


PAPER

Flocking dynamics of agents moving with a constant speed and a randomly switching topology

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Received: 28 October 2023; **Revised:** 15 February 2024; **Accepted:** 04 April 2024

Keywords: asymptotic flocking; Cucker–Smale; directed graph; multi-agent system; randomly switching topology; unit-speed constraint

2020 Mathematics Subject Classification: 34A34 (Primary); 34D05, 34F05 (Secondary)

Abstract

In this paper, we present a sufficient framework to exhibit the sample path-wise asymptotic flocking dynamics of the Cucker–Smale model with unit-speed constraint and the randomly switching network topology. We employ a matrix formulation of the given equation, which allows us to evaluate the diameter of velocities with respect to the adjacency matrix of the network. Unlike the previous result on the randomly switching Cucker–Smale model, the unit-speed constraint disallows the system to be considered as a nonautonomous linear ordinary differential equation on velocity vector, which forces us to get a weaker form of the flocking estimate than the result for the original Cucker–Smale model.

1. Introduction

The jargon *flocking* describes a phenomenon wherein agents within a self-propelled system organise themselves into cohesive groups and demonstrate structured motion based on local information and simple governing principles. After the seminal work by Vicsek et al. [32], the mathematical community also has shown keen interest in developing mathematical models to elucidate flocking dynamics over the past decades. Among them, the Cucker–Smale (CS) model, initially introduced in [12], is considered as one of the most simple and successful mathematical representations of flocking behaviour. Notable directions include the studies on the CS model on general digraph [16], unit-speed constraint [6], the impact of considering time delay [4, 14], temperature field [22], collision avoidance [9, 27], emergence of bi-cluster flocking [11], Riemannian manifold framework [17], the mean-field limit [19], time discretisation [15], hydrodynamic descriptions [23], hierarchical rooted leadership structures [21, 26, 28, 30] and alternating leaders [20]. We also refer to a comprehensive survey paper [8] for those who may be interested in this topic.

In this paper, we are interested in generalising the flocking model with the unit-speed constraint presented in [6] to be more practical. In [6], the unit-speed constrained model was given by the following second-order ordinary differential equations (ODEs) for the *position–velocity* configuration $\{(x_i, v_i)\}_{i=1}^N$, motivated from the well-known CS model:

$$\begin{cases} \frac{dx_i}{dt} = v_i, & t > 0, \quad i \in [N] := \{1, \dots, N\}, \\ \frac{dv_i}{dt} = \frac{1}{N} \sum_{j=1}^N \phi(\|x_i - x_j\|) \left(v_j - \frac{\langle v_j, v_i \rangle v_i}{\|v_i\|^2} \right), \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1}, \end{cases} \quad (1.1)$$

where N is the number of agents, $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^d , and $\| \cdot \|$ is the standard ℓ^2 -norm, respectively. In addition, we assume the communication weight $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+ (:= [0, \infty))$ is locally Lipschitz continuous function satisfying

$$0 \leq \phi(r) \leq \phi(0), \quad (\phi(r_1) - \phi(r_2))(r_1 - r_2) \leq 0, \quad r, r_1, r_2 \geq 0.$$

The manifold \mathbb{S}^{d-1} denotes a $(d - 1)$ -dimensional unit sphere, which is isometrically embedded to \mathbb{R}^d , that is,

$$\mathbb{S}^{d-1} := \left\{ x := (x^1, \dots, x^d) \mid \sum_{i=1}^d |x^i|^2 = 1 \right\}.$$

Similar to the CS model proposed in [12], the model (1.1) and its variants have garnered considerable attention. For instance, the study on the bi-cluster flocking was presented in [10], and the exploration of multi-cluster flocking and critical coupling strength was discussed in [18]. The time-delay effect was also considered in [5, 7], and considerations regarding the temperature field have been explored in [1, 2]. Moreover, the work in [29] has addressed the system within the context of a general digraph. In [29], the authors introduced a static network topology into (1.1) and provided a sufficient framework to exhibit the asymptotic flocking. The objective was to analyse how the connectivity among the system’s agents influences its behaviour, where the adjacency matrix of the network topology was given by

$$\chi_{ij} = \begin{cases} 1, & j\text{-th agent conveys information to } i\text{-th agent or } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Then, they provided a sufficient framework pertaining to initial data and system parameters that facilitate the emergence of flocking dynamics when the network topology (χ_{ij}) contains a directed spanning tree.

However, even if the interaction network is expressed as a directed graph, there are still things that can be done to more realistically model the behaviour of actual flocks. Namely, the interaction network in (1.1) is assumed to be constant regardless of time, which may be a rather unrealistic assumption considering that some unpredictable external factors can interfere with the interaction. Since we want to observe whether the flocking phenomenon occurs asymptotically over infinite time, it is natural to think of a mathematical model that allows the network to change over time. Beyond the Cucker–Smale model, the introduction of random effects into the network topology of a multi-agent system has been explored in various literature, such as the studies on random geometric graphs [3] and random link failures [24, 25], etc. Among them, we introduce a formulation of a many-body system with a randomly switching network topology inspired from [13], building upon the structure defined in (1.1). This system is governed by the Cauchy problem

$$\begin{cases} \frac{dx_i}{dt} = v_i, & t > 0, \quad i \in [N], \\ \frac{dv_i}{dt} = \frac{1}{N} \sum_{j=1}^N \chi_{ij}^\sigma \phi(\|x_i - x_j\|) \left(v_j - \frac{\langle v_j, v_i \rangle v_i}{\|v_i\|^2} \right), \\ (x_i(0, w), v_i(0, w)) = (x_i^0(w), v_i^0(w)) \in \mathbb{R}^d \times \mathbb{S}^{d-1}, \end{cases} \quad (1.2)$$

with the assumptions for N , ϕ , $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ and \mathbb{S}^{d-1} remain consistent with those in (1.1). In [13], the authors generalised the static network topology to a right-continuous stochastic process

$$\sigma : \mathbb{R}_+ \times \Omega \rightarrow [N_G] = \{1, 2, \dots, N_G\}.$$

Then, χ^σ denotes the adjacency matrix of \mathcal{G}_σ , where each \mathcal{G}_i is a directed graph with vertices $\mathcal{V} = \{\beta_1, \dots, \beta_N\}$. Due to this right continuity condition, the authors were able to constrain the process σ to have a piecewise continuous sample path that gives the opportunity to change its value at some random switching instants $0 = t_0 < t_1 < t_2 < \dots$, so that $\sigma(\cdot, \omega): [t_k, t_{k+1}) \rightarrow [N_G]$ is constant for each $k \in \mathbb{N} \cup \{0\}$.

In this paper, we adopt the above constructions on σ to the proposed system (1.2), and we are mainly concerned with the following issue:

What is the probability to emerge the flocking of the system (1.2)?

More specifically, the asymptotic flocking, for which we want to find sufficient conditions in terms of initial data and system parameters, is defined as follows:

Definition 1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a generic probability space and $Z = \{(x_i, v_i)\}_{i=1}^N$ be the solution process of the system (1.2).

1. The configuration Z exhibits group formation for $w \in \Omega$ if

$$\sup_{t \in \mathbb{R}_+} \max_{i,j \in [N]} \|x_i(t, w) - x_j(t, w)\| < \infty.$$

2. The configuration Z exhibits asymptotic velocity alignment for $w \in \Omega$ if

$$\lim_{t \rightarrow \infty} \max_{i,j \in [N]} \|v_i(t, w) - v_j(t, w)\| = 0.$$

3. The configuration Z exhibits asymptotic flocking for $w \in \Omega$ if

$$\sup_{t \in \mathbb{R}_+} \max_{i,j \in [N]} \|x_i(t, w) - x_j(t, w)\| < \infty, \quad \lim_{t \rightarrow \infty} \max_{i,j \in [N]} \|v_i(t, w) - v_j(t, w)\| = 0.$$

The rest of the paper is organised as follows. In Section 2, we reformulate the proposed system (1.2) into a matrix-valued ODE and briefly introduce some theoretical backgrounds related to our work. In addition, we provide preparatory estimates which will be crucially used to derive the desired flocking of the system (1.2) in Section 3. In Section 3, we demonstrate several sufficient frameworks concerning initial data and system parameters to exhibit the asymptotic flocking of the system (1.2). Finally, Section 5 is devoted to summarise the main results of this paper and discuss future work.

Notation. Throughout the paper, we employ the following notation for simplicity:

$$\begin{aligned} X &:= (x_1, \dots, x_N)^T, & V &:= (v_1, \dots, v_N)^T, & A_V &:= \min_{i,j \in [N]} (v_i, v_j), & \mathbb{R}_+ &:= [0, \infty), \\ D_Z &:= \max_{i,j \in [N]} \|z_i - z_j\| & \text{for } Z &= (z_1, \dots, z_N) \in \{X, V\}, & [N] &:= \{1, \dots, N\}, \\ M_{ij} &:= \text{the } (i, j)\text{-th entry of } M \in \mathbb{R}^{m \times n}, & x^i &:= \text{the } i\text{-th component of } x \in \mathbb{R}^d, \\ A \geq B, & A, B \in \mathbb{R}^{m \times n} & \iff & A_{ij} \geq B_{ij} & \text{for all } i \in [m], j \in [n]. \end{aligned}$$

2. Preliminaries

In this section, we reformulate the system (1.2) into a suitable matrix-valued ODE and introduce some theoretical backgrounds related to matrix analysis of graphs.

2.1. Graph theory

We first briefly introduce basic notions in graph theory. A direct graph (in short digraph) $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ consists of a finite set $\mathcal{V}(\mathcal{G}) = \{\beta_1, \dots, \beta_N\}$ of vertices and a set $\mathcal{E}(\mathcal{G}) \subset \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$ of arcs. If a pair (β_j, β_i) is contained in $\mathcal{E}(\mathcal{G})$, β_j is said to be a neighbour of β_i , and we denote the neighbour set of the vertex β_i by $\mathcal{N}_i := \{j : (\beta_j, \beta_i) \in \mathcal{E}(\mathcal{G})\}$. For given digraph \mathcal{G} , we define its corresponding adjacency matrix $\chi(\mathcal{G}) = (\chi_{ij})(\mathcal{G})$ as

$$\chi_{ij} = \begin{cases} 1 & \text{if } j \in \mathcal{N}_i \cup \{i\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since it is obvious that this correspondence is a one-to-one, we can say that given a matrix A consisting of zeros and ones and with diagonal components all equal to one, we can also uniquely determine its corresponding digraph, which we write it as $\mathcal{G}(A)$. A path in a digraph \mathcal{G} from β_{i_0} to β_{i_p} is a finite

sequence $\{\beta_{i_0}, \dots, \beta_{i_p}\}$ of distinct vertices such that each successive pair of vertices is an arc of \mathcal{G} . The integer p which is the number of its arcs is said to be the *length* of the path. If there is a path from β_i to β_j , then vertex β_j is said to be *reachable* from vertex β_i . We say \mathcal{G} contains a *spanning tree* if there exists a vertex such that any other vertex of \mathcal{G} is reachable from it. In this case, this vertex is said to be a *root*.

2.2. Conservation and dissipation law

In this subsection, we show that the system (1.2) has a conservation of each speed of agent and dissipation of their velocity diameter.

Lemma 2.1. (Conservation of speeds) *Suppose that (X, V) is a solution to the system (1.2). Then, it follows that for each $w \in \Omega$,*

$$\|v_i(t, w)\| = 1, \quad t \geq 0, \quad i \in [N].$$

Proof. Once we take an inner product v_i with (1.2)₂, the following relation holds:

$$\left\langle v_j - \frac{\langle v_j, v_i \rangle v_i}{\|v_i\|^2}, v_i \right\rangle = 0, \quad i, j \in [N].$$

Therefore, we have $\frac{d}{dt} \|v_i\|^2 = 0$ for all $i \in [N]$, which is our desired result. □

This means that the difference in velocities is determined entirely by the angle between them. However, one can also verify that for each $w \in \Omega$, the maximal angle

$$A_V(t, w) := \min_{i, j \in [N]} \langle v_i(t, w), v_j(t, w) \rangle$$

is monotonically increasing in $t \geq 0$, provided that $A_V(0, w) := \min_{i, j \in [N]} \langle v_i^0(w), v_j^0(w) \rangle$ is strictly positive.

Lemma 2.2. (Monotonicity of A_V) *Let (X, V) be a solution to the system (1.2) satisfying*

$$A_V(0, w) > 0$$

for some $\omega \in \Omega$. Then, the smallest inner product between heading angles $A_V(\cdot, w)$ is monotonically increasing.

Proof. For fixed $t \geq 0$, we denote $i_t, j_t \in [N]$ the two indices satisfying

$$A_V(t, w) := \langle v_{i_t}(t, w), v_{j_t}(t, w) \rangle.$$

If we define a set \mathcal{S} as

$$\mathcal{S} =: \{t \geq 0 \mid A_V(t, w) \leq 0\},$$

it follows from the condition $A_V(0, w) > 0$ and the continuity of $A_V(\cdot, w)$ that there exists $T_1 \in (0, \infty]$ satisfying

$$[0, T_1) \cap \mathcal{S} = \emptyset.$$

Now, define $\mathcal{T} := \inf \mathcal{S}$ and suppose we have $\mathcal{T} < \infty$. Then, from the continuity of A_V , we have

$$\lim_{t \rightarrow \mathcal{T}^-} A_V(t, w) = 0. \tag{2.1}$$

On the other hand, the locally Lipschitz function $A_V(\cdot, \omega)$ is differentiable at almost every $t \in (0, \mathcal{T})$. Then, we use (1.2)₂ to obtain

$$\begin{aligned} \frac{dA_V}{dt} &= \langle \dot{v}_i, v_j \rangle + \langle \dot{v}_j, v_i \rangle \\ &= \frac{1}{N} \sum_{k=1}^N \chi_{i,k}^\sigma \phi(\|x_k - x_i\|) (\langle v_k, v_j \rangle - \langle v_k, v_i \rangle \langle v_i, v_j \rangle) \\ &\quad + \frac{1}{N} \sum_{k=1}^N \chi_{j,k}^\sigma \phi(\|x_k - x_j\|) (\langle v_k, v_i \rangle - \langle v_k, v_j \rangle \langle v_i, v_j \rangle) \\ &\geq 0, \end{aligned}$$

where we used

$$\begin{aligned} 1 \geq \langle v_k, v_i \rangle \geq 0 \quad \text{and} \quad \langle v_k, v_j \rangle \geq \langle v_i, v_j \rangle \geq 0, \\ 1 \geq \langle v_k, v_j \rangle \geq 0 \quad \text{and} \quad \langle v_k, v_i \rangle \geq \langle v_i, v_j \rangle \geq 0, \end{aligned}$$

in the last inequality. Hence, $A_V(\cdot, \omega)$ is monotonically increasing in $[0, \mathcal{T}]$, which leads a contradiction to (2.1). Therefore, we have $\mathcal{T} = \infty$ and obtain the monotone increasing property of $A_V(\cdot, \omega)$. \square

As a direct consequence of Lemma 2.2, the velocity diameter D_V is monotonically decreasing, due to the relation between D_V and A_V .

Corollary 2.1. (Monotonicity of D_V) Assume that (X, V) is a solution to the system (1.2) with

$$D_V(0, w) < \sqrt{2}$$

for some $\omega \in \Omega$. Then, the velocity diameter $D_V(\cdot, w)$ is monotonically decreasing in time.

Proof. From Lemma 2.1, we have

$$\|v_i - v_j\|^2 = 2 - 2\langle v_i, v_j \rangle.$$

Therefore, one can obtain the monotone decreasing property of

$$D_V^2 = 2 - 2A_V$$

by using Lemma 2.2. \square

2.3. Matrix formulation

Now, we reorganise the system (1.2) to a matrix formulation. For this, we first fix $w \in \Omega$ and

$$X := (x_1, \dots, x_N)^T, \quad V := (v_1, \dots, v_N)^T,$$

and we use the result of Lemma 2.1, that is,

$$\|v_i(t, w)\| = 1, \quad t > 0, \quad i \in [N].$$

Then, the right-hand side of (1.2)₂ can be rewritten as

$$\begin{aligned} &\sum_{j=1}^N \chi_{ij}^\sigma \phi(\|x_i - x_j\|) (v_j - \langle v_j, v_i \rangle v_i) \\ &= \sum_{j=1}^N \chi_{ij}^\sigma \phi(\|x_i - x_j\|) (v_j - v_i) + \sum_{j=1}^N \chi_{ij}^\sigma \phi(\|x_i - x_j\|) (v_i - \langle v_j, v_i \rangle v_i) \\ &= \sum_{j=1}^N \chi_{ij}^\sigma \phi(\|x_i - x_j\|) (v_j - v_i) + \frac{1}{2} \sum_{j=1}^N \chi_{ij}^\sigma \phi(\|x_i - x_j\|) \|v_i - v_j\|^2 v_i. \end{aligned}$$

On the other hand, we also consider the Laplacian matrices

$$\mathcal{L}_l := \mathcal{D}_l - \mathcal{A}_l, \quad 1 \leq l \leq N_G,$$

where $\mathcal{A}_l \in \mathbb{R}^{N \times N}$ and $\mathcal{D}_l \in \mathbb{R}^{N \times N}$ are defined as

$$(\mathcal{A}_l)_{ij} := \chi_{ij}^l \phi(\|x_i - x_j\|), \quad \mathcal{D}_l := \text{diag}(d_1^l, \dots, d_N^l), \quad d_i^l := \sum_{j=1}^N \chi_{ij}^l \phi(\|x_i - x_j\|), \quad i, j \in [N].$$

Then, (1.2) can be represented by the following matrix form:

$$\begin{cases} \dot{X} = V, \\ \dot{V} = -\frac{1}{N} \mathcal{L}_\sigma V + \frac{1}{N} \mathcal{R}_\sigma, \end{cases} \tag{2.2}$$

where the vector \mathcal{R}_l is defined as

$$\mathcal{R}_l := (r_1^l, \dots, r_N^l)^T \quad \text{and} \quad r_i^l := \frac{1}{2} \sum_{j=1}^N \chi_{ij}^l \phi(\|x_i - x_j\|) \|v_i - v_j\|^2 v_i, \quad 1 \leq l \leq N_G.$$

2.4. Matrix analysis

Now, we review some basic concepts on the matrix analysis to be used in Section 3.

Definition 2.1. Let $A \in \mathbb{R}^{N \times N}$ be a matrix whose entries are all non-negative.

1. A is called a stochastic matrix if its row-sum is one:

$$\sum_{j=1}^N A_{ij} = 1, \quad i \in [N].$$

2. A is called a scrambling matrix if for $i, j \in [N]$, there exists $k \in [N]$ such that

$$A_{ik} > 0 \quad \text{and} \quad A_{jk} > 0.$$

3. A is called an adjacency matrix of a digraph $\mathcal{G} = \mathcal{G}(A)$ if

$$A_{ij} > 0 \iff (j, i) \in \mathcal{E}(\mathcal{G}).$$

In addition, the following ergodicity coefficient μ of $A \in \mathbb{R}^{N \times N}$ also plays a key role in the analysis in Section 3:

$$\mu(A) := \min_{i,j \in [N]} \sum_{k=1}^N \min(A_{ik}, A_{jk}).$$

For instance, one can easily verify the following two facts on the ergodicity coefficient:

1. $A \in \mathbb{R}^{N \times N}$ is a scrambling matrix $\iff \mu(A) > 0$.
2. Assume that $A, B \in \mathbb{R}^{N \times N}$. Then, one has

$$A \geq B \geq 0 \implies \mu(A) \geq \mu(B).$$

In what follows, we offer two results concerning stochastic matrices and scrambling matrices, which will be used to derive the asymptotic flocking of (1.2). The proof of the following lemma starts by applying the ideas of [15]. An improvement over the result in [15] is that the remaining term B now plays a role by D_B (the diameter between its columns) rather than the Frobenius norm $\|B\|$, which allows us to apply Lemma 2.3 recursively. Consequently, we can lower the order of N by one in the sufficient framework (F5) – (F6), compared to the result which uses the lemma in [15] directly.

Lemma 2.3. Let $A \in \mathbb{R}^{N \times N}$ be a non-negative stochastic matrix and let $B, Z, W \in \mathbb{R}^{N \times d}$ be matrices satisfying

$$W = AZ + B. \tag{2.3}$$

Then, for every norm $\|\cdot\|$ on \mathbb{R}^d , we have

$$D_W \leq (1 - \mu(A))D_Z + D_B,$$

where D_W, D_Z and D_B are defined as

$$\begin{aligned} B &:= (b_1, \dots, b_N)^T, & Z &:= (z_1, \dots, z_N)^T, & W &:= (w_1, \dots, w_N)^T, \\ b_i &:= (b_i^1, \dots, b_i^d)^T, & z_i &:= (z_i^1, \dots, z_i^d)^T, & z_i &:= (z_i^1, \dots, z_i^d)^T, \\ D_W &:= \max_{i,k \in [N]} \|w_i - w_k\|, & D_Z &:= \max_{i,k \in [N]} \|z_i - z_k\|, & D_B &:= \max_{i,k \in [N]} \|b_i - b_k\|. \end{aligned}$$

Proof. First of all, the condition that A is stochastic matrix implies

$$\sum_{j=1}^N \max\{0, a_{ij} - a_{kj}\} + \sum_{j=1}^N \min\{0, a_{ij} - a_{kj}\} = \sum_{j=1}^N (a_{ij} - a_{kj}) = 0. \tag{2.4}$$

Then, the direct calculation from (2.3) yields

$$\begin{aligned} \|w_i - w_k\| &= \left\| \sum_{j=1}^N a_{ij}z_j + b_i - \sum_{j=1}^N a_{kj}z_j - b_k \right\| \\ &= \left\| \sum_{j=1}^N (a_{ij} - a_{kj})z_j + (b_i - b_k) \right\| \\ &\leq \left\| \sum_{j=1}^N (a_{ij} - a_{kj})z_j \right\| + \|b_i - b_k\| \\ &= \sup_{\substack{\phi \in (\mathbb{R}^d)^* \\ \|\phi\|=1}} \sum_{j=1}^N (a_{ij} - a_{kj})\phi(z_j) + \|b_i - b_k\| \\ &= \sup_{\substack{\phi \in (\mathbb{R}^d)^* \\ \|\phi\|=1}} \left[\sum_{j=1}^N \max\{0, a_{ij} - a_{kj}\}\phi(z_j) + \sum_{j=1}^N \min\{0, a_{ij} - a_{kj}\}\phi(z_j) \right] + \|b_i - b_k\| \\ &\leq \sup_{\substack{\phi \in (\mathbb{R}^d)^* \\ \|\phi\|=1}} \left[\sum_{j=1}^N \max\{0, a_{ij} - a_{kj}\} \max_{n \in [N]} \phi(z_n) + \sum_{j=1}^N \min\{0, a_{ij} - a_{kj}\} \min_{n \in [N]} \phi(z_n) \right] \\ &\quad + \|b_i - b_k\|. \end{aligned} \tag{2.5}$$

Now, we substitute (2.4) to (2.5) to get

$$\begin{aligned} \|w_i - w_k\| &\leq \sup_{\substack{\phi \in (\mathbb{R}^d)^* \\ \|\phi\|=1}} \sum_{j=1}^N \max\{0, a_{ij} - a_{kj}\} \max_{n,m \in [N]} \phi(z_n - z_m) + \|b_i - b_k\| \\ &= \sum_{j=1}^N \max\{0, a_{ij} - a_{kj}\} \max_{n,m \in [N]} \|z_n - z_m\| + \|b_i - b_k\| \\ &= \sum_{j=1}^N (a_{ij} - \min\{a_{ij}, a_{kj}\}) \max_{n,m \in [N]} \|z_n - z_m\| + \|b_i - b_k\| \\ &\leq (1 - \mu(A))D_Z + D_B, \end{aligned}$$

where we used the definition of $\mu(A)$ and the fact

$$\sum_{j=1}^N a_{ij} = 1$$

in the last inequality, which implies our desired result. □

Lemma 2.4. [33] *For each $i \in [N - 1]$, let $A_i \in \mathbb{R}^{N \times N}$ be a non-negative matrix whose all diagonal components are strictly positive and $\mathcal{G}(A_i)$ has a spanning tree. Then, $A_1 \cdots A_{N-1}$ is a scrambling matrix.*

Finally, we review several properties of state-transition matrix. Let $t_0 \in \mathbb{R}$ and $A : [t_0, \infty) \rightarrow \mathbb{R}^{N \times N}$ be a right-continuous matrix-valued function. We consider a linear ODE governed by the following Cauchy problem:

$$\frac{d\xi(t)}{dt} = A(t)\xi(t), \quad t > t_0. \tag{2.6}$$

Then, the solution of (2.6) can be written as

$$\xi(t) = \Phi(t, t_0)\xi(t_0), \quad t \geq t_0, \tag{2.7}$$

where the A -dependent matrix $\Phi(t, t_0)$ is said to be the *state-transition matrix*, which can be represented by using the Peano–Baker series [31]:

$$\Phi(t, t_0) = I_N + \sum_{k=1}^{\infty} \int_{t_0}^t \int_{t_0}^{s_1} \cdots \int_{t_0}^{s_{k-1}} A(s_1) \cdots A(s_k) ds_k \cdots ds_1. \tag{2.8}$$

Now, fix $c \in \mathbb{R}$ and consider the following two Cauchy problems:

$$\frac{d\xi(t)}{dt} = A(t)\xi(t), \quad \frac{d\xi(t)}{dt} = (A(t) + cI_N)\xi(t), \quad t > t_0. \tag{2.9}$$

If $\Phi(t, t_0)$ and $\Psi(t, t_0)$ are the state-transition matrices for the two linear ODEs in (2.9), respectively, then the authors of [15] obtained the following relation between $\Phi(t, t_0)$ and $\Psi(t, t_0)$:

$$\Phi(t, t_0) = \exp(-c(t - t_0))\Psi(t, t_0), \quad t > t_0. \tag{2.10}$$

3. Emergence of stochastic flocking

In this section, we present the sufficient framework for the flocking model with unit-speed constraint and randomly switching network topology to exhibit asymptotic flocking. Recall that the model we are interested in this paper is

$$\begin{cases} \frac{dx_i}{dt} = v_i, & t > 0, \quad i \in [N], \\ \frac{dv_i}{dt} = \frac{1}{N} \sum_{j=1}^N \chi_{ij}^\sigma \phi(\|x_i - x_j\|) \left(v_j - \frac{\langle v_j, v_i \rangle v_i}{\|v_i\|^2} \right), \\ (x_i(0, w), v_i(0, w)) = (x_i^0(w), v_i^0(w)) \in \mathbb{R}^d \times \mathbb{S}^{d-1}. \end{cases} \tag{3.1}$$

3.1. Sufficient frameworks

In this subsection, we describe suitable sufficient frameworks in terms of initial data and system parameters to guarantee the asymptotic flocking of the system (3.1). For this, motivated from the methodologies studied in [13], we provide

- (F1) There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of independent and identically distributed (i.i.d) random variables $\tau_i: \Omega \rightarrow \mathbb{R}, i \in \mathbb{N}$ such that

$$\exists m, M > 0, \quad \mathbb{P}(m \leq \tau_i \leq M) = 1, \quad i \in \mathbb{N}.$$

- (F2) Define the sequence of random variables $\{t_n: \Omega \rightarrow \mathbb{R}\}_{n \in \mathbb{N} \cup \{0\}}$ as

$$t_n = \begin{cases} 0 & n = 0, \\ \sum_{i=1}^n \tau_i & n \geq 1. \end{cases}$$

Then for each $\omega \in \Omega$ and $n \in \mathbb{N}$, the sample path $\sigma(\cdot, \omega)$ satisfies

$$\sigma(t, \omega) = \sigma(t_n(\omega), \omega) \in [N_G], \quad \forall t \in [t_n(\omega), t_{n+1}(\omega)),$$

and this implies that the process σ is right continuous. In addition, we use the following notation for simplicity:

$$\sigma_n(\omega) := \sigma(t_n(\omega), \omega), \quad \omega \in \Omega, \quad n \in \mathbb{N} \cup \{0\}.$$

- (F3) $\{\sigma_{t_n}: \Omega \rightarrow [N_G]\}_{n \in \mathbb{N} \cup \{0\}}$ is the sequence of i.i.d random variables, where

$$\mathcal{P}_l := \mathbb{P}(\sigma_{t_n} = l) > 0 \quad \text{for all } l \in [N_G] \quad \text{and } n \in \mathbb{N} \cup \{0\}.$$

- (F4) The union of all admissible network topologies $\{\mathcal{G}_1, \dots, \mathcal{G}_{N_G}\}$ contains a spanning tree with vertices $\mathcal{V} = \{\beta_1, \dots, \beta_N\}$.

- (F5) There exist $R > 1$ and $\bar{n} \in \mathbb{N}$ such that

$$r := \frac{R}{\min_{1 \leq l \leq N_G} \{-\log(1 - \mathcal{P}_l)\}} < \frac{1}{M(N-1)\phi(0)}, \quad \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n}} \leq 1 - \frac{1}{R},$$

$$\left(\frac{m\phi(0) \exp(-\phi(0)M\bar{n})(N-1)^{-rM\phi(0)}}{N} \right)^{N-1} \leq \log 2,$$

- (F6) There exist $\bar{M} > 0$ and a sufficiently large number $D_X^\infty > 0$ such that for \mathbb{P} - almost every $w \in \Omega$,

$$D_V(0, w) < \min \left\{ \frac{N \log \bar{M}}{(N-1)\phi(0)\bar{M}C_0}, \sqrt{2} \right\}, \quad D_X(0, w) + \bar{M}C_0 D_V(0, w) < D_X^\infty,$$

where we set

$$C := \left(\frac{m\phi(D_X^\infty) \exp(-\phi(0)M\bar{n})(N-1)^{-rM\phi(0)}}{N} \right)^{N-1},$$

$$C_0 := M(N-1) \sum_{l=1}^{\infty} \left[(\bar{n} + r \log l(N-1)) \exp \left(-\frac{C(l^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \right].$$

Here, conditions (F1) – (F4) mean that the interaction network is randomly and repeatedly determined after a period of time within an appropriate range, and the probability of each network topology being selected is the same regardless of time. Each of these network topologies may not have sufficient connectivity, but the fact that the union of graphs that can be selected contains some spanning tree allows all agents to send or receive information without being isolated in the long run. In addition, (F5) – (F6) quantitatively represents the initial conditions to ensure the flocking phenomenon. In summary, these mean that the essential supremum of the initial velocity diameter ($= \text{ess sup}_{\omega \in \Omega} D_V(0, \omega)$) has to be sufficiently small.

Note that for $\sigma_t(w) \in [N_G]$, the adjacency matrix $\chi_{ij}^{\sigma_t(w)}$ is defined by

$$\chi_{ij}^{\sigma_t(w)} = \begin{cases} 1, & \text{if } (j, i) \in \mathcal{E}(\mathcal{G}_{\sigma_t(w)}), \\ 0, & \text{if } (j, i) \notin \mathcal{E}(\mathcal{G}_{\sigma_t(w)}), \end{cases}$$

and in particular, we have

$$(\beta_i, \beta_i) \in \mathcal{E}(\mathcal{G}_k), \quad i \in [N],$$

which makes each vertex β_i in the graph \mathcal{G}_k ‘seems’ to have a self loop.

Remark 3.1. If we consider the network topology as a graph which connects each pair of points sufficiently close to each other (as in the Vicsek model), it is natural to assume that the scale of time interval $\tau_i \in [m, M]$ between network switching becomes smaller when the number of particles N becomes larger. For example, it is widely known that the mean free time of gas molecules in a fixed volume of space is proportional to $1/N$.

3.2. Asymptotic flocking dynamics

First of all, we will consider the union of the network topology on time intervals of the form $[a, b)$, which we denote

$$\mathcal{G}([a, b))(w) := \left(\mathcal{V}, \bigcup_{t \in [a, b)} \mathcal{E}(\mathcal{G}_{\sigma(t, w)}) \right).$$

In addition, we also define a sequence $\{n_k = n_k(\bar{n})\}_{k \in \mathbb{N} \cup \{0\}}$ as

$$n_k(\bar{n}) := k\bar{n} + \sum_{l=1}^k \lfloor r \log l \rfloor, \quad k \in \mathbb{N} \cup \{0\},$$

where $\lfloor a \rfloor$ denotes the largest integer less than or equal to a . Then, the following lemma provides a lower bound estimate of the probability to the random digraph

$$\omega \mapsto \mathcal{G}([t_{n_k}, t_{n_{k+1}}))(w)$$

to have a spanning tree for all $k \in \mathbb{N} \cup \{0\}$.

Lemma 3.1. *Let (X, V) be a solution process to (3.1) satisfying (F1) – (F5). Then, the following probability estimate holds:*

$$\mathbb{P} \left(\bigcup_{k=0}^{\infty} \left\{ w : \mathcal{G}([t_{n_k}, t_{n_{k+1}}))(w) \text{ has a spanning tree} \right\} \right) \geq \exp \left(-\frac{R^2 \log R}{(R-1)^2} \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n}-1} \right).$$

Proof. Since $\{\sigma_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a i.i.d sequence of random variables, we have

$$\begin{aligned} & \mathbb{P} (w : \mathcal{G}([t_{n_k}, t_{n_{k+1}}))(w) \text{ does not contain a spanning tree}) \\ & \leq \mathbb{P} (w : \exists l \in \{1, \dots, N_G\} \text{ such that } \sigma_{t_{n_k+i}}(w) \neq l \text{ for all } 0 \leq i < n_{k+1} - n_k) \\ & \leq \sum_{l=1}^{N_G} \mathbb{P} (w : \sigma_{t_{n_k+i}}(w) \neq l \text{ for all } 0 \leq i < n_{k+1} - n_k) \\ & = \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{n_{k+1} - n_k} = \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n} + \lfloor r \log(k+1) \rfloor} \leq \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n}} \leq 1 - \frac{1}{R}. \end{aligned}$$

Then, the probability to contain a spanning tree is

$$\begin{aligned}
 & \mathbb{P} \left(\bigcup_{k=0}^{\infty} \left\{ w : \mathcal{G}([t_{n_k}, t_{n_{k+1}}))(w) \text{ has a spanning tree} \right\} \right) \\
 & \geq \prod_{k=0}^{\infty} \left(1 - \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n} + \lceil r \log(k+1) \rceil} \right) \\
 & \geq \prod_{k=0}^{\infty} \exp \left(- \frac{R \log R}{R-1} \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n} + \lceil r \log(k+1) \rceil} \right) \\
 & = \exp \left(- \frac{R \log R}{R-1} \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n}} \sum_{k=0}^{\infty} (1 - \mathcal{P}_l)^{\lceil r \log(k+1) \rceil} \right),
 \end{aligned} \tag{3.2}$$

where we used the following relation for $x \in [0, 1 - \frac{1}{R}]$:

$$1 - x \geq R^{-\frac{Rx}{R-1}} = \exp \left(- \frac{R \log R}{R-1} x \right).$$

In addition, since we have

$$1 < R = \min_{1 \leq l \leq N_G} \{-r \log(1 - \mathcal{P}_l)\} = -r \max_{1 \leq l \leq N_G} \log(1 - \mathcal{P}_l),$$

the convergence of p -series yields

$$\begin{aligned}
 \sum_{k=0}^{\infty} (1 - \mathcal{P}_l)^{\lceil r \log(k+1) \rceil} & \leq \sum_{k=0}^{\infty} (1 - \mathcal{P}_l)^{r \log(k+1) - 1} \\
 & = \frac{1}{1 - \mathcal{P}_l} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{-r \log(1 - \mathcal{P}_l)}} \\
 & < \frac{1}{1 - \mathcal{P}_l} \cdot \frac{-r \log(1 - \mathcal{P}_l)}{-r \log(1 - \mathcal{P}_l) - 1} \\
 & \leq \frac{1}{1 - \mathcal{P}_l} \cdot \frac{R}{R-1},
 \end{aligned} \tag{3.3}$$

where we used

$$\begin{aligned}
 (1 - \mathcal{P}_l)^{r \log(k+1)} & = (1 - \mathcal{P}_l)^{r \log(k+1)} = \exp(r \log(k+1) \log(1 - \mathcal{P}_l)) = (k+1)^{\log(1 - \mathcal{P}_l)}, \\
 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{-r \log(1 - \mathcal{P}_l)}} & < 1 + \int_1^{\infty} \frac{1}{x^{-r \log(1 - \mathcal{P}_l)}} dx \\
 & = 1 + \frac{1}{-r \log(1 - \mathcal{P}_l) - 1} \left[-\frac{1}{x^{-r \log(1 - \mathcal{P}_l) - 1}} \right]_1^{\infty} \\
 & = 1 + \frac{1}{-r \log(1 - \mathcal{P}_l) - 1},
 \end{aligned}$$

in the equality and the second inequality, respectively. Finally, we apply the inequality (3.3) to (3.2) to obtain the desired result:

$$\mathbb{P} \left(\bigcup_{k=0}^{\infty} \left\{ w : \mathcal{G}([t_{n_k}, t_{n_{k+1}}))(w) \text{ has a spanning tree} \right\} \right) \geq \exp \left(- \frac{R^2 \log R}{(R-1)^2} \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n}-1} \right). \quad \square$$

Now, we recall the matrix formulation of (3.1). According to (2.2), the matrix formulation of (3.1) was given as

$$\begin{cases} \dot{X}^{(1)} = V^{(1)}, \\ \dot{V}^{(1)} = -\frac{1}{N}\mathcal{L}_\sigma(X^{(1)})V^{(1)} + \frac{1}{N}\mathcal{R}_\sigma(X^{(1)}, V^{(1)}). \end{cases} \tag{3.4}$$

Here, we take into account the homogeneous part of (3.4), which becomes

$$\dot{V}^{(2)} = -\frac{1}{N}\mathcal{L}_\sigma(X^{(1)})V^{(2)}. \tag{3.5}$$

If we denote the state-transition matrix corresponding to (3.5) as $\bar{\Phi}$, then it follows from (2.7) that for $a \geq b \geq 0$,

$$V^{(2)}(a) = \bar{\Phi}(a, b)V^{(2)}(b). \tag{3.6}$$

Then, we have the following explicit form of the solution to (3.4):

$$V^{(1)}(a) = \bar{\Phi}(a, b)V^{(1)}(b) + \frac{1}{N} \int_b^a \bar{\Phi}(a, s)\mathcal{R}_{\sigma_s}(X^{(1)}(s), V^{(1)}(s))ds. \tag{3.7}$$

In the following lemma, we present a lower bound estimate of the ergodicity coefficient for the state-transition matrix $\bar{\Phi}$ when the sample path $(X, V)(\omega)$ satisfies

$$\mathcal{G}([t_{n_k}, t_{n_{k+1}}])(\omega) \text{ has a spanning tree for every } k \in \mathbb{N} \cup \{0\}, \tag{3.8}$$

to apply Lemma 2.3 to (3.7). For this, we assume some *a priori* conditions for a moment; there exists a non-negative number $D_X^\infty \geq 0$ and a positive number $\bar{M} > 1$ such that

$$\begin{aligned} &\circ \sup_{t \in \mathbb{R}_+} D_X(t, w) \leq D_X^\infty < \infty, \\ &\circ \frac{(N-1)^2\phi(0)M}{N} \sup_{k \in \mathbb{N}} \sum_{l=1}^k (\bar{n} + r \log(N-1) + r \log l) D_V(t_{n_{(k-1)(N-1)}}) \leq \log \bar{M}. \end{aligned} \tag{3.9}$$

Then, the following lemma allows to analyse the flocking dynamics of (3.1).

Lemma 3.2. *Let $w \in \Omega$ be an event satisfying (3.8), and assume the sample path $(X, V)(\omega)$ of the system (3.1) satisfies (F1) – (F5) and (3.9)₁. Then, we obtain the following assertions:*

1. *For every $k \in \mathbb{N} \cup \{0\}$, the ergodicity coefficient of $\bar{\Phi}(t_{n_{k(N-1)}}, t_{n_{(k-1)(N-1)}})$ satisfies*

$$\mu(\bar{\Phi}(t_{n_{k(N-1)}}, t_{n_{(k-1)(N-1)}})) \geq \left(\frac{m\phi(D_X^\infty)}{N}\right)^{N-1} \exp(-\phi(0)(t_{n_{k(N-1)}} - t_{n_{(k-1)(N-1)}})).$$

2. *For every $T_1 \geq T_2 \geq 0$, $\bar{\Phi}(T_1, T_2)$ is a stochastic matrix.*

Proof. (1) We employ the method used in [13]. First, for $k \in \mathbb{N} \cup \{0\}$ and $q \in [N-1]$, we denote $\{k_{qa}\}_{a=1}^{N_q+1}$ the unique integer-valued increasing sequence such that

$$\begin{aligned} n_{(k-1)(N-1)+q-1} &= k_{q1} < \dots < k_{qN_q+1} = n_{(k-1)(N-1)+q}, \\ \sigma_t &= \sigma_{i_{k_{qa}}} \neq \sigma_{i_{k_{qa+1}}}, \quad \forall t \in [t_{k_{qa}}, t_{k_{qa+1}}), \quad a \in [N_q], \end{aligned}$$

and we use the following notation for simplicity:

$$g_{qa} := \sigma_{i_{k_{qa}}} \in [N_G], \quad \forall a \in [N_q].$$

Now, we apply (2.6)–(2.10) to write the state-transition matrix for (3.5)–(3.6) in terms of Laplacian matrix \mathcal{L}_l . Since ϕ is monotonically decreasing and $\chi_{ij}^l \in \{0, 1\}$, the condition (3.9)₁ implies

$$\phi(\|x_i - x_j\|) \geq \phi(D_X^\infty) \quad \text{and} \quad 0 \leq d_i^l \leq N\phi(0), \quad i, j \in [N].$$

Then, we have

$$-\frac{1}{N}\mathcal{L}_{g_{qa}} = \frac{1}{N}(\mathcal{A}_{g_{qa}} - \mathcal{D}_{g_{qa}}) \geq \frac{1}{N}\underline{\mathcal{A}}_{g_{qa}} - \phi(0)I_N, \tag{3.10}$$

where each (i, j) -th entry of the matrix $\underline{\mathcal{A}}_{g_{qa}}$ is given as

$$(\underline{\mathcal{A}}_{g_{qa}})_{ij} := \begin{cases} \chi_{ij}^{g_{qa}} \phi(D_X^\infty), & i \neq j, \\ \phi(0), & i = j. \end{cases}$$

Thus, if $\bar{\Psi}(t_{k_{qa+1}}, t_{k_{qa}})$ is the state-transition matrix corresponding to $-\frac{1}{N}\mathcal{L}_{g_{qa}} + \phi(0)I_N$, the relation (2.10) yields

$$\bar{\Phi}(t_{k_{qa+1}}, t_{k_{qa}}) = \exp(-\phi(0)(t_{k_{qa+1}} - t_{k_{qa}}))\bar{\Psi}(t_{k_{qa+1}}, t_{k_{qa}}),$$

and we apply (F1) and (3.10) to the Peano–Baker series representation for $\bar{\Psi}(t_{k_{qa+1}}, t_{k_{qa}})$ to obtain

$$\begin{aligned} \bar{\Psi}(t_{k_{qa+1}}, t_{k_{qa}}) &= I_N + \sum_{k=1}^{\infty} \int_{t_{k_{qa}}}^{t_{k_{qa+1}}} \int_{t_{k_{qa}}}^{s_1} \cdots \int_{t_{k_{qa}}}^{s_{k-1}} \prod_{b=1}^k \left(-\frac{1}{N}\mathcal{L}_{g_{qa}}(s_b) + \phi(0)I_N \right) ds_k \cdots ds_1 \\ &\geq I_N + \sum_{k=1}^{\infty} \int_{t_{k_{qa}}}^{t_{k_{qa+1}}} \int_{t_{k_{qa}}}^{s_1} \cdots \int_{t_{k_{qa}}}^{s_{k-1}} \left(\frac{1}{N}\underline{\mathcal{A}}_{g_{qa}} \right)^k ds_k \cdots ds_1 \\ &= I_N + \sum_{k=1}^{\infty} \frac{1}{k!} (t_{k_{qa+1}} - t_{k_{qa}})^k \left(\frac{1}{N}\underline{\mathcal{A}}_{g_{qa}} \right)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (t_{k_{qa+1}} - t_{k_{qa}})^k \left(\frac{1}{N}\underline{\mathcal{A}}_{g_{qa}} \right)^k \\ &= \exp\left(\frac{1}{N}(t_{k_{qa+1}} - t_{k_{qa}})\underline{\mathcal{A}}_{g_{qa}} \right) \\ &\geq \exp\left(\frac{m}{N}\underline{\mathcal{A}}_{g_{qa}} \right). \end{aligned} \tag{3.11}$$

Therefore, the state-transition matrix $\bar{\Phi}(t_{k_{qN_q+1}}, t_{k_{q1}}) = \bar{\Phi}(t_{n_{(k-1)(N-1)+q}}, t_{n_{(k-1)(N-1)+q-1}})$ has a following lower bound:

$$\begin{aligned} \bar{\Phi}(t_{k_{qN_q+1}}, t_{k_{q1}}) &= \prod_{a=1}^{N_q} \bar{\Phi}(t_{k_{qa+1}}, t_{k_{qa}}) \\ &\geq \exp(-\phi(0)(t_{k_{qN_q+1}} - t_{k_{q1}})) \prod_{a=1}^{N_q} \exp\left(\frac{m}{N}\underline{\mathcal{A}}_{g_{qa}} \right) \\ &\geq \frac{m}{N} \exp(-\phi(0)(t_{k_{qN_q+1}} - t_{k_{q1}})) \sum_{a=1}^{N_q} \underline{\mathcal{A}}_{g_{qa}} \\ &\geq \frac{m}{N} \phi(D_X^\infty) \exp(-\phi(0)(t_{k_{qN_q+1}} - t_{k_{q1}})) A_q, \end{aligned} \tag{3.12}$$

where A_q is the adjacency matrix of $\mathcal{G}([t_{k_{q1}}, t_{k_{qN_q+1}}])$ for each $q \in [N - 1]$. Accordingly, we multiply (3.12) for $q = 1, 2, \dots, N - 1$ and obtain

$$\begin{aligned} &\bar{\Phi}(t_{n_{k(N-1)}}, t_{n_{(k-1)(N-1)}}) \\ &= \prod_{q=1}^{N-1} \bar{\Phi}(t_{n_{(k-1)(N-1)+q}}, t_{n_{(k-1)(N-1)+q-1}}) \\ &\geq \left(\frac{m}{N} \right)^{N-1} \phi(D_X^\infty)^{N-1} \exp(-\phi(0)(t_{n_{k(N-1)}} - t_{n_{(k-1)(N-1)}})) \prod_{q=1}^{N-1} A_q. \end{aligned} \tag{3.13}$$

Since $\mathcal{G}(A_q)$ contains a spanning tree for each $q \in [N - 1]$, Lemma 2.4 implies that $\prod_{q=1}^{N-1} A_q$ is a scrambling matrix and, in particular,

$$\mu \left(\prod_{q=1}^{N-1} A_q \right) \geq 1.$$

Therefore, we apply $A \geq B \geq 0 \implies \mu(A) \geq \mu(B)$ to (3.13) to obtain

$$\mu(\bar{\Phi}(t_{n_k(N-1)}, t_{n_{(k-1)(N-1)}})) \geq \left(\frac{m}{N}\right)^{N-1} \phi(D_X^\infty)^{N-1} \exp(-\phi(0)(t_{n_k(N-1)} - t_{n_{(k-1)(N-1)}})).$$

(2) Observe that the constant vector $V^{(2)} = [1, \dots, 1]^T$ can be a special solution to (3.5)₂ whatever $X^{(1)}$ is. Then, by (3.6), one has

$$[1, \dots, 1]^T = \bar{\Phi}(T_1, T_2)[1, \dots, 1]^T, \quad \forall T_1 \geq T_2 \geq 0.$$

Finally, we combine the above relation and the non-negativity of $\bar{\Psi}(T_1, T_2)$ obtained from the Peano–Baker series representation as (3.11) to see that

$$\bar{\Phi}(T_1, T_2) = \exp(-\phi(0)(T_1 - T_2))\bar{\Psi}(T_1, T_2) \geq 0$$

is a stochastic matrix, which is our desired result. □

Then, we apply Lemma 3.2 to (3.1)₂ to obtain the velocity alignment of the system (3.1) under *a priori* assumptions for some well-prepared initial data.

Lemma 3.3. (Velocity alignment) *Let $w \in \Omega$ be an event satisfying (3.8), and assume the sample path $(X, V)(\omega)$ of the system (3.1) satisfies (F1) – (F5) and (3.9)₁. If we further assume*

$$D_V(0, w) < \sqrt{2},$$

the following inequality holds for all $t \in [t_{n_k(N-1)}, t_{n_{(k+1)(N-1)}})$ and $k \in \mathbb{N} \cup \{0\}$:

$$D_V(t, w) \leq \bar{M}D_V(0, w) \exp\left(-\frac{C((k+1)^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)}\right),$$

where C is the constant defined in (F6), that is,

$$C := \left(\frac{m\phi(D_X^\infty) \exp(-\phi(0)M\bar{n})(N-1)^{-rM\phi(0)}}{N}\right)^{N-1}.$$

Proof. In this proof, we suppress w -dependence to simplify the notation. For every $t \in [t_{n_{(k-1)(N-1)}}, t_{n_k(N-1)})$, we apply the explicit formula (3.7) to obtain

$$V(t_{n_k(N-1)}) = \bar{\Phi}(t_{n_k(N-1)}, t_{n_{(k-1)(N-1)}})V(t_{n_{(k-1)(N-1)}}) + \frac{1}{N} \int_{t_{n_{(k-1)(N-1)}}}^{t_{n_k(N-1)}} \bar{\Phi}(t_{n_k(N-1)}, s)\mathcal{R}_{\sigma_s}(s)ds.$$

Since $\bar{\Phi}(t_{n_k(N-1)}, t_{n_{(k-1)(N-1)}})$ is a stochastic matrix (· Lemma 3.2), we apply Lemmas 2.3 and 3.2 to get the following estimate for D_V :

$$\begin{aligned} & D_V(t_{n_k(N-1)}) \\ & \leq (1 - \mu(\bar{\Phi}(t_{n_k(N-1)}, t_{n_{(k-1)(N-1)}}))) D_V(t_{n_{(k-1)(N-1)}}) + D_B \\ & \leq \left(1 - \left(\frac{m}{N}\right)^{N-1} \phi(D_X^\infty)^{N-1} \exp(-\phi(0)(t_{n_k(N-1)} - t_{n_{(k-1)(N-1)}}))\right) D_V(t_{n_{(k-1)(N-1)}}) + D_B \\ & =: \mathcal{I} + D_B, \end{aligned}$$

where the matrix B is defined as

$$B := \frac{1}{N} \int_{t_{n_{(k-1)(N-1)}}}^{t_{n_k(N-1)}} \bar{\Phi}(t_{n_k(N-1)}, s)\mathcal{R}_{\sigma_s}(s)ds.$$

In what follows, we estimate \mathcal{I} and D_B one by one.

- (Estimate of \mathcal{I}): By using (F1) and the definition of $n_k(\bar{n})$, we have

$$\mathcal{I} \leq \left(1 - \left(\frac{m\phi(D_X^\infty)}{N} \right)^{N-1} \exp \left(-\phi(0)M \left((N-1)\bar{n} + \sum_{l=(k-1)(N-1)+1}^{k(N-1)} [r \log l] \right) \right) \right) \times D_V(t_{n_{(k-1)(N-1)}}).$$

- (Estimate of D_B): Since D_B is the maximum distance between row vectors of B , one can easily verify that

$$D_B \leq \frac{1}{N} \int_{t_{n_{(k-1)(N-1)}}}^{t_{n_{k(N-1)}}} D_{\bar{\Phi}(t_{n_{k(N-1)}}, s) \mathcal{R}_{\sigma_s}(s)} ds.$$

Then, since $\bar{\Phi}(t_{n_{k(N-1)}}, s)$ is stochastic, we apply Lemma 2.3 to the integrand $D_{\bar{\Phi}(t_{n_{k(N-1)}}, s) \mathcal{R}_{\sigma_s}(s)}$ to obtain

$$\begin{aligned} D_B &\leq \frac{1}{N} \int_{t_{n_{(k-1)(N-1)}}}^{t_{n_{k(N-1)}}} D_{\bar{\Phi}(t_{n_{k(N-1)}}, s) \mathcal{R}_{\sigma_s}(s)} ds \\ &\leq \frac{1}{N} \int_{t_{n_{(k-1)(N-1)}}}^{t_{n_{k(N-1)}}} (1 - \mu(\bar{\Phi}(t_{n_{k(N-1)}}, s))) D_{\mathcal{R}_{\sigma_s}(s)} ds \\ &\leq \frac{1}{N} \int_{t_{n_{(k-1)(N-1)}}}^{t_{n_{k(N-1)}}} D_{\mathcal{R}_{\sigma_s}(s)} ds. \end{aligned}$$

Hence, it suffices to find an upper bound for $D_{\mathcal{R}_{\sigma_s}(s)}$, where the matrix $\mathcal{R}_{\sigma_s}(s)$ is given by

$$\mathcal{R}_{\sigma_s} := (r_1^{\sigma_s}, \dots, r_N^{\sigma_s})^T \quad \text{and} \quad r_i^{\sigma_s} := \frac{1}{2} \sum_{j=1}^N \chi_{ij}^{\sigma_s} \phi(\|x_i - x_j\|) \|v_i - v_j\|^2 v_i.$$

To do this, we use Lemma 2.1, $\phi(\cdot) \leq \phi(0)$, $\chi_{ij}^{\sigma_s} \in \{0, 1\}$ and Corollary 2.1 that for $s \in [t_{n_{(k-1)(N-1)}}, t_{n_{k(N-1)}})$,

$$\begin{aligned} \|r_i^{\sigma_s}\| &\leq \frac{1}{2} \sum_{j=1}^N \chi_{ij}^{\sigma_s} \phi(\|x_i - x_j\|) \|v_i - v_j\|^2 v_i \leq \frac{1}{2} \sum_{j=1}^N \phi(0) \|v_i - v_j\|^2 \|v_i\| \\ &\leq \frac{1}{2} \sum_{j=1}^N \phi(0) \|v_i - v_j\|^2 = \frac{1}{2} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \phi(0) \|v_i - v_j\|^2 \\ &\leq \frac{1}{2} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \phi(0) D_V^2(s) \\ &\leq \frac{1}{2} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \phi(0) D_V(t_{n_{(k-1)(N-1)}})^2 \\ &= \frac{(N-1)}{2} \phi(0) D_V(t_{n_{(k-1)(N-1)}})^2. \end{aligned}$$

Therefore, the diameter D_B has the following upper bound:

$$\begin{aligned}
 D_B &\leq \frac{1}{N} \int_{t_{n_{(k-1)(N-1)}}}^{t_{n_{k(N-1)}}} D_{\mathcal{R}_{\sigma_s}(s)} ds \\
 &\leq \frac{1}{N} \int_{t_{n_{(k-1)(N-1)}}}^{t_{n_{k(N-1)}}} (N-1)\phi(0)D_V(t_{n_{(k-1)(N-1)}})^2 ds \\
 &= \frac{N-1}{N} \phi(0)(t_{n_{k(N-1)}} - t_{n_{(k-1)(N-1)}})D_V(t_{n_{(k-1)(N-1)}})^2 \\
 &\leq \frac{N-1}{N} \phi(0)M \left((N-1)\bar{n} + \sum_{l=(k-1)(N-1)+1}^{k(N-1)} \lfloor r \log l \rfloor \right) D_V(t_{n_{(k-1)(N-1)}})^2.
 \end{aligned}$$

Thus, we combine two estimates of \mathcal{I} and D_B to obtain

$$\begin{aligned}
 &D_V(t_{n_{k(N-1)}}) \\
 &\leq \left(1 - \left(\frac{m\phi(D_X^\infty)}{N} \right)^{N-1} \exp \left(-\phi(0)M \left((N-1)\bar{n} + \sum_{l=(k-1)(N-1)+1}^{k(N-1)} \lfloor r \log l \rfloor \right) \right) \right) \\
 &\quad \times D_V(t_{n_{(k-1)(N-1)}}) \\
 &\quad + \frac{N-1}{N} \phi(0)M \left((N-1)\bar{n} + \sum_{l=(k-1)(N-1)+1}^{k(N-1)} \lfloor r \log l \rfloor \right) D_V(t_{n_{(k-1)(N-1)}})^2 \\
 &\leq \left(1 - \left(\frac{m\phi(D_X^\infty)}{N} \right)^{N-1} \exp \left(-\phi(0)M \left((N-1)\bar{n} + \sum_{l=(k-1)(N-1)+1}^{k(N-1)} r \log(k(N-1)) \right) \right) \right) \\
 &\quad \times D_V(t_{n_{(k-1)(N-1)}}) \\
 &\quad + \frac{N-1}{N} \phi(0)M \left((N-1)\bar{n} + \sum_{l=(k-1)(N-1)+1}^{k(N-1)} r \log(k(N-1)) \right) D_V(t_{n_{(k-1)(N-1)}})^2 \\
 &= \left(1 - \left(\frac{m\phi(D_X^\infty)}{N} \right)^{N-1} \exp \left(-\phi(0)M(N-1) \left(\bar{n} + r \log(k(N-1)) \right) \right) \right) D_V(t_{n_{(k-1)(N-1)}}) \\
 &\quad + \frac{(N-1)}{N} \phi(0)M \left((N-1) \left(\bar{n} + r \log(k(N-1)) \right) \right) D_V(t_{n_{(k-1)(N-1)}})^2 \\
 &= \left(1 - \left(\frac{m\phi(D_X^\infty) \exp(-\phi(0)M\bar{n})}{N} \right)^{N-1} (k(N-1))^{-rM(N-1)\phi(0)} \right) D_V(t_{n_{(k-1)(N-1)}}) \\
 &\quad + \frac{(N-1)^2}{N} \phi(0)M \left(\bar{n} + r \log(k(N-1)) \right) D_V(t_{n_{(k-1)(N-1)}})^2 \\
 &= \left[1 + \frac{(N-1)^2}{N} \phi(0)M \left(\bar{n} + r \log(N-1) + r \log k \right) D_V(t_{n_{(k-1)(N-1)}}) \right. \\
 &\quad \left. - \left(\frac{m\phi(D_X^\infty) \exp(-\phi(0)M\bar{n})(N-1)^{-rM\phi(0)}}{N} \right)^{N-1} k^{-rM(N-1)\phi(0)} \right] D_V(t_{n_{(k-1)(N-1)}}) \\
 &\leq \exp \left(\frac{(N-1)^2}{N} \phi(0)M \left(\bar{n} + r \log(N-1) + r \log k \right) D_V(t_{n_{(k-1)(N-1)}}) \right) \\
 &\quad \times \exp(-Ck^{-rM(N-1)\phi(0)}) D_V(t_{n_{(k-1)(N-1)}}),
 \end{aligned}$$

where we used the following relation in the last inequality:

$$(1 + \alpha) - x \leq \exp(\alpha - x), \quad \forall x \geq 0.$$

By iterating the above process, we apply the *a priori* condition (3.9)₂ to obtain the following inequality for $t \in [t_{n_k(N-1)}, t_{n_{(k+1)(N-1)}})$:

$$\begin{aligned}
 D_V(t) &\leq D_V(0) \exp\left(\frac{(N-1)^2}{N} \phi(0) M \sum_{l=1}^k (\bar{n} + r \log(N-1) + r \log l) D_V(t_{n_{(k-l)(N-1)}})\right) \\
 &\quad \times \exp\left(-C \sum_{l=1}^k l^{-rM(N-1)\phi(0)}\right) \\
 &\leq \bar{M} D_V(0) \exp\left(-C \frac{(k+1)^{1-rM(N-1)\phi(0)} - 1}{1 - rM(N-1)\phi(0)}\right),
 \end{aligned}$$

where we used

$$\sum_{l=1}^k l^{-rM(N-1)\phi(0)} \geq \int_1^{k+1} x^{-rM(N-1)\phi(0)} dx = \frac{(k+1)^{1-rM(N-1)\phi(0)} - 1}{1 - rM(N-1)\phi(0)}$$

in the last inequality, which completes the proof. □

Remark 3.2. In [13], the authors used a similar argument to provide a sufficient framework for the Cucker–Smale model to exhibit asymptotic flocking. For the Cucker–Smale model, the D_B term in Lemma 3.3 does not exist, so that the sufficient framework does not need to constrain the upper bound of initial D_V with respect to \bar{n} . Therefore, it was possible to take $\bar{n} \rightarrow \infty$ to show that the probability to exhibit asymptotic flocking is 1 for some well-prepared initial data and system parameters. In the model (3.1), however, the sufficient framework needs to constrain D_V with respect to \bar{n} (via C_0 in this paper) due to the existence of this D_B term (see Lemma 2.3).

From Lemma 3.3, we can estimate the decay rate of velocity diameter D_V in terms of \bar{M}, m, M, r, N . Therefore, we can determine a suitable sufficient condition in terms of initial data and system parameters to make the assumption (3.9)₂ imply the assumption (3.9)₁.

Lemma 3.4. (Group formation) *Let $w \in \Omega$ be an event satisfying (3.8), and assume the sample path $(X, V)(\omega)$ of the system (3.1) satisfies (F1) – (F5) and (3.9)₂. If we further assume*

$$\begin{aligned}
 D_V(0, w) &< \sqrt{2}, \\
 D_X(0, w) + \bar{M}C_0 D_V(0, w) &< D_X^\infty,
 \end{aligned}
 \tag{3.14}$$

the first *a priori* assumption (3.9)₁ also holds, that is,

$$\sup_{t \in \mathbb{R}_+} D_X(t, w) \leq D_X^\infty,$$

where C_0 is the constant defined in (F6).

Proof. First, we define S as

$$S =: \{t > 0 \mid D_X(s, w) \leq D_X^\infty, \forall s \in [0, t)\},$$

and claim:

$$t^* := \sup S = +\infty.$$

To see this, suppose that the contrary holds, that is, $t^* < +\infty$. Since D_X is continuous in t and (3.14) implies $D_X(0, \omega) < D_X^\infty$, we have $t^* > 0$ and

$$D_X(t^*-, w) = D_X^\infty.$$

Then, we integrate the result of Lemma 3.3 on $t \in [0, t^*]$ and apply (3.14) to obtain

$$\begin{aligned}
 D_X(t, w) &\leq D_X(0, w) + \int_0^{t^*} D_V(s, w) ds \\
 &\leq D_X(0, w) + \int_0^\infty D_V(s, w) ds \\
 &\leq D_X(0, w) + \overline{M} D_V(0, w) \sum_{k=0}^\infty \left[(t_{n_{(k+1)(N-1)}} - t_{n_{k(N-1)}}) \right. \\
 &\quad \left. \times \exp\left(-\frac{C((k+1)^{1-rM(N-1)\phi(0)} - 1)}{1-rM(N-1)\phi(0)}\right) \right] \\
 &\leq D_X(0, w) + \overline{M} M(N-1) D_V(0, w) \sum_{k=0}^\infty \left[(\bar{n} + r \log(k+1)(N-1)) \right. \\
 &\quad \left. \times \exp\left(-\frac{C((k+1)^{1-rM(N-1)\phi(0)} - 1)}{1-rM(N-1)\phi(0)}\right) \right] \\
 &= D_X(0, w) + \overline{M} C_0 D_V(0, w) < D_X^\infty.
 \end{aligned}$$

By taking $t \rightarrow t^*$, this inequality yields

$$D_X(t^*-, w) = \lim_{t \rightarrow t^*-} D_X(t, w) \leq D_X(0, w) + \overline{M} C_0 D_V(0, w) < D_X^\infty,$$

which contradicts to $D_X(t^*-, w) = D_X^\infty$. Therefore, we can conclude $t^* = +\infty$, which is our desired result. □

Finally, we show that the condition (F6) implies the assumption (3.9)₂, so that the asymptotic flocking occurs for $\omega \in \Omega$ satisfying (3.8).

Lemma 3.5. *Let $w \in \Omega$ be an event satisfying (3.8), and assume the sample path $(X, V)(\omega)$ of the system (3.1) satisfies (F1) – (F6). Then, the assumption (3.9)₂ holds, that is,*

$$\frac{(N-1)^2 \phi(0) M}{N} \cdot \sup_{k \in \mathbb{N}} \left[\sum_{l=1}^k (\bar{n} + r \log l(N-1)) D_V(t_{n_{(l-1)(N-1)}}) \right] \leq \log \overline{M}.$$

Proof. First, define \mathcal{S} as a subset of \mathbb{N} satisfying

$$\mathcal{S} =: \left\{ k \mid \frac{(N-1)^2 \phi(0) M}{N} \sum_{l=1}^k (\bar{n} + r \log l(N-1)) D_V(t_{n_{(l-1)(N-1)}}) \leq \log \overline{M} \right\}.$$

Since (F6) immediately implies $1 \in \mathcal{S}$, we can define $s^* =: \sup \mathcal{S} \in \mathbb{N} \cup \{\infty\}$. Then, we claim

$$s^* = +\infty.$$

To see this, suppose we have $s^* < +\infty$. Then,

$$\mathcal{J} := \frac{(N-1)^2 \phi(0) M}{N} \sum_{l=1}^{s^*+1} (\bar{n} + r \log l(N-1)) D_V(t_{n_{(l-1)(N-1)}}) > \log \overline{M}. \tag{3.15}$$

On the other hand, we can apply Corollary 2.1, Lemma 3.3 and (F6) to get

$$\begin{aligned} \mathcal{J} &\leq \frac{(N-1)^2 \phi(0) M \bar{M} D_V(0)}{N} \\ &\times \left(\sum_{l=1}^{s^*} (\bar{n} + r \log l(N-1)) \exp \left(-\frac{C(l^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \right. \\ &\left. + (\bar{n} + r \log(s^* + 1)(N-1)) \exp \left(-\frac{C((s^*)^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \right). \end{aligned} \tag{3.16}$$

By using $1 - rM(N-1)\phi(0) \in (0, 1)$ and $C \leq \log 2$ in (F5), we have

$$\begin{aligned} &\sum_{l=s^*+1}^{\infty} (\bar{n} + r \log l(N-1)) \exp \left(-\frac{C(l^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \\ &\geq (\bar{n} + r \log(s^* + 1)(N-1)) \exp \left(-\frac{C((s^*)^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \\ &\times \sum_{l=s^*+1}^{\infty} \exp \left(-\frac{C(l^{1-rM(N-1)\phi(0)} - (s^*)^{1-rM(N-1)\phi(0)})}{1 - rM(N-1)\phi(0)} \right) \\ &\geq (\bar{n} + r \log(s^* + 1)(N-1)) \exp \left(-\frac{C((s^*)^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \\ &\times \sum_{l=s^*+1}^{\infty} \exp(-C(l - s^*)) \\ &\geq (\bar{n} + r \log(s^* + 1)(N-1)) \exp \left(-\frac{C((s^*)^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \times \sum_{l=1}^{\infty} 2^{-l} \\ &= (\bar{n} + r \log(s^* + 1)(N-1)) \exp \left(-\frac{C((s^*)^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right). \end{aligned} \tag{3.17}$$

Therefore, we combine (3.16) and (3.17) to obtain

$$\begin{aligned} \mathcal{J} &\leq \frac{(N-1)^2 \phi(0) M \bar{M} D_V(0)}{N} \\ &\times \left[\sum_{l=1}^{\infty} (\bar{n} + r \log l(N-1)) \exp \left(-\frac{C(l^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \right] \\ &< \log \bar{M}, \end{aligned}$$

which leads a contradiction to (3.15). This implies that s^* must be infinity, which means the assumption (3.9)₂ holds. □

Now, we are ready to state our main result. By combining Lemmas 3.1–3.5, we can deduce the following result.

Theorem 3.1. (Probability of asymptotic flocking) *Suppose that (X, V) is a solution process of the system (3.1) satisfying (F1) – (F6). Then, we have*

$$\mathbb{P}(w \in \Omega: (X, V)(\omega) \text{ exhibits asymptotic flocking}) \geq \exp \left(-\frac{R^2 \log R}{(R-1)^2} \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n}-1} \right).$$

Proof. Lemma 3.1 shows that the probability to satisfy (3.8) is greater than or equal to $\exp \left(-\frac{R^2 \log R}{(R-1)^2} \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n}-1} \right)$. Then, we apply Lemmas 3.3–3.5 to obtain the desired result. □

To check whether this result is meaningful, we can compare the expected behaviour of the trivial solution with the result in Theorem 3.1. On the one hand, one can easily verify that the solution of (3.1) becomes the uniform linear motion of all agents with the same velocities when the event ω satisfies $D_V(0, \omega) = 0$. On the other hand, the following corollary shows that the probability to exhibit asymptotic flocking converges to 1 when $\sup_{\omega \in \Omega} D_V(0, \omega)$ converges to 0, which implies that the result in Theorem 3.1 is consistent with the uniform linear motion of the trivial solution.

Corollary 3.1. *Suppose that $(X^{(n)}, V^{(n)})$ is a sequence of the solution process of the system (3.1) satisfying (F1) – (F5) and*

$$\sup_{n \in \mathbb{N}} \text{ess sup}_{\omega \in \Omega} D_{X^{(n)}}(0, \omega) < \infty, \quad \lim_{n \rightarrow \infty} \text{ess sup}_{\omega \in \Omega} D_{V^{(n)}}(0, \omega) = 0. \tag{3.18}$$

Then, we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(w \in \Omega: (X^{(n)}, V^{(n)})(\omega) \text{ exhibits asymptotic flocking} \right) = 1.$$

Proof. Note that the initial velocity diameter $D_V(0, \omega)$ only affects to (F6). To meet the condition (F6), we set

$$\bar{M} = e, \quad D_X^\infty = \sup_{n \in \mathbb{N}} \text{ess sup}_{\omega \in \Omega} D_{X^{(n)}}(0, \omega) + \frac{1}{\phi(0)}.$$

Then, (F6) holds true for $(X^{(n)}, V^{(n)})$ if

$$\sup_{\omega \in \Omega} D_{V^{(n)}}(0, \omega) < \min \left\{ \frac{1}{e\phi(0)C_0}, \sqrt{2} \right\}, \tag{3.19}$$

where C_0 is the number determined by \bar{n} :

$$C = \left(\frac{m\phi(D_X^\infty) \exp(-\phi(0)M\bar{n})(N-1)^{-rM\phi(0)}}{N} \right)^{N-1},$$

$$C_0 = M(N-1) \sum_{l=1}^{\infty} \left[(\bar{n} + r \log l(N-1)) \exp \left(-\frac{C(l^{1-rM(N-1)\phi(0)} - 1)}{1 - rM(N-1)\phi(0)} \right) \right].$$

In fact, every sufficiently large \bar{n} is allowed in the condition (F5), and $C_0 = C_0(m, M, \phi, D_X^\infty, \bar{n})$ can be considered as an increasing function with respect to \bar{n} . By using the condition (3.18), one can see that for every sufficiently large \bar{n} , there exists $n_0 \in \mathbb{N}$ such that (3.19) holds for all $n \geq n_0$. Therefore, we apply Theorem 3.1 to get

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \mathbb{P} \left(w \in \Omega: (X^{(n)}, V^{(n)})(\omega) \text{ exhibits asymptotic flocking} \right) \\ & \geq \lim_{\bar{n} \rightarrow \infty} \exp \left(-\frac{R^2 \log R}{(R-1)^2} \sum_{l=1}^{N_G} (1 - \mathcal{P}_l)^{\bar{n}-1} \right) \\ & = 1, \end{aligned}$$

which implies our desired result. □

4. Numerical simulation

In this section, we performed a numerical simulation of the Cauchy problem (3.1), especially for cases where theoretical predictions are relatively easy due to the simple structure of the interaction network.

Consider a system with three points, as shown in Figure 1, where particle 2 only affects particles 1 and 3, and no other interaction exists. Additionally, assume the following deterministic initial data so that we can control the simulation results more easily:

$$\begin{aligned} x_1(0) &= (0, 1), \quad x_2(0) = (0, 0), \quad x_3(0) = (0, -1), \\ v_1(0) &= (\cos \varepsilon, \sin \varepsilon), \quad v_2(0) = (1, 0), \quad v_3(0) = (\cos \varepsilon, -\sin \varepsilon), \quad \varepsilon \in (0, 1). \end{aligned}$$

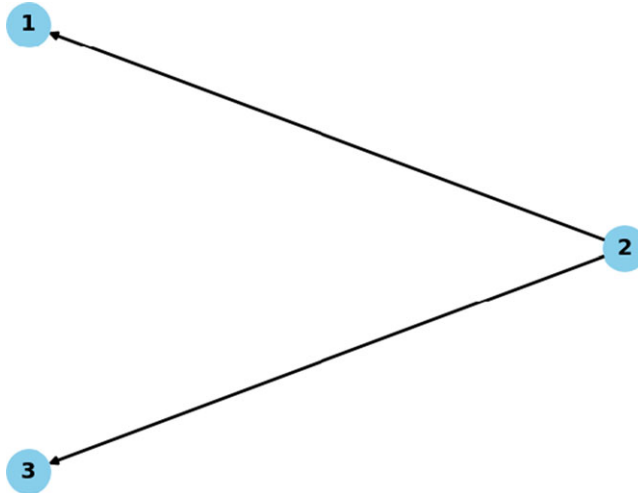


Figure 1. Interaction network.

Then, we have

$$\begin{cases} \frac{dv_1}{dt} = \frac{1}{3} \chi_{12}^\sigma \phi(\|x_1 - x_2\|) (v_2 - \langle v_1, v_2 \rangle v_1), \\ \frac{dv_2}{dt} = 0, \\ \frac{dv_3}{dt} = \frac{1}{3} \chi_{32}^\sigma \phi(\|x_3 - x_2\|) (v_2 - \langle v_3, v_2 \rangle v_3), \end{cases} \tag{4.1}$$

and the derivative of the inner product of velocities can be calculated as follows:

$$\frac{d}{dt} \langle v_i, v_2 \rangle = \frac{1}{3} \chi_{i2}^\sigma \phi(\|x_i - x_2\|) (1 - \langle v_i, v_2 \rangle^2), \quad i = 1, 3. \tag{4.2}$$

By using the primitive $\Phi(x) = \int_0^x \phi(y) dy$, the following simple inequality can be obtained from (4.2):

$$\begin{aligned} \frac{d}{dt} \|v_i - v_2\| &= \frac{1}{2\|v_i - v_2\|} \frac{d}{dt} \|v_i - v_2\|^2 \\ &= \frac{1}{2\|v_i - v_2\|} \frac{d}{dt} (2 - 2\langle v_i, v_2 \rangle) \\ &= -\frac{1}{3\|v_i - v_2\|} \chi_{i2}^\sigma \phi(\|x_i - x_2\|) (1 - \langle v_i, v_2 \rangle^2) \\ &= -\frac{1}{12} \chi_{i2}^\sigma \phi(\|x_i - x_2\|) \|v_i - v_2\| (4 - \|v_i - v_2\|^2) \\ &\leq -\frac{1}{12} \chi_{i2}^\sigma (4 - \|v_i - v_2\|^2) \phi(\|x_i - x_2\|) (v_i - v_2) \cdot \left(\frac{x_i - x_2}{\|x_i - x_2\|} \right) \\ &= -\frac{1}{12} \chi_{i2}^\sigma (4 - \|v_i - v_2\|^2) \frac{d}{dt} \Phi(\|x_i - x_2\|). \end{aligned} \tag{4.3}$$

If there was no random selection of digraph \mathcal{G} and $\chi_{12}^\sigma \equiv \chi_{32}^\sigma \equiv 1$ for all t , (4.3) yields

$$\frac{d}{dt} \left(\log \frac{2 + \|v_i - v_2\|}{2 - \|v_i - v_2\|} + \frac{1}{3} \Phi(\|x_i - x_2\|) \right) \leq 0, \tag{4.4}$$

and if initial data satisfies

$$\frac{1}{3} \int_{\|x_i(0) - x_2(0)\|}^\infty \phi(x) dx > \log \frac{2 + \|v_i(0) - v_2(0)\|}{2 - \|v_i(0) - v_2(0)\|}, \tag{4.5}$$

we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{3} \Phi(\|x_i(t) - x_2(t)\|) &\leq \log \frac{2 + \|v_i(0) - v_2(0)\|}{2 - \|v_i(0) - v_2(0)\|} + \frac{1}{3} \Phi(\|x_i(0) - x_2(0)\|) \\ &< \frac{1}{3} \lim_{x \rightarrow \infty} \Phi(x), \end{aligned}$$

which implies the existence of the finite upper bound D_X^∞ of $\|x_i - x_2\|$. Then, one can apply (4.3) to obtain

$$\begin{aligned} \frac{d}{dt} \log \|v_i - v_2\| &= -\frac{1}{12} \phi(\|x_i - x_2\|)(4 - \|v_i - v_2\|^2) \\ &\leq -\frac{1}{12} \phi(D_X^\infty)(4 - \|v_i(0) - v_2(0)\|^2), \end{aligned}$$

which shows the exponential convergence of $\|v_i - v_2\|$, so that the asymptotic flocking emerges. If $\|x_i(0) - x_2(0)\| = 1$ and $\phi(x) = \frac{1}{(1+x^2)^2}$, the left-hand side (4.5) is

$$\frac{1}{3} \cdot \left[\frac{1}{2} \left(\frac{x}{x^2 + 1} + \arctan x \right) \right]_1^\infty = \frac{\pi - 2}{24} \simeq 0.047566,$$

and (4.5) is equivalent to

$$2 - \frac{4}{1 + \exp\left(\frac{\pi-2}{24}\right)} > \|v_i(0) - v_2(0)\| = 2 \sin \frac{\varepsilon}{2} \iff \varepsilon < 0.0475618xxx.$$

Below, Figure 2 shows the trajectories of three particles when $\chi_{12}^\sigma \equiv \chi_{32}^\sigma \equiv 1$, $\|x_i(0) - x_2(0)\| = 1$ and $\phi(x) = \frac{1}{(1+x^2)^2}$. To perform the numerical experiment, we simply used the first order Euler method and plotted trajectories for a total of 100,000 s with a time interval $\Delta t = 0.1$ s. Although the horizontal axis in the plot is the x -coordinate rather than time, it can be seen as if the y -coordinates are drawn according to time, since the velocities of the three particles are close to $(1, 0)$. From these results, we can see that our theoretical prediction of the sufficient conditions for flocking to occur is nearly optimal, even with numerical errors.

On the other hand, the least connected way for the union of graphs to have a spanning tree is that $N_G = 2$ and $P_1 + P_2 = 1$, where \mathcal{G}_1 and \mathcal{G}_2 only contain one edge ($2 \rightarrow 1$) and ($2 \rightarrow 3$), respectively. In this case, since each χ_{ij}^σ is a component of \mathcal{G}_σ 's adjacency matrix, the sum of $\chi_{12}^{\sigma(t,\omega)}$ and $\chi_{32}^{\sigma(t,\omega)}$ must be 1 for all t and ω . However, even if all constants are set, it is very difficult to estimate the exact value of C_0 in (F6) because the series C_0 converges at a very slow rate. For example, if we have $N = 3$, $P_1 = P_2 = \frac{1}{2}$ and $m = M = 0.05$, then the conditions we get from (F5) are

$$r = \frac{R}{\log 2} < \frac{1}{0.05 \cdot 2 \cdot 1}, \frac{1}{2^{\bar{n}-1}} \leq 1 - \frac{1}{R}, \left(\frac{0.05 \cdot 1 \cdot \exp(-1 \cdot 0.05 \cdot \bar{n}) \cdot 2^{-r \cdot 0.05 \cdot 1}}{3} \right)^2 \leq \log 2,$$

where the last conditions holds for every $r > 0$ and $\bar{n} \in \mathbb{N}$. If we set $R = 5 \log 2$, any integer $\bar{n} \geq 2$ satisfies the condition (F5), and at this time, the probability guaranteed in Theorem 3.1 can be maximised by choosing the largest \bar{n} which satisfies (F6).

In Figure 3, we show the trajectories of three particles when $\chi_{12}^{\sigma_m} = 1 - \chi_{32}^{\sigma_m}$ is a m th sample from the distribution $\text{Bernoulli}(\frac{1}{2})$ and $\|x_i(0) - x_2(0)\| = 1$, $\phi(x) = \frac{1}{(1+x^2)^2}$ as in Figure 2. Unlike in Figure 2, we cannot explicitly find the exact value of ε at which the long-time behaviour starts to change, but at least, we can see that the distance between points diverges to infinity at $\varepsilon = 0.04$ and asymptotic flocking occurs at $\varepsilon = 0.02$.

The first feature that can be seen from the repeated experimental results at $\varepsilon = 0.02$ is that asymptotic flocking occurs with a higher probability than predicted by theory. Although not shown in Figure 4, in practice, flocking never failed to occur even once during the experiments. The second feature is that the diameters of the three points always converged to a value of (approximately) 4.5, regardless of whether particle 1 or particle 3 moved further away from particle 2. In fact, this is somewhat natural, since the

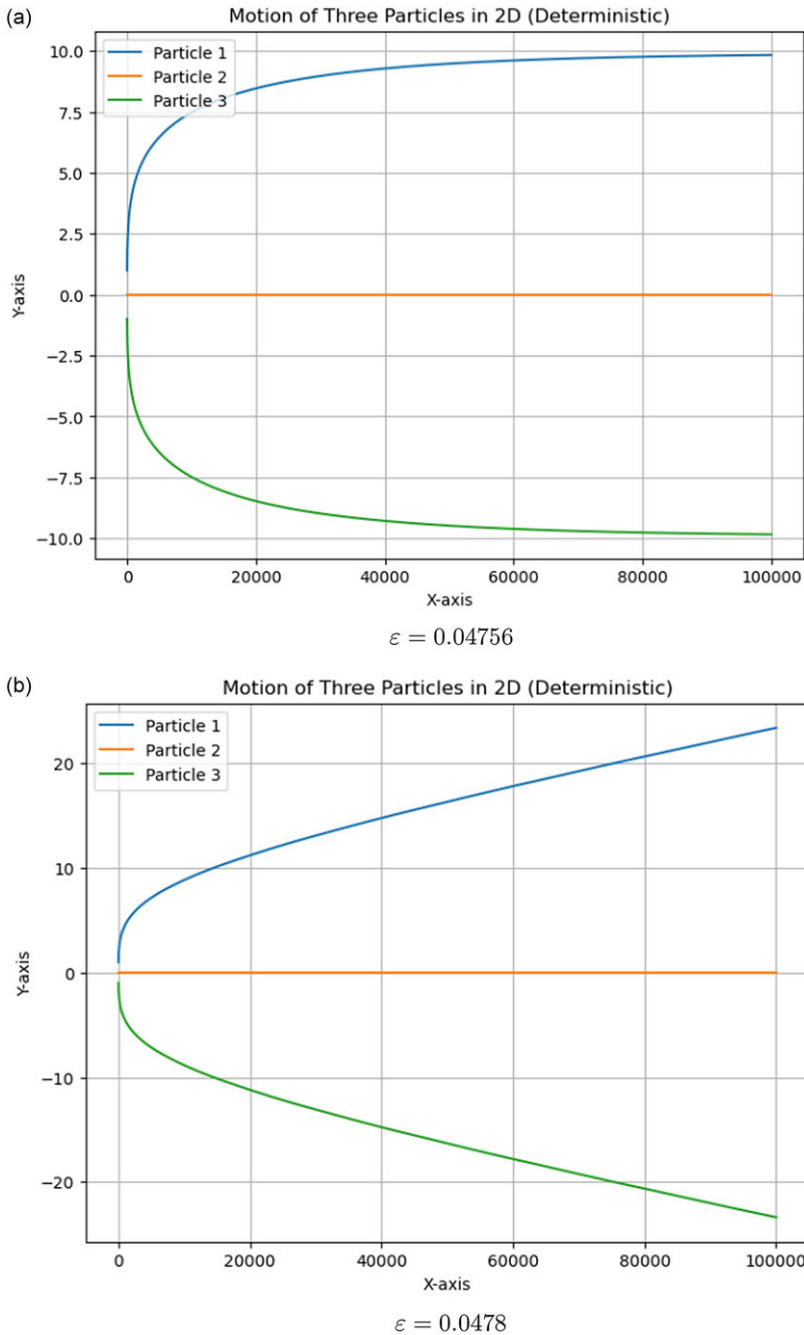


Figure 2. Trajectories of three particles, $\chi_{12}^\sigma \equiv \chi_{32}^\sigma \equiv 1$.

more times the interaction is turned off, the further away from particle 2 it is, and the sum of χ_{12}^σ and χ_{32}^σ is identical to the constant 1 in this system.

In Figure 5, we vary the size of $M = m$ while keeping all other conditions the same and plot their trajectories. From these three experiments, we can say that flocking tends to be harder to guarantee for larger $M = m$. In fact, this has some theoretical interpretation: the particle that has its interaction turned off for time M will move away from the other particle for a long time without interaction, and the

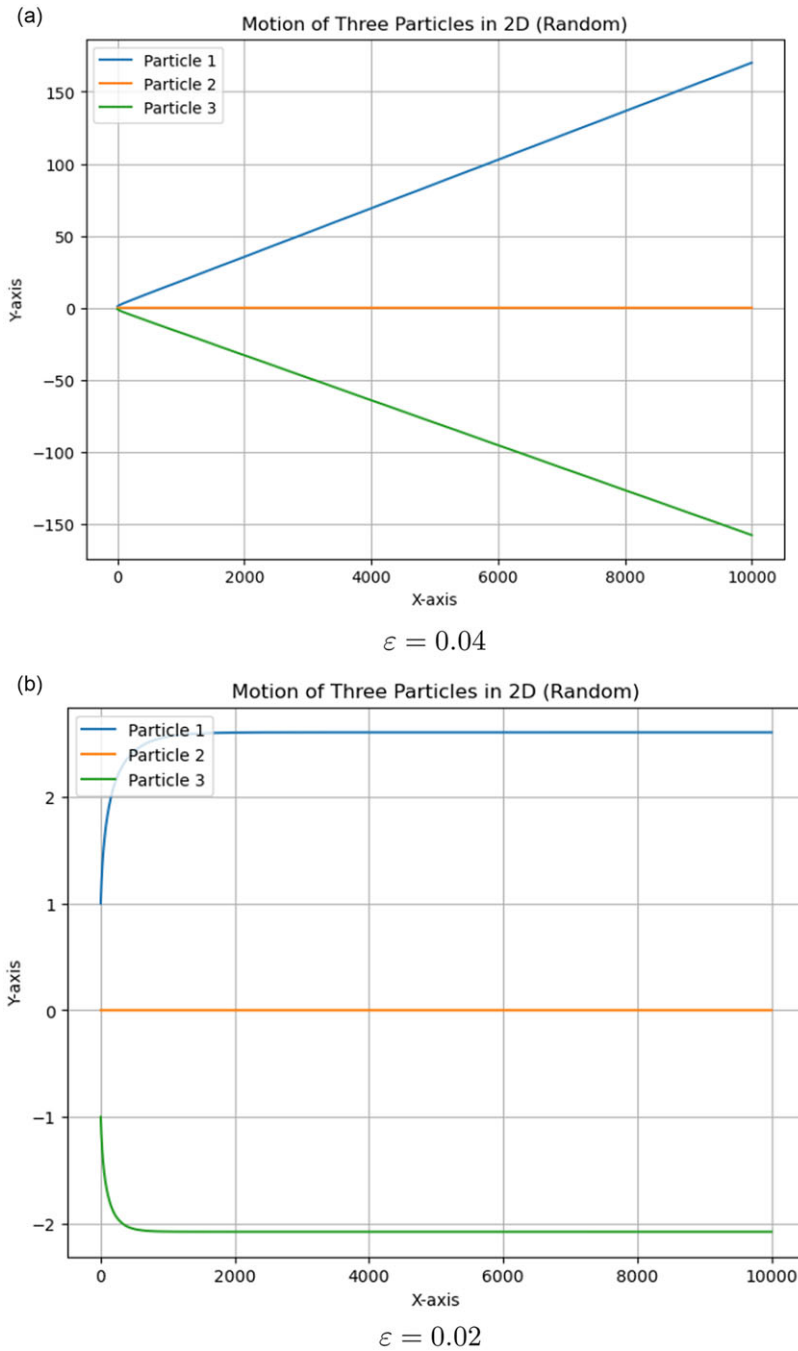


Figure 3. Trajectories of three particles, $\chi_{12}^{\sigma_{1n}} = 1 - \chi_{32}^{\sigma_{1n}} \stackrel{i.i.d}{\sim} \text{Bernoulli}(\frac{1}{2})$.

two particles that have already moved away will not be able to interact enough to reduce their velocity difference to cause flocking. Therefore, to guarantee flocking for $M > 0$, the interaction must be stronger than in the deterministic example in Figure 2, and this tendency increases as M increases. The sufficient condition (F5) we presented also has an upper bound on the value of M that can cause flocking, which is $M < \frac{\log 2}{2}$ in the current setting. Although flocking actually occurred even at a larger $M = 0.5$, it can be clearly confirmed that the presence of M affects whether flocking occurs.

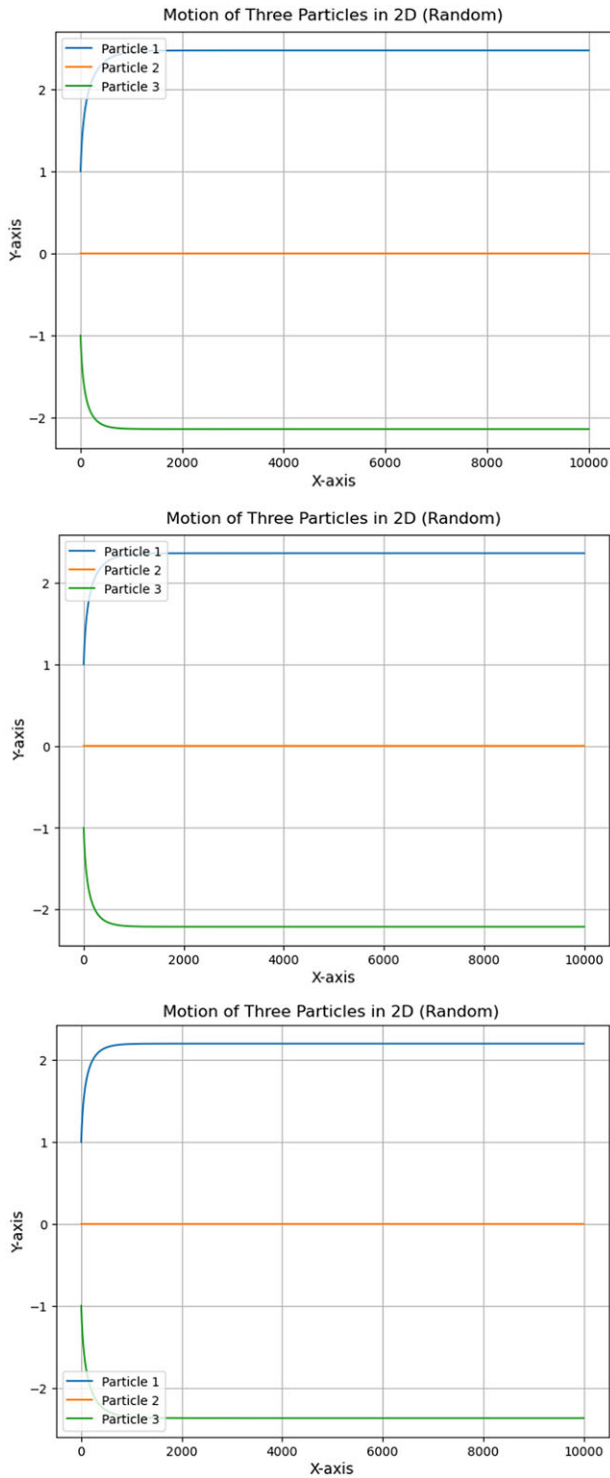


Figure 4. Three different simulations at $\epsilon = 0.02$.

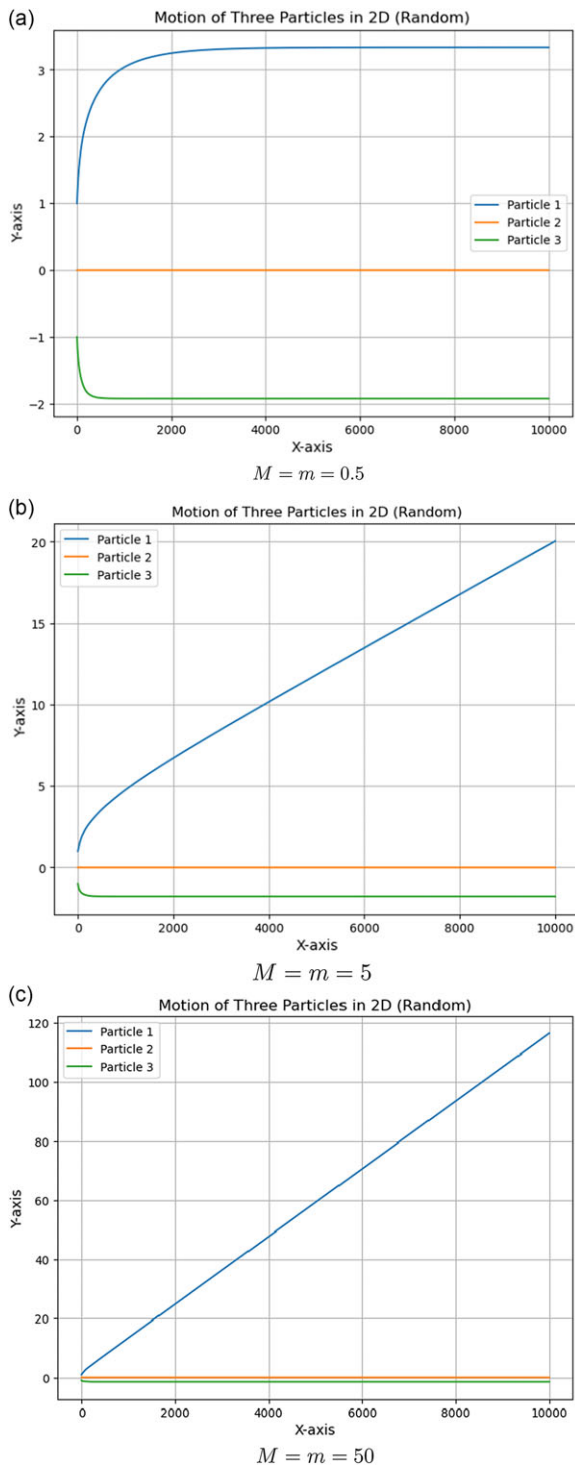


Figure 5. Three different simulations at different M .

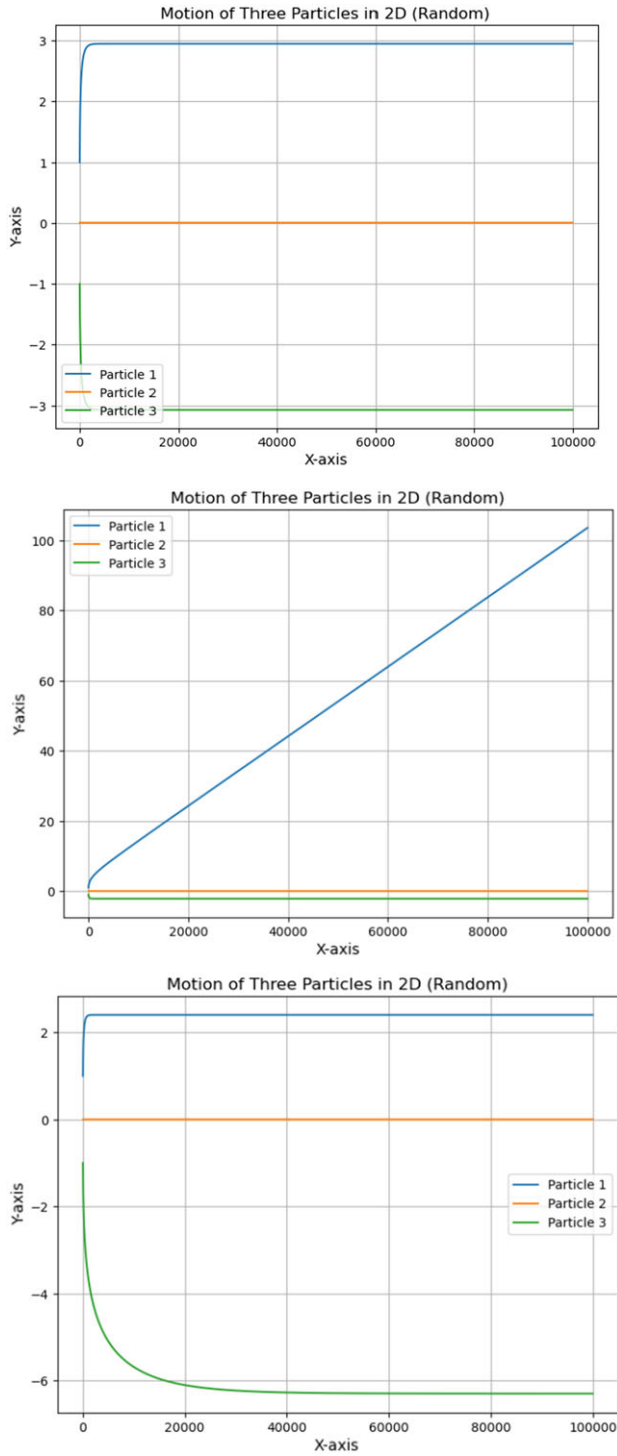


Figure 6. Three different simulations at $\epsilon = 0.022$.

Finally, we present experimental results that prove that flocking can either occur or not occur depending on sampling and that calculating the probability of flocking occurring as in our paper is indeed an appropriate form of outcome. In Figure 6, we ran three experiments with all parameters set the same as in Figure 4 except for ε , which was set to 0.022. Although flocking occurs under relatively lenient conditions compared with the sufficient conditions $(\mathcal{F}1) - (\mathcal{F}6)$ we have presented, we are open to the possibility that this is not just a technical limitation but also a special property of the examples used in our numerical experiments.

5. Conclusion

In this paper, we presented a sufficient framework concerning initial data and system parameters to exhibit the asymptotic flocking of the Cucker–Smale model with a unit-speed constraint and randomly switching topology. For this, we used the explicit form of the given dynamical system by using the state-transition matrix of its homogeneous counterpart. Then, we used the relation between the ergodicity coefficient and the diameter of velocity to show that the asymptotic flocking occurs when the event that the union of the network topology in some time interval contains a spanning tree occurs infinitely many times. Subsequently, we provided a lower bound estimate of the probability of such an event, which therefore becomes the lower bound of probability to exhibit asymptotic flocking. In particular, we verified that the probability to exhibit asymptotic flocking converges to 1 when the sufficient framework $(\mathcal{F}1) - (\mathcal{F}5)$ holds and $\sup_{\omega \in \Omega} D_V(0, \omega)$ converges to 0.

Acknowledgements. The work of H. Ahn was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (2022R1C1C2007321).

Competing interests. All authors declare that they have no conflicts of interest.

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