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ON A MULTIVARIATE GENERALIZED POLYA PROCESS WITHOUT REGULARITY PROPERTY

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Most of the multivariate counting processes studied in the literature are regular processes, which implies, ignoring the types of the events, the non-occurrence of multiple events. However, in practice, several different types of events may occur simultaneously. In this paper, a new class of multivariate counting processes which allow simultaneous occurrences of multiple types of events is suggested and its stochastic properties are studied. For the modeling of such kind of process, we rely on the tool of superposition of seed counting processes. It will be shown that the stochastic properties of the proposed class of multivariate counting processes are explicitly expressed. Furthermore, the marginal processes are also explicitly obtained. We analyze the multivariate dependence structure of the proposed class of counting processes.

Keywords: characterization of multivariate counting processes, complete intensity functions, dependence structure, generalized polya process, superposition

1. INTRODUCTION

In practice, the occurrences of some type of random events are counted and thus, in such cases, we are dealing with counting processes. A stochastic process $\{N(t),\ t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of events that occur by time t. Until now, most of the research is focused on the univariate counting process, for which their stochastic properties have been thoroughly studied. The Markov process, homogeneous and non-homogeneous Poisson processes (HPP and NHPP), and the renewal process are the most frequently applied univariate counting processes (Limnios and Oprişan [20] and Barbu and Limnios [3]).

As mentioned above, so far, most researchers have paid their attention to univariate counting process. However, stochastically dependent multivariate series of events can be commonly observed in many contexts (see Cha and Giorgio [11] for plenty of examples). In this regard, in Cha and Giorgio [11], a new multivariate point process model, called the multivariate generalized Polya process (MVGPP), has been developed.

One of the critical features of the MVGPP is that it is a “regular process”, which implies, ignoring the types of the events, the non-occurrence of multiple events. However, in some cases, the "regularity condition" in multivariate point processes needs to be relaxed because, in practice, several different types of events may occur simultaneously. For example, in insurance risk model, an insurance portfolio frequently consists of two or more insurance policies or subportfolios. In this case, the insurance portfolio may simultaneously face different types of claims arising from the same catastrophe such as a windstorm or a vehicle accident (see Chan, Hailiang Yang and Zhang [12], Cai and Li [5] and Woo [26]). As a simplest explanatory model, suppose that $\{(N_1(t), N_2(t)),\ t \geq 0\}$ is a bivariate claim process, where $N_i(t)$, $i = 1, 2$, represents the number of claim causing events in the ith risk process by time t. Furthermore, some events in the two risk processes are common ones and shared by $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$, whereas the other events are respective ones in the two processes. Denote by X_{ij} the jth loss at the ith risk process, $i = 1, 2,\ j = 1, 2, \cdots$, and by $L_i(t)$, $i = 1, 2$, the total claim cost by time t in the ith risk process, respectively. Then $L_i(t)$ is defined as $L_i(t) = \sum_{j=1}^{N_i(t)} X_{ij}$, $i = 1, 2$ and, in this case, $L_1(t)$ and $L_2(t)$ are stochastically dependent. In insurance risk analysis, the stochastic properties (e.g., joint and marginal distributions and moments such as mean, variance and covariance) of $L_1(t)$ and $L_2(t)$ are of practical interest. Multivariate processes having simultaneous multiple types of events may also occur frequently in engineering science. For example, in reliability, in addition to respective recurrent failures, simultaneous failures in two parts in a system may occur (i.e., two types of events can occur) due to external common fatal shocks (see Marshall and Olkin [21] and Kundu and Gupta [18,19]).

Therefore, in this paper, our aim is to develop a new multivariate point process without the regularity property. However, the "marginal regularity" will be retained as in most univariate point processes. More accurate mathematical definitions of the regularity and the marginal regularity in multivariate point processes will be given in Section 2. For a systematic modeling of such kind of process, we rely on the tool of superposition of seed counting processes. To increase the practical applicability of the developed multivariate point process, our prime aim will be to keep mathematical tractability and computability as simple as possible. Another very important goal of this paper will be to develop a class of multivariate counting processes which possesses positive dependence because, as illustrated in Cha and Giorgio [11], multivariate series of events occurring in practice are frequently positively dependent. While this paper basically follows Cha [7] and Cha and Giorgio [11], especially, the proofs of the positive dependence property for both bivariate and multivariate cases are significantly different from that in Cha and Giorgio [11] due to structural difference of the studied models. This will be one of the main contributions of this paper.

The structure of this paper is as follows. In Section 2, using the tool of superposition of processes, we define a new class of marginally regular bivariate counting processes. It will be shown that the marginal process of the proposed bivariate counting process becomes an existing counting process and it has explicit stochastic properties. In Section 3, we derive the stochastic properties of the proposed class of bivariate counting processes. For further characterization of it, we suggest an alternative definition for the proposed class of bivariate counting processes which relies on mixing and, based on it, we analyze the dependence structure of the developed bivariate process. In Section 4, we briefly discuss the generalization of the bivariate process to the multivariate case. Finally, in Section 5, some concluding remarks will be given.

2. STOCHASTIC MODELING OF BIVARIATE COUNTING PROCESS

First, we will start with the case of bivariate counting process and will extend our discussion to the multivariate case afterward. Let $\{\mathbf{N}(t),\ t \geq 0\}$, where $\mathbf{N}(t) = (N_1(t), N_2(t))$, be a bivariate process of our interest. We can then define the marginal counting processes $\{N_i(t),\ t \geq 0\}$ and, for convenience, they will be called type i counting process, $i = 1, 2$, respectively. Furthermore, the events from type i counting process $\{N_i(t),\ t \geq 0\}$ will also be called type i events. In this paper, our aim is to develop a class of *marginally regular* bivariate counting processes. A univariate counting process $\{N(t),\ t \geq 0\}$ is called "regular" (or "orderly") if

$$P(N(t + \Delta t) - N(t) > 1) = o(\Delta t), \quad \forall t \geq 0.$$

Regularity is intuitively the non-occurrence of multiple events in a small interval. Note that the regularity in a multivariate process should be more precisely defined (see also Cox and Lewis [14]). There are two types of regularity in multivariate counting processes: (i) marginal regularity and (ii) regularity. For a multivariate counting process, we say that the process is *marginally regular* if its marginal processes, considered as univariate counting processes, are all regular. The multivariate process is said to be *regular* if the "pooled" process is regular. This type of regularity, of course, implies the marginal regularity. Throughout this paper, we will assume that the bivariate process $\{\mathbf{N}(t),\ t \geq 0\}$ of our interest is "marginally regular" (but "not regular") process.

To stochastically model a new class of marginally regular bivariate counting processes, we first introduce a well-known method of generating dependency which is frequently used to develop bivariate distributions. The well-known "bivariate Poisson distribution" was proposed by Campbell [6] and Holgate [16]. Although Campbell [6] obtained it in a different and more complex way, Holgate [16] obtained the bivariate Poisson distribution (BPD) using the "trivariate reduction method". That is, let W_1, W_2, and W_3 be independent Poisson random variables with the parameters λ_1, λ_2, and λ_3, respectively. Then a discrete bivariate distribution can be defined by setting

$$X_1 \equiv W_1 + W_3 \quad \text{and} \quad X_2 \equiv W_2 + W_3. \tag{1}$$

The bivariate distribution defined by (X_1, X_2) in (1) is called the BPD. It can be easily shown that the joint probability mass function of (X_1, X_2) is given by

$$f(x_1, x_2) = e^{-(\lambda_1+\lambda_2+\lambda_3)} \sum_{u=0}^{\min\{x_1,x_2\}} \frac{\lambda_1^{x_1-u}\lambda_2^{x_2-u}\lambda_3^{u}}{(x_1-u)!(x_2-u)!u!}, \quad x_1 = 0, 1, 2, \ldots, x_2 = 0, 1, 2, \ldots.$$

It is also well known that the BPD possesses very convenient properties, for example, all the marginal distributions are given by the Poisson distributions.

To obtain a marginally regular dependent bivariate counting process, one can extend the relation between random variables suggested in (1) to that between counting processes. Thus, a natural extension of the BPD to the bivariate Poisson process can be defined as follows.

Definition 1 (Bivariate Poisson process (BPP)): *Let $\{W_i(t),\ t \geq 0\}$ be the NHPP with the intensity function $\lambda_i(t)$, $i = 1, 2, 3$, respectively, and assume that they are mutually independent. Define a bivariate process $\{(X_1(t), X_2(t)),\ t \geq 0\}$ as $X_1(t) \equiv W_1(t) + W_3(t)$ and $X_2(t) \equiv W_2(t) + W_3(t)$, for all $t \geq 0$. Then the bivariate process $\{(X_1(t), X_2(t)),\ t \geq 0\}$ is called the bivariate Poisson process with the set of parameters $(\lambda_1(t), \lambda_2(t), \lambda_3(t))$.*

The properties of the BPP $\{(X_1(t), X_2(t)),\ t \geq 0\}$ defined in Definition 1 can be derived by using those of univariate Poisson process. However, a crucial demerit of the BPP is that

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the degree of dependence is too weak. That is, for $t_2 > t_1$ and $s_2 > s_1$, if the two intervals $(t_1, t_2]$ and $(s_1, s_2]$ are not overlapping, then

$$P(X_1(t_2) - X_1(t_1) = n_1, \quad X_2(s_2) - X_2(s_1) = n_2)$$
$$= P(X_1(t_2) - X_1(t_1) = n_1)P(X_2(s_2) - X_2(s_1) = n_2),$$

for all n_1 and n_2, due to the independent increments property of the involved Poisson processes. That is, the numbers of events from $\{X_1(t),\ t \geq 0\}$ and $\{X_2(t),\ t \geq 0\}$ for any two non-overlapping intervals are independent. Thus, although the BPP defined in Definition 1 can be conveniently applied in practice, due to the described reason, it would not be suitable for modeling bivariate series of events that have a stronger dependence.

Thus, now we consider to generate a marginally regular bivariate counting process which has a stronger dependence. At the same time, the stochastic properties of the bivariate process and its marginal processes should be mathematically tractable. For this purpose, we will consider the generalized Polya process (GPP) as the seed counting processes. To introduce the Definition of the GPP, the concept of stochastic intensity is needed. For a univariate orderly counting process $\{N(t),\ t \geq 0\}$ and its past history (i.e., internal filtration) $\mathcal{H}_{t-}$ in the interval $[0, t)$, the stochastic intensity is defined by (see also Aven and Jensen [1] and Cha [7]),

$$\lambda_t \equiv \lim_{\Delta t \to 0} \frac{P\left(N(t, t + \Delta t) \geq 1 | \mathcal{H}_{t-}\right)}{\Delta t} = \lim_{\Delta t \to 0} \frac{P\left(N(t, t + \Delta t) = 1 | \mathcal{H}_{t-}\right)}{\Delta t},$$

where $N(t_1, t_2)$, $t_1 < t_2$, is the number of events in $[t_1, t_2)$. The definition of the GPP is as follows.

Definition 2 (Generalized Polya process (Cha [7])): *A counting process* $\{N(t),\ t \geq 0\}$ *is called the GPP with the set of parameters* $(\lambda(t), \alpha, \beta)$, $\alpha \geq 0$, $\beta > 0$, *if*

(i) $N(0) = 0$;
(ii) $\lambda_t = (\alpha N(t-) + \beta)\lambda(t)$.

Note that the GPP with $(\lambda(t), \alpha = 0, \beta = 1)$ *reduces to the NHPP with the intensity function* $\lambda(t)$ *and, accordingly, the GPP can be understood as a generalized version of the NHPP. See Cha [7] for more detailed stochastic properties of the GPP. As mentioned before, one of the important objectives of this study is to develop bivariate process such that the stochastic properties of the bivariate process and its marginal processes should be mathematically tractable. Thus, if we follow the modeling procedure suggested in Definition 1 based on the GPP seed processes, the superposition of two GPPs, which corresponds to the marginal process, should be obtained in a mathematically tractable form. For this purpose, before defining a new bivariate counting process based on the GPP seed processes, we study the condition under which the superposition of two GPPs results in a GPP again. The following theorem is about the result on the superposition of two independent GPPs.*

Theorem 1: *Let* $\{M_i(t),\ t \geq 0\}$ *be the GPP with the set of parameters* $(\lambda(t), \alpha, \beta_i)$, $i = 1, 2$, *respectively, and assume that they are independent. Define* $M(t) \equiv M_1(t) + M_2(t)$, $t \geq 0$. *Then* $\{M(t),\ t \geq 0\}$ *is the GPP with the set of parameters* $(\lambda(t), \alpha, \beta_1 + \beta_2)$.

Proof: Denote by λ_t^M the stochastic intensity function of $\{M(t),\ t \geq 0\}$. Then, for the past history (internal filtration) $\mathcal{H}_{t-}^M$ of the process $\{M(t),\ t \geq 0\}$,

$$\begin{aligned}
\lambda_t^M &= \lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | \mathcal{H}_{t-}^M\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | T_i,\ i = 1, 2, \ldots, M_1(t-) + M_2(t-); M_1(t-) + M_2(t-)\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | M_1(t-) + M_2(t-)\right)}{\Delta t},
\end{aligned}$$

where $T_1 \leq T_2 \leq T_3 \leq \cdots < t$ are the sequential arrival points of the events in $\{M(t),\ t \geq 0\}$ and the third equality holds due to the Markovian property of the GPP. Note that

$$\begin{aligned}
&\lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | M_1(t-) + M_2(t-)\right)}{\Delta t} \\
&\quad = E_{(M_1(t-)|M_1(t-)+M_2(t-))}\left[\lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | M_1(t-), M_1(t-) + M_2(t-)\right)}{\Delta t}\right] \\
&\quad = E_{(M_1(t-)|M_1(t-)+M_2(t-))}\left[\lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | M_1(t-), M_2(t-)\right)}{\Delta t}\right], \qquad \textbf{(2)}
\end{aligned}$$

where $E_{(M_1(t-)|M_1(t-)+M_2(t-))}[\cdot]$ stands for the expectation with respect to the conditional distribution of $(M_1(t-)|M_1(t-) + M_2(t-))$. Observe that, in (2),

$$\begin{aligned}
&\lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | M_1(t-), M_2(t-)\right)}{\Delta t} \\
&\quad = (\alpha M_1(t-) + \beta_1)\lambda(t) + (\alpha M_2(t-) + \beta_2)\lambda(t) \\
&\quad = (\alpha(M_1(t-) + M_2(t-)) + \beta_1 + \beta_2)\lambda(t),
\end{aligned}$$

which depends on $M_1(t-)$ only through $M_1(t-) + M_2(t-)$. Thus, the conditional expectation in (2) is just given by

$$\begin{aligned}
&E_{(M_1(t-)|M_1(t-)+M_2(t-))}\left[\lim_{\Delta t \to 0} \frac{P\left(M(t, t+\Delta t) = 1 | M_1(t-), M_2(t-)\right)}{\Delta t}\right] \\
&\quad = (\alpha(M_1(t-) + M_2(t-)) + \beta_1 + \beta_2)\lambda(t).
\end{aligned}$$

Therefore,

$$\lambda_t^M = (\alpha M(t-) + \beta_1 + \beta_2)\lambda(t),$$

and $\{M(t),\ t \geq 0\}$ is the GPP with the set of parameters $(\lambda(t), \alpha, \beta_1 + \beta_2)$. ■

Based on Theorem 1, we can now define a new marginally regular bivariate generalized Polya process (MR-BVGPP) which has the GPPs as the marginal processes.

Definition 3 (Marginally regular bivariate generalized Polya process): *Let $\{V_i(t),\ t \geq 0\}$ be the GPP with the set of parameters $(\lambda(t), \alpha, \beta_i)$, $i = 1, 2, 3$, respectively, and assume that they are mutually independent. Define a bivariate process $\{(N_1(t), N_2(t)),\ t \geq 0\}$ as $N_1(t) \equiv V_1(t) + V_3(t)$ and $N_2(t) \equiv V_2(t) + V_3(t)$, for all $t \geq 0$. Then the bivariate process $\{(N_1(t), N_2(t)),\ t \geq 0\}$ is called the MR-BVGPP with the set of parameters $(\lambda(t), \alpha, \beta_1, \beta_2, \beta_3)$.*

From Theorem 1, obviously, the marginal process $\{N_i(t),\ t \geq 0\}$ is given by the GPP, which is explicitly stated in the following proposition.

Proposition 1: *For the MR-BVGPP $\{(N_1(t), N_2(t)),\ t \geq 0\}$ with the set of parameters $(\lambda(t), \alpha, \beta_1, \beta_2, \beta_3)$, the marginal process $\{N_i(t),\ t \geq 0\}$ is the GPP with the set of parameters $(\lambda(t), \alpha, \beta_i + \beta_3)$, $i = 1, 2$, respectively.*

3. STOCHASTIC PROPERTIES

3.1. Restarting Property and the Joint Distribution of the Number of Events

In this section, we will derive stochastic properties of the MR-BVGPP $\{(N_1(t), N_2(t)),\ t \geq 0\}$. For this, first of all, the understanding of the "restarting property" of a process would be much helpful.

Definition 4 (Restarting property (Cha [7])): *Let $t > 0$ be an "arbitrary" time point. If the conditional future stochastic process from t, given the history until time t, follows the same type of stochastic process with possibly different set of process parameters, then the process is called to possess the restarting property. A stochastic process that enjoys the restarting property is called a restarting process.*

The restarting property is explained in a much more detail and several examples for univariate restarting processes are given in Cha [7]. It is shown that the GPP possesses this property and, in addition, it "unconditionally" restart (see Theorem 2 of Cha [7]). Note that the bivariate process $\{(N_1(t), N_2(t)),\ t \geq 0\}$ in Definition 3 also possesses this restarting property and it also "unconditionally" restarts, which is explicitly stated in the following theorem. Furthermore, relying on these properties, the joint distributions of the number of events in an arbitrary interval or disjoint intervals can be conveniently obtained. Denote by $\mathcal{H}_{it-}$ the history of Type i process in the interval $[0, t)$, $i = 1, 2$. For a constant $u \geq 0$, define $N_{ui}(t) \equiv N_i(u + t) - N_i(u)$, $i = 1, 2$, and $V_{ui}(t) \equiv V_i(u + t) - V_i(u)$, $i = 1, 2, 3$. Furthermore, throughout this paper, we define $\Lambda(t) \equiv \int_0^t \lambda(x)dx$, $t \geq 0$.

Theorem 2: *Let $0 \equiv u_0 < u_1 < u_2 < \cdots < u_m$.*

(i) Given $(\mathcal{H}_{1u-}, \mathcal{H}_{2u-})$, $\{(N_{u1}(t), N_{u2}(t)),\ t \geq 0\}$ is the MR-BVGPP with the set of parameters $(\lambda(u + t), \alpha, \alpha m_1 + \beta_1, \alpha m_2 + \beta_2, \alpha m_3 + \beta_3)$, where m_i is the realization of $V_i(u-)$, $i = 1, 2, 3$, respectively.

(ii) For any $u \geq 0$, $\{(N_{u1}(t), N_{u2}(t)),\ t \geq 0\}$ is 'unconditionally' MR-BVGPP with the set of parameters $(\varphi(t, u), \alpha, \beta_1, \beta_2, \beta_3)$, where

$$\varphi(t, u) = \frac{\lambda(u + t)\exp\{\alpha\Lambda(u + t)\}}{1 + \exp\{\alpha\Lambda(u + t)\} - \exp\{\alpha\Lambda(u)\}}.$$

(iii)

$$P(N_i(u_2) - N_i(u_1) = n_i,\ i = 1, 2)$$
$$= \sum_{j=0}^{\min\{n_1, n_2\}} \frac{\Gamma(j + \beta_3/\alpha)\Gamma(n_1 - j + \beta_1/\alpha)\Gamma(n_2 - j + \beta_2/\alpha)}{j!(n_1 - j)!(n_2 - j)!\Gamma(\beta_1/\alpha)\Gamma(\beta_2/\alpha)\Gamma(\beta_3/\alpha)}$$

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$$\times\left(\frac{1-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}{1+\exp\{-\alpha\Lambda(u_2)\}-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}\right)^{n_1+n_2-j}$$
$$\times\left(\frac{\exp\{-\alpha\Lambda(u_2)\}}{1+\exp\{-\alpha\Lambda(u_2)\}-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}\right)^{(\beta_1+\beta_2+\beta_3)/\alpha}.$$

(iv)

$$P(N_i(u_j)-N_i(u_{j-1})=n_{ij},\ i=1,2,j=1,2,\ldots,m)$$
$$=\sum_{j_1=0}^{\min\{n_{11},n_{21}\}}\sum_{j_2=0}^{\min\{n_{12},n_{22}\}}\cdots\sum_{j_m=0}^{\min\{n_{1m},n_{2m}\}}\prod_{i=1}^{m}\left[\frac{\Gamma(\sum_{k=1}^{i}j_k+\beta_3/\alpha)}{j_i!\Gamma(\sum_{k=1}^{i-1}j_k+\beta_3/\alpha)}\right.$$
$$\times\frac{\Gamma(\sum_{k=1}^{i}(n_{1k}-j_k)+\beta_1/\alpha)}{(n_{1i}-j_i)!\Gamma(\sum_{k=1}^{i-1}(n_{1k}-j_k)+\beta_1/\alpha)}\cdot\frac{\Gamma(\sum_{k=1}^{i}(n_{2k}-j_k)+\beta_2/\alpha)}{(n_{2i}-j_i)!\Gamma(\sum_{k=1}^{i-1}(n_{2k}-j_k)+\beta_2/\alpha)}$$
$$\times\Big(1-\exp\{-\alpha\Lambda(u_i-u_{i-1}|u_{i-1})\}\Big)^{n_{1i}+n_{2i}-j_i}$$
$$\left.\times\Big(\exp\{-\alpha\Lambda(u_i-u_{i-1}|u_{i-1})\}\Big)^{\sum_{k=1}^{i-1}(n_{1k}+n_{2k}-j_k)+(\beta_1+\beta_2+\beta_3)/\alpha}\right],$$

where $\sum_{k=1}^{i-1}(\cdot)\equiv 0$ *when* $i=1,\ \lambda(t|s)\equiv\lambda(t+s),\Lambda(t|s)\equiv\int_0^t\lambda(u|s)du.$

PROOF: Property (i): Note that if $(\mathcal{H}_{1u-},\mathcal{H}_{2u-})$ is given, then the corresponding histories of the seed processes $\{V_i(t),\ t\geq 0\}$, $i=1,2,3$, are also specified. Furthermore, given the histories of the seed processes $\{V_i(t),\ t\geq 0\}$, $i=1,2,3$, in the interval $[0,u)$, due to the Markovian property of the GPP, the future process $\{(N_{u1}(t),N_{u2}(t)),\ t\geq 0\}$ depends only on $(V_1(u-),V_2(u-),V_3(u-))$. Specifically, given $(V_1(u-)=m_1,V_2(u-)=m_2,V_3(u-)=m_3)$, the future process $\{V_{ui}(t),\ t\geq 0\}$ follows the GPP with the set of parameters $(\lambda(u+t),\alpha,\alpha m_i+\beta_i)$, $i=1,2,3$, respectively. Thus, we have the desired result.

Property (ii): From Theorem 2 of Cha [7], $\{V_{ui}(t),\ t\geq 0\}$ is the GPP with the set of parameters $(\varphi(t,u),\alpha,\beta_i)$, where

$$\varphi(t,u)=\frac{\lambda(u+t)\exp\{\alpha\Lambda(u+t)\}}{1+\exp\{\alpha\Lambda(u+t)\}-\exp\{\alpha\Lambda(u)\}}.$$

Thus, we have the desired result.

Property (iii): Note that $N_i(u_2)-N_i(u_1)=N_{u_1i}(u_2-u_1)=V_{u_1i}(u_2-u_1)+V_{u_13}(u_2-u_1)$, $i=1,2$, and thus, from Property (ii), we have

$$P(N_i(u_2)-N_i(u_1)=n_i,\ i=1,2)$$
$$=\sum_{j=0}^{\min\{n_1,n_2\}}\frac{\Gamma(j+\beta_3/\alpha)}{j!\Gamma(\beta_3/\alpha)}\left(\frac{1-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}{1+\exp\{-\alpha\Lambda(u_2)\}-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}\right)^{j}$$
$$\times\left(\frac{\exp\{-\alpha\Lambda(u_2)\}}{1+\exp\{-\alpha\Lambda(u_2)\}-\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}}\right)^{\beta_3/\alpha}$$

$$\times \frac{\Gamma(n_1 - j + \beta_1/\alpha)}{(n_1 - j)!\Gamma(\beta_1/\alpha)} \left(\frac{1 - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}}{1 + \exp\{-\alpha\Lambda(u_2)\} - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}} \right)^{n_1 - j}$$

$$\times \left(\frac{\exp\{-\alpha\Lambda(u_2)\}}{1 + \exp\{-\alpha\Lambda(u_2)\} - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}} \right)^{\beta_1/\alpha}$$

$$\times \frac{\Gamma(n_2 - j + \beta_2/\alpha)}{(n_2 - j)!\Gamma(\beta_2/\alpha)} \left(\frac{1 - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}}{1 + \exp\{-\alpha\Lambda(u_2)\} - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}} \right)^{n_2 - j}$$

$$\times \left(\frac{\exp\{-\alpha\Lambda(u_2)\}}{1 + \exp\{-\alpha\Lambda(u_2)\} - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}} \right)^{\beta_2/\alpha},$$

which results in the desired result.

Property (iv): Observe that

$$\begin{aligned}
&P(N_i(u_j) - N_i(u_{j-1}) = n_{ij},\ i = 1, 2, j = 1, 2, \ldots, m) \\
&= \sum_{j_1=0}^{\min\{n_{11}, n_{21}\}} \sum_{j_2=0}^{\min\{n_{12}, n_{22}\}} \cdots \sum_{j_m=0}^{\min\{n_{1m}, n_{2m}\}} P(N_1(u_k) - N_1(u_{k-1}) = n_{1k}, \\
&\qquad N_2(u_k) - N_2(u_{k-1}) = n_{2k}, V_3(u_k) - V_3(u_{k-1}) = j_k,\ k = 1, 2, \ldots, m) \\
&= \sum_{j_1=0}^{\min\{n_{11}, n_{21}\}} \sum_{j_2=0}^{\min\{n_{12}, n_{22}\}} \cdots \sum_{j_m=0}^{\min\{n_{1m}, n_{2m}\}} P(V_3(u_k) - V_3(u_{k-1}) = j_k, \\
&\qquad V_1(u_k) - V_1(u_{k-1}) = n_{1k} - j_k, V_2(u_k) - V_2(u_{k-1}) = n_{2k} - j_k,\ k = 1, 2, \ldots, m) \\
&= \sum_{j_1=0}^{\min\{n_{11}, n_{21}\}} \sum_{j_2=0}^{\min\{n_{12}, n_{22}\}} \cdots \sum_{j_m=0}^{\min\{n_{1m}, n_{2m}\}} \prod_{i=1}^{m} P(V_3(u_i) - V_3(u_{i-1}) = j_i, \\
&V_1(u_i) - V_1(u_{i-1}) = n_{1i} - j_i, V_2(u_i) - V_2(u_{i-1}) = n_{2i} - j_i | V_3(u_k) - V_3(u_{k-1}) = j_k, \\
&\qquad V_1(u_k) - V_1(u_{k-1}) = n_{1k} - j_k, V_2(u_k) - V_2(u_{k-1}) = n_{2k} - j_k, k = 1, 2, \ldots, i-1) \\
&= \sum_{j_1=0}^{\min\{n_{11}, n_{21}\}} \sum_{j_2=0}^{\min\{n_{12}, n_{22}\}} \cdots \sum_{j_m=0}^{\min\{n_{1m}, n_{2m}\}} \prod_{i=1}^{m} [P(V_3(u_i) - V_3(u_{i-1}) \\
&\qquad\qquad = j_i | V_3(u_k) - V_3(u_{k-1}) = j_k,\ k = 1, 2, \ldots, i-1) \\
&\quad \times P(V_1(u_i) - V_1(u_{i-1}) = n_{1i} - j_i | V_1(u_k) - V_1(u_{k-1}) = n_{1k} - j_k,\ k = 1, 2, \ldots, i-1) \\
&\quad \times P(V_2(u_i) - V_2(u_{i-1}) = n_{2i} - j_i | V_2(u_k) - V_2(u_{k-1}) = n_{2k} - j_k, k = 1, 2, \ldots, i-1)].
\end{aligned}$$

Due to the restarting property of the GPP, the conditional counting process

$$(V_l(t + u_{i-1}) - V_l(u_{i-1}) | V_l(u_k) - V_l(u_{k-1}) = m_k,\ k = 1, 2, \ldots, i-1),\ t \geq 0,$$

which counts the number of events from the time point u_{i-1}, follows the GPP with the set of parameters $(\lambda_l(u_{i-1} + t), \alpha, \alpha \sum_{k=1}^{i-1} m_k + \beta_l)$, $l = 1, 2, 3$, respectively. Then we have

$$
\begin{aligned}
&P(N_i(u_j) - N_i(u_{j-1}) = n_{ij},\ i = 1, 2, j = 1, 2, \ldots, m) \\
&\quad = \sum_{j_1=0}^{\min\{n_{11},n_{21}\}} \sum_{j_2=0}^{\min\{n_{12},n_{22}\}} \cdots \sum_{j_m=0}^{\min\{n_{1m},n_{2m}\}} \prod_{i=1}^{m} \left[\frac{\Gamma(\sum_{k=1}^{i} j_k + \beta_3/\alpha)}{j_i!\Gamma(\sum_{k=1}^{i-1} j_k + \beta_3/\alpha)} \right. \\
&\qquad \times \Big(1 - \exp\{-\alpha\Lambda(u_i - u_{i-1}|u_{i-1})\}\Big)^{j_i} \\
&\qquad \times \Big(\exp\{-\alpha\Lambda(u_i - u_{i-1}|u_{i-1})\}\Big)^{\sum_{k=1}^{i-1} j_k + \beta_3/\alpha} \\
&\qquad \times \frac{\Gamma(\sum_{k=1}^{i}(n_{1k} - j_k) + \beta_1/\alpha)}{(n_{1i} - j_i)!\Gamma(\sum_{k=1}^{i-1}(n_{1k} - j_k) + \beta_1/\alpha)} \Big(1 - \exp\{-\alpha\Lambda(u_i - u_{i-1}|u_{i-1})\}\Big)^{n_{1i}-j_i} \\
&\qquad \times \Big(\exp\{-\alpha\Lambda(u_i - u_{i-1}|u_{i-1})\}\Big)^{\sum_{k=1}^{i-1}(n_{1k}-j_k)+\beta_1/\alpha} \\
&\qquad \times \frac{\Gamma(\sum_{k=1}^{i}(n_{2k} - j_k) + \beta_2/\alpha)}{(n_{2i} - j_i)!\Gamma(\sum_{k=1}^{i-1}(n_{2k} - j_k) + \beta_2/\alpha)} \Big(1 - \exp\{-\alpha\Lambda(u_i - u_{i-1}|u_{i-1})\}\Big)^{n_{2i}-j_i} \\
&\qquad \left. \times \Big(\exp\{-\alpha\Lambda(u_i - u_{i-1}|u_{i-1})\}\Big)^{\sum_{k=1}^{i-1}(n_{2k}-j_k)+\beta_2/\alpha} \right],
\end{aligned}
$$

which results in the desired result. ■

3.2. Characterization Based on the Mixture of BPP

In developing new distributions or counting process models in insurance risk modeling, the tool of mixing has taken a crucial role (see, e.g., Willmot and Woo [24,25]). To derive further properties of MR-BVGPP, the following characterization of MR-BVGPP in terms of the mixture of BPP would take a crucial role. To show the equivalence of any two bivariate counting processes, we need to show that the two counting processes have the same complete intensity functions (Cox and Lewis [14], Aven and Jensen [1,2] and Cha and Giorgio [11]). As explained in Cha and Giorgio [11], a "marginally regular bivariate counting process" $\{(N_1(t), N_2(t)),\ t \geq 0\}$ can be specified by the following *complete intensity functions*:

$$
\begin{aligned}
\lambda_{1t} &\equiv \lim_{\Delta t \to 0} \frac{P\left(N_1(t, t+\Delta t) \geq 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t} = \lim_{\Delta t \to 0} \frac{P\left(N_1(t, t+\Delta t) = 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t}, \\
\lambda_{2t} &\equiv \lim_{\Delta t \to 0} \frac{P\left(N_2(t, t+\Delta t) \geq 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t} = \lim_{\Delta t \to 0} \frac{P\left(N_2(t, t+\Delta t) = 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t}, \\
\lambda_{12t} &\equiv \lim_{\Delta t \to 0} \frac{P\left(N_1(t, t+\Delta t)N_2(t, t+\Delta t) \geq 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t},
\end{aligned} \tag{3}
$$

where $\mathcal{H}_{it-} \equiv \{N_i(u),\ 0 \leq u < t\}$ is the history (internal filtration) of the marginal process $\{N_i(t),\ t \geq 0\}$, $i = 1, 2$, and $N_i(t_1, t_2)$, $t_1 < t_2$, represents the number of type i events in $[t_1, t_2)$, $i = 1, 2$, respectively. Note that $\mathcal{H}_{it-}$ can be completely defined in terms of $N_i(t-)$ and the sequential arrival points of the events $0 \leq T_{i1} \leq T_{i2} \leq \cdots \leq T_{iN_i(t-)} < t$ in $[0, t)$, $i = 1, 2$, where $N_i(t-)$ is the total number of events of type i point process in $[0, t)$, $i = 1, 2$.

Theorem 3: *Let $\{(N_1(t), N_2(t)),\ t \geq 0\}$ be the MR-BVGPP with the set of parameters $(\lambda(t), \alpha, \beta_1, \beta_2, \beta_3)$. Furthermore, let $\{(N_1^*(t), N_2^*(t)),\ t \geq 0\}$ be the mixture of the BPP* *with the set of parameters* $(z_1\lambda(t)\exp\{\alpha\Lambda(t)\}, z_2\lambda(t)\exp\{\alpha\Lambda(t)\}, z_3\lambda(t)\exp\{\alpha\Lambda(t)\})$ *(given* $Z_1 = z_1$, $Z_2 = z_2$, $Z_3 = z_3$*) and the corresponding mixing distributions (pdf) of* Z_1, Z_2, Z_3, *given by*

$$f_{Z_i}(z_i) = \frac{1}{\Gamma(\beta_i/\alpha)}\alpha^{-\beta_i/\alpha}z_i^{\beta_i/\alpha-1}\exp\{-z_i/\alpha\},\ 0 < z_i < \infty,\ i = 1,2,3,$$

respectively, where $\{Z_i,\ i = 1,2,3\}$, *are assumed to be mutually independent. Then the bivariate counting processes* $\{(N_1(t), N_2(t)),\ t \geq 0\}$ *and* $\{(N_1^*(t), N_2^*(t)),\ t \geq 0\}$ *share the same stochastic properties.*

Proof: We will derive the complete intensity functions of $\{(N_1(t), N_2(t)),\ t \geq 0\}$ and $\{(N_1^*(t), N_2^*(t)),\ t \geq 0\}$ and will show that they are the same.

First, we obtain the complete intensity functions λ_{1t}, λ_{2t} and λ_{12t} of $\{(N_1(t), N_2(t)),\ t \geq 0\}$ in Definition 3. Observe that

$$\begin{aligned}\lambda_{1t} &= \lim_{\Delta t\to 0}\frac{P\left(N_1(t,t+\Delta t) = 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{P\left(N_1(t,t+\Delta t) = 1|V_1(t-),V_2(t-),V_3(t-)\right)}{\Delta t},\end{aligned}$$

where note that $V_1(t-), V_2(t-), V_3(t-)$ are determined if $\mathcal{H}_{1t-}$ and $\mathcal{H}_{2t-}$ are given and the equality holds due to the Markovian property of the GPP. Due to the relationship $N_1(t) = V_1(t) + V_3(t)$,

$$\begin{aligned}\lambda_{1t} &= (\alpha V_1(t-) + \beta_1)\lambda(t) + (\alpha V_3(t-) + \beta_3)\lambda(t) = (\alpha(V_1(t-) + V_3(t-)) + \beta_1 + \beta_3)\lambda(t)\\ &= (\alpha N_1(t-) + \beta_1 + \beta_3)\lambda(t).\end{aligned}$$

By symmetry, $\lambda_{2t} = (\alpha N_2(t-) + \beta_2 + \beta_3)\lambda(t)$. Furthermore, similarly,

$$\begin{aligned}\lambda_{12t} &= \lim_{\Delta t\to 0}\frac{P\left(N_1(t,t+\Delta t)N_2(t,t+\Delta t) = 1|\mathcal{H}_{1t-};\mathcal{H}_{2t-}\right)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{P\left(N_1(t,t+\Delta t)N_2(t,t+\Delta t) = 1|V_1(t-),V_2(t-),V_3(t-)\right)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{P\left(V_3(t,t+\Delta t) = 1|V_3(t-)\right)}{\Delta t}\\ &= (\alpha V_3(t-) + \beta_3)\lambda(t).\end{aligned}$$

Now, we obtain the stochastic intensity functions λ_{1t}^*, λ_{2t}^*, λ_{12t}^* of $\{(N_1^*(t), N_2^*(t)),\ t \geq 0\}$. Let $\mathcal{H}_{1t-}^*$ and $\mathcal{H}_{2t-}^*$ be the corresponding histories of the marginal processes of $\{(N_1^*(t), N_2^*(t)),\ t \geq 0\}$. In $\{(N_1^*(t), N_2^*(t)),\ t \geq 0\}$, define $V_i^*(t)$ as the number of events in which only type i event occurs in $(0,t]$, $i = 1,2$, respectively, and $V_3^*(t)$ as the number of events in which both type 1 and type 2 events occur simultaneously. Then, clearly, $N_i^*(t) = V_i^*(t) + V_3^*(t)$, $i = 1,2$. Define $\mathcal{G}_{it-}^*$ as the history of the process $\{V_i^*(t),\ t \geq 0\}$, $i = 1,2,3$. Then, $(\mathcal{H}_{it-}^*,\ i = 1,2)$ specifies $(\mathcal{G}_{it-}^*,\ i = 1,2,3)$, and vice versa. Thus,

$$\begin{aligned}\lambda_{1t}^* &= \lim_{\Delta t\to 0}\frac{P\left(N_1^*(t,t+\Delta t) = 1|\mathcal{H}_{1t-}^*;\mathcal{H}_{2t-}^*\right)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{P\left(N_1^*(t,t+\Delta t) = 1|\mathcal{G}_{1t-}^*;\mathcal{G}_{2t-}^*;\mathcal{G}_{3t-}^*\right)}{\Delta t}\end{aligned}$$

$$= \lim_{\Delta t \to 0} \frac{P\left(N_1^*(t, t+\Delta t) = 1 | \mathcal{G}_{1t-}^*; \mathcal{G}_{3t-}^*\right)}{\Delta t}$$
$$= E_{(Z_1, Z_3 | \mathcal{G}_{1t-}^*; \mathcal{G}_{3t-}^*)}\left[\lim_{\Delta t \to 0} \frac{P\left(N_1^*(t, t+\Delta t) = 1 | Z_1, Z_3; \mathcal{G}_{1t-}^*; \mathcal{G}_{3t-}^*\right)}{\Delta t}\right],$$

where $E_{(Z_1, Z_3 | \mathcal{G}_{1t-}^*; \mathcal{G}_{3t-}^*)}[\ \cdot\ ]$ stands for the expectation with respect to the conditional distribution of $(Z_1, Z_3 | \mathcal{G}_{1t-}^*; \mathcal{G}_{3t-}^*)$ and

$$\lim_{\Delta t \to 0} \frac{P\left(N_1^*(t, t+\Delta t) = 1 | Z_1, Z_3; \mathcal{G}_{1t-}^*; \mathcal{G}_{3t-}^*\right)}{\Delta t} = (Z_1 + Z_3)\lambda(t)\exp\{\alpha\Lambda(t)\}.$$

Denote by $U_{i1} \leq U_{i2} \leq \cdots$ the sequential arrival points of the events in the process $\{V_i^*(t),\ t \geq 0\}$, $i = 1, 2, 3$. Note that the conditional joint distribution of

$$(\mathcal{G}_{1t-}^*; \mathcal{G}_{3t-}^* | Z_1, Z_3) = (U_{i1}, U_{i2}, \ldots, U_{iV_i^*(t-)}, V_i^*(t-), i = 1, 3 | Z_1, Z_3)$$

is given by (see, e.g., the proof of Theorem 1 in Cha and Finkelstein [9] and Cha and Finkelstein [10])

$$z_1\varphi(u_{11})z_1\varphi(u_{12})\cdots z_1\varphi(u_{1m_1})\exp\left\{-z_1\int_0^t \varphi(x)dx\right\}$$
$$\times z_3\varphi(u_{31})z_3\varphi(u_{32})\cdots z_3\varphi(u_{3m_3})\exp\left\{-z_3\int_0^t \varphi(x)dx\right\},$$

where $\varphi(t) \equiv \lambda(t)\exp\{\alpha\Lambda(t)\}$, u_{ij} represents the realization of U_{ij} and m_i represents that of $V_i^*(t-)$, $i = 1, 3$, respectively. Thus, the conditional joint distribution of $(Z_1, Z_3 | \mathcal{G}_{1t-}^* = \mathbf{g}_{1t-}; \mathcal{G}_{3t-}^* = \mathbf{g}_{3t-})$, where $\mathbf{g}_{it-} \equiv (u_{i1}, u_{i2}, \ldots, u_{im_i}, m_i)$ is the realization of $\mathcal{G}_{it-}^*$, $i = 1, 3$, is given by

$$\frac{z_1^{m_1} z_3^{m_3}\exp\{-(z_1 + z_3)\int_0^t \varphi(x)dx\} f_{Z_1}(z_1) f_{Z_3}(z_3)}{\int_0^\infty \int_0^\infty v_1^{m_1} v_3^{m_3}\exp\{-(v_1 + v_3)\int_0^t \varphi(x)dx\} f_{Z_1}(v_1) f_{Z_3}(v_3) dv_1 dv_3}.$$

Thus, given $(\mathcal{G}_{1t-}^* = \mathbf{g}_{1t-}; \mathcal{G}_{3t-}^* = \mathbf{g}_{3t-})$,

$$\lim_{\Delta t \to 0} \frac{P\left(N_1^*(t, t+\Delta t) = 1 | \mathcal{G}_{1t-}^* = \mathbf{g}_{1t-}; \mathcal{G}_{3t-}^* = \mathbf{g}_{3t-}\right)}{\Delta t}$$
$$= \lambda(t)\exp\{\alpha\Lambda(t)\}$$
$$\times \frac{\int_0^\infty \int_0^\infty (z_1 + z_3) z_1^{m_1} z_3^{m_3}\exp\{-(z_1 + z_3)\int_0^t \lambda(x)\exp\{\alpha\Lambda(x)\}dx\} f_{Z_1}(z_1) f_{Z_3}(z_3) dz_1 dz_3}{\int_0^\infty \int_0^\infty v_1^{m_1} v_3^{m_3}\exp\{-(v_1 + v_3)\int_0^t \lambda(x)\exp\{\alpha\Lambda(x)\}dx\} f_{Z_1}(v_1) f_{Z_3}(v_3) dv_1 dv_3}$$
$$= \lambda(t)\exp\{\alpha\Lambda(t)\}\Bigg(\int_0^\infty z_1^{m_1+1}\exp\{-z_1\int_0^t \lambda(x)\exp\{\alpha\Lambda(x)\}dx\} f_{Z_1}(z_1) dz_1$$
$$\times \int_0^\infty z_3^{m_3}\exp\{-z_3\int_0^t \lambda(x)\exp\{\alpha\Lambda(x)\}dx\} f_{Z_3}(z_3) dz_3$$
$$+ \int_0^\infty z_3^{m_3+1}\exp\{-z_3\int_0^t \lambda(x)\exp\{\alpha\Lambda(x)\}dx\} f_{Z_3}(z_3) dz_3$$

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$$\times \int_0^\infty z_1^{m_1} \exp\{-z_1 \int_0^t \lambda(x)\exp\{\alpha\Lambda(x)\}dx\} f_{Z_1}(z_1)dz_1\Bigg)$$

$$\times \Bigg(\int_0^\infty v_1^{m_1} \exp\{-v_1 \int_0^t \lambda(x)\exp\{\alpha\Lambda(x)\}dx\} f_{Z_1}(v_1)dv_1$$

$$\times \int_0^\infty v_3^{m_3} \exp\{-v_3 \int_0^t \lambda(x)\exp\{\alpha\Lambda(x)\}dx\} f_{Z_3}(v_3)dv_3\Bigg)^{-1}$$

$$= \lambda(t)\exp\{\alpha\Lambda(t)\}\Bigg\{\sum_{i=1,3}\Bigg[\Bigg(\int_0^\infty z_i^{m_i+1}\exp\{-z_i\frac{1}{\alpha}(\exp\{\alpha\Lambda(t)\}-1)\}f_{Z_i}(z_i)dz_i\Bigg)$$

$$\times\Bigg(\int_0^\infty v_i^{m_i}\exp\{-v_i\frac{1}{\alpha}(\exp\{\alpha\Lambda(t)\}-1)\}f_{Z_i}(v_i)dv_i\Bigg)^{-1}\Bigg]\Bigg\}$$

$$= (\alpha m_1+\beta_1)\lambda(t) + (\alpha m_3+\beta_3)\lambda(t)$$

$$= (\alpha n_1+\beta_1+\beta_3)\lambda(t),$$

where $n_1 \equiv m_1 + m_3$ is the realization of $N_1^*(t-) = V_1^*(t-) + V_3^*(t-)$, and the following calculation is used: for a non-negative integer k,

$$\int_0^\infty z_i^k \exp\{-z_i\frac{1}{\alpha}(\exp\{\alpha\Lambda(t)\}-1)\}f_{Z_i}(z_i)dz_i$$

$$= \int_0^\infty \frac{1}{\Gamma(\beta_i/\alpha)}\alpha^{-\beta_i/\alpha} z_i^{k+\beta_i/\alpha-1}\exp\left\{-z_i\frac{1}{\alpha}\exp\{\alpha\Lambda(t)\}\right\}dz_i$$

$$= \frac{\alpha^{-\beta_i/\alpha}}{\Gamma(\beta_i/\alpha)}\frac{\Gamma(k+\beta_i/\alpha)}{\left(\frac{1}{\alpha}\exp\{\alpha\Lambda(t)\}\right)^{k+\beta_i/\alpha}}$$

$$\times\int_0^\infty \frac{1}{\Gamma(k+\beta_i/\alpha)}\left(\frac{1}{\alpha}\exp\{\alpha\Lambda(t)\}\right)^{k+\beta_i/\alpha} z_i^{k+\beta_i/\alpha-1}\exp\left\{-z_i\frac{1}{\alpha}\exp\{\alpha\Lambda(t)\}\right\}dz_i$$

$$= \frac{\alpha^{-\beta_i/\alpha}}{\Gamma(\beta_i/\alpha)}\frac{\Gamma(k+\beta_i/\alpha)}{\left(\frac{1}{\alpha}\exp\{\alpha\Lambda(t)\}\right)^{k+\beta_i/\alpha}}, \quad i=1,3,$$

and

$$\Gamma(k+1+\beta_i/\alpha) = (k+\beta_i/\alpha)\Gamma(k+\beta_i/\alpha).$$

Thus,

$$\lambda_{1t} = (\alpha N_1^*(t-)+\beta_1+\beta_3)\lambda(t).$$

By symmetry, $\lambda_{2t}^* = (\alpha N_2^*(t-)+\beta_2+\beta_3)\lambda(t)$. Furthermore,

$$\lambda_{12t}^* = \lim_{\Delta t\to 0}\frac{P\left(N_1^*(t,t+\Delta t)N_2^*(t,t+\Delta t)=1|\mathcal{H}_{1t-}^*;\mathcal{H}_{2t-}^*\right)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{P\left(N_1^*(t,t+\Delta t)N_2^*(t,t+\Delta t)=1|\mathcal{G}_{1t-}^*;\mathcal{G}_{2t-}^*;\mathcal{G}_{3t-}^*\right)}{\Delta t}$$

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$$= \lim_{\Delta t \to 0} \frac{P\left(V_3^*(t, t+\Delta t) = 1 | \mathcal{G}_{3t-}^*\right)}{\Delta t}$$

$$= E_{(Z_3|\mathcal{G}_{3t-}^*)}\left[\lim_{\Delta t \to 0} \frac{P\left(V_3^*(t, t+\Delta t) = 1 | Z_3; \mathcal{G}_{3t-}^*\right)}{\Delta t}\right],$$

where

$$\lim_{\Delta t \to 0} \frac{P\left(V_3^*(t, t+\Delta t) = 1 | Z_3; \mathcal{G}_{3t-}^*\right)}{\Delta t} = Z_3 \lambda(t) \exp\{\alpha \Lambda(t)\}.$$

Then, by a similar procedure as those described above, it can be shown that

$$\lambda_{12t}^* = (\alpha V_3^*(t-) + \beta_3)\lambda(t).$$

This completes the proof. ■

Remark 1:

(a) From Proposition 1, we can see that the marginal distribution of $N_i(t)$ follows a negative binomial distribution, $i = 1, 2$, respectively. Note that a Poisson-gamma mixture is a negative binomial distribution. Thus, this relation can give a clue for the relation obtained in Theorem 3.

(b) In actuarial science (e.g., Buhlmann [4]), so-called "contagion models" have been studied in univariate counting process framework, where the corresponding stochastic intensity function is the same as the GPP but the parameter α can be both positive and negative. Thus, it would be very interesting to consider the corresponding bivariate counting process model with $\alpha < 0$.

3.3. Dependence Structure

As mentioned earlier, our aim in this paper is to develop a marginally regular process which possesses a strong positive dependence structure. We will now discuss the dependence structure of MR-BVGPP. In Cha and Giorgio [11], the following new concept of dependency for bivariate stochastic processes was defined. See also Cha and Giorgio [11] for some practical interpretations of this concept.

Definition 5 (Positive quadrant dependent bivariate process (PQDBP)): *A bivariate point process* $\{(Y_1(t), Y_2(t)),\ t \geq 0\}$ *is PQDBP if*

$$P(Y_1(t_2) - Y_1(t_1) > n_1, Y_2(s_2) - Y_2(s_1) > n_2)$$
$$\geq P(Y_1(t_2) - Y_1(t_1) > n_1) P(Y_2(s_2) - Y_2(s_1) > n_2),$$

for all $t_2 > t_1$, $s_2 > s_1$, n_1 *and* n_2.

Note that the PQDBP property implies a rather strong dependency between the two processes $\{Y_1(t),\ t \geq 0\}$ and $\{Y_2(t),\ t \geq 0\}$. For example, it implies positive covariance between the number of events of the two processes in any arbitrary time intervals: $\mathrm{Cov}(Y_1(t_2) - Y_1(t_1), Y_2(s_2) - Y_2(s_1)) \geq 0$, for all $t_2 > t_1$, $s_2 > s_1$ (see also Cha and Giorgio [11]). We will now analyze the dependence structure of MR-BVGPP. For this purpose, we need the following preliminary lemma.

LEMMA 1: *Let X be a random variable and $g(x)$, $h(x)$ be real valued functions.*

(i) If both $g(x)$ and $h(x)$ are increasing; or if both $g(x)$ and $h(x)$ are decreasing, then $E[g(X)h(X)] \geq E[g(X)]E[h(X)]$.

(ii) Let X, Y, Z be mutually independent random variables and $g(x, y)$, $h(x, z)$ be real valued functions. If both $g(x, y)$ and $h(x, z)$ are increasing in x; or if both $g(x, y)$ and $h(x, z)$ are decreasing in x, then $E[g(X, Y)h(X, Z)] \geq E[g(X, Y)]E[h(X, Z)]$.

PROOF: The proof of (i) can be found in Joe [17] and Cuadras [15]. We now prove (ii). Denote by $F_Y(y)$ and $F_Z(z)$ the distribution functions of Y and Z, respectively. For any $Y = y$ and $Z = z$,

$$E[g(X, y)h(X, z)] \geq E[g(X, y)]E[h(X, z)], \tag{4}$$

due to (i). Then, from (4),

$$\begin{aligned} E[g(X, Y)h(X, Z)] &= \int_0^\infty \int_0^\infty E[g(X, y)h(X, z)]dF_Y(y)dF_Z(z) \\ &\geq \int_0^\infty \int_0^\infty E[g(X, y)]E[h(X, z)]dF_Y(y)dF_Z(z) \\ &= E[g(X, Y)]E[h(X, Z)]. \end{aligned}$$

■

The following Theorem 4 states that MR-BVGPP is a PQDBP.

THEOREM 4: *Let $\{(N_1(t), N_2(t)),\ t \geq 0\}$ be the MR-BVGPP with the set of parameters $(\lambda(t), \alpha, \beta_1, \beta_2, \beta_3)$. Then $\{(N_1(t), N_2(t)),\ t \geq 0\}$ is a PQDBP:*

$$\begin{aligned} &P(N_1(t_2) - N_1(t_1) > n_1, N_2(s_2) - N_2(s_1) > n_2) \\ &\quad \geq P(N_1(t_2) - N_1(t_1) > n_1)P(N_2(s_2) - N_2(s_1) > n_2), \end{aligned} \tag{5}$$

for all $t_2 > t_1$, $s_2 > s_1$, n_1 and n_2.

PROOF: To show inequality (5) is equivalent to show

$$\begin{aligned} &P(N_1^*(t_2) - N_1^*(t_1) > n_1, N_2^*(s_2) - N_2^*(s_1) > n_2) \\ &\quad \geq P(N_1^*(t_2) - N_1^*(t_1) > n_1)P(N_2^*(s_2) - N_2^*(s_1) > n_2), \end{aligned} \tag{6}$$

due to Theorem 3. As in the proof of Theorem 3, for $\{(N_1^*(t), N_2^*(t)),\ t \geq 0\}$ (which was defined in Theorem 3), we define $V_i^*(t)$ as the number of events in which only type i event occurs in $(0, t]$, $i = 1, 2$, respectively, and $V_3^*(t)$ as the number of events in which both type 1 and type 2 events occur simultaneously. Then, it holds that $N_i^*(t) = V_i^*(t) + V_3^*(t)$, $i = 1, 2$, and thus it suffices to show that

$$\begin{aligned} &P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1, V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2) \\ &\quad \geq P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1) \\ &\quad\quad \times P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2), \end{aligned} \tag{7}$$

for all $t_2 > t_1$, $s_2 > s_1$, n_1, and n_2. Without loss of generality, assume $t_1 \leq s_1$. We will consider three cases depending on whether the intervals $(t_1, t_2]$ and $(s_1, s_2]$ are overlapping

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(partially or fully) or completely separated: Case (i) $t_2 \leq s_1$; Case (ii) $s_1 < t_2 < s_2$; Case (iii) $s_1 < s_2 \leq t_2$.
Case (i) $t_2 \leq s_1$: In this case, the two intervals are not overlapping and due to the independent increments property of the NHPP,

$$\begin{aligned}
&P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1, V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2) \\
&\quad = E[P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1, \\
&\qquad V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2 | Z_1, Z_2, Z_3)] \\
&\quad = E[P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1 | Z_1, Z_2, Z_3) \\
&\qquad \times P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2 | Z_1, Z_2, Z_3)] \\
&\quad = E[P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1 | Z_1, Z_3) \\
&\qquad \times P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2 | Z_2, Z_3)]
\end{aligned}$$

Observe that, given $Z_1 = z_1, Z_3 = z_3$, $(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) | Z_1 = z_1, Z_3 = z_3)$ follows the Poisson distribution with mean value $(z_1 + z_3) \int_{t_1}^{t_2} \lambda(u) \exp\{\alpha\Lambda(u)\} du$. Thus, as the survival function of a Poisson distribution increases if its mean value increases (see, e.g., the proof of Theorem 1 in Cha [8]), $P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1 | Z_1, \{Z_3 = v\})$ is increasing function of v. Similarly, $P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2 | Z_2, \{Z_3 = v\})$ is also increasing function of v. Then, by Lemma 1-(ii),

$$\begin{aligned}
&E[P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1 | Z_1, Z_3) \\
&\qquad \times P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2 | Z_2, Z_3)] \\
&\quad \geq E[P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1 | Z_1, Z_3)] \\
&\qquad \times E[P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2 | Z_2, Z_3)] \\
&\quad = P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(t_2) - V_3^*(t_1) > n_1) \\
&\qquad \times P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(s_1) > n_2).
\end{aligned}$$

Case (ii) $s_1 < t_2 < s_2$: In this case, the two intervals have the common part $(s_1, t_2]$ and we need to use some adequate conditioning in order to cleverly use Lemma 1-(ii). In this case, the inequality (7) can be written as

$$\begin{aligned}
&P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1, \\
&\quad V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2) \\
&\qquad \geq P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1) \\
&\qquad\quad \times P(V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2).
\end{aligned}$$

Observe that

$$\begin{aligned}
&P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1, \\
&\quad V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2) \\
&\qquad = E[P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1, \\
&\qquad\quad V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2 | Z_1, Z_2, Z_3)],
\end{aligned}$$

and

$$\begin{aligned}
&P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1, \\
&\quad V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2 | Z_1, Z_2, Z_3) \\
&\qquad = E_{(V_3^*(t_2) - V_3^*(s_1) | Z_1, Z_2, Z_3)}[P(V_1^*(t_2) - V_1^*(t_1) \\
&\qquad\quad + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1, \\
&\quad V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] \\
&\qquad > n_2 | Z_1, Z_2, Z_3, V_3^*(t_2) - V_3^*(s_1))] \\
&\qquad = E_{(V_3^*(t_2) - V_3^*(s_1) | Z_1, Z_2, Z_3)}\Big[P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(s_1) - V_3^*(t_1) \\
&\qquad\quad > n_1 - [V_3^*(t_2) - V_3^*(s_1)] | Z_1, Z_2, Z_3, V_3^*(t_2) - V_3^*(s_1)) \\
&\qquad\quad \times P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(t_2) \\
&\qquad\quad > n_2 - [V_3^*(t_2) - V_3^*(s_1)] | Z_1, Z_2, Z_3, V_3^*(t_2) - V_3^*(s_1))\Big],
\end{aligned}$$

where "$E_{(V_3^*(t_2) - V_3^*(s_1) | Z_1, Z_2, Z_3)}[\ \cdot\ ]$" stands for the expectation with respect to the conditional distribution of $(V_3^*(t_2) - V_3^*(s_1) | Z_1, Z_2, Z_3)$. Note that, given $\{Z_1, Z_2, Z_3\}$, the processes $\{V_i^*(t),\ t \geq 0\}$, $i = 1, 2, 3$, are mutually independent NHPPs. Furthermore, the intervals $(t_1, s_1]$ and $(s_1, t_2]$ are not overlapping. Thus, $V_1^*(t_2) - V_1^*(t_1)$ is independent of $V_3^*(t_2) - V_3^*(s_1)$ and $V_3^*(s_1) - V_3^*(t_1)$ is also independent of $V_3^*(t_2) - V_3^*(s_1)$. Accordingly, given $\{Z_1, Z_2, Z_3\}$, $V_1^*(t_2) - V_1^*(t_1) + V_3^*(s_1) - V_3^*(t_1)$ is independent of $V_3^*(t_2) - V_3^*(s_1)$ and we have

$$\begin{aligned}
&P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(s_1) - V_3^*(t_1) > n_1 - v | Z_1, Z_2, Z_3, \{V_3^*(t_2) - V_3^*(s_1) = v\}) \\
&\quad = P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(s_1) - V_3^*(t_1) > n_1 - v | Z_1, Z_2, Z_3).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(t_2) > n_2 - v | Z_1, Z_2, Z_3, \{V_3^*(t_2) - V_3^*(s_1) = v\}) \\
&\quad = P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(t_2) > n_2 - v | Z_1, Z_2, Z_3).
\end{aligned}$$

Therefore, both

$$P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(s_1) - V_3^*(t_1) > n_1 - v | Z_1, Z_2, Z_3, \{V_3^*(t_2) - V_3^*(s_1) = v\})$$

and

$$P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(t_2) > n_2 - v | Z_1, Z_2, Z_3, \{V_3^*(t_2) - V_3^*(s_1) = v\})$$

are increasing in v. Thus, due to Lemma 1, for all fixed $Z_i = z_i$, $i = 1, 2, 3$,

$$\begin{aligned}
&E_{(V_3^*(t_2) - V_3^*(s_1) | Z_1, Z_2, Z_3)}\Big[P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(s_1) - V_3^*(t_1) \\
&\quad > n_1 - [V_3^*(t_2) - V_3^*(s_1)] | Z_1, Z_2, Z_3, V_3^*(t_2) - V_3^*(s_1)) \\
&\qquad \times P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(t_2) \\
&\qquad > n_2 - [V_3^*(t_2) - V_3^*(s_1)] | Z_1, Z_2, Z_3, V_3^*(t_2) - V_3^*(s_1))\Big] \\
&\quad \geq E_{(V_3^*(t_2) - V_3^*(s_1) | Z_1, Z_2, Z_3)}\Big[P(V_1^*(t_2) - V_1^*(t_1) + V_3^*(s_1) - V_3^*(t_1)
\end{aligned}$$

$$
\begin{aligned}
&> n_1 - [V_3^*(t_2) - V_3^*(s_1)]|Z_1, Z_2, Z_3, V_3^*(t_2) - V_3^*(s_1))\Big] \\
&\times E_{(V_3^*(t_2)-V_3^*(s_1)|Z_1,Z_2,Z_3)}\Big[P(V_2^*(s_2) - V_2^*(s_1) + V_3^*(s_2) - V_3^*(t_2) \\
&> n_2 - [V_3^*(t_2) - V_3^*(s_1)]|Z_1, Z_2, Z_3, V_3^*(t_2) - V_3^*(s_1))\Big] \\
&= P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1|Z_1, Z_2, Z_3) \\
&\quad \times P(V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2|Z_1, Z_2, Z_3) \\
&= P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1|Z_1, Z_3) \\
&\quad \times P(V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2|Z_2, Z_3).
\end{aligned}
$$

From the above discussion, we have

$$
\begin{aligned}
&P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1, \\
&\quad V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2|Z_1, Z_2, Z_3) \\
&\quad\quad \geq P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1|Z_1, Z_3) \\
&\quad\quad\quad \times P(V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2|Z_2, Z_3). \quad \mathbf{(8)}
\end{aligned}
$$

Now taking expectations both sides of (8), we have

$$
\begin{aligned}
&P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1, \\
&\quad V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2) \\
&\quad\quad \geq E\Big[P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1|Z_1, Z_3) \\
&\quad\quad\quad \times P(V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2|Z_2, Z_3)\Big] \\
&\quad\quad \geq E\Big[P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1|Z_1, Z_3)\Big] \\
&\quad\quad\quad \times E\Big[P(V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2|Z_2, Z_3)\Big] \\
&\quad\quad = P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1) \\
&\quad\quad\quad \times P(V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2),
\end{aligned}
$$

where the last inequality holds due to Lemma 1-(ii) as both

$$P(V_1^*(t_2) - V_1^*(t_1) + [V_3^*(t_2) - V_3^*(s_1) + V_3^*(s_1) - V_3^*(t_1)] > n_1|Z_1, \{Z_3 = v\})$$

and

$$P(V_2^*(s_2) - V_2^*(s_1) + [V_3^*(s_2) - V_3^*(t_2) + V_3^*(t_2) - V_3^*(s_1)] > n_2|Z_2, \{Z_3 = v\})$$

are increasing in v by the same reason as that stated in Case (i). Thus, we have shown the desired result.

Case (iii) $s_1 < s_2 \leq t_2$: This case can be proved similarly to Case (ii). ■

4. GENERALIZATION TO THE MULTIVARIATE PROCESS

The bivariate process $\{(N_1(t), N_2(t)),\ t \geq 0\}$ studied in the previous sections can be generalized to the multivariate case $\{\mathbf{N}(t),\ t \geq 0\} = \{(N_1(t), N_2(t), \ldots, N_m(t)),\ t \geq 0\}$ applying

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similar procedure. A natural extension of the MR-BVGPP in Definition 3 would be as follows.

DEFINITION 6 (Marginally regular multivariate generalized Polya process (MR-MVGPP)): *Let* $\{V_i(t),\ t \geq 0\}$ *be the GPP with the set of parameters* $(\lambda(t), \alpha, \beta_i)$, $i = 1, 2, \ldots, m+1$, *respectively, and assume that they are mutually independent. Define a multivariate process* $\{(N_1(t), N_2(t), \ldots, N_m(t)),\ t \geq 0\}$ *as* $N_i(t) \equiv V_i(t) + V_{m+1}(t)$, $i = 1, 2, \ldots, m$, *for all* $t \geq 0$. *Then the multivariate process* $\{(N_1(t), N_2(t), \ldots, N_m(t)),\ t \geq 0\}$ *is called the MR-MVGPP with the set of parameters* $(\lambda(t), \alpha, \beta_1, \beta_2, \cdots, \beta_{m+1})$.

Obviously, from Theorem 1, the marginal process $\{N_i(t),\ t \geq 0\}$ is given by the GPP with the set of parameters $(\lambda(t), \alpha, \beta_i + \beta_{m+1})$, $i = 1, 2, \ldots, m$. The main results for the MR-MVGPP can be obtained by applying similar arguments as those described in Sections 2 and 3. For example, when $m = 3$ (trivariate process), for $u_2 > u_1$,

$$
\begin{aligned}
&P(N_i(u_2) - N_i(u_1) = n_i,\ i = 1, 2, 3) \\
&= \sum_{j=0}^{\min\{n_1, n_2, n_3\}} P(V_i(u_2) - V_i(u_1) = n_i - j, i = 1, 2, 3, V_4(u_2) - V_4(u_1) = j) \\
&= \sum_{j=0}^{\min\{n_1, n_2, n_3\}} \frac{\Gamma(j + \beta_4/\alpha)\Gamma(n_1 - j + \beta_1/\alpha)\Gamma(n_2 - j + \beta_2/\alpha)\Gamma(n_3 - j + \beta_3/\alpha)}{j!(n_1 - j)!(n_2 - j)!(n_3 - j)!\Gamma(\beta_1/\alpha)\Gamma(\beta_2/\alpha)\Gamma(\beta_3/\alpha)\Gamma(\beta_4/\alpha)} \\
&\quad \times \left(\frac{1 - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}}{1 + \exp\{-\alpha\Lambda(u_2)\} - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}}\right)^{n_1 + n_2 + n_3 - 2j} \\
&\quad \times \left(\frac{\exp\{-\alpha\Lambda(u_2)\}}{1 + \exp\{-\alpha\Lambda(u_2)\} - \exp\{-\alpha[\Lambda(u_2) - \Lambda(u_1)]\}}\right)^{(\beta_1 + \beta_2 + \beta_3 + \beta_4)/\alpha}.
\end{aligned}
$$

As in the bivariate case, the MR-MVGPP can be characterized in terms of mixture of the multivariate Poisson process (MPP), which is defined in the following definition.

DEFINITION 7 (Multivariate Poisson process): *Let* $\{W_i(t),\ t \geq 0\}$ *be the NHPP the intensity function* $\lambda_i(t)$, $i = 1, 2, \ldots, m+1$, *respectively, and assume that they are mutually independent. Define a multivariate process* $\{\mathbf{X}(t),\ t \geq 0\} = \{(X_1(t), X_2(t), \ldots, X_m(t)),\ t \geq 0\}$ *as* $X_i(t) \equiv W_i(t) + W_{m+1}(t)$, $i = 1, 2, \ldots, m$, *for all* $t \geq 0$. *Then the multivariate process* $\{\mathbf{X}(t),\ t \geq 0\}$ *is called the MPP with the set of parameters* $(\lambda_1(t), \lambda_2(t), \ldots, \lambda_{m+1}(t))$.

By showing the equivalence of the complete intensity functions, similarly as before, we can have the following proposition.

PROPOSITION 2: *Let* $\{(N_1(t), N_2(t), \cdots, N_m(t)),\ t \geq 0\}$ *be the MR-MVGPP with the set of parameters* $(\lambda(t), \alpha, \beta_1, \beta_2, \cdots, \beta_{m+1})$. *Furthermore, let* $\{(N_1^*(t), N_2^*(t), \cdots, N_m^*(t)),\ t \geq 0\}$ *be the mixture of the MPP with the set of parameters* $(z_i\lambda(t)\exp\{\alpha\Lambda(t)\}, i = 1, 2, \cdots, m+1))$ *(given* $Z_i = z_i$, $i = 1, 2, \cdots, m$*) by using the corresponding mixing distributions (pdf) of*

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Z_i, given by

$$f_{Z_i}(z_i) = \frac{1}{\Gamma(\beta_i/\alpha)} \alpha^{-\beta_i/\alpha} z_i^{\beta_i/\alpha - 1} \exp\{-z_i/\alpha\}, \quad 0 < z_i < \infty,\ i = 1, 2, \ldots, m+1,$$

respectively, where $\{Z_i,\ \ i = 1, 2, \ldots, m+1\}$ are assumed to be mutually independent. Then the multivariate counting processes $\{(N_1(t), N_2(t), \ldots, N_m(t)),\ \ t \geq 0\}$ and $\{(N_1^*(t), N_2^*(t), \ldots, N_m^*(t)),\ t \geq 0\}$ share the same stochastic properties.

A new dependence concept for multivariate point processes has been defined in Cha and Giorgio [11].

Definition 8 (Positive upper orthant dependent multivariate process (PUODMP)): *A multivariate point process $\{(Y_1(t), Y_2(t), \ldots, Y_m(t)),\ t \geq 0\}$ is PUODMP if*

$$P(Y_i(t_{i2}) - Y_i(t_{i1}) > n_i, i = 1, 2, \ldots, m) \geq \Pi_{i=1}^m P(Y_i(t_{i2}) - Y_i(t_{i1}) > n_i), \tag{9}$$

for all $t_{i2} > t_{i1}$ and n_i, $i = 1, 2, \ldots, m$.

The interpretation of inequality (9) is similar to that in the bivariate case, that is, for any fixed $t_{i2} > t_{i1}$, $i = 1, 2, \ldots, m$, the m random variables $Y_i(t_{i2}) - Y_i(t_{i1})$, $i = 1, 2, \ldots, m$, are more likely simultaneously to have large values, compared with m independent random variables with the same univariate marginal distributions.

We will now show that the MR-MVGPP in Definition 6 is a PUODMP. For this, we first need two preliminary lemmas.

Lemma 2: *Let $\{U(t),\ t \geq 0\}$ be the HPP with intensity 1. Define $\Phi(t) \equiv \int_0^t \phi(s)ds$, $t \geq 0$, for a non-negative function $\phi(t)$, $t \geq 0$. Then the following properties hold.*

(i) Define $W(t) \equiv U(\Phi(t))$. Then $\{W(t),\ \ t \geq 0\}$ is the NHPP with intensity function $\phi(t)$.

(ii) For $t_1 < t_2$, $U(t_2) - U(t_1) =_D U(t_2 - t_1)$, where "$=_D$" stands for equality in distribution.

Proof: (i) The process $\{W(t),\ t \geq 0\}$ satisfies the two conditions of the NHPP with mean function $\Phi(t)$ in Definition 1.8 in Çinlar [13] as $\{U(t),\ t \geq 0\}$ is a HPP with intensity 1 and $\Phi(\cdot)$ is a non-negative increasing function. Property (ii) obviously holds for a HPP as it has the stationary increments property. ■

Lemma 3: *Let $(K_1, K_2, \ldots, K_l)$ and $(L_1, L_2, \ldots, L_r)$ be random vectors, where $L_1, L_2, \ldots, L_r$ are mutually independent. If the components of $(K_1, K_2, \ldots, K_l)$ are respectively increasing functions of $(L_1, L_2, \ldots, L_r)$ then $(K_1, K_2, \ldots, K_l)$ is an associated random vector.*

Proof: See cases (iv) and (v) of Theorem 3.10.5 in Müller and Stoyan [22]. ■

Theorem 5: *Let $\{(N_1(t), N_2(t), \ldots, N_m(t)),\ \ t \geq 0\}$ be the MR-MVGPP with the set of parameters $(\lambda(t), \alpha, \beta_1, \beta_2, \ldots, \beta_{m+1})$. Then $\{(N_1(t), N_2(t), \ldots, N_m(t)),\ \ t \geq 0\}$ is a PUODMP:*

$$P(N_i(t_{i2}) - N_i(t_{i1}) > n_i, i = 1, 2, \ldots, m) \geq \Pi_{i=1}^m P(N_i(t_{i2}) - N_i(t_{i1}) > n_i),$$

for all $t_{i2} > t_{i1}$ and n_i, $i = 1, 2, \ldots, m$.

Proof: Define $\phi(t) \equiv \lambda(t)\exp\{\alpha\Lambda(t)\}$, $t > 0$, $\Phi(t) \equiv \int_0^t \phi(s)ds$, and $\Delta_i \equiv \Phi(t_{i2}) - \Phi(t_{i1}) = \int_{t_{i1}}^{t_{i2}} \phi(s)ds$, for $i = 1, 2, \ldots, m$. Furthermore, let $(t^*_{(1)}, t^*_{(2)}, \ldots, t^*_{(2m)})$ be the increasing arrangement of $2m$ dimensional vector $(t_{11}, t_{12}, t_{21}, t_{22}, \ldots, t_{m1}, t_{m2})$. (If there are ties in $(t_{i1}, t_{i2}, i = 1, 2, \ldots, m)$ and, due to this, if there are more than one increasing arrangements, then $(t^*_{(1)}, t^*_{(2)}, \ldots, t^*_{(2m)})$ can be any one of them.) Let $\{U_i(t),\ t \geq 0\}$ be the HPP with intensity 1, $i = 1, 2, \ldots, m+1$, and they are mutually independent. Then, due to Proposition 2 and Lemma 2-(i), $N_i(t)$ can be represented as $N_i(t) =_D U_i(\Phi(t)Z_i) + U_{m+1}(\Phi(t)Z_{m+1})$, $i = 1, 2, \ldots, m$, where $\{U_i(t),\ t \geq 0\}$, $i = 1, 2, \ldots, m+1$, and $\{Z_1, \ldots Z_{m+1}\}$ are independent. Furthermore, due to Lemma 2-(ii), $N_i(t_{i2}) - N_i(t_{i1}) =_D U_i(\Delta_i Z_i) + U_{m+1}(\Delta_i Z_{m+1})$, $i = 1, 2, \ldots, m$. Then

$$\begin{aligned} &P(N_i(t_{i2}) - N_i(t_{i1}) > n_i, i = 1, 2, \ldots, m) \\ &\quad = E_{Z_1,\ldots,Z_{m+1}}[P(U_i(\Delta_i Z_i) + U_{m+1}(\Delta_i Z_{m+1}) > n_i,\ i = 1, \ldots, m | Z_1, \ldots Z_{m+1})]. \end{aligned} \quad \textbf{(10)}$$

In (10), as $\{U_i(t),\ t \geq 0\}$, $i = 1, 2, \ldots, m+1$, and $\{Z_1, \ldots Z_{m+1}\}$ are independent,

$$\begin{aligned} &P(U_i(\Delta_i Z_i) + U_{m+1}(\Delta_i Z_{m+1}) > n_i,\ i = 1, \ldots, m | Z_i = z_i, i = 1, 2, \ldots, m+1) \\ &\quad = P(U_i(\Delta_i z_i) + U_{m+1}(\Delta_i z_{m+1}) > n_i,\ i = 1, \ldots, m). \end{aligned} \quad \textbf{(11)}$$

Furthermore,

$$\begin{aligned} &P(U_i(\Delta_i z_i) + U_{m+1}(\Delta_i z_{m+1}) > n_i,\ i = 1, \ldots, m | U_i(\Delta_i z_i) = u_i,\ i = 1, \ldots, m) \\ &\quad = P(U_{m+1}(\Delta_i z_{m+1}) > n_i - u_i,\ i = 1, \ldots, m). \end{aligned} \quad \textbf{(12)}$$

Now, for our proof, we will show that

$$(U_{m+1}(\Delta_1 z_{m+1}), \ldots, U_{m+1}(\Delta_m z_{m+1}))$$

is a PUOD (positive upper orthant dependent) random vector. For a fixed i, suppose that the ranks of t_{i1} and t_{i2}, where $t_{i_1} < t_{i_2}$, in the increasing arrangement $(t^*_{(1)}, t^*_{(2)}, \ldots, t^*_{(2m)})$ are r_1 and r_2, where $r_1 < r_2$, that is, $t_{i_1} = t^*_{(r_1)}$ and $t_{i_2} = t^*_{(r_2)}$. Define

$$\begin{aligned} &Q_1 \equiv U_{m+1}(\Phi(t^*_{(2)})z_{m+1}) - U_{m+1}(\Phi(t^*_{(1)})z_{m+1}), \\ &Q_2 \equiv U_{m+1}(\Phi(t^*_{(3)})z_{m+1}) - U_{m+1}(\Phi(t^*_{(2)})z_{m+1}), \\ &\quad \ldots, Q_{2m-1} \equiv U_{m+1}(\Phi(t^*_{(2m)})z_{m+1}) - U_{m+1}(\Phi(t^*_{(2m-1)})z_{m+1}). \end{aligned}$$

Then $Q_1, Q_2, \ldots, Q_{2m-1}$ are mutually independent due to the independent increments property of the HPP and

$$\begin{aligned} \sum_{j=r_1}^{r_2-1} Q_j &= U_{m+1}(\Phi(t^*_{(r_2)})z_{m+1}) - (U_{m+1}(\Phi(t^*_{(r_1)})z_{m+1}) \\ &= U_{m+1}(\Phi(t_{i_2})z_{m+1}) - (U_{m+1}(\Phi(t_{i_1})z_{m+1}) \\ &=_D U_{m+1}((\Phi(t_{i_2}) - \Phi(t_{i_1}))z_{m+1}) = U_{m+1}(\Delta_i z_{m+1}), \end{aligned}$$

due to Lemma 2-(ii). Thus, any component in $(U_{m+1}(\Phi(t_{i_2})z_{m+1}) - U_{m+1}(\Phi(t_{i_1})z_{m+1}), i = 1, 2, \ldots, m)$ is increasing function of a mutually independent random vector and, due to Lemma 3, $(U_{m+1}(\Phi(t_{i_2})z_{m+1}) - U_{m+1}(\Phi(t_{i_1})z_{m+1}), i = 1, 2, \ldots, m) =_D (U_{m+1}(\Delta_i z_{m+1}), i = 1, 2, \ldots, m)$ is an associated random vector. As the association implies the PUOD property

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(see, e.g., Theorem 2.4 in Joe (1997)), the random vector $(U_{m+1}(\Delta_i z_{m+1}), i = 1, 2, \ldots, m)$ is PUOD. This also implies

$$P(U_{m+1}(\Delta_i z_{m+1}) > n_i - u_i,\ i = 1, \ldots, m) \geq \prod_{i=1}^{m} P(U_{m+1}(\Delta_i z_{m+1}) > n_i - u_i), \qquad \textbf{(13)}$$

for all u_i, $i = 1, \ldots, m$, and, from (12) and (13),

$$\begin{aligned}
&P(U_i(\Delta_i z_i) + U_{m+1}(\Delta_i z_{m+1}) > n_i,\ i = 1, \ldots, m) \\
&\quad = E_{U_1(\Delta_1 z_1),\ldots,U_m(\Delta_m z_m)}[P(U_{m+1}(\Delta_i z_{m+1}) > n_i - U_i(\Delta_i z_i),\ i = 1, \ldots, m)] \\
&\quad \geq E_{U_1(\Delta_1 z_1),\ldots,U_m(\Delta_m z_m)}\left[\prod_{i=1}^{m} P(U_{m+1}(\Delta_i z_{m+1}) > n_i - U_i(\Delta_i z_i))\right] \\
&\quad = \prod_{i=1}^{m} E_{U_i(\Delta_i z_i)}[P(U_{m+1}(\Delta_i z_{m+1}) > n_i - U_i(\Delta_i z_i))]. \qquad \textbf{(14)}
\end{aligned}$$

From (10), (11), and (14),

$$\begin{aligned}
&P(N_i(t_{i2}) - N_i(t_{i1}) > n_i, i = 1, 2, \ldots, m) \\
&\quad = E_{Z_1,\ldots,Z_{m+1}}[P(U_i(\Delta_i Z_i) + U_{m+1}(\Delta_i Z_{m+1}) > n_i,\ i = 1, \ldots, m)] \\
&\quad \geq E_{Z_1,\ldots,Z_{m+1}}\left[\prod_{i=1}^{m} E_{U_i(\Delta_i Z_i)}[P(U_{m+1}(\Delta_i Z_{m+1}) > n_i - U_i(\Delta_i Z_i))]\right] \\
&\quad = E_{Z_1,\ldots,Z_{m+1}}\left[\prod_{i=1}^{m} E_{U_i(\Delta_i Z_i)}[P(U_i(\Delta_i Z_i) > n_i - U_{m+1}(\Delta_i Z_{m+1}))]\right] \\
&\quad = E_{Z_1,\ldots,Z_m}\left[E_{Z_{m+1}}[\prod_{i=1}^{m} G_i(Z_i, Z_{m+1})]\right], \qquad \textbf{(15)}
\end{aligned}$$

where

$$\begin{aligned}
G_i(z_i, z_{m+1}) &\equiv E_{U_i(\Delta_i z_i)}[P(U_i(\Delta_i z_i) > n_i - U_{m+1}(\Delta_i z_{m+1}))] \\
&= \int_{u_i \in \mathbf{R}} P(u_i > n_i - U_{m+1}(\Delta_i z_{m+1}))P(U_i(\Delta_i z_i) \in du_i).
\end{aligned}$$

Observe that $U_{m+1}(\Delta_i z_{m+1})$ is increasing in z_{m+1} in the usual stochastic order (Shaked and Shanthikumar [23]) and thus $P(u_i > n_i - U_{m+1}(\Delta_i z_{m+1}))$ is increasing in z_{m+1}. This implies that $G_i(z_i, z_{m+1})$ is increasing in z_{m+1}. Then, by extending Lemma 1-(ii),

$$E_{Z_{m+1}}\left[\prod_{i=1}^{m} G_i(Z_i, Z_{m+1})] \geq \prod_{i=1}^{m} E_{Z_{m+1}}[G_i(Z_i, Z_{m+1})\right].$$

Therefore, from (15),

$$
\begin{aligned}
&P(N_i(t_{i2}) - N_i(t_{i1}) > n_i, i = 1, 2, \ldots, m) \\
&\geq E_{Z_1,\ldots,Z_m}\left[\prod_{i=1}^{m} E_{Z_{m+1}}\left[E_{U_i(\Delta_i Z_i)}[P(U_i(\Delta_i Z_i) > n_i - U_{m+1}(\Delta_i Z_{m+1}))]\right]\right] \\
&= \prod_{i=1}^{m} E_{Z_i, Z_{m+1}}\left[E_{U_i(\Delta_i Z_i)}[P(U_i(\Delta_i Z_i) + U_{m+1}(\Delta_i Z_{m+1}) > n_i)]\right] \\
&= \prod_{i=1}^{m} P(N_i(t_{i2}) - N_i(t_{i1}) > n_i),
\end{aligned}
$$

which completes the proof. ■

5. CONCLUDING REMARKS

Multivariate counting processes are practically very useful tools for modeling random occurrences of multivariate series of events arising over time intervals. However, until now, very few practically available multivariate counting processes have been developed and, accordingly, there has been a great discrepancy between desired practical applications and available useful models. Furthermore, most of the multivariate counting processes studied in the literature are regular processes, which implies, ignoring the types of the events, the non-occurrence of multiple events. However, in practice, several different types of events may occur simultaneously. In this regard, our aim of this paper was to develop a new class of multivariate counting processes which is not regular and allows mathematical tractability in various applications.

The multivariate counting process model developed in this paper has many merits from "application point of view". First of all, most of the results, including joint distributions of the number of events, are obtained explicitly. This aspect is practically of great importance because it allows explicit expression of the likelihood function in estimation procedure. Furthermore, as studied in this paper, the developed model possesses the restarting property. Due to this property, one can analyze the counting process observed starting from any positive time point $u > 0$ in the same manner as the original process which starts from 0 and, accordingly, the properties can also be explicitly expressed in this case. This also makes estimation procedure feasible based on the observation which has started from any positive time point $u > 0$. In addition, the marginal counting processes of the multivariate process are the univariate GPPs and one can conveniently use the properties of the GPP when analyzing marginal processes in the model. Furthermore, as illustrated in the examples on bivariate or multivariate counting processes that can occur in different areas, multivariate series of events occurring in practice are frequently positively dependent. It has been shown that the developed multivariate process possesses a strong type of positive dependence. Due to these reasons, the developed class of multivariate processes could be applied in various applications.

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