Bull. Aust. Math. Soc. **106** (2022), 18–27
doi:10.1017/S0004972721000885

COUNTEREXAMPLES TO THE HASSE PRINCIPLE IN FAMILIES

NGUYEN XUAN THO

(Received 24 August 2021; accepted 15 September 2021; first published online 5 November 2021)

Abstract

We generalise two quartic surfaces studied by Swinnerton-Dyer to give two infinite families of diagonal quartic surfaces which violate the Hasse principle. Standard calculations of Brauer–Manin obstructions are exhibited.

2020 *Mathematics subject classification*: primary 11D25; secondary 11D88.

Keywords and phrases: Brauer–Manin obstruction, Hasse principle, diagonal quartic surfaces.

1. Introduction

For a variety V defined over $\mathbb{Q}$, the Hasse principle says that if $V(\mathbb{R}) \neq \emptyset$ and $V(\mathbb{Q}_p) \neq \emptyset$ for all prime numbers p, then $V(\mathbb{Q}) \neq \emptyset$. This principle is true for quadratic forms (see Serre [16, Section 3.2, Theorem 8]) but it is not true in general. The classical counterexamples are $2y^2 = 1 - 17x^4$ (Lind [10], Reichardt [14]), $3x^3 + 4y^3 + 5y^3 = 0$ (Selmer [15]) and $5x^3 + 9y^3 + 10z^3 + 12w^3 = 0$ (Cassels and Guy [5]). For more examples, see Colliot-Thélène *et al.* [6], Skorobogatov [18], Poonen [12], Quan [13] and Hirakawa [8]. This paper focuses on counterexamples to the Hasse principle in the class of diagonal quartic surfaces

$$\alpha x^4 + \beta y^4 + \gamma z^4 + \delta w^4 = 0, \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are nonzero integers such that $\alpha\beta\gamma\delta$ is a square. These surfaces were studied extensively by Swinnerton-Dyer [19] and Bright [2–4]. However, only a few examples of surfaces (1.1) are known to violate the Hasse principle. The surfaces

$$4x^4 + 9y^4 = 8z^4 + 8w^4, \tag{1.2a}$$

$$2x^4 + 9y^4 = 6z^4 + 12w^4 \tag{1.2b}$$

and the family

$$x^4 + 4y^4 = d(z^4 + w^4),$$

The author is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) (grant number 101.04-2019.314).

where $d \in \mathbb{Z}^+$, $d \equiv 2 \pmod{16}$, no prime $p \equiv 3 \pmod 4$ divides d, no prime $p \equiv 5 \pmod 8$ divides d to an odd power and $d = r^2 + s^2$ with $r \equiv \pm 3 \pmod 8$, appeared in Swinnerton-Dyer [19]. Bremner and Tho [1] found the family

$$x^4 + 7P^2y^4 = 14P^2Q^2z^4 + 18Q^2w^4, \tag{1.3}$$

where every prime divisor of PQ is congruent to 1 mod 24, if p is a prime divisor of P, then $2Q^2$ is a fourth power mod p and, if q is a prime divisor of Q, then $-7P^2$ is a fourth power mod q. Specialising to $P = Q = 1$ in (1.3) gives the surface

$$x^4 + 7y^4 = 14z^4 + 18w^4, \tag{1.4}$$

which has a solution $(2\theta^2 + 2\theta, 2\theta, \theta^2 + 1, \theta^2 - 1)$, where $\theta^3 + \theta^2 - 1 = 0$. Currently, the surface (1.4) is the only known example of type (1.1) which violates the Hasse principle but has nontrivial points in a cubic number field. It is an open question whether the surfaces (1.2a) or (1.2b) have points in cubic number fields. It is worth mentioning here the work of Manin [11] and Colliot-Thélène *et al.* [6], which is entirely devoted to the study of diagonal *cubic* surfaces

$$\alpha x^3 + \beta y^3 + \gamma z^3 + \delta w^3 = 0,$$

where $\alpha, \beta, \gamma, \delta$ are nonzero integers.

The principal results of this paper are Theorems 1.1 and 1.2. For an odd prime number p and an integer a, with $p \nmid a$, the symbol $(a/p)_4$ is $+1$ if a is a fourth power mod p and is -1 otherwise. The symbol (a/p) is defined similarly by replacing fourth powers mod p by squares mod p.

THEOREM 1.1. *Let a, b, c, d be square-free integers such that $-abcd$ is a square, $a \equiv -b \equiv c \equiv d \equiv \pm 1 \pmod 8$, $a \equiv c \equiv d \pmod 3$, $(3/p) = (2/p)_4 = 1$ for any prime divisor p of $abcd$ and, for any permutation (a_1, b_1, c_1, d_1) of (a, b, c, d) and any prime divisor p of a_1b_1 not dividing c_1d_1, we have $(c_1d_1/p) = 1$. Then the surface*

$$128a^2x^4 + 18b^2y^4 = c^2z^4 + d^2w^4 \tag{1.5}$$

(i) *is solvable in $\mathbb{Q}_p$ for all prime numbers p and*
(ii) *has no rational points.*

THEOREM 1.2. *Let a, b, c, d be square-free integers such that $-abcd$ is a square, $\gcd(a, b, c, d) = 1$, $a \equiv -b \equiv c \equiv d \equiv \pm 1 \pmod 8$, $a \equiv -b \equiv c \equiv d \pmod 3$, $(-1/p) = (2/p)_4 = (3/p)_4 = 1$ for any prime divisor p of $abcd$ and, for any permutation (a_1, b_1, c_1, d_1) of (a, b, c, d) and any prime divisor p of a_1b_1 not dividing c_1d_1, we have $(c_1d_1/p) = 1$. Then the surface*

$$27a^2x^4 + 24b^2y^4 = c^2z^4 + 2d^2w^4 \tag{1.6}$$

(i) *is solvable in $\mathbb{Q}_p$ for all prime numbers p and*
(ii) *has no rational points.*

To prove Theorems 1.1 and 1.2, we explicitly calculate the Brauer–Manin obstruction following the framework described by Swinnerton-Dyer in [19].

The conditions on a, b, c, d and p in Theorems 1.1 and 1.2 are imposed to guarantee the solubility of (1.5) and (1.6) in $\mathbb{Q}_p$ for all prime numbers p. There are infinitely many integers a, b, c, d satisfying these conditions. For example, we mimic the example in [6, Proposition 5] by letting $a = 1$, $b = -1$ and $c = d = q$, where q is a prime of the form $q = r^2 + 576s^2$, where $r, s \in \mathbb{Z}^+$. The fact that there are infinitely many prime numbers q of this form follows from Cox [7, Theorem 9.12]. The condition $(2/q)_4 = 1$ follows from Silverman [17, Ch. IX, Proposition 6.6]. From Lemmermeyer [9, page 159], we have $(-3/q)_4 = 1$. Since $q \equiv 1 \pmod{16}$, we have $(-1/q) = (-1/q)_4 = 1$. Hence, $(3/q)_4 = 1$.

Remark 1.3. When we specialise $b = -1$ and $a = c = d = 1$ in (1.5) and map $(x, y, z, w) \mapsto (x/2, y/2, z, w)$, we have the surface (1.2a). When we specialise $b = -1$ and $a = c = d = 1$ in (1.6) and map $(x, y, z, w) \mapsto (x/4, y/2, z, w)$, we have the surface (1.2b).

For the rest of this paper, for a prime number p, $(\cdot,\cdot)_p$ denotes the Hilbert symbol and $v_p(s)$ denotes the highest power of a prime number p dividing s. For a subset S of a field, we set $S^2 = \{x^2 : x \in S\}$, $-S = \{-x : x \in S\}$ and $-3S = \{-3x : x \in S\}$. We need some properties of the Hilbert symbol (see Serre [16, Ch. III]).

- Let a, b, $c \in \mathbb{Q}_p^*$. Then $(a, bc)_p = (a, b)_p (a, c)_p$.
- Let a, $b \in \mathbb{Q}^*$. Then $(a, b)_\infty \prod_{p \text{ prime}} (a, b)_p = 1$.
- Let q be an odd prime. Let a, b be units in $\mathbb{Z}_q$. Then $(a, b)_q = 1$.
- Let a, $b \in \mathbb{R}$. If $a > 0$, then $(a, b)_\infty = 1$.
- Let $a, b \in \mathbb{Q}^*$. Write $a = p^\epsilon a_1$ and $b = p^\delta b_1$, where $p \nmid a_1 b_1$. Then

$$(a, b)_p = (-1)^{(a_1-1)(b_1-1)/4 + \epsilon(b_1^2-1)/8 + \delta(a_1^2-1)/8} \quad \text{if } p = 2$$

and

$$(a, b)_p = (-1)^{\epsilon\delta(p-1)/2} \left(\frac{b_1}{p}\right)^\epsilon \left(\frac{a_1}{p}\right)^\delta \quad \text{if } p > 2.$$

2. Proof of Theorem 1.1

Lemma 5.2 in Bright [2] implies that Equation (1.5) has solutions in $\mathbb{Q}_p$ for all primes $p \notin \{2, 3, 5\}$ and $p \nmid abcd$. We consider the cases $p \in \{2, 3, 5\}$ or $p \mid abcd$.

Case 1: $p = 2$. Since

$$a \equiv -b \equiv c \equiv d \equiv \pm 1 \pmod{8},$$

we have $|a|, |b|, |c|, |d| \in \mathbb{Z}_2^2$. Then (1.5) has a solution in $\mathbb{Q}_2$, namely

$$\left(0, \frac{1}{\sqrt{|b|}}, \frac{\sqrt[4]{17}}{\sqrt{|c|}}, \frac{1}{\sqrt{|d|}}\right).$$

Case 2: $p > 2$. By Hensel's lemma [16, Section 2.2, Corollary 1], it is enough to show that (1.5) has a nontrivial solution mod p.

https://doi.org/10.1017/S0004972721000885 Published online by Cambridge University Press

- $p = 3$ and $p \nmid abcd$. Equation (1.5) has a solution $(1, 0, 1, 1)$ mod 3.
- $p = 5$ and $p \nmid abcd$. Equation (1.5) has a solution:
  - $(1, 1, 0, 0)$ mod 5 if $5 \mid a^2 + b^2$;
  - $(0, 0, 1, 1)$ mod 5 if $5 \mid c^2 + d^2$;
  - $(1, 1, 0, 1)$ mod 5 if $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \pmod 5$;
  - $(1, 0, 1, 1)$ mod 5 if $a^2 \equiv b^2 \equiv 1 \pmod 5$ and $c^2 \equiv d^2 \equiv 4 \pmod 5$;
  - $(1, 0, 1, 1)$ mod 5 if $a^2 \equiv b^2 \equiv 4 \pmod 5$ and $c^2 \equiv d^2 \equiv 1 \pmod 5$.
- $p \mid abcd$. We only consider the case $p \mid ac$ and $p \nmid bd$. The remaining cases can be proved similarly. Since $(bd/p) = 1$ by the given hypothesis,

$$\left(\frac{bd}{p}\right) = 1 = \left(\frac{3}{p}\right) = \left(\frac{2}{p}\right)_4 = 1,$$

so there exists $z_0 \in \mathbb{Z}$ such that $18b^2y_0^4 \equiv d^2 \pmod p$. Therefore, (1.5) has a solution $(0, y_0, 0, 1) \pmod p$.

We now show that (1.5) has no rational points. On the contrary, assume that (x, y, z, w) is a rational point on (1.5). We can further assume that $x, y, z, w \in \mathbb{Z}$ and $\gcd(x, y, z, w) = 1$. If $3 \mid x$, then considering (1.5) mod 3 gives $3 \mid z$ and $3 \mid w$. Hence, $3 \mid y$, which is impossible since $\gcd(x, y, z, w) = 1$. Therefore, $3 \nmid x$. Similarly, $3 \nmid z$ and $3 \nmid w$. If $2 \mid y$, considering (1.5) mod 4 gives $2 \mid z$ and $2 \mid w$. By letting $z = 2z_1$ and $w = 2w_1$, where $z_1, w_1 \in \mathbb{Z}$, (1.5) reduces to $8a^2x^4 = c^2z_1^4 + d^2w_1^4$, so that $2 \mid z_1$ and $2 \mid w_1$. Hence, $2 \mid x$. Therefore, $2 \mid \gcd(x, y, z, w)$, which is impossible. Thus, $2 \nmid y$. Similarly, $2 \nmid z$ and $2 \nmid w$. So, $3 \nmid xzw$ and $2 \nmid yzw$.

From (1.5),

$$(cz^2 - dw^2 - 6by^2)(cz^2 - dw^2 + 6by^2) = (16ax^2 - cz^2 - dw^2)(16ax^2 + cz^2 + dw^2).$$

Therefore, there exist coprime integers u, v such that

$$u(cz^2 - dw^2 + 6by^2) = v(16ax^2 - cz^2 - dw^2), \tag{2.1a}$$

$$v(cz^2 - dw^2 - 6by^2) = u(16ax^2 + cz^2 + dw^2). \tag{2.1b}$$

Eliminating x^2, y^2, z^2 and w^2 respectively gives

$$6Aby^2 + Bcz^2 - Cdw^2 = 0, \tag{2.2a}$$

$$16Aax^2 + Ccz^2 + Bdw^2 = 0, \tag{2.2b}$$

$$8Bax^2 - 3Cby^2 + Adw^2 = 0, \tag{2.2c}$$

$$8Cax^2 + 3Bby^2 + Acz^2 = 0, \tag{2.2d}$$

where $A = u^2 + v^2$, $B = u^2 + 2uv - v^2$ and $C = u^2 - 2uv - v^2$.

LEMMA 2.1. *For all odd primes p, we have $(B, 6)_p = 1$.*

https://doi.org/10.1017/S0004972721000885 Published online by Cambridge University Press

PROOF. We consider three cases.

Case 1: $p = 3$. Considering (2.1a) mod 3 gives $3 \mid v$. Hence, $3 \nmid u$. Therefore, $B \equiv u^2 \equiv 1 \pmod 3$, so that $B \in \mathbb{Z}_3^2$. Hence, $(B, 6)_3 = 1$.

Case 2: $p \mid abcd$. Since

$$\left(\frac{2}{p}\right)_4 = \left(\frac{3}{p}\right) = 1, \quad \text{we have} \quad \left(\frac{6}{p}\right) = 1.$$

Hence, $6 \in \mathbb{Z}_p^2$ and $(B, 6)_p = 1$.

Case 3: $p > 3$ and $p \nmid abcd$.

- $p \mid B$. Since any common divisor of B and AC divides the discriminant of ABC, which is -2^{28}, we have $p \nmid AC$. From (2.2a),

$$6ACbdy^2 + BCcdz^2 = (Cdw)^2. \tag{2.3}$$

Therefore, $(6ACbd, BCcd)_p = 1$. Since $6ACbd$ and Ccd are units in $\mathbb{Z}_p$, we have $(6ACbd, Ccd)_p = 1$. Thus, from (2.3),

$$(6ACbd, B)_p = 1. \tag{2.4}$$

From (2.2b),

$$-ACac(4x)^2 - BCcdw^2 = (Ccz)^2. \tag{2.5}$$

Hence, $(-ACac, -BCcd)_p = 1$. Since $-ACac$ and $-Ccd$ are units in $\mathbb{Z}_p$, we have $(-ACac, -Ccd)_p = 1$. Thus, from (2.5),

$$(-ACac, B)_p = 1. \tag{2.6}$$

From (2.4) and (2.6),

$$(-6abcd, B)_p = 1. \tag{2.7}$$

Since $-abcd$ is a square, we have $(-abcd, B)_p = 1$. Therefore, from (2.7), $(B, 6)_p = 1$.
- $p \nmid B$. In this case, B and 6 are units in $\mathbb{Z}_p$. Hence, $(B, 6)_p = 1$. □

LEMMA 2.2. *We have* $(B, 6)_2 = -1$ *and* $(B, 6)_\infty = 1$.

PROOF. Considering (2.1a) mod 8 gives $-2u \equiv -2v \pmod 8$ and so $4 \mid (u - v)$. Let $u - v = 4k$, where $k \in \mathbb{Z}$. Then

$$B = 2u^2 - (u - v)^2 = 2(u^2 - 8k^2).$$

Since $u^2 - 8k^2 \equiv 1 \pmod 8$, we have $B = 2\ell^2$, where $\ell \in \mathbb{Z}_2$. Hence,

$$(B, 6)_2 = (2, 6)_2 = (2, 2)_2(2, 3)_2 = (-1)^{(3^2-1)/8} = -1.$$

Since $6 > 0$, it follows that $(B, 6)_\infty = 1$. □

https://doi.org/10.1017/S0004972721000885 Published online by Cambridge University Press

From Lemmas 2.1 and 2.2,

$$(B,6)_\infty \times \prod_{p \text{ prime}} (B,6)_p = -1,$$

which contradicts the product formula for the Hilbert symbol. So, the surface (1.5) has no rational points.

3. Proof of Theorem 1.2

The solution of (1.6) in $\mathbb{Q}_p$ for each prime p is proved in the same way as in Theorem 1.1. We focus on the second part of Theorem 1.2. Assume that (x, y, z, w) is a rational point on (1.6). Then we can assume that $x, y, z, w \in \mathbb{Z}$ with $\gcd(x, y, z, w) = 1$. From (1.6), we also have $3 \nmid zw$, $2 \nmid zwx$ and

$$(9ax^2 - cz^2 - 2dw^2)(9ax^2 + cz^2 + 2dw^2) = 2(dw^2 - cz^2 - 6by^2)(dw^2 - cz^2 + 6by^2).$$

Therefore, there exist coprime integers u, v such that

$$u(9ax^2 - cz^2 - 2dw^2) = v(dw^2 - cz^2 + 6by^2), \tag{3.1a}$$

$$v(9ax^2 + cz^2 + 2dw^2) = 2u(dw^2 - cz^2 - 6by^2). \tag{3.1b}$$

Eliminating x^2, y^2, z^2 and w^2 respectively gives

$$6Aby^2 + Bcz^2 - Cdw^2 = 0, \tag{3.2a}$$

$$9Aax^2 - Ccz^2 - 2Bdw^2 = 0, \tag{3.2b}$$

$$3Bax^2 + 2Cby^2 - Adw^2 = 0, \tag{3.2c}$$

$$-3Cax^2 + 4Bby^2 + Acz^2 = 0, \tag{3.2d}$$

where $A = 2u^2 + v^2$, $B = 2u^2 + 2uv - v^2$ and $C = 2u^2 - 4uv - v^2$.

LEMMA 3.1. *For all primes $p > 3$:*

(i) $(A, 6)_p = 1$;
(ii) $(C, 3)_p = 1$.

PROOF. Let $p > 3$ be a prime.

(i) Any common prime divisor of A and BC divides the discriminant of ABC, which is $-2^{28} \cdot 3^{10}$. Therefore, p is not a common divisor of A and BC.

Case 1: $p \mid A$ and $p \nmid abcd$. From (3.2b), $ACac(3x)^2 - 2BCcdw^2 = (Ccz)^2$. Hence,

$$(ACac, -2BCcd)_p = 1. \tag{3.3}$$

Since Cac and $-2BCcd$ are units in $\mathbb{Z}_p$, we have $(Cac, -2BCcd)_p = 1$. Thus, from (3.3),

$$(A, -2BCcd)_p = 1. \tag{3.4}$$

https://doi.org/10.1017/S0004972721000885 Published online by Cambridge University Press

From (3.2c), $3ABadw^2 - 6BCaby^2 = (3Bax)^2$. Hence,

$$(3ABad, -6BCab)_p = 1. \tag{3.5}$$

Since $3Bad$ and $-6BCab$ are units in $\mathbb{Z}_p$, we have $(3Bad, -6BCab)_p = 1$. Thus, from (3.5),

$$(A, -6BCab)_p = 1. \tag{3.6}$$

From (3.4) and (3.6),

$$(A, 3abcd)_p = 1. \tag{3.7}$$

Since $-abcd$ is a square, from (3.7),

$$(A, -3)_p = 1.$$

Since $p \mid A = 2u^2 + v^2$ and $\gcd(u, v) = 1$, we have $(-2/p) = 1$. Thus, $-2 \in \mathbb{Z}_p^2$. Therefore, $(A, -2)_p = 1$. Hence,

$$(A, 6)_p = (A, -3)_p(A, -2)_p = 1.$$

Case 2: $p \mid A$ and $p \mid abcd$. Then

$$\left(\frac{-1}{p}\right) = \left(\frac{3}{p}\right)_4 = 1 \quad \text{and so} \quad \left(\frac{-3}{p}\right) = 1.$$

Therefore, $-3 \in \mathbb{Z}_p^2$ and so $(A, -3)_p = 1$. Since $p \mid A = 2u^2 + v^2$, we have $(-2/p) = 1$. Hence, $(A, -2)_p = 1$ and

$$(A, 6)_p = (A, -3)_p(A, -2)_p = 1.$$

Case 3: $p \nmid A$. Then A and 6 are units in $\mathbb{Z}_p$. Therefore,

$$(A, 6)_p = 1.$$

(ii) Any common prime divisor of C and AB divides the discriminant of ABC, which is $-2^{28} \cdot 3^{10}$. Therefore, p is not a common divisor of C and AB.

Case 1: $p \mid C$ and $p \nmid abcd$. From (3.2a), $BCcdw^2 - 6ABbcy^2 = (Bcz)^2$. Hence,

$$(BCcd, -6ABbc)_p = 1. \tag{3.8}$$

Since Bcd and $-6ABbc$ are units in $\mathbb{Z}_p$, we have $(Bcd, -6ABbc)_p = 1$. From (3.8),

$$(C, -6ABbc)_p = 1. \tag{3.9}$$

From (3.2b), $ACacz^2 + 2ABadw^2 = (3Aax)^2$. Hence,

$$(ACac, 2ABad)_p = 1. \tag{3.10}$$

Since Aac and $2ABad$ are units in $\mathbb{Z}_p$, we have $(Aac, 2ABad)_p = 1$. From (3.10),

$$(C, 2ABad)_p = 1. \tag{3.11}$$

https://doi.org/10.1017/S0004972721000885 Published online by Cambridge University Press

From (3.9) and (3.11),

$$(C, -3abcd)_p = 1. \tag{3.12}$$

Since $-abcd$ is a square, from (3.12),

$$(C, 3)_p = 1.$$

Case 2: $p \mid C$ and $p \mid abcd$. Then $(3/p)_4 = 1$, so $(3/p)_2 = 1$. Therefore, $3 \in \mathbb{Z}_p^2$ and so $(C, 3)_p = 1$.

Case 3: $p \nmid C$. Since $p > 3$, both C and 3 are units in $\mathbb{Z}_p$. Hence, $(C, 3)_p = 1$. □

LEMMA 3.2. *We have* $(A, 6)_2 = (C, 3)_2 = 1$.

PROOF. Since $a \equiv c \equiv d \pmod 4$, taking (3.1a) mod 4 gives

$$2u \equiv 2v \pmod 4.$$

Hence, $2 \mid (u - v)$. Therefore, u and v are odd. Thus, $A = 2u^2 + v^2 \equiv 3 \pmod 8$ and $C = 2(u - v)^2 - 3v^2 \equiv 1 \pmod 4$. Let $A = 8h + 3$ and $C = 4h_1 + 1$, where $h, h_1 \in \mathbb{Z}$. Then

$$(A, 6)_2 = (8h + 3, 6)_2 = (-1)^{(8h+3-1)(3-1)/4+(8h+3)^2-1/8} = 1$$

and

$$(C, 3)_2 = (4h_1 + 1, 3)_2 = (-1)^{(4h_1+1-1)(3-1)/4} = 1.$$

This completes the proof. □

LEMMA 3.3. *We have* $A \in \mathbb{Q}_3^2$ *or* $A \in -3\mathbb{Q}_3^2$. *Furthermore, if* $A \in \mathbb{Q}_3^2$, *then* $C \in -\mathbb{Q}_3^2$.

PROOF. We consider two cases.

Case 1: $3 \mid uv$.

If $3 \mid u$, then $3 \nmid v$. Since $A \equiv v^2 \pmod 3$ and $C \equiv -v^2 \pmod 3$, it follows that $A \in \mathbb{Q}_3^2$ and $C \in -\mathbb{Q}_3^2$.

Otherwise, $3 \mid v$ and $3 \nmid u$. Since $c \equiv d \pmod 3$, we have $9 \mid z^2 + 2w^2$ from (3.1a) and $9 \mid w^2 - z^2 - 6y^2$ from (3.1b). Therefore, $9 \mid 3w^2 - 6y^2$, so that $3 \mid w^2 - 2y^2$, which is impossible.

Case 2: $3 \nmid uv$. Then $3 \nmid u$ and $3 \nmid v$. Hence, $3 \mid A$.

Case 2.1: $3 \nmid u - v$. Then

$$B = 3u^2 - (u - v)^2 \equiv -1 \pmod 3,$$
$$C = 2(u - v)^2 - 3v^2 \equiv -1 \pmod 3.$$

Therefore, $B = -\beta^2$ and $C = -\gamma^2$, where β, $\gamma \in \mathbb{Z}_3$ and $3 \nmid \beta\gamma$. Then (3.2c) and (3.2d) become

$$3\beta^2x^2 + 2\gamma^2y^2 + Aw^2 = 0, \tag{3.13a}$$
$$3\gamma^2x^2 - 4\beta^2y^2 + 2Az^2 = 0. \tag{3.13b}$$

https://doi.org/10.1017/S0004972721000885 Published online by Cambridge University Press

Since $3 \nmid \beta\gamma$ and $3 \mid A$, from (3.13a) and (3.13b), $3 \mid y$. Let $y = 3y_1$, where $y_1 \in \mathbb{Z}$. Then (3.13a) gives $Aw^2 = -3(\beta^2x^2 + 6\gamma^2y_1^2)$. Since $3 \nmid \beta x$, we have $A \in -3\mathbb{Q}_3^2$.

Case 2.2: $3 \mid u - v$. Let $u - v = 3t$, where $t \in \mathbb{Z}$. Then

$$B = 3u^2 - (u - v)^2 = 3(u^2 - 3t^2),$$

$$C = 2(u - v)^2 - 3v^2 = 3(6t^2 - v^2).$$

Therefore, $B = 3\beta^2$ and $C = -3\gamma^2$, where $\beta, \gamma \in \mathbb{Z}_3$ and $3 \nmid \beta\gamma$. Equation (3.2a) becomes

$$6A_1by^2 + \beta^2cz^2 + \gamma^2dw^2 = 0, \tag{3.14}$$

where $A_1 = A/3$. Since $c \equiv d \pmod 3$ and $3 \nmid \beta\gamma cdzw$, (3.14) is impossible mod 3. □

Lemma 3.4. *We have:*

(i) $(C, 3)_3 = -1$ *if* $A \in \mathbb{Q}_3^2$;
(ii) $(A, 6)_3 = -1$ *if* $A \in -3\mathbb{Q}_3^2$;
(iii) $(A, 6)_\infty = (C, 3)_\infty = 1$.

Proof. (i) If $A \in \mathbb{Q}_3^2$, then $C \in -\mathbb{Q}_3^2$. Therefore,

$$(C, 3)_3 = (-1, 3)_3 = -1.$$

(ii) If $A \in -3\mathbb{Q}_3^2$, then

$$(A, 6)_3 = (-3, 6)_3 = (-3, 2)_3(-3, 3)_3 = -1. \tag{3.15}$$

(iii) Since $6 > 0$ and $3 > 0$,

$$(A, 6)_\infty = (C, 3)_\infty = 1.$$ □

Proof of Theorem 1.2. By Lemma 3.3, we need to consider two cases.

Case 1: $A \in \mathbb{Q}_3^2$. Then $C \in -\mathbb{Q}_3^2$. From Lemmas 3.1, 3.2 and 3.4,

$$(C, 3)_\infty \times \prod_{p \text{ prime}} (C, 3)_p = -1. \tag{3.16}$$

Case 2: $A \in -3\mathbb{Q}_3^3$. From Lemmas 3.1, 3.2 and 3.4,

$$(A, 6)_\infty \times \prod_{p \text{ prime}} (A, 6)_p = -1. \tag{3.17}$$

Both (3.16) and (3.17) contradict the product formula for the Hilbert symbol. So, the surface (1.6) has no rational points. □

Acknowledgments

The author is sincerely grateful to the referee for many valuable comments and suggestions, especially in the proof of Lemma 3.3, which make the current version of this paper much shorter than the previous one. Part of this work was finished during the author's stay at the Vietnam Institute of Advanced Study in Mathematics (VIASM). The author would like to thank the Institute for their support.

References

[1] A. Bremner and N. X. Tho, 'An interesting quartic surface, everywhere locally solvable, with cubic point but no global point', *Publ. Math. Debrecen* **93**(1–2) (2018), 253–260.
[2] M. Bright, *Computations on Diagonal Quartic Surfaces* (PhD Dissertation, University of Cambridge, 2002).
[3] M. Bright, 'Brauer groups of diagonal quartic surfaces', *J. Symbolic Comput.* **41** (2006), 544–558.
[4] M. Bright, 'The Brauer–Manin obstruction on a general diagonal quartic surface', *Acta Arith.* **147**(3) (2021), 291–302.
[5] J. W. S. Cassels and M. J. T. Guy, 'On the Hasse principle for cubic surfaces', *Mathematika* **13** (1966), 111–120.
[6] J.-L. Colliot-Thélène, D. Kanevsky and J.-J. Sansuc, 'Arithmétique des surfaces cubiques diagonales', in: *Diophantine Approximation and Transcendence Theory (Bonn, 1985)* (ed. G. Wüstholz), Lecture Notes in Mathematics, 1290 (Springer, Berlin, 1987), 1–108.
[7] D. A. Cox, *Primes of the Form $x^2 + ny^2$: Fermat, Class Field Theory and Complex Multiplication*, 2nd edn (John Wiley, Hoboken, NJ, 2013).
[8] Y. Hirakawa, 'Counterexamples to the local-global principle associated with Swinnerton-Dyer's cubic form', *Rocky Mountain J. Math.* **50**(6) (2020), 2097–2102.
[9] F. Lemmermeyer, *Reciprocity Laws: From Euler to Eisenstein* (Springer, Berlin, 2000).
[10] C. E. Lind, *Untersuchungen über die rationalen Punkte der ebenen kubischen Kurven vom Geschlecht Eins* (Thesis, University of Uppsala, 1940).
[11] Y. I. Manin, *Cubic Forms: Algebra, Geometry, Arithmetic* (North Holland, Amsterdam, 2012).
[12] B. Poonen, 'An explicit family of genus-one curves violating the Hasse principle', *J. Théor. Nombres Bordeaux* **13**(1) (2001), 263–274.
[13] N. N. D. Quan, 'On the Hasse principle for certain quartic hypersurfaces', *Proc. Amer. Math. Soc.* **139**(12) (2011), 4293–4305.
[14] H. Reichardt, 'Einige im Kleinen überall lösbare, im Grossen Unlösbare diophantische Gleichungen', *J. reine angew. Math.* **184** (1942), 12–18.
[15] E. S. Selmer, 'The Diophantine equation $ax^3 + by^3 + cz^3 = 0$', *Acta Math.* **85** (1951), 203–362.
[16] J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics, 7 (Springer, New York, 1996).
[17] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, 106 (Springer, New York, 1986).
[18] A. Skorobogatov, 'Beyond the Manin obstruction', *Invent. Math.* **135** (1999), 399–424.
[19] H. P. F. Swinnerton-Dyer, 'Arithmetic of diagonal quartic surfaces. II', *Proc. Lond. Math. Soc. (3)* **80**(3) (2000), 513–544.

NGUYEN XUAN THO, School of Applied Mathematics and Informatics,
Hanoi University of Science and Technology, Hanoi, Vietnam
e-mail: tho.nguyenxuan1@hust.edu.vn

https://doi.org/10.1017/S0004972721000885 Published online by Cambridge University Press