mathematical topics. He did, however, make major contributions to the celestial mechanics of the solar system. Much of his work in this respect is contained in successive volumes of *Mécanique Céleste* which were published between 1799 and 1825.

One of the many problems tackled by Laplace was that of the gravitational field of a non-spherical body. It seems now that this problem was first addressed by Legendre[14] who had restricted himself to the special case of a body that was a solid of revolution. This meant that the deviation from a simple sphere depended only on latitude, but was independent of longitude. In his investigations Legendre had introduced a class of polynomials that are now known by his name. Laplace extended this work by considering the more general case in which the mass distribution was a function of latitude and longitude. In this work he introduced a new function now known as spherical harmonics, in which the Legendre polynomials are modulated by series of functions composed of sines and cosines.

In his work on the gravitational fields of non-spherical bodies, Laplace also used the concept of gravitational potential from which one can derive the components of the gravitational field. Although the general idea of a potential function had already been used by Bernoulli in hydrodynamics, this was the first time it had been used in connection with a field in free space. It is now a powerful tool in the general theory of fields.

Laplace made a very detailed study of planetary perturbations, including the effects of the other planets on the orbit of our Earth. He was able to show that these perturbations would cause the eccentricity of the Earth's orbit to decrease over a period of about ten thousand years. This would result in an increase of the mean distance of the Earth from the Sun, and a consequent decrease in the effect of the Sun on the orbit of the Moon. The net result of this is that it explained, as precisely as it was then possible to do, the secular acceleration of the Moon, which was a problem that had baffled many of Laplace's predecessors.[15]

Although much of the work on celestial mechanics seemed removed from the original purpose of solving the longitude problem, it was entirely necessary in order to improve the accuracy of the *Nautical Almanac*, to which the Board of Longitude and the Admiralty were committed. Laplace's theory was used in the calculation of the *Almanac* for 1808. In later years John Couch Adams (1819–92), whose theoretical work on the perturbations of Uranus eventually led to the discovery of Neptune, was able to show that Laplace's work on the Moon did not explain all of the lunar secular acceleration.[16] However, at the time, this important result obtained by Laplace was seen as one of the great achievements of Newtonian celestial mechanics.

(e) *Bessell (1784–1846)*. Bessell was another mathematician who concerned himself with problems in mathematical astronomy. One biographer said that most of Bessell's mathematical work had some application to astronomical problems. One could well question whether this was the case, but his work on the functions now named after him was definitely inspired by astronomy.

The functions we now call Bessell functions first occurred in the astronomical work of Legendre, and in the work of Bernoulli and Euler on some problems in physics. However, the properties of these functions were more thoroughly investigated by Bessell in his work on planetary perturbations.

Bessell made his debut into the world of astronomy with his analysis of Harriot's observations of Halley's comet, which was discovered by Edmund Halley – second Astronomer Royal at Greenwich. He secured himself a place in the history of astronomy with his outstanding reduction of the observations of James Bradley – third Astronomer Royal – which was published in 1818 as *Fundamenta Astronomiae*.[17] Thus the work of the Greenwich Observatory and that of the astronomers that worked at this institution proved to be an inspiration to astronomers on the Continent.

This fact was well summed up by Delambre:

> One can truly say that, if in some great revolution the sciences came to be lost, and we this collection alone preserved... one would find in it the wherewithal to reconstruct almost in its entirety the whole edifice of modern astronomy....

The collection that Delambre[18] was referring to comprised the Greenwich observations made by Bradley and Maskelyne.

7. APPLIED MATHEMATICAL SPIN-OFF FROM CELESTIAL MECHANICS. Fourier, the French mathematician who invented Fourier analysis, once said that

> The profound study of nature is the most fecund source of mathematical discovery.[19]

In the decades following the publication of Newton's *Principia*, the application of mechanics to the dynamics of the solar system was one of the most important parts of the 'profound study of nature'. We have seen that the problems posed by this grand enterprise attracted many of the best mathematicians in Europe, and gave rise to mathematical functions and techniques that have proved of lasting value.

Newton's own work on differential calculus came from his need to deal with the mathematical problems posed by the elliptical orbits of Kepler's laws of planetary motion. His work on integral calculus was inspired, at least in part, by his investigations on the gravitational fields of extended spherical bodies – his law of gravitation being formulated only for point masses.

Pannekoek[20] offered a reasoned explanation for the interest of mathematicians in astronomical problems:

> ...But astronomy received a large share of their exertions, first, because of the difficulty of the problems posed, which was a stimulus to ingenuity, and secondly, because the results of a fascinating theory in solving time-honoured problems could be verified by accurate observations....

The methods of perturbation theory[21] are now used in a variety of engineering and scientific contexts, including the quantum theory of complex atoms. Legendre polynomials are used in certain problems in quantum mechanics, and spherical harmonics are used to discuss the shape of planets as well as their gravitational fields and the magnetic fields of those planets that have extended magneto-spheres. Seymour[22] used them to discuss the magnetic field of the Milky Way Galaxy. Bessel functions have been used to deal with problems in vibrating circular membranes[23] and in atomic theory.[24] This list is, of course, far from exhaustive, but it gives some indication of the mathematical legacy of celestial mechanics.

8. DISCUSSION AND CONCLUSION. The accuracy of astronomical observations and the theory developed to deal with problems posed by these vast improvements in previous observations came mainly from three separate but related circumstances.

The first came from the establishment of the Paris and Greenwich Observatories. These state-funded institutions provided a secure environment in which to develop long-term projects dedicated to the quest for accuracy. The Royal Observatory at Greenwich was specifically charged with improving the tables of motions of the heavens for 'perfecting navigation and astronomy'. This gave an important focus to the work of England's first scientific institution, and thus it became the most important centre for the study of positional astronomy, and of lunar and planetary motion.

Navigation also demanded more accurate measuring instruments for finding position at sea. The instrument makers who were to supply these needs were the same ones who improved equipment for the observatories. North,[25] in his book *The Fontana History of Astronomy and Cosmology*, pointed out that their work for observatories was not economically sufficient to keep the industry going, but they found good trade from other sources:

> It would be foolish to pretend that this trade was economically as important as...the London trade in clocks and watches of the same period, nor should one forget the rapid expansion in the trade in sextants used for astronomical navigation. Ramsden...with a staff of sixty artisans, had produced a thousand sextants by 1789....

For the first one hundred years of the Greenwich Observatory's existence, lunar tables were not accurate enough to make the lunar distance method a practical possibility for finding longitude at sea. The mathematicians who worked on the problems of the Moon's motion could not have been unaware of the

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possible practical applications of their work. Although this work continued long after the required accuracy had been achieved, there can be little doubt that the initial stimulus for observations on the Moon's motion, and the theories to explain these movements, came from navigational considerations.

Several people who worked at Greenwich and others who have written extensively on its history, have been aware of the guiding role which navigation played.

Maunder,[26] in his book *The Royal Observatory Greenwich*, published in 1900, said:

> Fundamentally, Greenwich Observatory was founded and has been maintained for distinctly practical purposes, chiefly for the improvement of the eminently practical science of navigation. Other enquiries relating to navigation, as, for instance, terrestrial magnetism and meteorology, have been added since. The pursuit of these objects has of necessity meant that the Observatory was equipped with powerful and accurate instruments, and the possession of these again has led to their use in fields which lay outside the domain of the purely utilitarian, fields from which the only harvest that could be reaped was that of the increase of our knowledge.

McCrea[27] said, in 1975:

> Astronomers interested in navigation were bound to be interested in geomagnetism, and the study of the geomagnetic field soon leads to that of solar-terrestrial relationships and to solar physics.

As we have already seen, North was well aware of the importance of navigation to the instrument-making industry. He also made the following comment on Delambre's assessment of the importance of the Greenwich observations of Bradley and Maskelyne:

> Delambre was exaggerating, but mildly, when he wrote that if all other materials of the kind were to be destroyed, the Greenwich records alone would suffice for the restoration of astronomy.

His inclusion of the phrase '...but mildly...' seems to indicate that he generally agreed with this statement. His book also shows that he was well aware of the reasons for the establishing of the Royal Observatory at Greenwich. It seems inconsistent then that he should make the following statement:

> The history of the astronomical part of navigation is one in which the most advanced of all the exact sciences offers help in solving a practical problem. There is a myth prevalent in some quarters that the debt was owed by astronomy, which was driven and refined in response to the practical needs of navigation... Astronomers were rarely driven by mariners' needs, except when they had hope of gain, moral or financial, from solving points of theory (say in regard to Jupiter's satellites or the Moon's longitude) that would help in the problem of finding terrestrial longitude.

North discusses at some length the astrological motive in the history of astronomy, and yet he underplays the role of navigation. Throughout its very long history, astronomers have responded to some of the needs of societies and cultures that patronized it. The main social motives before the great voyages of discovery were provided by calendar-making, time-keeping and astrology. However, none of these social applications of astronomy demand the high level of accuracy required to find longitude at sea. As Bernal[28] said in *Science in History*:

> These [the great voyages of discovery] were the fruit of the first conscious application of astronomical and geographical science to the service of glory and profit...The motions of the stars now had a cash value and astronomy stood in no danger of being neglected, even after astrology had gone out of fashion.

I believe Pannekoek[20] gives a more balanced appraisal of the impact of navigation on the history of theoretical science:

> Navigation no longer needs the Moon...The problem of the longitude at sea was an episode in the history of astronomy, but highly important for the progress of science now closed. It greatly stimulated celestial mechanics as an important branch in the general theoretical knowledge of mankind.

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