*Proceedings of the Royal Society of Edinburgh*, **136A**, 759–778, *2006*

# Asymptotic behaviour of positive steady states to a predator–prey model

**Yihong Du**
School of Mathematics, Statistics and Computer Science,
University of New England, Armidale, NSW 2351, Australia, and
Department of Mathematics, Qufu Normal University, Qufu,
Shandong 273165, People's Republic of China
(ydu@turing.une.edu.au)

**Mingxin Wang**
Department of Mathematics, Xuzhou Normal University,
Xuzhou 221116, People's Republic of China, and
Department of Mathematics, Southeast University,
Nanjing 210018, People's Republic of China
(mxwang@seu.edu.cn)

(MS received 14 July 2004; accepted 24 August 2005)

To understand the heterogeneous spatial effect on predator–prey models, we study the behaviour of the positive steady states of a predator–prey model as certain parameters are small or large. We compare the case when the model has a spatial degeneracy with the case when it does not have such a degeneracy. Our results show that the effect of the degeneracy can be clearly observed in one limiting case, but not in the others.

## 1. Introduction and main results

Let $\Omega_0, \Omega \subset \mathbb{R}^N$ be two smooth bounded domains that satisfy $\bar{\Omega}_0 \subset \Omega$, and let $a(x)$ be a continuous non-negative function satisfying

$$a(x) \equiv 0 \quad \text{in } \bar{\Omega}_0, \qquad a(x) > 0 \quad \text{in } \bar{\Omega} \setminus \bar{\Omega}_0.$$

For any $\Omega^* \subset \Omega$, we denote by $\lambda_1^{\mathrm{D}}(\Omega^*)$ the first eigenvalue of the Dirichlet problem

$$-\Delta u = \lambda u \quad \text{in } \Omega^*, \qquad u = 0 \quad \text{on } \partial\Omega^*.$$

Let $\lambda$, $\mu$ and $\beta$ be positive constants. In [3], the authors showed that the 'degenerate' predator–prey model

$$\left.\begin{aligned} -\Delta u &= \lambda u - a(x)u^2 - \beta uv && \text{in } \Omega, \\ -\Delta v &= \mu v\left(1 - \frac{v}{u}\right) && \text{in } \Omega, \\ \partial_\nu u &= \partial_\nu v = 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{1.1}$$





https://doi.org/10.1017/S0308210500004704 Published online by Cambridge University Press

can behave very differently from the perturbed non-degenerate model

$$\left.\begin{aligned} -\Delta u &= \lambda u - [a(x)+\varepsilon]u^2 - \beta uv && \text{in } \Omega, \\ -\Delta v &= \mu v\left(1 - \frac{v}{u}\right) && \text{in } \Omega, \\ \partial_\nu u &= \partial_\nu v = 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

where $\varepsilon$ is a positive constant, $\nu$ is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$. Here, following [3], we call (1.1) a degenerate model because the coefficient $a(x)$ vanishes on part of the domain $\Omega$. It was proved in [3] that (1.2) always has a positive solution, but, for (1.1), when

$$\lambda > \lambda_1^{\mathrm{D}}(\Omega_0) > \mu,$$

there is no positive solution for all small positive $\beta$. For parameters in these ranges, making use of this essential difference between (1.1) and (1.2), it was proved in [3] that the positive solutions $(u_\varepsilon, v_\varepsilon)$ of (1.2) can develop sharp spatial patterns when $\varepsilon > 0$ is small. On the other hand, it was shown in [3] that, under the condition

$$\mu > \lambda \geqslant \lambda_1^{\mathrm{D}}(\Omega_0), \tag{1.3}$$

problem (1.1) has at least one positive solution $(u, v)$ for any $\beta > 0$. Due to this fact and the *a priori* estimates established in [3], it is easy to show that, under (1.3), for small $\varepsilon > 0$, any positive solution $(u_\varepsilon, v_\varepsilon)$ of (1.2) is close to a positive solution $(u, v)$ of (1.1), and, hence, no sharp spatial patterns of $(u_\varepsilon, v_\varepsilon)$ can be observed.

In order to gain further understanding of (1.1) and (1.2) (in particular, to know whether (1.1) and (1.2) still exhibit any essential differences when (1.3) is satisfied), we consider in this paper several limiting cases of these systems. To be more specific, assuming (1.3), we shall discuss the limiting behaviour of the positive solutions of (1.1) and (1.2) for the cases $\beta \to 0^+$, $\beta \to \infty$ and $\mu \to \infty$, respectively. In ecological terms, these cases may be interpreted as, respectively, weak-predator, strong-predator and small-predator diffusion.

It emerges that, in the weak-predator case, the effect of the degeneracy can be clearly observed in the limit, where the positive solutions of (1.1) exhibit sharp spatial patterns, while those of (1.2) do not have such patterns; in the strong-predator case, the limiting behaviours of (1.1) and (1.2) are the same; in the small-predator diffusion case, the limiting behaviours of (1.1) and (1.2) are similar.

More precisely, suppose that (1.3) holds. The main results of this paper are then as given in the following theorem.

Theorem 1.1 (limiting behaviour as $\beta \to 0^+$). *Let $(u, v) \triangleq (u_\beta, v_\beta)$ be a positive solution of (1.1). We then draw the following conclusions:*

(a) $\lim_{\beta\to 0^+}(u_\beta(x), v_\beta(x)) = (\infty, \infty)$ *uniformly on $\bar{\Omega}_0$;*

(b) *along any sequence of $\beta$ decreasing to 0, there is a subsequence $\{\beta_n\}$ such that*

$$\lim_{n\to\infty}(u_{\beta_n}, v_{\beta_n}) = (u, v) \textit{ uniformly on any compact subset of } \bar{\Omega} \setminus \bar{\Omega}_0,$$

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*where* $(u, v)$ *is a positive solution of the system*

$$\left.\begin{aligned} -\Delta u &= \lambda u - a(x)u^2 \quad \text{in } \Omega \setminus \bar{\Omega}_0, \\ -\Delta v &= \mu v\left(1 - \frac{v}{u}\right) \quad \text{in } \Omega \setminus \bar{\Omega}_0, \\ \partial_\nu u|_{\partial\Omega} = \partial_\nu v|_{\partial\Omega} &= 0, \qquad u|_{\partial\Omega_0} = v|_{\partial\Omega_0} = \infty; \end{aligned}\right\} \tag{1.4}$$

(c) *if there exists* $\xi > 0$ *such that* $a(x)d(x, \Omega_0)^{-\xi}$ *is bounded for all* $x \in \Omega \setminus \bar{\Omega}_0$ *close to* $\partial\Omega_0$*, then (1.4) has a unique positive solution* $(u, v)$ *and the convergence in (b) holds for* $\beta \to 0^+$*;*

(d) *if* $(u, v) \triangleq (u_\beta, v_\beta)$ *is a positive solution of (1.2), then*

$$\lim_{\beta \to 0^+} (u_\beta(x), v_\beta(x)) = (U_\varepsilon, V_\varepsilon)$$

*uniformly over* $\bar{\Omega}$*, where* $U_\varepsilon$ *and* $V_\varepsilon$ *are the unique positive solutions to*

$$-\Delta U = \lambda U - [a(x) + \varepsilon]U^2, \quad \partial_\nu U|_{\partial\Omega} = 0,$$

*and*

$$-\Delta V = \mu V\left(1 - \frac{V}{U_\varepsilon}\right), \qquad \partial_\nu V|_{\partial\Omega} = 0,$$

*respectively.*

THEOREM 1.2 (limiting behaviour as $\beta \to +\infty$). *Let* $(u, v) \triangleq (u_\beta, v_\beta)$ *be a positive solution of (1.1). Then* $\lim_{\beta\to\infty}(u_\beta, v_\beta) = (0, 0)$ *uniformly on* $\bar{\Omega}$*. The same holds for positive solutions of (1.2).*

THEOREM 1.3 (limiting behaviour as $\mu \to +\infty$). *Let* $(u, v) \triangleq (u_\mu, v_\mu)$ *be a positive solution of (1.1). Then* $u_\mu \to w$ *in* $C^1(\bar{\Omega})$ *and* $v_\mu \to w$ *uniformly on any compact subset of* $\Omega$*, where* $w$ *is the unique positive solution of*

$$-\Delta w = \lambda w - [a(x) + \beta]w^2 \quad \text{in } \Omega, \quad \partial_\nu w = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

*A similar conclusion holds for the positive solutions of (1.2), except that the limiting function is the unique positive solution of*

$$-\Delta w = \lambda w - [a(x) + \varepsilon + \beta]w^2 \quad \text{in } \Omega, \quad \partial_\nu w = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

To better understand the behaviour of the positive solutions $(u_\beta, v_\beta)$, the limits

$$\lim_{\beta\to 0^+}\left(\frac{u_\beta}{\|u_\beta\|_\infty}, \frac{v_\beta}{\|v_\beta\|_\infty}\right) \quad \text{and} \quad \lim_{\beta\to\infty}\left(\frac{u_\beta}{\|u_\beta\|_\infty}, \frac{v_\beta}{\|v_\beta\|_\infty}\right)$$

will also be discussed.

In [3], the situation $0 < \lambda < \lambda_1^{\mathrm{D}}(\Omega_0)$ was also considered and it was shown that, in this case, both (1.1) and (1.2) have positive solutions for every $\mu > 0$ and $\beta > 0$. Using the techniques in [3], it can be shown that the corresponding limiting behaviours of (1.1) and (1.2) are the same in each of the three cases ($\beta \to 0^+$, $\beta \to \infty$ and $\mu \to \infty$); we leave the details to the interested reader.

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In [1], the Lotka–Volterra predator–prey model with certain degeneracies was examined. As pointed out in [3], the effects of the degeneracy on the Lotka–Volterra model differ considerably from those on the predator–prey model considered in [3] and here.

The rest of the paper is organized as follows. In § 2, we consider the weak-predator case, and theorem 1.1 is proved there. In § 3 we discuss the strong-predator case and give the proof of theorem 1.2 and related results. The small-predator diffusion case is studied in § 4, where theorem 1.3 is proved.

## 2. The weak-predator case: proof of theorem 1.1

In this section, we discuss the behaviour of (1.1) and (1.2) as $\beta$ decreases to 0, all other parameters being positive and fixed. We assume (1.3) throughout this paper. We shall discuss (1.1) first. As will become clear soon, the analysis is rather involved. In contrast, the asymptotic behaviour of (1.2) is easy to understand, and will be considered at the end of this section.

We start with a technical lemma.

LEMMA 2.1. *Assume that* $(\tilde{u}_i, \tilde{v}_i)$ *is a positive solution of (1.1) with* $\beta = \beta_i$. *Let* $\{k_i\}$, $\{\ell_i\}$ *be two sequences of positive numbers satisfying*

$$\frac{\ell_i}{k_i}, \frac{\tilde{u}_i}{k_i}, \frac{\tilde{v}_i}{\ell_i} \leqslant C$$

*for some positive constant* $C$ *and all* $i$. *If*

$$\left(\frac{\tilde{u}_i}{k_i}, \frac{\tilde{v}_i}{\ell_i}\right) \to (\tilde{u}, \tilde{v}) \text{ weakly in } H^1(\Omega) \times H^1(\Omega) \text{ and strongly in } L^2(\Omega) \times L^2(\Omega),$$

*and*

$$\tilde{u} \not\equiv 0 \quad \text{and} \quad \tilde{u}\tilde{v} \equiv 0 \quad \text{in } \Omega,$$

*then it must hold that* $\lambda \geqslant \mu$.

*Proof.* Define

$$u_i = \frac{\tilde{u}_i}{k_i} \quad \text{and} \quad v_i = \frac{\tilde{v}_i}{\ell_i}.$$

We then have

$$\left.\begin{aligned} -\Delta u_i &= \lambda u_i - a(x) k_i u_i^2 - \beta_i \ell_i u_i v_i \leqslant \lambda u_i && \text{in } \Omega, \\ -\Delta v_i &= \mu v_i \left(1 - \frac{\ell_i}{k_i} \frac{v_i}{u_i}\right) && \text{in } \Omega, \\ \partial_\nu u_i &= \partial_\nu v_i = 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.1}$$

Multiplying the first equation of (2.1) by $u_i$ and integrating over $\Omega$, we deduce that

$$\int_\Omega |\nabla u_i|^2 \, dx \leqslant \lambda \int_\Omega u_i^2. \tag{2.2}$$

From the second equation of (2.1) we see that $\mu$ is the first eigenvalue of the problem

$$-\Delta w + \frac{\mu \ell_i v_i}{k_i u_i} w = \mu w \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega,$$

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and $v_i$ is the corresponding eigenfunction. It follows from the variational characterization of the first eigenvalue that

$$\int_\Omega \left[|\nabla\phi|^2 + \frac{\mu\ell_i v_i}{k_i u_i}\phi^2\right] \geqslant \mu\int_\Omega \phi^2, \quad \forall\phi\in H^1(\Omega).$$

Taking $\phi = u_i$, we have

$$\int_\Omega \left[|\nabla u_i|^2 + \frac{\mu\ell_i u_i v_i}{k_i}\right] \geqslant \mu\int_\Omega u_i^2.$$

Therefore, by (2.2) we can deduce that

$$\lambda\int_\Omega u_i^2 + \int_\Omega \frac{\mu\ell_i u_i v_i}{k_i} \geqslant \mu\int_\Omega u_i^2.$$

As $\ell_i/k_i \leqslant C$, and $u_i v_i \to \tilde{u}\tilde{v} \equiv 0$ in $L^1(\Omega)$, and $u_i \to \tilde{u}$ in $L^2(\Omega)$, we have

$$\int_\Omega \frac{\mu\ell_i u_i v_i}{k_i} \to 0 \quad \text{as } i\to\infty,$$

and, hence,

$$\lambda\int_\Omega \tilde{u}^2 \geqslant \mu\int_\Omega \tilde{u}^2. \tag{2.3}$$

Since $\tilde{u}\not\equiv 0$ by assumption, it follows from (2.3) that $\lambda\geqslant\mu$. □

Lemma 2.2. *Let $(u_\beta, v_\beta)$ be a positive solution of (1.1). Then*

$$\lim_{\beta\to 0^+}\|u_\beta\|_\infty = \lim_{\beta\to 0^+}\|v_\beta\|_\infty = \infty.$$

*Proof.* By a simple comparison argument to the differential equation of $v_\beta$ we obtain

$$\|v_\beta\|_\infty \leqslant \|u_\beta\|_\infty. \tag{2.4}$$

We first prove that $\lim_{\beta\to 0^+}\|u_\beta\|_\infty = \infty$. Assume on the contrary that, along a certain sequence of $\beta$ decreasing to 0, $\|u_\beta\|_\infty$ is bounded from above; then, by (2.4), $\|v_\beta\|_\infty$ is also bounded from above. In this case, we claim that $\min_{\bar\Omega} u_\beta \geqslant \delta$ for some positive constant $\delta$ and all $\beta$ in that sequence. In fact, if $\min_{\bar\Omega} u_\beta \to 0$ as $\beta\to 0^+$ along some subsequence, by applying the Harnack inequality to the differential equation of $u_\beta$, it follows that

$$\|u_\beta\|_\infty = \max_{\bar\Omega} u_\beta \leqslant C\min_{\bar\Omega} u_\beta \to 0$$

for some positive constant $C$ that depends only on $\lambda$, $\Omega$ and the bounds of $\|u_\beta\|_\infty$ and $\|a\|_\infty$.

Define

$$\hat{u}_\beta = \frac{u_\beta}{\|u_\beta\|_\infty}.$$

Then $\hat{u}_\beta$ satisfies

$$-\Delta\hat{u}_\beta = \lambda\hat{u}_\beta - a(x)\hat{u}_\beta u_\beta - \beta\hat{u}_\beta v_\beta \quad \text{in } \Omega, \quad \partial_\nu\hat{u}_\beta = 0 \quad \text{on } \partial\Omega.$$

Note that $0 \leqslant \hat{u}_\beta \leqslant 1$ and, by the assumptions, $a(x)u_\beta \to 0$, $\beta v_\beta \to 0$ as $\beta \to 0^+$ (along the sequence). By standard regularity results for elliptic problems (see [7]) we find that $\hat{u}_\beta \to \hat{u}$ in $C^1(\bar{\Omega})$ along a further subsequence for some non-negative function $\hat{u}$, and $\hat{u}$ satisfies

$$-\Delta \hat{u} = \lambda \hat{u} \quad \text{in } \Omega, \quad \partial_\nu \hat{u} = 0 \quad \text{on } \partial\Omega.$$

Since $\lambda > 0$, this implies that $\hat{u} \equiv 0$, which is a contradiction, as $\max_{\bar{\Omega}} \hat{u} = 1$. This proves our claim that $\min_{\bar{\Omega}} u_\beta \geqslant \delta$.

By standard regularity results for elliptic problems, we now find that, along a subsequence, $u_\beta \to u$ in $C^1(\bar{\Omega})$ for some positive function $u$, and $u$ satisfies

$$-\Delta u = \lambda u - a(x)u^2 \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Since $\lambda \geqslant \lambda_1^{\mathrm{D}}(\Omega_0)$, we know that problem (2.5) has no positive solution. This contradiction completes our proof of the fact that

$$\lim_{\beta \to 0^+} \|u_\beta\|_\infty = \infty.$$

Note that $\hat{u}_\beta$ satisfies

$$-\Delta \hat{u}_\beta = \lambda \hat{u}_\beta - a(x)\hat{u}_\beta u_\beta - \beta \hat{u}_\beta v_\beta \leqslant \lambda \hat{u}_\beta \quad \text{in } \Omega, \quad \partial_\nu \hat{u}_\beta = 0 \quad \text{on } \partial\Omega. \tag{2.6}$$

As $0 \leqslant \hat{u}_\beta \leqslant 1$, it follows that

$$\int_\Omega (|\nabla \hat{u}_\beta|^2 + \hat{u}_\beta^2) \leqslant (1+\lambda) \int_\Omega \hat{u}_\beta^2 \leqslant (1+\lambda)|\Omega|.$$

Therefore, given any sequence of $\beta$ decreasing to 0, we can extract a subsequence along which $\hat{u}_\beta$ converges to some $\hat{u} \in H^1(\Omega)$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Since $\|\hat{u}_\beta\|_\infty = 1$, we also have $\hat{u}_\beta \to \hat{u}$ in $L^p(\Omega)$ for all $p > 1$. It is obvious that $0 \leqslant \hat{u} \leqslant 1$. From (2.6) we also have

$$0 \leqslant \hat{u}_\beta \leqslant (1+\lambda)(I - \Delta)^{-1}\hat{u}_\beta.$$

Since $\hat{u}_\beta \to \hat{u}$ in $L^p(\Omega)$ for all $p > 1$, the above inequality implies that $\hat{u}_\beta \to 0$ in $L^\infty(\Omega)$ if $\hat{u} \equiv 0$. Since $\|\hat{u}_\beta\|_\infty = 1$, this is impossible and, hence, $\hat{u} \not\equiv 0$.

For any compact subset $K \subset \Omega \setminus \Omega_0$, from the equation for $\hat{u}_\beta$ we obtain

$$\begin{aligned} \lambda \int_\Omega \hat{u}_\beta &= \int_\Omega a(x)\|u_\beta\|_\infty \hat{u}_\beta^2 + \beta \int_\Omega v_\beta \hat{u}_\beta \\ &\geqslant \|u_\beta\|_\infty \int_K a(x)\hat{u}_\beta^2 \\ &\geqslant a_K \|u_\beta\|_\infty \int_K \hat{u}_\beta^2, \end{aligned}$$

where $a_K = \min_K a(x) > 0$. Since $\|u_\beta\|_\infty \to \infty$, we deduce from the above inequality that $\int_K \hat{u}^2 = \lim \int_K \hat{u}_\beta^2 = 0$. Consequently,

$$\hat{u} \equiv 0 \quad \text{in } \Omega \setminus \Omega_0. \tag{2.7}$$

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For later use, let us note that, since $\partial\Omega_0$ is smooth, (2.7) implies that $\hat{u}|_{\Omega_0} \in H_0^1(\Omega_0)$. We are now ready to show that $\lim_{\beta\to 0^+} \|v_\beta\|_\infty = \infty$. If this is not true, then, along a sequence of $\beta$ converging to 0, we have

$$\int_\Omega (|\nabla v_\beta|^2 + v_\beta^2) \leqslant (1+\mu)\int_\Omega v_\beta^2 \leqslant C$$

for some positive constant $C$ that does not depend on $\beta$. Therefore, subject to a subsequence, $v_\beta$ converges to some $v \in H^1(\Omega)$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Since $\|v_\beta\|_\infty$ is bounded in $\beta$, we also find that $v_\beta \to v$ in $L^p(\Omega)$ for all $p > 1$.

Since $\hat{u}_\beta$ satisfies

$$-\Delta\hat{u}_\beta = \lambda\hat{u}_\beta - \beta v_\beta\hat{u}_\beta \quad \text{in } \Omega_0,$$

in view of $\beta v_\beta \to 0$ in $L^\infty(\Omega)$, and $\hat{u}_\beta \to \hat{u}$ in $L^p(\Omega)$ for all $p > 1$ and (2.7), we conclude that $\hat{u}_\beta \to \hat{u}$ uniformly on any compact subset of $\Omega_0$, and $\hat{u}$ satisfies (in the weak sense)

$$-\Delta\hat{u} = \lambda\hat{u} \quad \text{in } \Omega_0, \quad \hat{u} = 0 \quad \text{on } \partial\Omega_0.$$

Due to (2.7) and $\hat{u} \not\equiv 0$, we obtain $\lambda = \lambda_1^{\mathrm{D}}(\Omega_0)$ and $\hat{u} > 0$ in $\Omega_0$. Hence,

$$u_\beta \to \infty \text{ uniformly on any compact subset of } \Omega_0. \tag{2.8}$$

For any $\varphi \in C_0^\infty(\Omega_0)$, we have

$$\int_{\Omega_0} \nabla v_\beta \nabla\varphi = \mu\int_{\Omega_0} v_\beta\varphi - \mu\int_{\Omega_0} \frac{v_\beta^2}{u_\beta}\varphi.$$

In view of (2.8), it follows that

$$-\Delta v = \mu v \quad \text{in } \Omega_0, \quad v \geqslant 0 \quad \text{on } \partial\Omega_0.$$

Since $\mu > \lambda_1^{\mathrm{D}}(\Omega_0)$, this is possible only if $v \equiv 0$ on $\bar{\Omega}_0$. This fact, combined with (2.7), yields $\hat{u}v \equiv 0$ in $\Omega$. Taking $k_\beta = \|u_\beta\|_\infty$, $\ell_\beta = 1$ in lemma 2.1, we deduce that $\lambda \geqslant \mu$, since $\hat{u} \not\equiv 0$. This is a contradiction to (1.3). Lemma 2.2 is thus proved. □

LEMMA 2.3. *Let $(u_\beta, v_\beta)$ be a positive solution of (1.1). Then, along any sequence of $\beta$ decreasing to 0, there exists a subsequence $\{\beta_n\}$ such that $u_{\beta_n} \to \infty$ uniformly on any compact subset of $\Omega_0$, and $u_{\beta_n} \to u$ in $C^1(\bar{\Omega} \setminus \Omega_0^\delta)$ for any $0 < \delta \ll 1$, where $\Omega_0^\delta = \{x \in \Omega : d(x, \Omega_0) < \delta\}$ and $u$ is a non-negative function satisfying*

$$-\Delta u = \lambda u - a(x)u^2 \quad \text{in } \Omega \setminus \Omega_0, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega.$$

*Proof.* Define

$$\hat{u}_\beta = \frac{u_\beta}{\|u_\beta\|_\infty} \quad \text{and} \quad \hat{v}_\beta = \frac{v_\beta}{\|v_\beta\|_\infty}.$$

Then $-\Delta\hat{u}_\beta \leqslant \lambda\hat{u}_\beta$ and $-\Delta\hat{v}_\beta \leqslant \mu\hat{v}_\beta$. Therefore, similar to the proof of lemma 2.2, there exist two non-negative and non-trivial functions $\hat{u}, \hat{v} \in H^1(\Omega)$ such that, along a subsequence of the given sequence of $\beta$,

$$(\hat{u}_\beta, \hat{v}_\beta) \rightharpoonup (\hat{u}, \hat{v}) \quad \text{in } [H^1(\Omega)]^2, \quad (\hat{u}_\beta, \hat{v}_\beta) \to (\hat{u}, \hat{v}) \quad \text{in } [L^p(\Omega)]^2, \quad \forall p > 1.$$

Moreover, $\hat{v} \not\equiv 0$ in $\Omega_0$. In fact, if $\hat{v} \equiv 0$ in $\Omega_0$, then $\hat{u}\hat{v} \equiv 0$ in $\Omega$ by (2.7). Since $\|v_\beta\|_\infty \leqslant \|u_\beta\|_\infty$ and $\hat{u} \not\equiv 0$ in $\Omega$, taking $k_\beta = \|u_\beta\|_\infty$, $\ell_\beta = \|v_\beta\|_\infty$ in lemma 2.1, we obtain $\lambda \geqslant \mu$. This is a contradiction.

It is obvious that $\hat{u}_\beta$ satisfies

$$-\Delta \hat{u}_\beta = \lambda \hat{u}_\beta - \beta \|v_\beta\|_\infty \hat{u}_\beta \hat{v}_\beta \quad \text{in } \Omega_0. \tag{2.9}$$

We claim that $\beta\|v_\beta\|_\infty$ is bounded. If this is not true, subject to a subsequence, we may assume that $\beta\|v_\beta\|_\infty \to \infty$. For any $\varphi \in C_0^\infty(\Omega_0)$, by (2.9) we have

$$\int_{\Omega_0} \nabla \hat{u}_\beta \nabla \varphi = \lambda \int_{\Omega_0} \hat{u}_\beta \varphi - \beta \|v_\beta\|_\infty \int_{\Omega_0} \hat{u}_\beta \hat{v}_\beta \varphi.$$

It follows that $\int_{\Omega_0} \hat{u}\hat{v}\varphi = 0$. Therefore, $\hat{u}\hat{v} \equiv 0$ in $\Omega_0$. Using (2.7) we obtain $\hat{u}\hat{v} \equiv 0$ in $\Omega$. Similarly, we can apply lemma 2.1 to the above and conclude that this is impossible.

Subject to a subsequence, we may assume that $\beta\|v_\beta\|_\infty \to b \in [0, \infty)$. Then $\hat{u}_\beta$ converges to $\hat{u}$ uniformly on any compact subset of $\Omega_0$ and $(\hat{u}, \hat{v})$ satisfies (in the weak sense)

$$-\Delta \hat{u} = \lambda \hat{u} - b\hat{v}\hat{u} = (\lambda - b\hat{v})\hat{u} \quad \text{in } \Omega_0, \quad \hat{u} = 0 \quad \text{on } \partial\Omega_0. \tag{2.10}$$

Using $\hat{u} \geqslant 0$, $\hat{u} \not\equiv 0$ and (2.7), we conclude by Harnack's inequality that $\hat{u} > 0$ in $\Omega_0$. Therefore, $u_\beta \to \infty$ on any compact subset of $\Omega_0$. As $\hat{v} \not\equiv 0$ in $\Omega_0$, from (2.10), we conclude that $b = 0$ if and only if $\lambda = \lambda_1^{\mathrm{D}}(\Omega_0)$.

By [4], the boundary blow-up problem

$$-\Delta U = \lambda U - a(x)U^2, \quad x \in \Omega \setminus \Omega_0, \quad \partial_\nu U|_{\partial\Omega} = 0, \quad U|_{\partial\Omega_0} = \infty$$

has a minimal positive solution, which we denote by $U_\lambda$. Since $u_\beta$ satisfies

$$\begin{gathered} -\Delta u_\beta = \lambda u_\beta - a(x)u_\beta^2 - \beta u_\beta v_\beta \leqslant \lambda u_\beta - a(x)u_\beta^2, \quad x \in \Omega \setminus \Omega_0, \\ \partial_\nu u_\beta|_{\partial\Omega} = 0, \quad u_\beta|_{\partial\Omega_0} < \infty, \end{gathered}$$

by [4, lemma 2.1] we have

$$u_\beta(x) \leqslant U_\lambda(x) \quad \text{in } \Omega \setminus \Omega_0. \tag{2.11}$$

Therefore,

$$\begin{gathered} -\Delta v_\beta = \mu v_\beta \left(1 - \frac{v_\beta}{u_\beta}\right) \leqslant \mu v_\beta \left(1 - \frac{v_\beta}{U_\lambda}\right), \quad x \in \Omega \setminus \Omega_0, \\ \partial_\nu v_\beta|_{\partial\Omega} = 0, \quad v_\beta|_{\partial\Omega_0} < \infty. \end{gathered}$$

Let $V_\lambda$ be the minimal positive solution of

$$\begin{gathered} -\Delta V = \mu V(1 - U_\lambda^{-1} V), \quad x \in \Omega \setminus \Omega_0, \\ \partial_\nu V|_{\partial\Omega} = 0, \quad V|_{\partial\Omega_0} = \infty. \end{gathered}$$

By [4, lemma 2.1] we have

$$v_\beta(x) \leqslant V_\lambda(x) \quad \text{in } \Omega \setminus \Omega_0. \tag{2.12}$$

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Since $\|v_\beta\|_\infty \to \infty$, (2.12) implies that

$$\hat{v}_\beta \to 0 \text{ uniformly on any compact subset of } \bar{\Omega} \setminus \bar{\Omega}_0, \text{ and, hence, } \hat{v} \equiv 0 \text{ in } \Omega \setminus \Omega_0. \tag{2.13}$$

By (2.11), (2.12) and the standard regularity result for elliptic equations, we see that, for any sequence of $\beta$ decreasing to 0, there exists a subsequence $\{\beta_n\}$ such that, for any $0 < \delta \ll 1$, $u_{\beta_n} \to u$ in $C^1(\bar{\Omega} \setminus \Omega_0^\delta)$ for some non-negative function $u$, and $u$ satisfies

$$-\Delta u = \lambda u - a(x)u^2 \quad \text{in } \Omega \setminus \Omega_0, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega.$$

This completes the proof. □

We shall show that the limit function $u$ in lemma 2.3 satisfies

$$u > 0 \quad \text{in } \Omega \setminus \Omega_0, \quad u = \infty \quad \text{on } \partial\Omega_0.$$

As a consequence, $u$ is a positive solution of the following problem:

$$\left.\begin{aligned} -\Delta u = \lambda u - a(x)u^2, \quad x \in \Omega \setminus \bar{\Omega}_0, \\ \partial_\nu u|_{\partial\Omega} = 0, \quad u|_{\partial\Omega_0} = \infty. \end{aligned}\right\} \tag{2.14}$$

We now need another technical lemma.

LEMMA 2.4. *Let $\{h_i(x)\}$ and $\{a_i(x)\}$ be two sequences of continuous functions satisfying*

(i) $\|h_i\|_\infty, \|a_i\|_\infty \leqslant C$ *for some positive constant* $C$ *and all* $i$,

(ii) $a_i(x) > 0$ *on* $\bar{\Omega} \setminus \bar{\Omega}_0$ *for all* $i$, *and* $a_i(x) \to 0$ *uniformly on* $\bar{\Omega}_0$.

*Assume that $u_i$ is a positive solution of the problem*

$$-\Delta u_i = h_i(x)u_i - a_i(x)u_i^2 \quad \text{in } \Omega, \quad \partial_\nu u_i = 0 \quad \text{on } \partial\Omega.$$

*If*

$$u_i \to \infty \text{ uniformly on any compact subset of } \Omega_0, \tag{2.15}$$

*then*

$$u_i \to \infty \text{ uniformly on } \bar{\Omega}_0 \text{ as } i \to \infty.$$

*Proof.* We adapt some of the techniques given in [4,5]. As $\Omega_0$ is smooth, there exists $R > 0$ such that, for any $x \in \partial\Omega_0$, there is an interior tangent ball of $\Omega_0$ at $x$ with radius $R$, i.e. $B_x(y; R) \subset \Omega_0$ and $\partial B_x(y; R) \cap \partial\Omega_0 = \{x\}$, where $y \in \Omega_0$ and $R > 0$ are the centre and radius of $B_x(y; R)$, respectively.

Define $u_i(x_i) = \min_{\bar{\Omega}_0} u_i$. To prove our result, it is sufficient to prove that $u_i(x_i) \to \infty$ as $i \to \infty$. We assume on the contrary that there exists a subsequence of $\{u_i(x_i)\}$, still with the same notation, such that $u_i(x_i) \leqslant C$ for some positive constant $C$. Using (2.15), it is easy to see that, subject to a subsequence, either $x_i \in \partial\Omega_0$ or $x_i \to x_0 \in \partial\Omega_0$ as $i \to \infty$. Choose $\Omega_i \subset \Omega_0$ as follows:

$$\Omega_i = \begin{cases} \Omega_0 & \text{if } x_i \in \partial\Omega_0, \\ \{x \in \Omega_0 : d(x, \partial\Omega_0) > d(x_i, \partial\Omega_0)\} & \text{if } x_i \in \Omega_0. \end{cases}$$

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Then $\Omega_i \subset \Omega_0$, and $\Omega_i$ has the same smoothness as that of $\Omega_0$ and, for all large $i$, there is an interior tangent ball of $\Omega_i$ at $x_i$ with radius $R$, which we denote by $B(y_i; R)$.

Claim. *There exist $\sigma > 0$ and $c_i \to \infty$ such that*

$$u_i(x) \geqslant u_i(x_i) + c_i\psi_i(x) \quad \text{for } \tfrac{1}{2}R \leqslant |x - y_i| \leqslant R, \tag{2.16}$$

*where* $\psi_i(x) = \mathrm{e}^{-\sigma|x-y_i|^2} - \mathrm{e}^{-\sigma R^2}$.

In fact, as $h_i(x)$ is uniformly bounded, we may choose a large constant $\sigma > 2NR^{-2}$ such that $\sigma^2R^2 - 2N\sigma - |h_i(x)| > 0$ on $\bar{\Omega}$ for all $i$. When $\frac{1}{2}R \leqslant |x-y_i| \leqslant R$, a direct calculation yields

$$\begin{aligned} \Delta\psi_i + h_i(x)\psi_i &= (4\sigma^2|x-y_i|^2 - 2N\sigma + h_i(x))\mathrm{e}^{-\sigma|x-y_i|^2} - h_i(x)\mathrm{e}^{-\sigma R^2} \\ &\geqslant (\sigma^2R^2 - 2N\sigma + h_i(x))\mathrm{e}^{-\sigma|x-y_i|^2} - h_i(x)\mathrm{e}^{-\sigma R^2} \\ &> (\sigma^2R^2 - 2N\sigma)\mathrm{e}^{-\sigma R^2} \\ &> 0. \end{aligned} \tag{2.17}$$

Choose a compact subset

$$K \subset \Omega_0 \quad \text{such that} \bigcup_i B(y_i; \tfrac{1}{2}R) \subset K.$$

In view of (2.15), there exist $c_i^* \to \infty$ such that $u_i(x) > u_i(x_i) + c_i^*(\mathrm{e}^{-\sigma R^2/4} - \mathrm{e}^{-\sigma R^2})$ for all $x \in B(y_i; \frac{1}{2}R) \subset K$. Define $\varepsilon_i = \max_{\bar{\Omega}_0} a_i(x)$. By hypothesis (ii), $\varepsilon_i \to 0$ as $i \to \infty$. Let $c_i = \min\{c_i^*, \varepsilon_i^{-1/2}\}$; then $c_i \to \infty$ and

$$u_i(x) > u_i(x_i) + c_i(\mathrm{e}^{-\sigma R^2/4} - \mathrm{e}^{-\sigma R^2}), \quad \forall x \in B(y_i; \tfrac{1}{2}R) \subset K. \tag{2.18}$$

We now consider the following problem:

$$\left.\begin{aligned} -\Delta w = h_i(x)w - a_i(x)w^2, \quad x \in B_{x_i}(y_i; R) \setminus \bar{B}(y_i; \tfrac{1}{2}R), \\ w|_{\partial B_{x_i}(y_i;R)} = u_i(x_i), \qquad w|_{\partial B(y_i;R/2)} = u_i(x_i) + c_i(\mathrm{e}^{-\sigma R^2/4} - \mathrm{e}^{-\sigma R^2}). \end{aligned}\right\} \tag{2.19}$$

In view of (2.18), $u_i$ is a super-solution of (2.19). Define

$$w_i(x) = u_i(x_i) + c_i\psi_i(x).$$

Then $w_i$ satisfies the boundary conditions of (2.19). Since $\|h_i\|_\infty, u_i(x_i) \leqslant C$, $0 < \psi_i(x) \leqslant 1$, and $a_i(x) \leqslant \varepsilon_i$, by (2.17) we find that, for $x \in B_{x_i}(y_i; R) \setminus \bar{B}(y_i; \frac{1}{2}R)$,

$$\begin{aligned} \Delta w_i + h_i(x)w_i - a_i(x)w_i^2 &\geqslant c_i(\sigma^2R^2 - 2N\sigma)\mathrm{e}^{-\sigma R^2} + h_i(x)u_i(x_i) - a_i(x)w_i^2 \\ &\geqslant c_i(\sigma^2R^2 - 2N\sigma)\mathrm{e}^{-\sigma R^2} - C^2 - 2\varepsilon_i(C^2 + c_i^2) \\ &\geqslant c_i(\sigma^2R^2 - 2N\sigma)\mathrm{e}^{-\sigma R^2} - (1 + 2\varepsilon_i)C^2 - 2 \\ &> 0 \quad \text{for } i \gg 1 \text{ (since } c_i \to \infty). \end{aligned}$$

Therefore, for large $i$, $w_i$ is a sub-solution of (2.19). By [6, lemma 2.1], this implies that $u_i \geqslant w_i$ in $\bar{B}_{x_i}(y_i; R) \setminus B(y_i; \frac{1}{2}R)$, i.e. (2.16) holds. This proves our claim.

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Choose a positive constant $k$ such that $\|a_i\|_\infty \leqslant k$ for all $i$ and consider the following problem:

$$\left.\begin{aligned} -\Delta u = h_i(x)u - ku^2, \quad x \in \Omega \setminus \bar{\Omega}_i, \\ \partial_\nu u|_{\partial\Omega} = 0, \qquad u|_{\partial\Omega_i} = u_i(x_i). \end{aligned}\right\} \tag{2.20}$$

Since $u_i(x_i) = \min_{\bar{\Omega}_i} u_i$, 0 and $u_i$ are sub- and super-solutions of (2.20), respectively. So, (2.20) has a non-negative solution $\tilde{u}_i$. In fact, $\tilde{u}_i$ is positive by the strong maximum principle. By [6, lemma 2.1] we may deduce that

$$u_i(x) \geqslant \tilde{u}_i(x) \quad \text{in } \bar{\Omega} \setminus \Omega_i.$$

We consider next a further auxiliary problem:

$$\left.\begin{aligned} -\Delta u = Cu - ku^2, \quad x \in \Omega \setminus \bar{\Omega}_i, \\ \partial_\nu u|_{\partial\Omega} = 0, \qquad u|_{\partial\Omega_i} = C, \end{aligned}\right\} \tag{2.21}$$

where $\|h_i\|_\infty, u_i(x_i) \leqslant C$. By [4, lemma 2.3], problem (2.21) has a unique positive solution $u_i^*$. It is obvious that $u_i^*$ is uniformly bounded on $\bar{\Omega} \setminus \Omega_i$ with respect to $i$. Since $\|h_i\|_\infty, u_i(x_i) \leqslant C$, $u_i^*$ is a super-solution of (2.20). Therefore, $\tilde{u}_i \leqslant u_i^*$ on $\bar{\Omega} \setminus \Omega_i$ by [6, lemma 2.1]. In particular, $\|\tilde{u}_i\|_{L^\infty(\Omega\setminus\Omega_i)}$ is bounded in $i$. Then the $L^p$-estimates and the Sobolev imbedding theorems imply that $\|\tilde{u}_i\|_{C^1(\bar{\Omega}\setminus\Omega_i)}$ depends only on the structure of $\Omega_i$ and the bounds of $\|h_i\|_\infty$, $\|\tilde{u}_i\|_{L^\infty(\Omega\setminus\Omega_i)}$, $u_i(x_i)$ and $|\Omega \setminus \Omega_i|$. Because

$$\|h_i\|_\infty, \|\tilde{u}_i\|_{L^\infty(\Omega\setminus\Omega_i)}, u_i(x_i) \leqslant C,$$

$|\Omega \setminus \Omega_0| \leqslant |\Omega \setminus \Omega_i| \leqslant 2|\Omega \setminus \Omega_0|$, $\Omega_i \to \Omega_0$, and $\Omega_i$ has the same smoothness as that of $\Omega_0$, it follows that

$$\|\tilde{u}_i\|_{C^1(\bar{\Omega}\setminus\Omega_i)} \leqslant C$$

for some positive constant $C$ and all $i$. In particular, $|\nabla \tilde{u}_i(x_i)| \leqslant C$ for all $i$. Since

$$u_i(x) \geqslant \tilde{u}_i(x) \quad \text{in } \bar{\Omega} \setminus \Omega_i, \qquad u_i(x_i) = \tilde{u}_i(x_i),$$

we have

$$\frac{\partial u_i(x_i)}{\partial \eta_i} \leqslant \frac{\partial \tilde{u}_i(x_i)}{\partial \eta_i} \leqslant C,$$

where $\eta_i = (y_i - x_i)/|y_i - x_i|$.

On the other hand, as $u_i(x) \geqslant w_i(x) \equiv u_i(x_i) + c_i\psi_i(x)$ in $\bar{B}_{x_i}(y_i; R) \setminus B(y_i; \frac{1}{2}R)$ and $u_i(x_i) = w_i(x_i)$, we have

$$\frac{\partial u_i(x_i)}{\partial \eta_i} \geqslant \frac{\partial w_i(x_i)}{\partial \eta_i} = c_i \frac{\partial \psi(x_i)}{\partial \eta_i} = c_i[2\sigma R e^{-\sigma R^2}] \to \infty,$$

since $c_i \to \infty$. This contradiction finishes the proof. □

THEOREM 2.5. *Let $(u, v) \triangleq (u_\beta, v_\beta)$ be a positive solution of (1.1). We then draw the following conclusions:*

(a) $\lim_{\beta\to 0^+}(u_\beta(x), v_\beta(x)) = (\infty, \infty)$ *uniformly on* $\bar{\Omega}_0$*;*

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(b) *along any sequence of* $\beta$ *decreasing to* 0*, there is a subsequence* $\{\beta_n\}$ *such that*

$$\lim_{n\to\infty}(u_{\beta_n}, v_{\beta_n}) = (u, v) \text{ uniformly on any compact subset of } \bar{\Omega}\setminus\bar{\Omega}_0,$$

*where* $(u, v)$ *is a positive solution of the system*

$$\left.\begin{aligned} -\Delta u &= \lambda u - a(x)u^2 \qquad \text{in } \Omega\setminus\bar{\Omega}_0,\\ -\Delta v &= \mu v\left(1-\frac{v}{u}\right) \qquad \text{in } \Omega\setminus\bar{\Omega}_0,\\ \partial_\nu u|_{\partial\Omega} = \partial_\nu v|_{\partial\Omega} &= 0, \qquad u|_{\partial\Omega_0} = v|_{\partial\Omega_0} = \infty; \end{aligned}\right\} \tag{2.22}$$

(c) *if there exists* $\xi > 0$ *such that* $a(x)d(x, \Omega_0)^{-\xi}$ *is bounded for all* $x \in \Omega\setminus\bar{\Omega}_0$ *close to* $\partial\Omega_0$*, then (2.22) has a unique positive solution* $(u, v)$ *and the convergence in (b) holds for* $\beta \to 0^+$.

*Proof.* Note that $u_\beta$ satisfies

$$-\Delta u_\beta = [\lambda - \beta\|v_\beta\|_\infty \hat{v}_\beta(x)]u_\beta - a(x)u_\beta^2 \quad \text{in } \Omega, \qquad \partial_\nu u_\beta = 0 \quad \text{on } \partial\Omega.$$

Define $h_\beta(x) = \lambda - \beta\|v_\beta\|_\infty \hat{v}_\beta(x)$. Along any sequence of $\beta$ decreasing to 0, from the proof of lemma 2.3, we already know that $\beta\|v_\beta\|_\infty$ is bounded. We may assume that $\beta\|v_\beta\|_\infty \to b \in [0,\infty)$. Therefore, $\|h_\beta\|_\infty$ is bounded in $\beta$. For any sequence $\beta_i \to 0^+$, taking $h_i(x) = h_{\beta_i}(x)$, $a_i(x) = a(x)$ and $u_i(x) = u_{\beta_i}(x)$ in lemma 2.4, and making use of lemma 2.3, we obtain $u_i \to \infty$ uniformly on $\bar{\Omega}_0$ as $i \to \infty$. Consequently,

$$u_\beta \to \infty \text{ uniformly on } \bar{\Omega}_0 \text{ as } \beta \to 0^+. \tag{2.23}$$

For fixed large positive constant $M$, consider the problem

$$\left.\begin{aligned} -\Delta w = h_\beta(x)w - \|a\|_\infty w^2, \quad x \in \Omega\setminus\bar{\Omega}_0,\\ \partial_\nu w|_{\partial\Omega} = 0, \qquad w|_{\partial\Omega_0} = M. \end{aligned}\right\} \tag{2.24}$$

Analogously to the discussion to problem (2.20) we find that (2.23) has a unique positive solution $w_M(x)$. By continuity we see that there exists $\varepsilon = \varepsilon(M,\beta) > 0$ such that $w_M(x) \geqslant \frac{1}{2}M$ for all $x \in \{x \in \Omega\setminus\Omega_0 : d(x,\partial\Omega_0) < \varepsilon\}$. By (2.23) and [6, lemma 2.1] we deduce that $u_\beta \geqslant w_M$ on $\bar{\Omega}\setminus\Omega_0$, provided that $\beta \ll 1$.

Let $u$ be the limit of $u_\beta$ in lemma 2.3. Then $u = \lim_{\beta_n\to 0^+} u_{\beta_n} \geqslant w_M > 0$ in $\bar{\Omega}\setminus\Omega_0$ and $u(x) \geqslant \frac{1}{2}M$ for all $x \in \{x \in \Omega\setminus\Omega_0 : d(x,\partial\Omega_0) < \varepsilon\}$. As $M$ is arbitrary, it follows that $u|_{\partial\Omega_0} = \infty$ and, hence, $u$ is a positive solution of (2.14).

We now consider $v_{\beta_n}$, where $\{\beta_n\}$ is any sequence decreasing to 0. A simple sub- and super-solution argument shows that the problem

$$-\Delta v = \mu v\left(1 - \frac{v}{\min_{\bar{\Omega}_0} u_{\beta_n}}\right) \quad \text{in } \Omega_0, \quad v = 0 \quad \text{on } \partial\Omega_0$$

has a unique positive solution $v_n$ and, due to (2.23), it can easily be seen that $v_n \to \infty$ uniformly on any compact subset of $\Omega_0$. By [6, lemma 2.1], we can easily see that $v_{\beta_n} \geqslant v_n$ in $\Omega_0$. We can now apply lemma 2.4 to the equation for $v_{\beta_n}$ with $h_n(x) = \mu$, $a_n(x) = \mu u_{\beta_n}^{-1}(x)$, to conclude that $v_{\beta_n} \to \infty$ uniformly in $\bar{\Omega}_0$. Hence, $v_\beta \to \infty$ uniformly in $\bar{\Omega}_0$ as $\beta \to 0^+$.

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Suppose now that $\{\beta_n\}$ is the sequence along which $u_\beta$ converges to $u$. Since $u_{\beta_n}(x) \to u(x) > 0$ and $v_{\beta_n}(x) \leqslant V_\lambda(x)$ in $\Omega \setminus \bar{\Omega}_0$, analogously to the discussion to $u_\beta$, we can deduce that, subject to a subsequence, $v_{\beta_n} \to v$ in $C^1_{\mathrm{loc}}(\bar{\Omega} \setminus \bar{\Omega}_0)$ and $v$ is a positive solution of the following problem

$$\left.\begin{aligned} -\Delta v = \lambda v\left(1 - \frac{v}{u}\right) \quad \text{in } \Omega \setminus \bar{\Omega}_0,\\ \partial_\nu v|_{\partial\Omega} = 0, \qquad v|_{\partial\Omega_0} = \infty. \end{aligned}\right\} \tag{2.25}$$

This proves conclusions (a) and (b) in the theorem.

Conclusion (c) follows from [2]. Indeed, under the extra condition on $a(x)$, by [2, theorem 3.2], (2.14) has a unique positive solution. By [2, lemma 2.2], the unique positive solution $u(x)$ of (2.14) satisfies

$$C_1 d(x, \Omega_0)^{-\gamma} \leqslant u(x) \leqslant C_2 d(x, \Omega_0)^{-\gamma}, \quad x \in \Omega \setminus \bar{\Omega}_0,$$

for some positive constants $\gamma$, $C_1$ and $C_2$. This in turn implies that $\tilde{a}(x) = \mu u(x)^{-1}$ satisfies a condition similar to that for $a(x)$ near $\partial\Omega_0$ and, hence, we can apply [2, theorem 3.2] to (2.25) to conclude that it has a unique positive solution $v$. Therefore, (2.22) has a unique positive solution. This implies that the convergence of $(u_\beta, v_\beta)$ in conclusion (b) holds for $\beta \to 0^+$. □

THEOREM 2.6. *If $(u, v) \triangleq (u_\beta, v_\beta)$ is a positive solution of (1.2), then*

$$\lim_{\beta \to 0^+} (u_\beta(x), v_\beta(x)) = (U_\varepsilon, V_\varepsilon)$$

*uniformly over $\bar{\Omega}$, where $U_\varepsilon$ and $V_\varepsilon$ are the unique positive solutions to*

$$-\Delta U = \lambda U - [a(x) + \varepsilon]U^2, \qquad \partial_\nu U|_{\partial\Omega} = 0,$$

*and*

$$-\Delta V = \mu V\left(1 - \frac{V}{U_\varepsilon}\right), \qquad \partial_\nu V|_{\partial\Omega} = 0,$$

*respectively.*

*Proof.* The proof of this theorem is easy. Let $\{\beta_n\}$ be an arbitrary sequence decreasing to 0, and denote $(u_n, v_n) = (u_{\beta_n}, v_{\beta_n})$. A simple sub- and super-solution argument shows that

$$u_n \leqslant \frac{\varepsilon}{\lambda}, \qquad v_n \leqslant \frac{\varepsilon}{\lambda}.$$

Using these estimates and a sub- and super-solution argument again, we deduce

$$u_n, v_n \geqslant \frac{[\lambda - \beta(\varepsilon/\lambda)]}{\|a\|_\infty + \varepsilon}.$$

We can now apply standard regularity theory for elliptic equations to conclude that, subject to a subsequence, $(u_n, v_n)$ converges to some $(U, V)$ in $[C^1(\bar{\Omega})]^2$ and, by our estimates for $u_n$ and $v_n$, $U$ and $V$ are positive solutions to

$$-\Delta U = \lambda U - [a(x) + \varepsilon]U^2, \quad \partial_\nu U|_{\partial\Omega} = 0,$$

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and

$$-\Delta V = \mu V\left(1 - \frac{V}{U}\right), \qquad \partial_\nu V|_{\partial\Omega} = 0,$$

respectively. It is well known that the equation for $U$ (and, hence, that for $V$) has a unique positive solution. Therefore, $(u_\beta, v_\beta) \to (U, V)$ as $\beta \to 0^+$. □

We now consider the asymptotic behaviour of

$$\hat{u}_\beta = \frac{u_\beta}{\|u_\beta\|_\infty} \quad \text{and} \quad \hat{v}_\beta = \frac{v_\beta}{\|v_\beta\|_\infty}.$$

From the proof of lemma 2.3, we already know that, for any given sequence of $\beta$ decreasing to 0, there exist two non-negative and non-trivial functions $\hat{u}, \hat{v} \in H^1(\Omega)$ such that, along a subsequence of the given sequence of $\beta$,

$$(\hat{u}_\beta, \hat{v}_\beta) \rightharpoonup (\hat{u}, \hat{v}) \quad \text{in } [H^1(\Omega)]^2, \qquad (\hat{u}_\beta, \hat{v}_\beta) \to (\hat{u}, \hat{v}) \quad \text{in } [L^p(\Omega)]^2,\ \forall p > 1.$$

Moreover, $\hat{v}$ satisfies (2.13), $\hat{v} \not\equiv 0$ in $\Omega_0$, $\hat{u} \equiv 0$ on $\Omega \setminus \Omega_0$ and $\hat{u}$ is a positive solution to (2.10).

As $\|v_\beta\|_\infty \leqslant \|u_\beta\|_\infty$, subject to a subsequence, we may assume that

$$\frac{\|v_\beta\|_\infty}{\|u_\beta\|_\infty} \to \xi \in [0, 1].$$

By the interior estimate for elliptic problems and the Sobolev imbedding theorem we find that $\hat{u}_\beta \to \hat{u}$ in $C^1_{\text{loc}}(\Omega_0)$. Since $\hat{u} > 0$ in $\Omega_0$ and $\hat{v}_\beta \to \hat{v}$ in $L^p(\Omega)$, it follows from

$$-\Delta \hat{v}_\beta = \lambda \hat{v}_\beta\left(1 - \frac{\|v_\beta\|_\infty}{\|u_\beta\|_\infty}\frac{\hat{v}_\beta}{\hat{u}_\beta}\right) \quad \text{in } \Omega_0$$

and from (2.13) that $\hat{v}$ satisfies

$$-\Delta \hat{v} = \mu \hat{v}\left(1 - \xi\frac{\hat{v}}{\hat{u}}\right) \quad \text{in } \Omega_0, \quad \hat{v} = 0 \quad \text{on } \partial\Omega_0. \tag{2.26}$$

Since $\hat{v} \not\equiv 0$, by the Harnack inequality we see from (2.24) that $\hat{v} > 0$ in $\Omega_0$. As $\mu > \lambda_1^{\mathrm{D}}(\Omega_0)$, we must have $\xi > 0$.

Therefore, we have the following result.

THEOREM 2.7. *Let $(u, v) \triangleq (u_\beta, v_\beta)$ be a positive solution of (1.1). Then, along any sequence of $\beta$ decreasing to 0, there is a subsequence $\{\beta_n\}$ along which*

$$\left(\frac{u_\beta}{\|u_\beta\|_\infty}, \frac{v_\beta}{\|v_\beta\|_\infty}\right) \to (\hat{u}, \hat{v})$$

*weakly in $[H^1(\Omega)]^2$ and strongly in $[L^p(\Omega)]^2$ for any $p > 1$, where $\hat{u}$ and $\hat{v}$ are positive in $\Omega_0$ and $\hat{u} = \hat{v} \equiv 0$ in $\bar{\Omega} \setminus \Omega_0$. Moreover, $(\hat{u}, \hat{v})$ satisfies*

$$\left.\begin{aligned} -\Delta \hat{u} &= \lambda \hat{u} - b\hat{u}\hat{v} && \text{in } \Omega_0, \\ -\Delta \hat{v} &= \mu \hat{v}\left(1 - \xi\frac{\hat{v}}{\hat{u}}\right) && \text{in } \Omega_0, \\ \hat{u} &= \hat{v} = 0 && \text{on } \partial\Omega_0, \end{aligned}\right\} \tag{2.27}$$

*for some constants $b \geqslant 0$, $\xi \in (0, 1]$; $b = 0$ if and only if $\lambda = \lambda_1^{\mathrm{D}}(\Omega_0)$.*

REMARK 2.8. The constants $b$ and $\xi$ in (2.27) can be uniquely determined when the spatial dimension $N = 1$. In this case, it is easy to adapt the arguments in [8] to show that (2.27) has a unique positive solution. Using this fact and $\|\hat{u}\|_\infty = \|\hat{v}\|_\infty = 1$, one can uniquely determine $b$ and $\xi$. We omit the details for the sake of brevity.

## 3. The strong-predator case: proof of theorem 1.2

Since our proofs for (1.1) and (1.2) are the same, we combine the two problems into one by agreeing that $\varepsilon = 0$ is possible in (1.2). Therefore, we assume $\varepsilon \geqslant 0$ in this section.

Let $(u_\beta, v_\beta)$ be a positive solution to (1.2) with $\varepsilon \geqslant 0$. We need to show that

$$\lim_{\beta\to\infty} (\|u_\beta\|_\infty, \|v_\beta\|_\infty) = (0, 0).$$

First, a simple comparison argument gives $\|v_\beta\|_\infty \leqslant \|u_\beta\|_\infty$ for all $\beta > 0$. Define

$$\hat{u}_\beta = \frac{u_\beta}{\|u_\beta\|_\infty}, \qquad \hat{v}_\beta = \frac{v_\beta}{\|v_\beta\|_\infty};$$

then, from the inequalities $-\Delta\hat{u}_\beta \leqslant \lambda\hat{u}_\beta$ and $-\Delta\hat{v}_\beta \leqslant \mu\hat{v}_\beta$, we deduce, by arguments similar to those following (2.6), that, given any sequence of $\beta$ increasing to $\infty$, there exist two non-negative and non-trivial functions $\hat{u}, \hat{v} \in H^1(\Omega)$ such that, along a subsequence,

$$(\hat{u}_\beta, \hat{v}_\beta) \rightharpoonup (\hat{u}, \hat{v}) \quad \text{in } [H^1(\Omega)]^2, \qquad (\hat{u}_\beta, \hat{v}_\beta) \to (\hat{u}, \hat{v}) \quad \text{in } [L^p(\Omega)]^2 \quad \forall p > 1.$$

From the inequality

$$\lambda|\Omega| \geqslant \lambda \int_\Omega \hat{u}_\beta = \int_\Omega [a(x) + \varepsilon]\hat{u}_\beta u_\beta + \beta\|v_\beta\|_\infty \int_\Omega \hat{u}_\beta\hat{v}_\beta \geqslant \beta\|v_\beta\|_\infty \int_\Omega \hat{u}_\beta\hat{v}_\beta,$$

we deduce that if $\beta\|v_\beta\|_\infty$ is unbounded in $\beta$, then, subject to a subsequence,

$$\int_\Omega \hat{u}\hat{v} = \lim_{\beta\to\infty} \int_\Omega \hat{u}_\beta\hat{v}_\beta = 0.$$

Hence, $\hat{u}\hat{v} \equiv 0$ on $\Omega$ and, by lemma 2.1, we arrive at a contradiction: $\lambda \geqslant \mu$. Therefore, $\beta\|v_\beta\|_\infty$ is bounded in $\beta$ and, consequently,

$$\|v_\beta\|_\infty \to 0 \quad \text{as } \beta \to \infty.$$

We claim that $\|u_\beta\|_\infty$ is bounded in $\beta$. If this assertion is not true, then, by passing to a subsequence, $\|u_\beta\|_\infty \to \infty$. Analogously to the proof of lemma 2.2, $\hat{u}$ satisfies (2.7). Since $\beta\|v_\beta\|_\infty$ is bounded in $\beta$, we may assume that, subject to a subsequence, $\beta\|v_\beta\|_\infty \to b \in [0, \infty)$. Then $\hat{u}_\beta$ converges to $\hat{u}$ uniformly on any compact subset of $\Omega_0$ and $(\hat{u}, \hat{v})$ satisfies

$$-\Delta\hat{u} = \lambda\hat{u} - b\hat{u}\hat{v} \quad \text{in } \Omega_0, \quad \hat{u} = 0 \quad \text{on } \partial\Omega_0.$$

Since $\hat{u} \equiv 0$ in $\Omega \setminus \Omega_0$, and $\hat{u} \not\equiv 0$ in $\Omega$, the Harnack inequality implies that $\hat{u} > 0$ in $\Omega_0$. Note that

$$-\Delta\hat{v}_\beta = \mu\hat{v}_\beta\left(1 - \frac{v_\beta}{\|u_\beta\|_\infty\hat{u}_\beta}\right) \quad \text{in } \Omega_0;$$

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it follows from $\|u_\beta\|_\infty \to \infty$, $v_\beta \to 0$ and $\hat{u}_\beta \to \hat{u} > 0$ in $\Omega_0$ that

$$-\Delta \hat{v} = \mu \hat{v} \quad \text{in } \Omega_0, \quad \hat{v} \geqslant 0 \quad \text{on } \partial\Omega_0.$$

As $\mu > \lambda_1^{\mathrm{D}}(\Omega_0)$, it must hold that $\hat{v} \equiv 0$ in $\Omega_0$. This fact, combined with (2.7), yields $\hat{u}\hat{v} \equiv 0$ in $\Omega$. By lemma 2.1 we again deduce a contradiction.

We now prove $\lim_{\beta\to\infty} \|u_\beta\|_\infty = 0$. We assume on the contrary that, by passing to a subsequence, $\|u_\beta\|_\infty \geqslant \delta > 0$. Since $\|u_\beta\|_\infty$ and $\beta\|v_\beta\|_\infty$ are bounded in $\beta$, from similar considerations as above, there exist two non-negative and non-trivial functions $u, \hat{v} \in H^1(\Omega)$ such that, by passing to a subsequence,

$$u_\beta \to u \quad \text{in } C^1(\bar{\Omega}), \quad \hat{v}_\beta \rightharpoonup \hat{v} \quad \text{in } H^1(\Omega), \quad \hat{v}_\beta \to \hat{v} \quad \text{in } L^p(\Omega)\ \forall p > 1,$$

and $\|u\|_\infty \geqslant \delta$. Moreover, if $\beta\|v_\beta\|_\infty \to b$, then it can easily be seen that $(u, \hat{v})$ satisfies

$$-\Delta u = \lambda u - [a(x) + \varepsilon]u^2 - b\hat{v}(x)u \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega.$$

The Harnack inequality gives $u > 0$ on $\bar{\Omega}$. In view of $\|v_\beta\|_\infty \to 0$ and $u_\beta \to u > 0$ on $\bar{\Omega}$, it follows from

$$-\Delta \hat{v}_\beta = \mu \hat{v}_\beta \left(1 - \frac{\|v_\beta\|_\infty \hat{v}_\beta}{u_\beta}\right) \quad \text{in } \Omega, \qquad \partial_\nu \hat{v}_\beta = 0 \quad \text{on } \partial\Omega,$$

that $\hat{v}$ satisfies

$$-\Delta \hat{v} = \mu \hat{v} \quad \text{in } \Omega, \quad \partial_\nu \hat{v} = 0 \quad \text{on } \partial\Omega.$$

This is impossible since $\mu > 0$ and $\hat{v} \geqslant 0$, $\hat{v} \not\equiv 0$. Hence, $\lim_{\beta\to\infty} \|u_\beta\|_\infty = 0$. This completes the proof of theorem 1.2.

Remark 3.1. As before, define

$$\hat{u}_\beta = \frac{u_\beta}{\|u_\beta\|_\infty}, \qquad \hat{v}_\beta = \frac{v_\beta}{\|v_\beta\|_\infty};$$

then along any sequence of $\beta$ increasing to $\infty$, there is a subsequence along which

$$(\hat{u}_\beta, \hat{v}_\beta) \rightharpoonup (\hat{u}, \hat{v}) \quad \text{in } [H^1(\Omega)]^2, \qquad (\hat{u}_\beta, \hat{v}_\beta) \to (\hat{u}, \hat{v}) \quad \text{in } [L^p(\Omega)]^2\ \forall p > 1,$$

and $(\hat{u}, \hat{v})$ is a positive solution of the system

$$\left.\begin{aligned} -\Delta \hat{u} &= \lambda \hat{u} - b\hat{u}\hat{v} && \text{in } \Omega, \\ -\Delta \hat{v} &= \mu \hat{v}\left(1 - \xi\frac{\hat{v}}{\hat{u}}\right) && \text{in } \Omega, \\ \partial_\nu \hat{u} &= \partial_\nu \hat{v} = 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where $\xi \in (0, 1]$, $b > 0$.

Remark 3.2. The positive constants $b$ and $\xi$ in (3.1) can be uniquely determined when the spatial dimension $N = 1$. In this case a simple variation of the arguments in [8] shows that (3.1) has a unique positive solution, which is necessarily the constant solution

$$(\hat{u}, \hat{v}) = \left(\xi\frac{\lambda}{b}, \frac{\lambda}{b}\right).$$

Since $\|\hat{u}\|_\infty = \|\hat{v}\|_\infty = 1$, we must have $(\hat{u}, \hat{v}) = (1, 1)$ and, hence, $b = \lambda$, $\xi = 1$.

## 4. The small-predator diffusion case: proof of theorem 1.3

As in § 3, we again consider (1.1) and (1.2) simultaneously by regarding $\varepsilon$ in (1.2) as only non-negative. Suppose that $(u_\mu, v_\mu)$ is a positive solution of (1.2) with $\varepsilon \geqslant 0$ fixed.

As before, by a simple comparison argument we have

$$\|v_\mu\|_\infty \leqslant \|u_\mu\|_\infty, \quad \forall \mu. \tag{4.1}$$

STEP 1 (there exists $0 < C < \infty$ such that $\|u_\mu\|_\infty, \|v_\mu\|_\infty \leqslant C$ for all $\mu \geqslant \mu_0$). We may assume that $\varepsilon = 0$; the conclusion for $\varepsilon > 0$ follows from a standard comparison argument. By (4.1), we need only to prove $\|u_\mu\|_\infty \leqslant C$ for all $\mu \geqslant \mu_0$. Assume on the contrary that, by passing to a subsequence, $\|u_\mu\|_\infty \to \infty$ as $\mu \to \infty$. Set

$$\tilde{u}_\mu = \frac{u_\mu}{\|u_\mu\|_\infty}, \qquad \tilde{v}_\mu = \frac{v_\mu}{\|v_\mu\|_\infty}.$$

Then due to the inequality $-\Delta \tilde{u}_\mu \leqslant \lambda \tilde{u}_\mu$ and the boundedness of $\tilde{v}_\mu$ in $L^2(\Omega)$, there exist two non-negative functions $\tilde{u}$ and $\tilde{v}$ such that, by passing to a subsequence:

(i) $\tilde{u}_\mu \to \tilde{u}$ weakly in $H^1(\Omega)$, and strongly in $L^p(\Omega)$ for all $p > 1$;

(ii) $\tilde{v}_\mu \to \tilde{v}$ weakly in $L^2(\Omega)$;

(iii) $\tilde{u} \equiv 0$ in $\Omega \setminus \Omega_0$, $\tilde{u} \not\equiv 0$ in $\Omega$.

Moreover, taking $k_i = \|u_\mu\|_\infty$, $\ell_i = \|v_\mu\|_\infty$ in the proof of lemma 2.1, we obtain the inequality (2.2) with $u_i = \tilde{u}_\mu$ and

$$\int_\Omega |\nabla \tilde{u}_\mu|^2 + \mu \frac{\|v_\mu\|_\infty}{\|u_\mu\|_\infty} \int_\Omega \tilde{u}_\mu \tilde{v}_\mu \geqslant \mu \int_\Omega \tilde{u}_\mu^2. \tag{4.2}$$

Since $0 \leqslant \tilde{u}_\mu, \tilde{v}_\mu \leqslant 1$, by passing to a subsequence, we may assume that $\tilde{u}_\mu \tilde{v}_\mu \rightharpoonup \tilde{u}\tilde{v}$ in $L^2(\Omega)$.

We claim that $\lim_{\mu\to\infty} \|v_\mu\|_\infty = \infty$. If this is not true, by passing to a subsequence, we may assume that $\|v_\mu\|_\infty \to m \in [0, \infty)$. For any $\varphi \in C_0^\infty(\Omega_0)$, in view of

$$-\Delta \tilde{u}_\mu = \lambda \tilde{u}_\mu - \beta \|v_\mu\|_\infty \tilde{u}_\mu \tilde{v}_\mu \quad \text{in } \Omega_0,$$

we obtain

$$\int_{\Omega_0} \nabla \tilde{u}_\mu \nabla \varphi = \lambda \int_{\Omega_0} \tilde{u}_\mu \varphi - \beta \|v_\mu\|_\infty \int_{\Omega_0} \tilde{u}_\mu \tilde{v}_\mu \varphi.$$

Therefore,

$$\int_{\Omega_0} \nabla \tilde{u} \nabla \varphi = \lambda \int_{\Omega_0} \tilde{u} \varphi - m\beta \int_{\Omega_0} \tilde{u}\tilde{v}\varphi.$$

Since $\tilde{u} \equiv 0$ in $\Omega \setminus \Omega_0$, we see that $\tilde{u}$ is a weak solution of

$$-\Delta \tilde{u} = \lambda \tilde{u} - m\beta \tilde{v}\tilde{u} \quad \text{in } \Omega_0, \qquad \tilde{u} = 0 \quad \text{on } \partial\Omega_0.$$

The Harnack inequality asserts that $\tilde{u} > 0$ in $\Omega_0$. Since $0 \leqslant \tilde{u}_\mu(x), \tilde{v}_\mu(x) \leqslant 1$, we can also prove that $\tilde{u}_\mu \to \tilde{u}$ uniformly on any compact subset of $\Omega_0$.

For any $\varphi \in C_0^\infty(\Omega)$, from the second equation of (1.2) we have

$$-\frac{1}{\mu}\int_{\Omega_0} \tilde{v}_\mu \Delta\varphi = \int_{\Omega_0} \tilde{v}_\mu \varphi\biggl(1 - \frac{\|v_\mu\|_\infty}{\|u_\mu\|_\infty}\frac{\tilde{v}_\mu}{\tilde{u}_\mu}\biggr). \tag{4.3}$$

Using $\|v_\mu\|_\infty \leqslant C$, $\|u_\mu\|_\infty \to \infty$ and $\tilde{u}_\mu \to \tilde{u} > 0$ uniformly on any compact subset of $\Omega_0$, we obtain from (4.3) that

$$\int_{\Omega_0} \tilde{v}\varphi = 0, \text{ which implies that } \tilde{v} \equiv 0 \quad \text{in } \Omega_0.$$

Therefore, $\tilde{u}\tilde{v} \equiv 0$ in $\Omega$, and, hence,

$$\lim_{\mu\to\infty}\int_\Omega \tilde{u}_\mu \tilde{v}_\mu = 0. \tag{4.4}$$

In view of $\|v_\mu\|_\infty \leqslant \|u_\mu\|_\infty$ and $\tilde{u}_\mu \to \tilde{u}$ in $L^2(\Omega)$, it follows from (2.2), (4.2) and (4.4) that

$$0 \leqslant \int_\Omega \tilde{u}^2 = \lim_{\mu\to\infty}\int_\Omega \tilde{u}_\mu^2 \leqslant \lim_{\mu\to\infty}\int_\Omega \tilde{u}_\mu\tilde{v}_\mu = 0. \tag{4.5}$$

This is a contradiction, since $\tilde{u}$ is not identically zero.

Therefore, $\|v_\mu\|_\infty \to \infty$ as $\mu \to \infty$. Then, from

$$\lambda|\Omega| \geqslant \lambda\int_\Omega \tilde{u}_\mu^2 = \int_\Omega a(x)\tilde{u}_\mu u_\mu + \beta\|v_\mu\|_\infty \int_\Omega \tilde{u}_\mu\tilde{v}_\mu \geqslant \beta\|v_\mu\|_\infty \int_\Omega \tilde{u}_\mu\tilde{v}_\mu,$$

it follows that

$$\int_\Omega \tilde{u}\tilde{v} = \lim_{\mu\to\infty}\int_\Omega \tilde{u}_\mu\tilde{v}_\mu = 0,$$

i.e. (4.4) holds. We can obtain (4.5) using a method similar to the above. This gives a contradiction. This proves step 1.

STEP 2 (there exists $c > 0$ such that $\min_{\bar{\Omega}} v_\mu, \min_{\bar{\Omega}} u_\mu \geqslant c$, for all $\mu \geqslant \mu_0$). We first prove $\min_{\bar{\Omega}} u_\mu \geqslant c$ for all $\mu \geqslant \mu_0$. Assume on the contrary that, by passing to a subsequence, $\min_{\bar{\Omega}} u_\mu \to 0$ as $\mu \to \infty$. In step 1 showed that $\|u_\mu\|_\infty, \|v_\mu\|_\infty \leqslant C$ for some positive constant $C$ and all $\mu \geqslant \mu_0$. Applying the Harnack inequality to the first equation of (1.2), we find that $\max_{\bar{\Omega}} u_\mu \leqslant M \min_{\bar{\Omega}} u_\mu$ for some positive constant $M$ that is independent of $\mu \geqslant \mu_0$. Therefore, $\|u_\mu\|_\infty \to 0$ as $\mu \to \infty$, and so does $\|v_\mu\|_\infty$ by (4.1). This shows that $u_\mu, v_\mu \to 0$ uniformly on $\bar{\Omega}$ as $\mu \to \infty$.

As before, we define

$$\tilde{u}_\mu = \frac{u_\mu}{\|u_\mu\|_\infty}.$$

Then $\tilde{u}_\mu \to \tilde{u}$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for all $p > 1$, and $\tilde{u}$ is a non-negative and non-trivial solution of the problem

$$-\Delta\tilde{u} = \lambda\tilde{u} \quad \text{in } \Omega, \qquad \partial_\nu \tilde{u} = 0 \quad \text{on } \partial\Omega.$$

This is impossible since $\lambda > 0$. The proof for $\min_{\bar{\Omega}} u_\mu \geqslant c$ is complete.

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We show next that $\min_{\bar{\Omega}} v_\mu \geqslant \min_{\bar{\Omega}} u_\mu$. Indeed, suppose $\min_{\bar{\Omega}} v_\mu = v_\mu(x_0)$, $x_0 \in \bar{\Omega}$. Then, by [9, lemma 2.1], we find from the equation for $v_\mu$ that

$$\mu v_\mu(x_0)\left(1 - \frac{v_\mu(x_0)}{u_\mu(x_0)}\right) \leqslant 0.$$

Therefore,

$$\min_{\bar{\Omega}} v_\mu = v_\mu(x_0) \geqslant u_\mu(x_0) \geqslant \min_{\bar{\Omega}} u_\mu.$$

This completes the proof for step 2.

Step 3 (completion of the proof). From step 1, the function $\lambda u_\mu - [a(x)+\varepsilon]u_\mu^2 - \beta u_\mu v_\mu$ is uniformly bounded on $\bar{\Omega}$ with respect to $\mu$. Applying the elliptic estimate and the Sobolev imbedding theorem to the differential equation of $u_\mu$, it follows that, by passing to a subsequence, $u_\mu \to u$ in $C^{1,\alpha}(\bar{\Omega})$ for some positive function $u$ satisfying $c \leqslant u \leqslant C$. Since $v_\mu$ is bounded in $L^2(\Omega)$, by passing to a subsequence, $v_\mu \rightharpoonup v$ in $L^2(\Omega)$ for some positive function $v$ satisfying $c \leqslant v \leqslant C$. In the following we shall prove that $v = u$. This will be done by proving $v_\mu \to u$ uniformly on any compact subset of $\Omega$.

Fix $x_0 \in \Omega$. As $u \in C(\bar{\Omega})$ and $u(x) \geqslant c$ on $\bar{\Omega}$, for any fixed small $\sigma : 0 < \sigma < c$, there exists a $\delta > 0$ sufficiently small such that

$$u(x_0) - \tfrac{1}{2}\sigma < u(x) < u(x_0) + \tfrac{1}{2}\sigma \quad \text{in } \bar{B}(x_0, \delta), \tag{4.6}$$

where $B(x_0,\delta) = \{x \in \Omega, |x - x_0| < \delta\}$. In view of $u_\mu \to u$ uniformly on $\bar{B}(x_0,\delta)$ we have that

$$u(x) - \tfrac{1}{2}\sigma < u_\mu(x) < u(x) + \tfrac{1}{2}\sigma \quad \text{for all } x \in \bar{B}(x_0,\delta) \text{ and } \mu \gg 1.$$

Consequently,

$$u(x_0) - \sigma < u_\mu(x) < u(x_0) + \sigma \quad \text{for all } x \in \bar{B}(x_0,\delta) \text{ and } \mu \gg 1. \tag{4.7}$$

We now consider the following two auxiliary problems:

$$-\Delta w_\mu = \mu w_\mu \left(1 - \frac{w_\mu}{u(x_0) - \sigma}\right) \quad \text{in } B(x_0,\delta), \quad w_\mu = 0 \quad \text{on } \partial B(x_0,\delta). \tag{4.8}$$

$$-\Delta z_\mu = \mu z_\mu \left(1 - \frac{z_\mu}{u(x_0) + \sigma}\right) \quad \text{in } B(x_0,\delta), \quad z_\mu = \infty \quad \text{on } \partial B(x_0,\delta). \tag{4.9}$$

Using [6, lemmas 2.2 and 2.3], problems (4.8) and (4.9) have unique positive solutions $w_\mu$ and $z_\mu$, respectively, and

$$w_\mu \to u(x_0) - \sigma, \qquad z_\mu \to u(x_0) + \sigma \tag{4.10}$$

uniformly on any compact subset of $B(x_0,\delta)$. From (4.7) we see that $v_\mu$ is a super-solution of (4.8) and a sub-solution of (4.9). By [6, lemma 2.1], $w_\mu \leqslant v_\mu \leqslant z_\mu$ in $B(x_0,\delta)$. Therefore, by (4.10),

$$u(x_0) - \sigma = \lim_{\mu\to\infty} w_\mu(x) \leqslant \underline{\lim}_{\mu\to\infty} v_\mu(x) \leqslant \overline{\lim}_{\mu\to\infty} v_\mu(x) \leqslant \lim_{\mu\to\infty} z_\mu(x) = u(x_0) + \sigma$$

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uniformly on any compact subset of $B(x_0, \delta)$. By the arbitrariness of $\sigma$ we see that $\lim_{\mu\to\infty} v_\mu(x_0) = u(x_0)$. Since the convergence in (4.10) is uniform in $x_0$, we easily see that $v_\mu \to u$ uniformly on any compact subset of $\Omega$. Therefore, $v = u$.

From the first equation of (1.2) we now see that $u$ satisfies (1.6). Since $a(x)+\varepsilon+\beta \geqslant \beta > 0$, it is well known that (1.6) has a unique positive solution $w$. Therefore, the entire family of functions $u_\mu$ and $v_\mu$ converge to $w$. The proof of theorem 1.3 is complete.

Remark 4.1. In fact, we can show that $v_\mu \to w$ uniformly in $\bar{\Omega}$. To see this, for any $x_0 \in \partial\Omega$, we can extend $u_\mu$ and $v_\mu$ (by reflection across $\partial\Omega$) to $B(x_0, \delta)$ such that $v_\mu$ is a positive solution of

$$-Lv = \mu v\left(1 - \frac{v}{u_\mu}\right), \quad x \in B(x_0, \delta),$$

where $L$ is a second-order elliptic operator independent of $\mu$ and

$$L = \Delta \quad \text{on } B(x_0, \delta) \cap \Omega.$$

We can then argue as above to show that $v_\mu \to u$ uniformly on any compact subset of $B(x_0, \delta)$. By a standard finite covering argument, we see that $v_\mu \to u$ uniformly on $\bar{\Omega}$.

## Acknowledgments

Y.D. was partly supported by the Australian Research Council, and M.X.W. was partly supported by the Ministry of Education of China Science and Technology Major Projects, Grant no. 104090.

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(*Issued 11 August 2006*)