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RANDOMNESS IN THE HIGHER SETTING

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Abstract. We study the strengths of various notions of higher randomness: (i) strong Π^1_1-ML-randomness is separated from Π^1_1-ML-randomness; (ii) the hyperdegrees of Π^1_1-random reals are closed downwards (except for the trivial degree); (iii) the reals z in $NCR_{\Pi^1_1}$ are precisely those satisfying $z \in L_{\omega_1^z}$; and (iv) lowness for Δ^1_1-randomness is strictly weaker than that for Π^1_1-randomness.

§1. Introduction. Randomness in the higher setting refers to the study of algorithmic randomness properties of reals from the point of view of effective descriptive set theory. Until recently, the study of algorithmic randomness has been focused on reals in the arithmetical hierarchy. The only exception was a paper by Martin-Löf [13], in which he showed the intersection of a sequence of Δ^1_1-sets of reals to be Σ^1_1 (Sacks [19] introduced the notion of Π^1_1 and Δ^1_1-randomness in two exercises). The first systematic study of higher randomness appeared in Hjorth and Nies [10] where the notion of Π^1_1-Martin-Löf randomness was defined and the key properties investigated. The paper also studied the stronger notion of Π^1_1-randomness and showed the existence of a universal test for Π^1_1-random reals. In Chong, Nies and Yu [2] the authors examined the relative strengths of Π^1_1-Martin-Löf randomness, Π^1_1 and Δ^1_1-randomness, as well as their associated notions of lowness.

Effective descriptive set theory offers a natural and different platform for the study of algorithmic randomness. Since the Gandy–Spector Theorem injects a new perspective to Π^1_1-sets of natural numbers, viewing them as Σ_1-definable subsets of $L_{\omega_1^{\mathrm{CK}}}$ and therefore recursively enumerable (r.e.) in the larger universe, the tools of hyperarithmetic theory are readily available for the investigation of random reals in the generalized setting. Just as arithmetical randomness has drawn new insights into the structure of Turing degrees below $\mathbf{0}^{(\mathbf{n})}$ (for $n < \omega$), the study of higher randomness properties has enhanced our understanding of hyperdegrees and Π^1_1-sets of reals, a point which we hope results presented in this paper will convey.

We consider several basic notions of randomness (see the next section for the definitions). In [2] it was shown that Π^1_1-Martin-Löf randomness, Π^1_1 and Δ^1_1-randomness are equivalent for reals x if and only if $\omega_1^x = \omega_1^{\mathrm{CK}}$. In [15], Nies introduced another notion called *strong* Π^1_1-Martin-Łöf randomness which is an analog of weak 2-randomness in the literature. We prove (Theorem 3.5) that every

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hyperdegree greater than or equal to the hyperdegree of Kleene's $\mathcal{O}$ contains a real that is Π^1_1 but not strongly Π^1_1-Martin-Löf random, thus separating these two notions of randomness. In Theorem 4.4, we show that every nontrivial hyperdegree below the hyperdegree of a Π^1_1-random real contains a Π^1_1-random real. Such a downward closure property is not shared by weaker notions such as Π^1_1-Martin-Löf randomness. In fact, every nontrivial real below a Π^1_1-random is Π^1_1-random relative to a measure (Corollary 4.5), so that such reals are still essentially random. This result is strengthened in Theorem 5.1: We characterize the class $NCR_{\Pi^1_1}$ of reals x which are not Π^1_1-random relative to any representation of a continuous measure to be precisely those which satisfy $x \in L_{\omega_1^x}$. Our final result (Theorem 6.5) separates the notion of low for Δ^1_1-randomness from that of low for Π^1_1-randomness. To obtain this, we prove a general theorem about hyperdegrees (Theorem 6.3) which states that any two uncountable Σ^1_1-set of reals generate the cone of hyperdegrees with base the hyperdegree of Kleene's $\mathcal{O}$. The latter has its root in a result of Martin [12] that every uncountable Δ^1_1-set of reals contains a member of each hyperdegrees greater than or equal to the degree of $\mathcal{O}$. The paper concludes with a list of questions.

§**2. Preliminaries.** We assume that the reader is familiar with hyperarithmetic theory and randomness theory. For a general reference, refer to [6], [15], [19] or [3]. The notations adopted are standard. Reals are denoted $x, y, z, \dots$ A tree T is a subset of $2^{<\omega}$ or $\omega^{<\omega}$. $[T]$ denotes the set of infinite paths on T. By abuse of notation, we also write $x \in T$ (or $x \in U$ for $U \subseteq 2^{<\omega}$) if the context is clear. We use $k \gg n$ to express the fact that the number k is "much bigger than" n. If λ is a measure on the Cantor space 2^ω, and $\sigma \in 2^{<\omega}$, denote $\lambda(\sigma)$ to be the measure of λ on the basic open set $\{x \mid \sigma \prec x\}$. We also let $[\sigma]$ denote the set of binary strings extending σ.

Definition 2.1. Given a measure λ on 2^ω, a real $\hat{\lambda}$ *represents* λ *if for any* $\sigma \in 2^{<\omega}$ *and rational numbers* p, q, $\langle \sigma, p, q\rangle \in \hat{\lambda} \Leftrightarrow p < \lambda(\sigma) < q$.

Given a representation $\hat{\lambda}$ of a measure λ, one may define the notion of a $\hat{\lambda}$-Martin-Löf test as usual. More details can be found in [16]. We use μ to denote the Lebesgue measure throughout this paper.

Definition 2.2. (i) A Π^1_1-*ML-test* is a sequence $\{U_m\}_{m\in\omega}$ of uniformly Π^1_1-open sets such that $\forall m(\mu(U_m) < 2^{-m})$. We say that x is Π^1_1-*ML random* if $x \notin \bigcap_m U_m$ for every such collection $\{U_m\}$, i.e. if x passes every Π^1_1-ML-test.

(ii) ([15, Problem 9.2.17]) $\{U_m\}$ is a Π^1_1-*generalized ML-test* if $\{U_m\}$ is a sequence of uniformly Π^1_1-open sets and $\lim_m \mu(U_m) = 0$. We say that x is *strongly* Π^1_1-*ML-random* if x passes every generalized Π^1_1-ML-test.

Definition 2.2 (ii) is an analog of the notion of weak-2-randomness for reals, where Π^1_1 is replaced by r.e. One may refine Definition 2.2 (i) as follows. A Δ^1_1-ML-test is obtained when Π^1_1 in the definition is replaced by Δ^1_1. Indeed, if $\{U_n\}_{n\in\omega}$ is a Δ^1_1-ML-test, then there is a recursive ordinal α such that $\{U_n\}_{n\in\omega}$ is uniformly $\emptyset^{(\alpha)}$-r.e. We call such a test a $\emptyset^{(\alpha)}$-ML-test. A real x is Δ^1_1-ML-random if it passes every Δ^1_1-ML-test. If x is not Δ^1_1-ML-random, then there is an $\alpha < \omega_1^{\mathrm{CK}}$ and an $\emptyset^{(\alpha)}$-ML-test in which x fails. This fact will be used in Section 4.

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DEFINITION 2.3 (Hjorth and Nies in [10]). A real x is Π^1_1-*random* if it does not belong to any null Π^1_1-set of reals.

Clearly $\bigcap_{m\in\omega} U_m$ is Π^1_1 for any sequence of uniformly Π^1_1-open sets $\{U_m\}_{m\in\omega}$, so that Π^1_1-randomness implies strong Π^1_1-ML-randomness. We say that a real x is Δ^1_1-dominated if every function hyperarithmetic in x is dominated by a hyperarithmetic function. As usual, ω_1^x is the least ordinal which is not an x-recursive ordinal, and Church–Kleene ω_1 is $\omega_1^{\emptyset}$ which is always denoted ω_1^{CK}. By a result in [2], we have the following proposition.

PROPOSITION 2.4 (Chong, Nies and Yu). *If $\omega_1^x = \omega_1^{\mathrm{CK}}$, then x is Π^1_1-ML-random if and only if it is Π^1_1-random. Moreover, each Π^1_1-random real is Δ^1_1-dominated.*

The Gandy Basis Theorem plays an important role in our present study:

THEOREM 2.5 (Gandy [8]). *If $A \subseteq 2^\omega$ is a nonempty Σ^1_1-set, then there is an $x \in A$ such that $\omega_1^x = \omega_1^{\mathrm{CK}}$.*

Let L_α be the Gödel constructibility hierarchy at level α. The following is a set-theoretic characterization of Π^1_1-sets.

THEOREM 2.6 (Spector[21] , Gandy [9]). *A set $A \subseteq 2^\omega$ is Π^1_1 if and only if there is a Σ_1-formula φ such that $x \in A \Leftrightarrow L_{\omega_1^x}[x] \models \varphi$.*

We use $\leq_h$ to denote hyperarithmetic reduction. $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ is the structure for the ramified analytical hierarchy relative to x. For more details concerning the ramified analytical hierarchy, see [19].

If T is a tree that is $\Pi_1(L_{\omega_1^{\mathrm{CK}}})$-definable, then there is an effective enumeration over $L_{\omega_1^{\mathrm{CK}}}$ of the nodes not in T. For any $\gamma < \omega_1^{\mathrm{CK}}$, let $T[\gamma]$ be the Δ_1-tree which is an approximation of T at stage γ. Then $T = \bigcap_{\gamma<\omega_1^{\mathrm{CK}}} T[\gamma]$.

§3. Strong Π^1_1-ML-randomness.

In Nies [15], Problem 9.2.17 asks

QUESTION 3.1. *Is strong Π^1_1-ML-randomness equivalent to Π^1_1-ML-randomness*?

The question was motivated by the following consideration. In the standard argument separating weak 2-randomness from ML-randomness, one exploits the fact that the rate of convergence of $\mu(U_n)$ to 0 can be coded by the "size of the space" available to U_n, where $\{U_n\}_{n\in\omega}$ is a test designed to exhibit the separation (the technical details can be found in [5]). Such an approach is no longer possible in the present setting, since U_n is now enumerated in ω_1^{CK}, instead of ω, -many stages. The following result leads to a negative solution.

THEOREM 3.2.[1] *If x is the leftmost path of a Σ^1_1-closed set of reals, then x is not strongly Π^1_1-ML-random.*

The proof is measure-theoretic. More than separating the two notions of randomness, a measure-theoretic proof extracts useful information about the distribution of strong Π^1_1-ML-random reals in the hyperdegrees. We first give a criterion for a sequence of uniformly Π^1_1-open sets to be a generalized Π^1_1-ML-test. This lemma will also be applied to show Theorem 3.5.

[1] Bienvenu, Greenberg and Monin [1] have a shorter proof of this theorem.

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LEMMA 3.3. *Suppose that* $\{U_n\}_{n\in\omega}$ *is a sequence of uniformly* Π^1_1*-open sets. If there is a* $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$ *enumeration* $\{\hat{U}_{n,\gamma}\}_{n<\omega,\gamma<\omega_1^{\mathrm{CK}}}$ *of the sequence with two numbers* k *and* $m\geq 1$ *such that for every* n, $U_n=\bigcup_{\gamma<\omega_1^{\mathrm{CK}}}\hat{U}_{n,\gamma}$ *and for every* $\gamma<\omega_1^{\mathrm{CK}}$:

(a) $\hat{U}_{n+1,\gamma}\subseteq\hat{U}_{n,\gamma}$ *and each string in* $\hat{U}_n$ *has length at least* $k\cdot n$,
(b) $\forall\sigma\in 2^{k\cdot n-m}(\mu(\hat{U}_{n,\gamma}\cap[\sigma])<2^{-1+m-k\cdot n})$, *and*
(c) *For* $\gamma<\omega_1^{\mathrm{CK}}$ *and any real* z, *if* $z\in\hat{U}_{n,<\gamma}\setminus\hat{U}_{n,\gamma}$, *where* $\hat{U}_{n,<\gamma}=\bigcup_{\beta<\gamma}\hat{U}_{n,\beta}$, *then* $z\notin\hat{U}_{n,\beta}$ *for any* $\beta\geq\gamma$.

Then $\{U_n\}_{n\in\omega}$ *is a generalized* Π^1_1*-ML-test.*

PROOF. Note that by (c) the enumeration $\{\hat{U}_{n,\gamma}\}$ of U_n is not cumulative. Assume $\mu(\bigcap_{n\in\omega}U_n)>0$ for a contradiction. We will exhibit an infinite descending sequence of ordinals $\{\gamma_n\}_{n<\omega}$ for a contradiction. First of all, by the Lebesgue Density Theorem, the assumption implies that there is a σ_0 such that

$$\mu(\bigcap_{n\in\omega}U_n\cap[\sigma_0])>2^{-|\sigma_0|}\cdot(1-2^{-3})=\frac{7}{8}\cdot 2^{-|\sigma_0|}.$$

Moreover, we may assume that k divides $|\sigma_0|+m$. Let $n_0=\frac{|\sigma_0|+m}{k}$. Then there is a least $\gamma_0<\omega_1^{\mathrm{CK}}$ such that

$$\mu(\hat{U}_{n_0,\leq\gamma_0}\cap[\sigma_0])>\frac{7}{8}\cdot 2^{-|\sigma_0|}.$$

By (b),

$$\mu((\hat{U}_{n_0,<\gamma_0}\setminus\hat{U}_{n_0,\gamma_0})\cap[\sigma_0])>2^{-|\sigma_0|}\cdot(\frac{7}{8}-\frac{1}{2})=\frac{3}{8}\cdot 2^{-|\sigma_0|}.$$

By (a) and (c), the set of strings which appear after the ordinal γ_0 is contained in the complement of $(\hat{U}_{n_0,<\gamma_0}\setminus\hat{U}_{n_0,\gamma_0})\cap[\sigma_0]$, and so

$$\mu(\bigcap_{n>n_0}\hat{U}_{n,<\gamma_0}\cap[\sigma_0])>(\frac{7}{8}-\frac{5}{8})\cdot 2^{-|\sigma_0|}=\frac{1}{4}\cdot 2^{-|\sigma_0|}.$$

Hence there is a $\sigma_1\succ\sigma_0$ such that

$$\mu(\bigcap_{n>n_0}\hat{U}_{n,<\gamma_0}\cap[\sigma_1])>\frac{7}{8}\cdot 2^{-|\sigma_0|}.$$

We may assume that k divides $|\sigma_1|+m$ and $|\sigma_1|\gg|\sigma_0|$. Let $n_1=\frac{|\sigma_1|+m}{k}\gg n_0$. Then there is a least $\gamma_1<\gamma_0$ such that

$$\mu(\hat{U}_{n_1,\leq\gamma_1}\cap[\sigma_1])>\frac{7}{8}\cdot 2^{-|\sigma_1|}.$$

Repeating the argument, we obtain an infinite descending sequence $\gamma_0>\gamma_1>\cdots$, which is not possible. ⊣

PROOF. (of Theorem 3.2).
Let $T\subseteq 2^{<\omega}$ be a Σ^1_1-tree. For any $n<\omega$ and $\gamma<\omega_1^{\mathrm{CK}}$, let

$$\hat{U}_{n,\gamma}=\{z\restriction n+1\mid z\text{ is the leftmost path in }T[\gamma]\}.$$

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Define

$$\hat{U}_{n,<\gamma} = \bigcup_{\beta<\gamma} \hat{U}_{n,\beta}$$

and

$$U_n = \bigcup_{\gamma<\omega_1^{\mathrm{CK}}} \hat{U}_{n,\gamma}.$$

The following facts are immediate.

(1) For any n and $\gamma < \omega_1^{\mathrm{CK}}$, $\hat{U}_{n+1,\gamma} \subseteq \hat{U}_{n,\gamma}$ and every string in $\hat{U}_n$ has length at least n;
(2) $\forall \sigma \in 2^{n-1}(\mu(\hat{U}_{n,\gamma} \cap [\sigma]) \leq 2^{-n-1} < 2^{-n})$;
(3) For any n, $\gamma < \omega_1^{\mathrm{CK}}$ and real z, if $z \in \hat{U}_{n,<\gamma} \setminus \hat{U}_{n,\gamma}$, then $z \notin \hat{U}_{n,\beta}$ for any $\beta \geq \gamma$.

Clearly $\{U_n\}_{n\in\omega}$ is uniformly Π^1_1. By (1)–(3) and setting $k = m = 1$ in Lemma 3.3, i $\{U_n\}_{n\in\omega}$ is a generalized Π^1_1-ML-test. Obviously $x \in \bigcap_{n\in\omega} U_n$. We conclude that x is not strongly Π^1_1-ML-random. ⊣

Corollary 3.4. *Π^1_1-ML-randomness is strictly weaker than strong Π^1_1-ML-randomness.*

Proof. By a result in [10], there is a Σ^1_1-tree T such that $[T]$ is not empty and consists entirely of Π^1_1-ML-random reals. According to Theorem 3.2, the leftmost path in T is not strongly Π^1_1-ML-random. ⊣

We give another application of Lemma 3.3. The following theorem may be proved by combining results in [1] and [10]. We give a direct proof here.

Theorem 3.5. *For any real $x \geq_h \mathcal{O}$, there is a Π^1_1-ML-random $y \equiv_h x$ which is not strongly Π^1_1-ML-random.*

Proof. Given a tree T, let $\mathcal{T}(T)$ be the smallest subtree of T such that

- $\emptyset \in \mathcal{T}(T)$, and
- For $\sigma \in \mathcal{T}(T)$, let $V_\sigma = \{\nu \mid \nu \succ \sigma \wedge |\nu| = |\sigma| + 4 \wedge [\nu] \cap T \text{ is infinite}\}$. If τ is the leftmost or rightmost string in V_σ, then $\tau \in \mathcal{T}(T)$.

Now let $T \subseteq 2^{<\omega}$ be a Σ^1_1-tree of positive measure so that $[T]$ consists entirely of Π^1_1-ML-random reals. Note that T has no isolated infinite paths.

For any $\gamma < \omega_1^{\mathrm{CK}}$, let

$$\hat{U}_{n,\gamma} = \bigcup_{\sigma\in\mathcal{T}(T[\gamma])\wedge|\sigma|=4n+4} ([\sigma] \cap \mathcal{T}(T[\gamma])).$$

Define

$$\hat{U}_{n,<\gamma} = \bigcup_{\beta<\gamma} \hat{U}_{n,\beta}$$

and

$$U_n = \bigcup_{\gamma<\omega_1^{\mathrm{CK}}} \hat{U}_{n,\gamma}.$$

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The following facts are immediate.

(1) For any n and $\gamma < \omega_1^{\mathrm{CK}}$, $\hat{U}_{n+1,\gamma} \subseteq \hat{U}_{n,\gamma}$ and every string in $\hat{U}_{n,\gamma}$ has length at least $4n$ (in fact $4n+4$);
(2) $\forall \sigma \in 2^{4n}(\mu(\hat{U}_{n,\gamma} \cap [\sigma]) \leq 2 \cdot 2^{-4n-4} < 2^{-4n-1})$;
(3) For any n, $\gamma < \omega_1^{\mathrm{CK}}$ and real z, if $z \in \hat{U}_{n,<\gamma} \setminus \hat{U}_{n,\gamma}$, then $z \notin \hat{U}_{n,\beta}$ for any $\beta \geq \gamma$.

By (1)–(3) and Lemma 3.3 by setting $k = 4$ and $m = 0$, $\{U_n\}_{n<\omega}$ is a generalized Π^1_1-ML-test. It is obvious that $\bigcap_{n\in\omega} U_n$ contains a perfect subset of $[T]$. Furthermore, $\mathcal{O}$ hyperarithmetically computes a perfect tree S with $[S] \subseteq \bigcap_{n\in\omega} U_n$ so that no path in S is strongly Π^1_1-ML-random. Hence no path in S is Π^1_1-random and by Proposition 2.4, any $y \in [S]$ satisfies $\omega_1^y > \omega_1^{\mathrm{CK}}$ and so $\mathcal{O} \leq_h y$. Such a y exists in every hyperdegree above the degree of $\mathcal{O}$. Theorem 3.5 is proved. ⊣

§4. Hyperdegrees of Π^1_1-random reals. While the hyperdegrees of Δ^1_1-random reals cover the cone of hyperdegrees above the hyperjump, it is not difficult to see that the situation is quite different outside this cone:

Proposition 4.1. *If x is Δ^1_1-random and $\omega_1^x = \omega_1^{\mathrm{CK}}$, then there is a real $y \geq_h x$ with $\omega_1^y = \omega_1^{\mathrm{CK}}$ whose hyperdegree contains no Δ^1_1-random real.*

Proof. Suppose that x is Δ^1_1-random and $\omega_1^x = \omega_1^{\mathrm{CK}}$. Let

$$H(x) = \{y \mid y \geq_T x \wedge \exists f \leq_T y \forall g \leq_h x (g \text{ is dominated by } f)\}.$$

By Theorem 2.6, $H(x)$ is $\Sigma^1_1(x)$. Since $\mathcal{O}^x \in H(x)$, $H(x)$ is not empty. Relativizing Gandy's Basis Theorem 2.5 to x, there is a real $y \in H(x)$ with $\omega_1^y = \omega_1^x = \omega_1^{\mathrm{CK}}$. Thus y is not Δ^1_1-dominated and so by Proposition 2.4, no real $z \equiv_h y$ is Δ^1_1-random. ⊣

By contrast, the hyperdegrees of Π^1_1-random reals are downward closed.

Lemma 4.2.[2] *If x is Π^1_1-random and $y \leq_h x$, then there is a recursive ordinal γ such that $y \leq_T x \oplus \emptyset^{(\gamma)}$.*

Proof. Suppose that x is Π^1_1-random and $y \leq_h x$. Then $\omega_1^x = \omega_1^{\mathrm{CK}}$ and there is a formula $\varphi(\dot{x}, n)$ with rank $\alpha_0 < \omega_1^{\mathrm{CK}}$ such that

$$n \in y \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(x, n).$$

Recall that for a ranked sentence ψ, the relation "$\mu(\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi) > 0$" is Π^1_1 (Theorem 1.3.IV of [19]). Hence by the admissibility of ω_1^{CK}, there is a recursive ordinal $\beta > \alpha_0$ such that

$$A_{\alpha_0} = \{\ulcorner \psi \urcorner \mid \psi \text{ has rank at most } \alpha_0 \wedge \mu(\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi\}) > 0\}$$

is recursive in $\emptyset^{(\beta)}$. Then there is a recursive $\alpha_1 \geq \beta$ such that for any natural number i and formula ψ of rank at most β, there is a formula ψ' of rank at most α_1 such that $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi'\}$ is a $\Pi^0_1(\emptyset^{(\alpha_1)})$-subset of $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi\}$ and the difference in measure between these two sets is less than 2^{-i}.

Repeating this, we obtain a Δ_1-definable ω-sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ in $L_{\omega_1^{\mathrm{CK}}}$ whose supremum $\gamma = \bigcup_{i<\omega} \alpha_i$ satisfies the following two properties: for any $\beta < \gamma$,

[2]The lemma was also proved by Bienvenu, Greenberg and Monin [1] independently.

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(i) The set

$$A_\beta = \{\ulcorner\varphi\urcorner \mid \varphi \text{ has rank at most } \beta \wedge \mu(\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \varphi\}) > 0\}$$

is recursive in $\emptyset^{(\gamma)}$; and

(ii) For any natural number i and formula ψ with rank at most β, there is a formula ψ' of rank less than γ such that for some $\beta' < \gamma$, $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi'\}$ is a $\Pi^0_1(\emptyset^{(\beta')})$-subset of $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi\}$ and the difference in measure between these two sets is less than 2^{-i}.

Note that by Π^1_1-randomness, for any ranked formula ψ, if $x \in P_\psi = \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi\}$, then P_ψ has positive measure.

By Proposition 2.4, x is Δ^1_1-dominated and so there is a hyperarithmetic function $f : \omega \to \omega$ such that for any $n \in \mathcal{O}$ with $|n| < \gamma$ and any e for which $\Phi_e^{\emptyset^{(|n|)}}$ computes a tree $T_{e,n}$, if $x \notin [T_{e,n}]$, then $x \upharpoonright f(\langle e, n\rangle) \notin T_{e,n}$. This allows us to implement the following construction.

Recursively in $x \oplus \emptyset^{(\gamma)} \oplus f$, first find a ψ_0 with rank less than γ such that $P_0 = \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi_0\}$ contains x, has positive measure, and is a closed subset of either $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \varphi(z, 0)\}$ or $\{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \neg\varphi(z, 0)\}$. Since x is Π^1_1-random, by (ii), such a ψ_0 exists. Note that $x \oplus \emptyset^{(\gamma)} \oplus f$ is able to decide if $x \in P_0$. In general, for any n recursively in $x \oplus \emptyset^{(\gamma)} \oplus f$ choose the formula ψ_{n+1} with rank less than γ such that $P_{n+1} = \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \psi_{n+1}\}$ contains x, has positive measure, and is a closed subset of either $P_n \cap \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \varphi(z, n)\}$ or $P_n \cap \{z \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, z) \models \neg\varphi(z, n)\}$. Since x is Π^1_1-random, by (ii) there is such a ψ_{n+1}.

Thus $y \leq_T x \oplus \emptyset^{(\gamma)} \oplus f$. Without loss of generality, we may assume that $f \leq_T \emptyset^{(\gamma)}$. Then $y \leq_T x \oplus \emptyset^{(\gamma)}$. ⊣

Corollary 4.3. *For any Π^1_1-random x and $y \leq_h x$, there is a recursive ordinal α, a function $f \leq_T \emptyset^{(\alpha)}$ and an oracle function Φ such that for every n, $y(n) = \Phi^{x \oplus \emptyset^{(\alpha)} \upharpoonright f(n)}(n)[f(n)]$. In other words, $x \oplus \emptyset^{(\alpha)}$ Turing computes y via the function Φ with both use and time bounded by f.*

Proof. Suppose that x is Π^1_1-random and $y \leq_h x$. By Lemma 4.2, there is a recursive ordinal γ and an oracle function Φ such that for every n, $y(n) = \Phi^{x \oplus \emptyset^{(\gamma)}}(n)$. Let $g <_h x$ such that for every n, $y(n) = \Phi^{x \oplus \emptyset^{(\gamma)} \upharpoonright g(n)}(n)[g(n)]$. Since x is Δ^1_1-dominated, there is a hyperarithmetic h such that for all n, $h(n) > g(n)$. Hence there is a recursive ordinal $\alpha \geq \gamma$ such that h is many-one reducible to $\emptyset^{(\alpha)}$. Then it is not difficult to define an $f \leq_T \emptyset^{(\alpha)}$ and an oracle function Ψ such that for every n, $y(n) = \Psi^{x \oplus \emptyset^{(\alpha)} \upharpoonright f(n)}(n)[f(n)]$. ⊣

Theorem 4.4. *If x is Π^1_1-random and $\emptyset <_h y \leq_h x$, then there is a Π^1_1-random $z \equiv_h y$.*

Proof. Suppose that x is Π^1_1-random and $y \leq_h x$ is not hyperarithmetic. Then there is a recursive ordinal α, a nondecreasing function $f \leq_T \emptyset^{(\alpha)}$ and an oracle functional Ψ such that $\lim_{n\to\infty} f(n) = \infty$ and for every n,

$$y(n) = \Psi^{x \oplus \emptyset^{(\alpha)} \upharpoonright f(n)}(n)[f(n)].$$

We use a technique which is essentially due to Demuth [4]. For any $\tau \in 2^{<\omega}$, let

$$C(\tau) = \{\sigma \mid \sigma \in 2^{f(|\tau|)} \wedge \Psi^{\sigma \oplus \emptyset^{(\alpha)} \upharpoonright f(|\tau|)}[f(|\tau|)] \succeq \tau\}.$$

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In other words, $C(\tau)$ is the clopen set (generated by the set) of strings of length $f(|\tau|)$, which output extensions of τ via Ψ.

For strings τ and u, let $\tau <_\ell u$ mean "τ is to the left of u". Define $\emptyset^{(\alpha)}$-recursive functions:

$$l(u) = \sum_{\tau \in 2^{|u|} \wedge \tau <_\ell u} \Big(\sum_{\sigma \in C(\tau)} 2^{-|\sigma|} \Big)$$

and

$$r(u) = l(u) + \sum_{\sigma \in C(u)} 2^{-|\sigma|}.$$

Note that $l(u)$ is the measure of the set (generated by the set) of strings of length $f(|\tau|)$ which output strings to the left of τ through the functional Ψ, and $r(u)$ is $l(u)$ plus the measure of the set of strings of length $f(|u|)$ which outputs extensions of u, also through Ψ.

One may view $\sum_{\sigma \in C(\tau)} 2^{-|\sigma|}$ as a "measure" of τ, see Demuth [4]. For each n, let

$$l_n = l(y \upharpoonright n), \text{ and } r_n = r(y \upharpoonright n).$$

Then $l_n \leq l_{n+1} \leq r_{n+1} \leq r_n$ for every n.

Since y is not hyperarithmetic, by Sacks's result in [18] that the set of reals hyperarithmetically above a nonhyperarithmetic set is null, we have $\lim_{n\to\infty}(r_n - l_n) = 0$. Hence there is a unique real

$$z = \bigcap_{n \in \omega} (l_n, r_n).$$

Obviously $z \leq_T y \oplus \emptyset^{(\alpha)}$.

To prove $z \geq_h y$, note that for any n, one can $\emptyset^{(\alpha)}$-recursively find a string u of length n such that z lies in the interval $(l(u), r(u))$. Then it must be the case that $u = y \upharpoonright n$. So $y \leq_T z \oplus \emptyset^{(\alpha)}$. And thus $z \equiv_h y$. We claim that z is Δ^1_1-random.

Suppose the claim is false. Then there is a recursive ordinal $\beta < \omega_1^{\mathrm{CK}}$ and a $\emptyset^{(\beta)}$-ML-test $\{V_n\}_{n\in\omega}$ such that $z \in \bigcap_{n\in\omega} V_n$. Let

$$\hat{V}_n = \{u \mid \exists v \exists k (v \text{ is the } k\text{-th string in } V_n \wedge \exists p, q \in \mathbb{Q}(p < l(u) < r(u) < q \wedge [v] \subseteq (p,q) \wedge q - p < 2^{-n-k-2} + 2^{-|v|}))\}.$$

Since $z \in V_n$, we have $y \in \hat{V}_n$ for every n. Note that $\{\hat{V}_n\}_{n\in\omega}$ is $\emptyset^{(\beta+1+\alpha)}$-r.e. To see this, observe that if we define $\lambda(\tau) = \sum_{\sigma \in C(\tau)} 2^{-|\sigma|}$, then λ may be viewed as a measure over 2^ω. Then

$$\lambda(\hat{V}_{n+1}) = \sum_{u \in \hat{V}_{n+1}} \sum_{\sigma \in C(u)} 2^{-|\sigma|} \leq \mu(V_{n+1}) + \sum_{k\in\omega} 2^{-n-1-k-2+1} \leq 2^{-n-1} + 2^{-n-1} \leq 2^{-n}.$$

In other words, $\{\hat{V}_n\}_{n\geq 1}$ is a $\emptyset^{(\beta)}$-ML-test relative to λ.

Let

$$U_n = \{\sigma \mid \exists \tau \in \hat{V}_n(|\sigma| = f(|\tau|) \wedge \Phi^{\sigma \oplus \emptyset^{(\alpha)} \upharpoonright f(|\tau|)}[f(|\tau|)] \succeq \tau)\} = \bigcup_{u \in \hat{V}_n} C(u).$$

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Then $\{U_n\}_{n\in\omega}$ is $\emptyset^{(\beta+1+\alpha)}$-r.e and $x \in \bigcap_{n\in\omega} U_n$. Note that for every n,

$$\mu(U_n) \le \sum_{u\in\hat{V}_n}\sum_{\tau\in C(u)} 2^{-|\tau|} = \sum_{u\in\hat{V}_n} r(u) - l(u) \le \mu(V_n) + \sum_{k\in\omega} 2^{-n-k-2+1} < 2^{-n} + 2^{-n} = 2^{-n+1}.$$

Then $\{U_{n+1}\}_{n\in\omega}$ is a $\emptyset^{(\beta+1+\alpha)}$-ML-test. So x is not a Δ^1_1-random, a contradiction. ⊣

By combining the techniques used in the proof of Proposition 5.7 in [16] and Theorem 4.4:

Corollary 4.5. *For any* Π^1_1*-random* x*, if* $\emptyset <_h y \le_h x$ *then* y *is* Π^1_1*-random relative to a hyperarithmetic continuous measure.*

A further result is discussed in Theorem 5.1.

§5. On $NCR_{\Pi^1_1}$.

This section is inspired by the work of Reimann and Slaman in [16] and [17], where they investigated reals not Martin-Löf random relative to any continuous measure. They prove that NCR_1, the collection of such reals, is countable. In fact their proof shows that for any recursive ordinal α, the collection NCR_α of reals not $\emptyset^{(\alpha)}$-ML-random relative to any continuous measure is countable. Hence a natural question to ask is how far the countability property extends. We set an upper limit for this by proving Theorem 5.1.

Given a representation $\hat{\lambda}$ of a measure λ over 2^ω, define a real x to be Π^1_1*-random relative to* $\hat{\lambda}$ if it does not belong to a λ-null set which is $\Pi^1_1(\hat{\lambda})$. Define

$$NCR_{\Pi^1_1} = \{x \mid x \text{ is not } \Pi^1_1\text{-random relative to any representation } \hat{\lambda} \text{ of a continuous measure}\}.$$

Let $\mathcal{C} = \{x \in 2^\omega \mid x \in L_{\omega_1^x}\}$. It is known that $\mathcal{C}$ is the largest Π^1_1-thin set.

Theorem 5.1. $NCR_{\Pi^1_1} = \mathcal{C}$.

We decompose the proof of Theorem 5.1 into a sequence of lemmas.

Lemma 5.2. $NCR_{\Pi^1_1}$ *does not contain a perfect subset.*

Proof. The proof is essentially due to Reimann and Slaman [16]. Suppose that there is a perfect tree $T \subseteq 2^{<\omega}$ such that every member of $[T]$ is $NCR_{\Pi^1_1}$. Define a measure λ as follows:

$$\lambda(\emptyset) = 1, \text{ and}$$

$$\lambda([\sigma^\frown i]) = \begin{cases} \lambda([\sigma]) & \text{If } \sigma^\frown(1-i) \notin T; \\ \frac{1}{2}\lambda([\sigma]) & \text{Otherwise.} \end{cases}$$

Then λ is a continuous measure so that $\lambda([T]) = 1$. Thus $[T]$ must contain a Π^1_1-random relative to any representation $\hat{\lambda}$ of λ. ⊣

Lemma 5.3. $NCR_{\Pi^1_1}$ *is a thin* Π^1_1*-set, and hence* $NCR_{\Pi^1_1} \subseteq \mathcal{C}$.

Proof. By Lemma 5.2, $NCR_{\Pi^1_1}$ does not contain a perfect subset.

Relative to any representation $\hat{\lambda}$ of a continuous measure λ, we may perform the same proofs as in [18] so that all the results remain valid upon replacing Lesbegue

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measure μ by $\hat{\lambda}$. Then the set $\{z \mid \omega_1^{z\oplus\hat{\lambda}} > \omega_1^{\hat{\lambda}}\}$ is $\Pi^1_1(\hat{\lambda})$ and λ-null. Hence as in [2], there is a Π^1_1 set $\mathcal{Q} \subseteq (2^\omega)^2$ such that for each real $\hat{\lambda}$ representing a continuous measure, the set $\mathcal{Q}_{\hat{\lambda}} = \{y \mid (\hat{\lambda}, y) \in \mathcal{Q}\}$ is the largest $\Pi^1_1(\hat{\lambda})$ λ-null set. Then, as in Reimann and Slaman [17],

$$z \in NCR_{\Pi^1_1} \Leftrightarrow \forall\hat{\lambda}(\hat{\lambda} \text{ represents a continuous measure } \rightarrow z \in \mathcal{Q}_{\hat{\lambda}}).$$

Thus $NCR_{\Pi^1_1}$ is Π^1_1. ⊣

LEMMA 5.4. *If $x \in L_{\omega_1^x}$ and $z \not\geq_h x$, then $z \oplus x \geq_h \mathcal{O}^z$.*

PROOF. Suppose that $x \in L_{\omega_1^x}$ and $z \not\geq_h x$. Then $\omega_1^z < \omega_1^x$. So $\omega_1^{x\oplus z} > \omega_1^z$. Thus $z \oplus x \geq_h \mathcal{O}^z$. ⊣

LEMMA 5.5. *If $x \in \mathcal{C}$, then $x \in NCR_{\Pi^1_1}$.*

PROOF. Let λ be a continuous measure with representation $\hat{\lambda}$ and $x \in \mathcal{C}$. If $x \leq_h \hat{\lambda}$, then x is clearly not Π^1_1-random relative to $\hat{\lambda}$. Otherwise, by Lemma 5.4, $x \oplus \hat{\lambda} \geq_h \mathcal{O}^{\hat{\lambda}}$. But $\{z \mid z \oplus \hat{\lambda} \geq \mathcal{O}^{\hat{\lambda}}\}$ is a $\Pi^1_1(\hat{\lambda})$ λ-null set. This implies that x is not Π^1_1-random relative to $\hat{\lambda}$. ⊣

§**6. Separating lowness for higher randomness notions.** In [2], Chong, Nies, and Yu investigated lowness properties for Δ^1_1 and Π^1_1-randomness. It is unknown whether there is a nonhyperarithmetic real low for Π^1_1-random. However, there is a characterization of reals which are low for Π^1_1-randomness.

PROPOSITION 6.1 (Harrington, Nies and Slaman [2]). *Being low for Π^1_1-randomness is equivalent to being low for Δ^1_1-randomness and not cuppable above $\mathcal{O}$ by a Π^1_1-random.*

We may apply Proposition 6.1 to separate lowness for Δ^1_1-randomness from lowness for Π^1_1-randomness. Recall that given a class of sets of reals $\mathbf{\Gamma}$, a real x is $\mathbf{\Gamma}$-Kurtz random if it does not belong to any $\mathbf{\Gamma}$-closed null set.

In [11], Kjos-Hanssen, Nies, Stephan, and Yu investigated lowness for Δ^1_1-Kurtz randomness and lowness for Π^1_1-Kurtz randomness. They proved that lowness for Π^1_1-Kurtz randomness implies lowness for Δ^1_1-randomness. We show that the implication cannot be reversed.

In [22], Yu gave a new proof of the following theorem.

THEOREM 6.2 (Martin [12] and Friedman). *Every Σ^1_1-tree T with uncountably many infinite paths has a member of each hyperdegree $\geq_h \mathcal{O}$ as a path.*

We apply the technique introduced in [22] to prove the following result.

THEOREM 6.3. *Let A_0 and A_1 be uncountable Σ^1_1-sets of reals. For any $z \geq_h \mathcal{O}$, there are reals $x_0 \in A_0$ and $x_1 \in A_1$ such that $x_0 \oplus x_1 \equiv_h z$.*

PROOF. Fix a real $z \geq_h \mathcal{O}$ and two uncountable Σ^1_1-sets A_0 and A_1. Then there are two recursive trees $T_0, T_1 \subseteq 2^{<\omega} \times \omega^{<\omega}$ such that for $i \leq 1$, $A_i = \{x \mid \exists f \forall n(x \upharpoonright n, f \upharpoonright n) \in T_i\}$. We may assume that neither A_0 nor A_1 contains a hyperarithmetic real. Also assume that if $(\sigma, \tau) \in T_i$, $i \leq 1$, then $|\sigma| = |\mathrm{Dom}(\tau)|$. Let $T_2 \subseteq \omega^{<\omega}$ be recursive so that $[T_2]$ is uncountable and does not contain a hyperarithmetic infinite path. Let $f_{\mathcal{O}}$ be the leftmost path in T_2. Then $f_{\mathcal{O}} \equiv_h \mathcal{O}$.

For $i \leq 1$, let $[T_i] = \{(x, f) \mid \forall n((x \upharpoonright n, f \upharpoonright n) \in T_i)\}$. Our plan is to define $x_i \in A_i$ such that $z \equiv_h x_0 \oplus x_1$. To this end, a procedure of coding z and $f_{\mathcal{O}}$

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into x_0 and decoding them from $x_0 \oplus x_1$ will be introduced. Construction of x_i will be carried out in $L_{\omega_1^{\mathrm{CK}}}[z]$ on the recursive tree T_i, hence hyperarithmetically in z (since $z \geq_h \mathcal{O}$). Since A_i is Σ^1_1, x_{1-i} will also code in the function f_i which is the leftmost path in the second component of $[T_i]$ serving as a witness to x_i being in A_i (i.e. $(x_i, f_i) \in [T_i]$ and for any f, if $(x_i, f) \in [T_i]$ and $f_i \neq f$, then the least n where $f_i(n) \neq f(n)$ satisfies $f_i(n) < f(n)$). Since A_i has no hyperarithmetic member, for any $(\sigma, \tau) \in T_i$, if $\sigma \prec x$ and $\tau \prec f$ for some $(x, f) \in [T_i]$, then there exist incompatible extensions σ' and σ'' of σ, and (possibly compatible) extensions τ', τ'' of τ, so that (σ', τ') and (σ'', τ'') both have extensions in $[T_i]$. This "splitting property" of $[A_i]$ allows the coding of z, f_1 and $f_{\mathcal{O}}$ in x_0 and the coding of f_0 in x_1. More specifically, the branch to be selected by x_0 at a splitting node when $z(s)$ is to be coded (at stage $s+1$ of the construction) will follow the value of $z(s)$, so that a "left turn" is taken if $z(s) = 0$ and a right turn is taken if $z(s) = 1$. The coding at stage $s+1$ of $\tau_{1,s}$, which denotes the initial segment of f_1 defined at the end of stage s, is accomplished by taking $\tau_{1,s}(t)$-many consecutive left turns at splitting nodes, for each $t \in \mathrm{Dom}(\tau_{1,s})$. The coding of $f_{\mathcal{O}}(s)$ at stage $s+1$ of the construction is carried out by taking left turns at $f_{\mathcal{O}}(s)$-many consecutive splitting nodes.

For the purpose of decoding, one has to delineate different types of action taken during the coding phase. Since $\tau_{1,s}$ is a finite function, the end of coding the value $\tau_{1,s}(t)$ and the beginning of coding the value $\tau_{1,s}(t+1)$, for $t < |\mathrm{Dom}(\tau_{1,s})|$, is separated by a right turn at the splitting node between the two codings (of course since the construction is executed stage by stage, one may assume that at the beginning of stage $s+1$, the coding of $\tau_{1,s-1}$ is already completed. This means that at stage $s+1$, one only needs to code the values $\tau_{1,s}(t)$ for $t \in \mathrm{Dom}(\tau_{1,s}) \setminus \mathrm{Dom}(\tau_{1,s-1})$). A "right turn" is chosen at the next splitting node to signify the end of coding $\tau_{1,s}$, and the beginning of the coding of $f_{\mathcal{O}}(s)$. Finally, a right turn is taken at the next splitting node after coding $f_{\mathcal{O}}(s)$ to mark the end of the coding action for x_0 at stage $s+1$. This initial segment of x_0 coded at stage $s+1$ is denoted $\sigma_{0,s+1}$. Then $\tau_{0,s+1} \prec f_0$ is a finite string in $\omega^{<\omega}$ such that $|\mathrm{Dom}(\tau_{0,s+1})| = |\sigma_{0,s+1}|$, $\tau_{0,s+1}$ extends $\tau_{0,s}$, and is the leftmost such string. The coding of the initial segment $\tau_{0,s+1}$ of f_0 in x_1 at stage $s+1$, denoted $\sigma_{1,s+1}$, proceeds in a similar fashion. The definition of $\tau_{1,s+1}$, an initial segment of f_1 constructed at stage $s+1$, follows the same format.

We now describe the construction of x_0 and x_1 and the associated strings in detail. For $i \leq 1$ and $\sigma, \tau \in T_i$, let $T_i(\sigma, \tau)$ be the set of strings in T_i compatible with σ and τ), i.e.

$$T_i(\sigma, \tau) = \{(\sigma', \tau') \in T_i \mid \exists (\sigma'', \tau'') \in T_i(\sigma' \preceq \sigma'' \wedge \tau' \preceq \tau'' \wedge \sigma \preceq \sigma'' \wedge \tau \preceq \tau'')\}.$$

Note that it is unnecessary that $|\sigma| = |\tau|$ in the definition above.

We say that a string (or node) $\sigma^* \in 2^{<\omega}$ is *splitting over* (σ, τ) in T_i if $\sigma^* \succeq \sigma$ and for $j \leq 1$,

$$T_{i, \sigma^{*\frown} j}(\sigma, \tau) = \{(\sigma', \tau') \mid \sigma' \succeq \sigma^{*\frown} j \wedge \tau' \succeq \tau \wedge (\sigma', \tau') \in T_i\}$$

contains an infinite path. $T_{i, \sigma^{*\frown} j}(\sigma, \tau)$ is the subtree of T_i with root $\sigma^{*\frown} j$ in its first component (note that $\sigma^* \succeq \sigma$). Since A_i has no hyperarithmetic path, for each $j \leq 1$, there is a string that splits over some (σ', τ') in $T_{i, \sigma^{*\frown} j}(\sigma, \tau)$. Note that σ^* does not lie on T_i but some pair (σ^*, τ') does and we call (σ^*, τ') a *splitting node on* T_i.

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For each $i \leq 1$, we construct a sequence $(\sigma_{i,0}, \tau_{i,0}) \prec (\sigma_{i,1}, \tau_{i,1}) \prec \cdots$ on T_i and let $x_i = \bigcup_j \sigma_{i,j}$. Again, the idea is to apply a "mutual coding" technique so that x_0 codes the leftmost witness function $f_1 = \cup_s \tau_{1,s}$ (in the Σ^1_1-definition) for x_1 and x_1 codes the leftmost witness function $f_0 = \cup_s \tau_{0,s}$ for x_0. In addition, we also assign x_0 the responsibility of coding z as well as $f_{\mathcal{O}}$. More precisely, for each $s \in \omega$ we use $\sigma_{0,s}$ to code $z(s)$, $f_{\mathcal{O}}(s)$ and $\tau_{1,s-1}$, and use $\sigma_{1,s}$ to code $\tau_{0,s}$.

At stage 0, let $(\sigma_{i,0}, \tau_{i,0}) = (\emptyset, \emptyset)$ for $i \leq 1$. Without loss of generality, assume that $(\emptyset, \emptyset)$ is a splitting node in both T_0 and T_1.

The construction at stage $s + 1$ proceeds as follows:

Substage (i). First let σ^* be the shortest splitting node over $(\sigma_{0,s}, \tau_{0,s})$ in T_0. Thus $T_{0,\sigma^* \frown j}(\sigma_{0,s}, \tau_{0,s})$ contains an infinite path for $j \leq 1$. Let $\sigma^0_{0,s+1}$ be the leftmost splitting node over $(\sigma_{0,s}, \tau_{0,s})$ extending $\sigma^* \frown z(s)$ in T_0. Thus $z(s)$ is coded here. Next we code $\tau_{1,s}$. Let $n^0_{s+1} = |\mathrm{Dom}(\tau_{1,s})| - |\mathrm{Dom}(\tau_{1,s-1})|$ (let $|\mathrm{Dom}(\tau_{1,s-1})| = 0$ if $s = 0$). Inductively, for any $k \in [1, n^0_{s+1}]$, let $\sigma^k_{0,s+1}$ be the leftmost splitting node over $(\sigma_{0,s}, \tau_{0,s})$ extending $(\sigma^{k-1}_{0,s+1})\frown 1$ in T_0 so that there are $\tau_{1,s}(k + |\mathrm{Dom}(\tau_{1,s-1})|)$-many splitting nodes over $(\sigma_{0,s}, \tau_{0,s})$ in T_0 between $\sigma^{k-1}_{0,s+1}$ and $\sigma^k_{0,s+1}$. This completes the coding of $\tau_{1,s}$. To code $f_{\mathcal{O}}(s)$, let $\sigma^{n^0_{s+1}+1}_{0,s+1}$ be the leftmost splitting node extending $(\sigma^{n^0_{s+1}}_{0,s+1})\frown 1$ over $(\sigma_{0,s}, \tau_{0,s})$ in T_0 so that there are $f_{\mathcal{O}}(s)$-many splitting nodes in T_0 over $(\sigma_{0,s}, \tau_{0,s})$ between $\sigma^{n^0_{s+1}}_{0,s+1}$ and $\sigma^{n^0_{s+1}+1}_{0,s+1}$. For $j \leq 1$, let $\sigma^{n^0_{s+1}+1+j+1}_{0,s+1}$ be the next splitting node in T_0 over $(\sigma_{0,s}, \tau_{0,s})$ extending $(\sigma^{n^0_{s+1}+1+j}_{0,s+1})\frown 1$. This coding tells us that the action at this substage for the "x_0 side" is completed. Define $\sigma_{0,s+1} = \sigma^{n^0_{s+1}+3}_{0,s+1}$. Let $\tau_{0,s+1}$ extend $\tau_{0,s}$ so that $|\mathrm{Dom}(\tau_{0,s+1})| = |\sigma_{0,s+1}|$ and $\tau_{0,s+1}$ is the leftmost node such that the tree $T_0(\sigma_{0,s+1}, \tau_{0,s+1})$ has an infinite path.

Remark. At stage $s + 1$, the coding of x_0 in T_0 by way of $\sigma_{0,s+1}$ applies the method in [22], treating the Σ^1_1-set A_0 as a "closed set". This means that one first ignores the second component (the "τ side") and applies the coding construction in [22] to the closed set $X_{\sigma_{0,s},\tau_{0,s}} = \{x \succ \sigma_{0,s} \mid \forall n > |\tau_{0,s}| \exists y \exists f \succ \tau_{0,s}(x \upharpoonright n = y \upharpoonright n \wedge (y, f) \in [T_0])\}$. Once the coding of $\sigma_{0,s+1}$ (an initial segment of x_0) is completed, one pairs it with a finite string τ which has an infinite extension to guarantee that x_0 belongs to A_0. Since x_0 does not know the right τ to select, x_1 is designed to help decode the correct τ. The mutual coding strategy is crucial for this purpose.

Substage (ii). Let $\sigma^0_{1,s+1} = \sigma_{1,s}$ and $n^1_{s+1} = |\mathrm{Dom}(\tau_{0,s+1})| - |\mathrm{Dom}(\tau_{0,s})|$. Inductively for any $k \in [1, n^1_{s+1}]$, let $\sigma^k_{1,s+1}$ be the leftmost splitting node over $(\sigma_{1,s}, \tau_{1,s})$ extending $(\sigma^{k-1}_{1,s+1})\frown 1$ so that in T_1 there are $\tau_{0,s+1}(k + |\mathrm{Dom}(\tau_{0,s})|)$-many splitting nodes over $(\sigma_{1,s}, \tau_{1,s})$ between $\sigma^{k-1}_{1,s+1}$ and $\sigma^k_{1,s+1}$. Hence $\tau_{0,s+1}$ is coded. For $j \leq 1$, let $\sigma^{n^1_{s+1}+j+1}_{1,s+1}$ be the next splitting node over $(\sigma_{1,s}, \tau_{1,s})$ in T_1 extending $(\sigma^{n^1_{s+1}+j}_{1,s+1})\frown 1$. This coding tells us that the action of coding $\tau_{0,s+1}$ at this substage for the "x_1 side" is completed. Define $\sigma_{1,s+1} = \sigma^{n^1_{s+1}+2}_{1,s+1}$. Thus we have coded $\tau_{0,s+1}$ into $\sigma_{1,s+1}$. Let $\tau_{1,s+1}$ extend $\tau_{1,s}$ so that $|\mathrm{Dom}(\tau_{1,s+1})| = |\sigma_{1,s+1}|$ and $\tau_{1,s+1}$ is the leftmost finite string such that the tree $T_1(\sigma_{1,s+1}, \tau_{1,s+1})$ has an infinite path.

This ends the construction at stage $s + 1$.

Let $x_i = \bigcup_{s<\omega} \sigma_{i,s}$ for $i \leq 1$. Note that the construction is carried out over $L_{\omega_1^{\mathrm{CK}}+1}[z]$ since $\mathcal{O} \in L_{\omega_1^{\mathrm{CK}}+1}$ and whether $(\sigma, \tau) \in T_i$ has an infinite path extension in T_i is decided by stage $L_{\omega_1^{\mathrm{CK}}+1}$. As $z \geq_h \mathcal{O}$ we have $\omega_1^z > \omega_1^{\mathrm{CK}}$ and so $z \geq_h x_0 \oplus x_1$.

We now use x_0 and x_1 to decode the coding construction and hence hyperarithmetically recover $\mathcal{O}$ and z from $x_0 \oplus x_1$. The decoding is achieved via a finite injury method similar to that used in [22] to prove Theorem 6.2. However, a correct decoding requires use of the witness functions f_0 and f_1. Without these witness functions, x_0 would still code z and would belong to the closure of A_0. The entire construction is then reduced to that in the proof of Martin's theorem in [22]. However, in this case one cannot conclude that x_0 belongs to A_0. This difficulty is resolved by a procedure of mutual coding, in which x_0 also codes f_1 and x_1 codes f_0. The coding of z is weaved into the mutual coding strategy in the course of the construction.

We now point out the key decoding steps and leave the details to the reader. As in [22], we may fix a Σ_1-enumeration $\{T_i[\alpha]\}_{i\leq 2, \alpha<\omega_1^{\mathrm{CK}}}$ over $L_{\omega_1^{\mathrm{CK}}}$ such that for $i \leq 2$,

- $T_i[0] = T_i$,
- $T_i[\alpha] \subseteq T_i[\beta]$ if $\alpha \geq \beta$,
- $T_i[\omega_1^{\mathrm{CK}}] = \bigcap_{\alpha<\omega_1^{\mathrm{CK}}} T_i[\alpha]$,
- $T_i[\omega_1^{\mathrm{CK}}]$ has no dead end nodes, and
- $A_i = \{x \mid \exists f \forall n (x \upharpoonright n, f \upharpoonright n) \in T_i[\omega_1^{\mathrm{CK}}]\}$.

Since $[T_i]$ does not contain a hyperarithmetic infinite path, we have $[T_i[\omega_1^{\mathrm{CK}}]] = [T_i]$ to be a perfect tree (which as noted earlier enabled the coding procedure to be implemented). Clearly (x_i, f_i) is an infinite path in $T_i[\alpha]$ for each α. As was the case for T_i, $i \leq 1$, for each $\alpha < \omega_1^{\mathrm{CK}}$ one may define the notion of a string σ' being a splitting node over (σ, τ) in $T_i[\alpha]$. In particular, for $(\sigma, \tau) \in T_i[\alpha]$ such that $\sigma \prec x_i$, it makes sense to say that $x_i \upharpoonright n$ *splits over* (σ, τ) in $T_i[\alpha]$. This means that there are strings $\tau_0^i, \tau_1^i \succeq \tau$ such that $(x_i \upharpoonright n^\frown 0, \tau_0^i), (x_i \upharpoonright n^\frown 1, \tau_1^i) \in T_i[\alpha]$. In this case we also say that $x_i \upharpoonright n$ is a *locally splitting node* over (σ, τ) in $T_i[\alpha]$. Here "local" refers to the fact that there is no guarantee that $(x_i \upharpoonright n^\frown j, \tau_j^i)$ has an infinite extension in $T_i[\alpha]$ for $j = 0, 1$.

Since for each $i \leq 1$ and $\alpha < \omega_1^{\mathrm{CK}}$, $\langle x_i, f_i \rangle$ is a path on $T_i[\alpha]$, one may use $x_0 \oplus x_1$ to approximate the values of $f_{\mathcal{O}}(s)$ and $f_i(s)$ by simulating in $T_i[\alpha]$ the construction of x_i. This is achieved by attempting to retrieve the sequences $\{(\sigma_{i,s}, \tau_{i,s})\}_{s<\omega}$, $i \leq 1$, using $x_0 \oplus x_1$. Of course, since it is yet to be established that z and $f_{\mathcal{O}}$ are hyperarithmetic in $x_0 \oplus x_1$, the retrieval is only by way of approximating the original construction. Using $x_0 \oplus x_1$, one may mimic the construction of $\{(\sigma_{i,s}, \tau_{i,s})\}_{s<\omega}$ to define $\{(\sigma_{i,s}[\alpha], \tau_{i,s}[\alpha])\}_{s<\omega}$, for $i \leq 1$, so that $\sigma_{i,s+1}[\alpha]$ is an initial segment of x_i (in ascending order of length) and a local splitting node over $(\sigma_{i,s}[\alpha], \tau_{i,s}[\alpha])$ in $T_i[\alpha]$. Furthermore, for each i, $\tau_{i,s+1}[\alpha]$ is an approximation of $f_i \upharpoonright s$ at stage α. In other words, $\tau_{i,s+1}[\alpha]$ is the leftmost finite string so that $(\sigma_{i,s+1}[\alpha], \tau_{i,s+1}[\alpha]) \in T_i[\alpha](\sigma_{i,s}, \tau_{i,s})$.

Note that for each n and $\alpha < \alpha' < \omega_1^{\mathrm{CK}}$, the number of splitting nodes along $(x_i \upharpoonright n, f_i \upharpoonright n)$ in $T_i[\alpha]$ is greater than or equal to that in $T_i[\alpha']$.

One may use $T_2[\alpha]$ to define a $\Sigma_1(L_{\omega_1^{\mathrm{CK}}})$-approximation $f'_{\mathcal{O}}$ of $f_{\mathcal{O}}$ "from the left": $f'_{\mathcal{O}}(0)[\alpha]^\frown \cdots ^\frown f'_{\mathcal{O}}(n)[\alpha]$ is the leftmost string in $T_2[\alpha]$ (it is possible that there exist only finitely many such n's.). Then for $\alpha < \omega_1^{\mathrm{CK}}$, $f'_{\mathcal{O}}(0)[\alpha] \leq f_{\mathcal{O}}(0)$.

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Suppose $n > 0$ and $f'_{\mathcal{O}}(m)[\alpha] = f_{\mathcal{O}}(m)$ for all $m < n$. Then $f'_{\mathcal{O}}(n)[\alpha] \leq f_{\mathcal{O}}(n)$, and $\lim_{\alpha\to\omega_1^{\mathrm{CK}}} f'_{\mathcal{O}}(n)[\alpha] = f_{\mathcal{O}}(n)$.

The algorithm we adopt for decoding proceeds as follows. Construct a sequence of ordinals $\{\alpha_s\}_{s<\omega}$ which is Δ_1-definable in $L_{\omega_1^{x_0\oplus x_1}}[x_0\oplus x_1]$ so that $\lim_{s\to\omega}\alpha_s = \omega_1^{\mathrm{CK}}$, and use it as a parameter to decode the real z, and thereby conclude that $x_0\oplus x_1 \geq_h z$.

Let $\hat{g}_{\mathcal{O},0}(0) = f_{\mathcal{O}}(0)$ and $\alpha_0 = 0$.

We say that $x_0 \oplus x_1$ *consistently computes $f_{\mathcal{O}}$ up to j at stage α* if for every $k < j$,

(1) $\tau_{1,k}[\alpha]$ agrees with that coded in x_0 using $\sigma_{0,k+1}[\alpha]$. In other words, let $\sigma^*_{k+1}[\alpha]$ be the shortest locally splitting node over $(\sigma_{0,k}[\alpha], \tau_{0,k}[\alpha])$ in $T_0[\alpha]$. Then there is a $\sigma^0_{0,k+1}[\alpha]$ which is the leftmost locally splitting node over $(\sigma_{0,k}[\alpha], \tau_{0,k}[\alpha])$ extending $\sigma^*_{k+1}[\alpha]^\frown i$, for some $i \in \{0,1\}$, Moreover, inductively for each $l \in [1, n^0_{k+1}[\alpha]]$ where $n^0_{k+1}[\alpha] = |\mathrm{Dom}(\tau_{1,k}[\alpha])| - |\mathrm{Dom}(\tau_{1,k-1}[\alpha])|$, let $\sigma^l_{0,k+1}[\alpha]$ be the leftmost locally splitting node over $(\sigma_{0,k}[\alpha], \tau_{0,k}[\alpha])$ extending $(\sigma^{l-1}_{0,k}[\alpha])^\frown 1$ in $T_0[\alpha]$. Then there are $\tau_{1,k}[\alpha](l + |\mathrm{Dom}(\tau_{1,k-1}[\alpha])|)$-many locally splitting nodes over $(\sigma_{0,k}[\alpha], \tau_{0,k}[\alpha])$ in $T_0[\alpha]$ between $\sigma^{l-1}_{0,k+1}[\alpha]$ and $\sigma^l_{0,k+1}[\alpha]$;

(2) The approximation of $f_{\mathcal{O}}$ via T_2 using $f'_{\mathcal{O}}$ at stage α agrees with that via $\{\sigma_{0,l}[\alpha]\}_{l\leq k}$ up to the decoding of $f_{\mathcal{O}} \upharpoonright k$. In other words, let $\sigma^{n^0_{k+1}[\alpha]+1}_{0,k}[\alpha]$ be the leftmost locally splitting node extending $(\sigma^{n^0_{k+1}[\alpha]}_{0,k+1}[\alpha])^\frown 1$ in $T_0[\alpha]$ over $(\sigma_{0,k}[\alpha], \tau_{0,k}[\alpha])$ so that there are $f'_{\mathcal{O}}(k)[\alpha]$-many nodes which are locally splitting over $(\sigma_{0,k}[\alpha], \tau_{0,k}[\alpha])$ in $T_0[\alpha]$ between $\sigma^{n^0_{k+1}[\alpha]}_{0,k+1}[\alpha]$ and $\sigma^{n^0_{k+1}[\alpha]+1}_{0,k+1}[\alpha]$. For $i \leq 1$, let $\sigma^{n^0_{k+1}[\alpha]+1+i+1}_{0,k+1}[\alpha]$ be the next local splitting node in $T_0[\alpha]$ over $(\sigma_{0,k}[\alpha], \tau_{0,k}[\alpha])$ extending $(\sigma^{n^0_{k+1}[\alpha]+1+i}_{0,k+1}[\alpha])^\frown 1$. Then $\sigma_{0,k+1}[\alpha] = \sigma^{n^0_{k+1}[\alpha]+3}_{0,k+1}[\alpha]$, and

(3) $\tau_{0,k+1}[\alpha]$ agrees with that coded in x_1 using $\sigma_{1,k+1}[\alpha]$. Thus inductively for each $l \in [1, n^1_{k+1}[\alpha]]$, where $n^1_{k+1}[\alpha] = |\mathrm{Dom}(\tau_{0,k+1}[\alpha])| - |\mathrm{Dom}(\tau_{0,k}[\alpha])|$, let $\sigma^l_{1,k+1}[\alpha]$ be the leftmost locally splitting node over $(\sigma_{1,k}[\alpha], \tau_{1,k}[\alpha])$ extending $(\sigma^{l-1}_{1,k+1}[\alpha])^\frown 1$ in $T_1[\alpha]$ so that there are $\tau_{0,k+1}[\alpha](l + |\mathrm{Dom}(\tau_{0,k}[\alpha])|)$-many locally splitting nodes over $(\sigma_{1,k}[\alpha], \tau_{1,k}[\alpha])$ in $T_1[\alpha]$ between $\sigma^{l-1}_{1,k+1}[\alpha]$ and $\sigma^l_{1,k+1}[\alpha]$. For $i \leq 1$, let $\sigma^{n^1_{k+1}[\alpha]+i+1}_{1,k+1}[\alpha]$ be the next local splitting node over $(\sigma_{1,k}[\alpha], \tau_{1,k}[\alpha])$ in $T_1[\alpha]$. Then $\sigma_{1,k+1}[\alpha] = \sigma^{n^0_{k+1}[\alpha]+2}_{1,k+1}[\alpha]$.

Note that at any stage α, we may assume that $x_0 \oplus x_1$ always computes $f_{\mathcal{O}}$ consistently up to 0. Furthermore, for each $\alpha < \omega_1^{\mathrm{CK}}$ and $j \in \omega$, there is always an $\alpha' \geq \alpha$ such that $\alpha' < \omega_1^{\mathrm{CK}}$ and $x_0 \oplus x_1$ consistently computes $f_{\mathcal{O}}$ up to j at stage α'.

Suppose that α_{t-1} is defined where $t \geq 1$. Search for the least stage $\alpha > \alpha_{t-1}$ so that $x_0 \oplus x_1$ consistently computes $f_{\mathcal{O}}$ up to t at α. Let $\alpha_t = \alpha$ and $\hat{g}_{\mathcal{O},t}(j) = f'_{\mathcal{O}}(j)[\alpha_t]$ for each $j \leq t$.

This completes the construction at step t.

We verify that the decoding construction yields an algorithm to hyperarithmetically compute z from $x_0 \oplus x_1$.

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Let $\gamma = \bigcup_{t\in\omega} \alpha_t$.

Clearly $\gamma \le \omega_1^{\mathrm{CK}}$ and is a limit ordinal. Furthermore $\gamma < \omega_1^{x_0\oplus x_1}$. We prove that $\gamma = \omega_1^{\mathrm{CK}}$.

Suppose for the sake of contradiction that $\gamma < \omega_1^{\mathrm{CK}}$. Using the fact that neither x_0 nor x_1 is hyperarithmetic, we will show that for $i \le 1$, each sequence of parameters $n_t^i[\alpha_t]$, $\sigma_{i,t}[\alpha_t]$ and $\tau_{i,t}[\alpha_t]$ introduced in the decoding procedure above eventually stabilizes as $t \to \omega$. Suppose this is false. Then there is a least j_0 and a corresponding least t_0 such that $f_{\mathcal{O}}$ is consistently computed up to $j_0 - 1$, but not j_0, at α_t for all $t \ge t_0$. We argue that such a situation does not occur.

The proof proceeds by induction in the order of the introduction of the parameters at stage α_{j_0}. For convenience, we only show that $\sigma_{0,j_0}^{n_{j_0}^0[\alpha_t]+1}[\alpha_t]$ reaches a limit after some t (essentially the same argument applies to show that the other sequences of parameters also stabilise). This means that $\hat{g}_{\mathcal{O},t}(j_0) \ne \hat{g}_{\mathcal{O},t+1}(j_0)$ for at most finitely many t's.

Assume that $\sigma_{0,j_0}^{n_{j_0}^0[\alpha_t]}[\alpha_t]$ does not change from stage α_{t_0} onwards ($\sigma_{0,k+1}^{n_{k+1}^0[\alpha_t]}[\alpha_t]$ is as defined above before the decoding of $f_{\mathcal{O}} \restriction k$ in (2)). If $\sigma_{0,j_0}^{n_{j_0}^0[\alpha_t]+1}[\alpha_t]$ changes infinitely often, then $\hat{g}_{\mathcal{O},t}(j_0) \ne \hat{g}_{\mathcal{O},t+1}(j_0)$ for infinitely many t's. Then by the decoding construction, we have the following claim.

Claim 1. *$\hat{g}_{\mathcal{O},t+1}(j_0) \ge \hat{g}_{\mathcal{O},t}(j_0)$ for any $t > t_0 + 1$ and hence $\lim_{t\to\omega} \hat{g}_{\mathcal{O},t}(j_0) = +\infty$.*

Proof. If we assign a new value k to $\hat{g}_{\mathcal{O},t}(j_0)$ at some stage $\alpha_t > \alpha_{t_0+1}$, it must be the case that $\hat{g}_{\mathcal{O},t}(0)^\frown \cdots ^\frown \hat{g}_{\mathcal{O},t}(j_0 - 1)^\frown k \in T_2[\alpha_t]$. However, $\hat{g}_{\mathcal{O},t}(0)^\frown \cdots ^\frown \hat{g}_{\mathcal{O},t}(j_0 - 1) \in T_2[\alpha_{t_0}]$ by the assumption on j_0, so that $k > \hat{g}_{\mathcal{O},t-1}(j_0)$. ⊣

Claim 1 implies that at infinitely many t's, left turns were selected at locally splitting nodes in $T_0[\alpha_t]$ during the decoding phase for the purpose of approximating $f_{\mathcal{O}}(j_0)$.

Let $\bar{\sigma} = \sigma_{0,j_0}^{n_{j_0}^0[\alpha_{t_0}]}[\alpha_{t_0}]$ and $\bar{\tau} = \tau_{0,j_0}[\alpha_{t_0}]$. Note that $|\bar{\sigma}| > |\bar{\tau}|$. Define

$$X = \{x \succ \bar{\sigma} \mid \forall n \forall \beta < \gamma \exists \beta' \exists \tau' \succ \bar{\tau} (\beta \le \beta' < \gamma \wedge |\tau'| = n \wedge (x \restriction n, \tau') \in T_0[\beta'])\}.$$

If we let $\hat{p}(T) = \{\sigma \mid \exists \tau (\sigma, \tau) \in T\}$ be a "local projection" of a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$, then it is not difficult to see that $\bigcap_{\beta<\gamma} \hat{p}(T_0(\bar{\sigma}, \bar{\tau})[\beta])$ is a tree (Note: $T_0(\bar{\sigma}, \bar{\tau})[\beta]$ denotes the analog of $T_0(\bar{\sigma}, \bar{\tau})$ for $T_0[\beta]$). Moreover,

$$X = [\bigcap_{\beta<\gamma} \hat{p}(T_0(\bar{\sigma}, \bar{\tau})[\beta])].$$

In other words, X consists of all the reals x "potentially" with an accompanying witness f so that (x, f) is an infinite path in $T_0(\bar{\sigma}, \bar{\tau})[\gamma]$. Note that since $\gamma < \omega_1^{\mathrm{CK}}$, X is a Δ_1^1 closed subset of 2^ω.

Claim 2. *x_0 is the leftmost path in X.*

Proof. Suppose not. Then there exist y_0 and σ such that $\bar{\sigma} \prec \sigma$, $y_0 \succ \sigma^\frown 0$, $x_0 \succ \sigma^\frown 1$, and $\forall \beta < \gamma \exists \beta' \exists \tau' \succ \bar{\tau} (\beta \le \beta' < \gamma \wedge |\tau'| = |\sigma| + 1 \wedge (y_0 \restriction |\sigma| + 1, \tau') \in T_0[\beta'])$.

Now by Claim 1 let $t_1 > t_0$ be such that $\hat{g}_{\mathcal{O},t}(j_0) > |\sigma| + 1$ for all $t \ge t_1$. Then there are τ, τ' of length $|\sigma|$ so that $(x_0 \restriction |\sigma| + 1, \tau), (y_0 \restriction |\sigma| + 1, \tau') \in T_0[\alpha_{t_1+1}]$.

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However, since y_0 is to the left of x_0, the value of $\hat{g}_{\mathcal{O},t_1+1}(j_0)$ coded by x_0 is at most $|\sigma|+1$. Thus $\hat{g}_{\mathcal{O},t_1+1}(j_0) \leq |\sigma|+1$, a contradiction. ⊣

Claim 2 implies that x_0 is hyperarithmetic which is a contradiction. Hence there is a t^0 such that for any $t \geq t^0$, $\hat{g}_{\mathcal{O},t}(j_0) = \hat{g}_{\mathcal{O},t+1}(j_0)$ and so $\sigma_{0,j_0}^{n_{j_0}^0[\alpha_t]+1}[\alpha_t]$ is a constant for all $t > t^0$.

We leave it to the reader to verify the stabilization of the other sequences of parameters.

It follows that for each j, there is a t_j such that $\hat{g}_{\mathcal{O},t_j}(j) = \hat{g}_{\mathcal{O},t}(j) = f'_{\mathcal{O}}(j)[\alpha_t]$ for $t \geq t_j$. Define $\hat{g}(j) = \lim_{t\to\omega} \hat{g}_{\mathcal{O},t}(j)$ for $j \in \omega$. Then $\hat{g}(j) = \lim_{t\to\omega} f'_{\mathcal{O}}(j)[\alpha_t]$ and $\hat{g} \in L_{\gamma+1}$. Now $\hat{g}$ is the leftmost path in $T_2[\gamma]$ and so in T_2 as well. Thus $\hat{g} = f_{\mathcal{O}} \in L_{\gamma+1}$. This contradicts the assumption that $\gamma < \omega_1^{\mathrm{CK}}$. Hence $\gamma = \omega_1^{\mathrm{CK}}$ and so $f_{\mathcal{O}} \leq_h x_0 \oplus x_1$.

Using this, one may decode the entire construction in $T_i[\omega_1^{\mathrm{CK}}]$ and conclude that $z \leq_h x_0 \oplus x_1$, completing the proof of the Theorem. ⊣

Let $\mathcal{F}$ be the collection of all finite subsets of ω. A real x is Δ^1_1-*traceable* if for any function $f \leq_h x$, there is a Δ^1_1-function $g : \omega \to \mathcal{F}$ such that for every n, $|g(n)| = n$ and $f(n) \in g(n)$.

Lemma 6.4. *There is an uncountable Σ^1_1-set A in which every member is Δ^1_1-traceable.*

Proof. This is precisely what was proved in Theorem 4.7 of [20]. ⊣

By [2] and [11], each Δ^1_1-traceable real is low for Δ^1_1-randomness and hence low for Δ^1_1-Kurtz randomness. By [10], the Π^1_1-random reals form a Σ^1_1-set. Then by Lemma 6.4 and Theorem 6.3, there is an x which is low for Δ^1_1-randomness and $x \oplus y \equiv_h \mathcal{O}$ for some Π^1_1-random y. So y is a $\Pi^1_1(x)$-singleton. We thus conclude:

Theorem 6.5.

(i) *Lowness for Δ^1_1-randomness $\not\Rightarrow$ Lowness for Π^1_1-randomness.*
(ii) *Lowness for Δ^1_1-Kurtz-randomness $\not\Rightarrow$ Lowness for Π^1_1-Kurtz-randomness.*

Remark. Theorem 6.3 may be used to answer Question 58 in [7] and Question 3 in [20], whose solutions were announced by Friedman and Harrington but have remain unpublished.

We end this paper with two problems.

It is still unknown whether strong Π^1_1-ML-randomness coincides with Π^1_1-randomness. To separate these two notions, one way is to investigate the Borel ranks of different notions of randomness. Obviously the collection of Π^1_1-ML-random reals is $\utilde{\mathbf{\Pi}}^0_3$ and it can be shown that it is not $\utilde{\mathbf{\Sigma}}^0_3$ (see Part 2, [23]) . Moreover, it is not hard to see that the collection of Π^1_1-random reals is neither $\utilde{\mathbf{\Sigma}}^0_2$ nor $\utilde{\mathbf{\Pi}}^0_2$. Its exact Borel rank remains unknown. We have the following conjecture.

Conjecture 6.6.[3] *The collection of Π^1_1-random reals is not $\utilde{\mathbf{\Pi}}^0_3$.*

Also the question whether lowness for Π^1_1-randomness coincides with hyperarithmeticity remains open. In view of Theorem 6.1, we have the following question.

[3]The conjecture was refuted by Monin [14] recently by showing that the collection of Π^1_1-random reals is proper $\utilde{\mathbf{\Pi}}^0_3$.

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QUESTION 6.7. *Is it true that for any nonhyperarithmetic x and uncountable Σ^1_1-set $A \subseteq 2^\omega$, there is a $y \in A$ such that $x \oplus y \geq_h \mathcal{O}$?*

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