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Existence of solution for quasilinear equations involving local conditions

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In this paper, we study the existence of weak solutions of the quasilinear equation

$$\begin{cases} -\mathrm{div}(a(|\nabla u|^2)\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a : \mathbb{R} \to [0,\infty)$ is C^1 and a nonincreasing continuous function near the origin, the nonlinear term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying certain superlinear conditions only at zero, and λ is a positive parameter. The existence of the solution relies on C^1-estimates and variational arguments.

Keywords: Existence solution; quasilinear equations; variational methods.

2010 *Mathematics subject classification:* 35J93; 35B09; 35J50; 35J62

1. Introduction

In this paper, we deal with the question of the existence of weak solutions of the problem

$$\begin{cases} -\mathrm{div}(a(|\nabla u|^2)\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N > 2$, $a : \mathbb{R} \to [0,\infty)$ is a continuous function, λ is a positive parameter, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which grows as $|u|^{p-2}u$ near zero for $2 < p < 2^*$.

Since quasilinear equations serve as model of a wide class of differential operators, there has been a considerable amount of works on this subject (see [**2**–**4**, **16**] and references therein). For our purpose we consider two principal operators: $a(t) \equiv 1$ which reduces (1.1) to the Laplacian case, that is the scalar equation $-\Delta u = \lambda f(x,u)$; and $a(t) = (1+t)^{-1/2}$ where we obtain the mean curvature operator.

Notice that in the case of the Laplacian operator, important results have been obtained for elliptic problems involving superlinear terms, see for example [**1**, **10**, **25**]. In [**1**], Ambrosetti and Rabinowitz established an existence of nontrivial solution when f is subcritical and superlinear at zero. In that paper, they introduced the standard condition

(AR) There exist $\theta > 2$ and $u_0 > 0$ such that

$$0 < \theta F(x,u) \leqslant uf(x,u)\,,\ |u| \geqslant u_0 \text{ for all } x \in \Omega,$$

where $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$.

which has been later frequently used in the literature. The condition (AR) is quite natural and important not only to ensure that the Euler–Lagrange functional associated with problem has a mountain pass geometry, but also to guarantee that Palais–Smale sequence of the Euler–Lagrange functional is bounded. In this sense, several authors have weakened the (AR) condition to encompass these nonlinearities and proved that the Euler–Lagrange functional associated satisfies the Palais–Smale condition (see [**5**, **8**, **9**, **11**, **12**, **16**, **17**, **22**, **26**, **27**] and references therein).

On the other hand, concerning with the mean curvature operator under suitable conditions at zero and at infinity on the nonlinearity f, results of existence and non-existence of solutions have been obtained in different works, see for instance [**2**, **4**, **7**, **13**–**15**, **20**, **23**] and references therein. Comparing with the Laplacian case, the previous studies consider the superlinear case and they impose a kind of superquadraticity condition at infinity, which is implied by the (AR)-condition.

Concerning with local assumptions, we mention that in [**19**], Nakao considered linear case with the condition $2a'(t^2)t + a(t) > 0$ for any $t \in \mathbb{R}$ and, by using degree argument, they studied the existence of global branches from the least eigenvalue of $-\Delta$ and the trivial solution. Meanwhile, a more general setting was considered by Lorca and Ubilla in [**16**]. Under certain local hypotheses on a and f, both independent of the x-variable, and by imposing local monotonicity assumptions at zero, they showed the existence of solutions.

The purpose of this paper is to obtain existence results of weak solutions by imposing only local conditions at zero on the functions f and a. Therefore, we have to deal with major problems, to give some structure on the equation to use Partial Differential Equation (PDE's) tools, and, by the local nature of the equation, to control the C^1 norm of the eventual solutions. More precisely, our hypotheses are:

(H_1) There are $R_0 > 0$ and $\nu > 0$, such that $a \in C^1([0,R_0];(0,\infty))$ is a nonincreasing function and

$$2a'(t)t + a(t) > \nu\,,\ \text{for all } t \in [0,R_0).$$

(H_2) There are $p \in (2, 2^* - 1)$ and a nontrivial continuous function $\phi \in L^\infty(\Omega)$, which is positive in an open subset Ω_0 of Ω, with positive measure, such that $\phi(x) \geqslant \phi_0$ for all $x \in \Omega_0$ and some $\phi_0 > 0$ and

$$\lim_{u\to 0} \frac{f(x,u)}{|u|^{p-2}u} = \phi(x),$$

for $x \in \Omega$.

(H_3) There exist $q \in (2, 2^*)$, $q \leqslant p$, and $s_0 > 0$ such that

$$qF(x,u) - uf(x,u) \leqslant 0, \qquad \text{for all } \ u \in (0, s_0] \text{ and } x \in \Omega,$$

where $F(x,u)$ denotes the primitive of f, given by $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$.

Let us comment on the hypotheses above, (H_1) is needed to obtain $C^{1,\alpha}$-regularity on the eventual solutions. Moreover, it is necessary to show that gradient of the eventual solutions are bounded by a constant depending on theirs L^∞-norm. (H_2) and (H_3) allow us to construct an energy functional of a certain auxiliary problem, truncated of f and a for large values. Also, our hypotheses ensure that the associated funcional has the Mountain Pass Geometry, and to show that the Mountain Pass level goes to 0 when λ tends to infinity. More precisely, (H_2) allow us to obtain that 0 is a local minimum. Meanwhile (H_3) is a sort of Ambrosetti–Rabinowitz type of condition, but θ can be different from p and the nonlinear term could be slightly superlinear (see for instance [**6**]). We recall that the standard (AR) condition was introduced in [**1**] to ensure the boundedness of the Palais–Smale sequence.

Let us state our main result.

THEOREM 1.1. *Assume that conditions* (H_1), (H_2) *and* (H_3) *are satisfied. Then, Problem* (1.1) *possesses at least one nontrivial solution provided that the parameter* $\lambda > 0$ *is sufficiently large.*

EXAMPLES 1.2. *Some models where we may apply our main result are:*

(1) *A generalized mean curvature operator:*

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{(1+g(|\nabla u|^2))^\gamma}\right) = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $g : \mathbb{R} \to [0,\infty)$ *is a continuous and nondecreasing function,* $\gamma \geqslant 0$ *and* $f : \Omega \times \mathbb{R} \to [0,\infty)$ *is a continuous function verifying* (H_2)–(H_3).

(2) *A slightly superlinear function:*

$$\begin{cases} -\mathrm{div}\left(a(|\nabla u|^2)\nabla u\right) = \lambda u \ln(|u|+1) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $a : [0,\infty) \to \mathbb{R}$ *is a function which verifies* (H_1). *Observe that in this case, for* $p = 3$ *and* $q \in (2,3)$, *it is easy to see that* f *verifies* (H_2) *and* (H_3), *but not the (AR)-condition.*

2. Preliminaries

We emphasize that in our result we are assuming only conditions on nonlinearity f and on the function a near to zero, independently of their growth at infinity. Then, the associated Euler–Lagrange functional of problem (1.1) could be not necessarily

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well defined. To overcome this difficulty we consider truncated functions. That is, for $R_0 > R > 0$ let $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(t) = (1/(2(R_0 - R)))(2R_0 t - t^2 - R^2)$ and the truncated function $a_R : \mathbb{R} \to \mathbb{R}$ given by

$$a_R(t) = \begin{cases} a(t) & \text{if } t \leqslant R, \\ a(\varphi(t)) & \text{if } R \leqslant t \leqslant R_0, \\ a\left(\dfrac{R_0 + R}{2}\right) & \text{if } t \geqslant R_0, \end{cases}$$

and $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x, s) = \begin{cases} f(x, s), & |s| \leqslant s_0; \\ s_0^{1-q} f(x, \operatorname{sgn}(s) s_0)|s|^{q-1}, & |s| > s_0. \end{cases}$$

Since $\varphi(R) = R$, $\varphi'(R) = 1$, $\varphi'(R_0) = 0$ and $\varphi'(t)t - \varphi(t) \leqslant 0$ for $t \in [R, R_0]$, then $a_R \in C^1(0, \infty)$,

$$a'_R(t) = \begin{cases} a'(t) & \text{if } t \leqslant R \\ a'(\varphi(t))\varphi'(t) & \text{if } R \leqslant t \leqslant R_0 \\ 0 & \text{if } t \geqslant R_0, \end{cases}$$

and for $t \geqslant 0$

$$2a'_R(t)t + a_R(t) \geqslant \nu_0 > 0,$$

where $\nu_0 = \min\{\nu, a((R_0 + R)/2)\}$.

To prove our result using variational method, we consider the following auxiliary problem:

$$\begin{cases} -\operatorname{div}(a_R(|\nabla u|^2)\nabla u) = \lambda\psi(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Remark 2.1. *Note that by definition of function ψ, we have*

- *There is $m_0 > 0$ such that $\psi(y, s) \geqslant m_0 s^{p-1}$ for all $(y, s) \in \Omega_0 \times [0, s_0)$, and*
- *Exists $L > 0$ so that $|\psi(x, t)| \leqslant L|t|^{q-1}$ for $(x, t) \in \Omega \times \mathbb{R}$.*

The following lemma is due to Stampacchia, see [**24**]. It allows us to get an estimate of the $C^{1,\alpha}$-norm over the eventual solutions, depending on the L^∞-norm of the nonlinearity.

LEMMA 2.2. *Let $A = A(\eta)$ be a given C^1 vector field in $\mathbb{R}^N$, and $f = f(x,s)$ be a bounded Carathéodory function on $\Omega \times \mathbb{R}$. Let $u \in H_0^1(\Omega)$ be a solution of*

$$\int_\Omega (A(\nabla u)\cdot \nabla v + f(x,u)v) = 0,$$

for all $v \in H_0^1(\Omega)$. Assume that there exist two real numbers $0 < \nu < K$ such that

$$\nu|\xi|^2 \leqslant \frac{\partial A_i}{\partial \eta_j}(\nabla u)\xi_i\xi_j, \quad \textit{and} \quad \left|\frac{\partial A_i}{\partial \eta_j}(\nabla u)\right| \leqslant K,$$

for all $i,j = 1,\dots,N$ and all $\xi \in \mathbb{R}^N$. Then $u \in W^{2,r}(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$ and all $r \in (1,\infty)$. Moreover,

$$\|u\|_{1,\alpha} \leqslant \mathcal{O}(\nu, K, \Omega, \|f(\cdot,u)\|_\infty).$$

REMARK 2.3. *Note that it is easy to prove that a_R is a C^1 function and $G_0 \leqslant a_R(t) \leqslant \gamma_0$ for all $t \in \mathbb{R}_0^+$, where $G_0 = \inf_{t\in[0,R]} a(t)$ and $\gamma_0 = \max_{t\in[0,R_0]} a_R(t)$.*

Moreover, if we let $A : \mathbb{R}^N \to \mathbb{R}^N$ given by $A(\eta) = a_R(|\eta|^2)\eta$, then

$$\frac{\partial A_i}{\partial \eta_j} = 2a_R'(|\eta|^2)\eta_i\eta_j + a_R(|\eta|^2)\delta_{ij}.$$

In this way, for any $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^N$ we have

$$\begin{aligned}\frac{\partial A_i}{\partial \eta_j}\xi_i\xi_j &= 2a_R'(|\eta|^2)\xi^t(\eta_i\eta_j)\xi + a_R(|\eta|^2)|\xi|^2\\ &= 2a_R'(|\eta|^2)\xi\cdot(\eta(\xi\cdot\eta)) + a_R(|\eta|^2)|\xi|^2\\ &= 2a_R'(|\eta|^2)(\xi\cdot\eta)^2 + a_R(|\eta|^2)|\xi|^2.\end{aligned}$$

Since $a_R' \leqslant 0$, using the Cauchy–Schwarz inequality we get

$$\begin{aligned}\frac{\partial A_i}{\partial \eta_j}\xi_i\xi_j &\geqslant 2a_R'(|\eta|^2)|\xi|^2|\eta|^2 + a_R(|\eta|^2)|\xi|^2\\ &= [2a_R'(|\eta|^2)|\eta|^2 + a_R(|\eta|^2)]|\xi|^2\end{aligned}$$

The following lemma is a variant of the well-known Moser iterative scheme, see for instance [**2**, **18**].

LEMMA 2.4. *Let $u \in W_0^{1,p}(\Omega)$ be a solution of problem (2.1), then there exists a positive constant $C_1 = C_1(\Omega, q)$ such that*

$$\|u\|_\infty \leqslant C_1(\lambda L G_0^{-1})^{1/(2^*-q)}\|u\|_{2^*}^{\frac{2^*-2}{2^*-q}}.$$

Proof. From lemma 2.2 we known that if u is a solution of (2.1), then $u \in C^{1,\alpha}(\overline{\Omega})$. Since, there is $L > 0$ so that $\psi(x,t) \leqslant L|t|^{q-1}$, taking $v = \mathrm{sgn}(u)|u|^{2k+1} = u^{2k}u$ as

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a test function in equation (2.1) we obtain

$$\begin{aligned}
\frac{(2k+1)G_0}{(k+1)^2}\int_\Omega |\nabla(u^{k+1})|^2 &= G_0(2k+1)\int_\Omega |\nabla u|^2|u|^{2k} \\
&\leqslant (2k+1)\int_\Omega a_R(|\nabla u|^2)|\nabla u|^2|u|^{2k} \\
&= (2k+1)\int_\Omega a_R(|\nabla u|^2)(\nabla u\cdot\nabla u)|u|^{2k} \\
&= \int_\Omega a_R(|\nabla u|)\nabla u\cdot\nabla(u|u|^{2k}) \\
&= \int_\Omega \lambda\,\psi(x,u)\mathrm{sgn}(u)|u|^{2k+1} \\
&\leqslant \lambda L\int_\Omega |u|^{2k+q}
\end{aligned} \tag{2.2}$$

By Poincaré inequality we have

$$\left(\int |u^{k+1}|^{2N/(N-2)}\right)^{(N-2)/N} \leqslant C\int_\Omega |\nabla(u^{k+1})|^2.$$

Therefore, using Hölder's inequality, (2.2) reads

$$\begin{aligned}
\left(\int |u|^{\frac{2N(k+1)}{N-2}}\right)^{(N-2)/N} &= \|u\|_{2^*(k+1)}^{2(k+1)} \\
&\leqslant \frac{\lambda LC(k+1)^2}{(2k+1)G_0}\int_\Omega |u|^{2k+q} \\
&\leqslant \frac{\lambda LC(k+1)^2}{(2k+1)G_0}\|u\|_{2^*}^{q-2}\|u\|_{\frac{2\cdot 2^*(k+1)}{2^*-q+2}}^{2(k+1)}.
\end{aligned} \tag{2.3}$$

Then

$$\|u\|_{2^*(k+1)} \leqslant \left[\frac{\lambda LC(k+1)^2}{(2k+1)G_0}\right]^{\frac{1}{2(k+1)}}\|u\|_{2^*}^{\frac{q-2}{2(k+1)}}\|u\|_{\frac{2\cdot 2^*(k+1)}{2^*-q+2}}.$$

We define k_1 such that $2^*\cdot 2(k_1+1)/(2^*-q+2)=2^*$. Note that $k_1+1=1+(2^*-q)/2$. Then

$$\|u\|_{2^*(k_1+1)} \leqslant \left[\frac{\lambda LC(k_1+1)^2}{(2k_1+1)G_0}\right]^{\frac{1}{2(k_1+1)}}\|u\|_{2^*}^{\frac{q-2}{2(k_1+1)}}\|u\|_{2^*}.$$

Define by induction $2^*\cdot 2(k_n+1)/(2^*-q+2)=2^*(k_{n-1}+1)$, so $k_n+1=(2^*(2^*-q+2)/2)^n$ and

$$\begin{aligned}
\|u\|_{2^*(k_n+1)} &\leqslant \left[\frac{\lambda LC(k_n+1)^2}{(2k_n+1)G_0}\right]^{\frac{1}{2(k_n+1)}}\|u\|_{2^*}^{\frac{q-2}{2(k_n+1)}}\|u\|_{2^*(k_{n-1}+1)} \\
&\leqslant \left[\prod_{i=1}^n\left(\frac{\lambda LC(k_i+1)^2}{(2k_i+1)G_0}\right)^{\frac{1}{2(k_i+1)}}\right]\|u\|_{2^*}^{1+\frac{q-2}{2}\sum_{i=1}^n\frac{1}{k_i+1}}.
\end{aligned} \tag{2.4}$$

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Setting

$$C_1 = C^{1/(2^*-q)} \lim_{n\to\infty} \prod_{i=1}^{n} \left(\frac{(k_i+1)^2}{(2k_i+1)} \right)^{1/(2(k_i+1))}$$

and letting $n \to \infty$ in (2.4), we obtain

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1 (\lambda L G_0^{-1})^{\frac{1}{2^*-q}} \|u\|_{2^*}^{\frac{2^*-2}{2^*-q}}.$$

□

3. The Mp-geometry

Let $A_R(t) = \int_0^t a_R(s)\,\mathrm{d}s$ and $\Psi(x,u) = \int_0^u \psi(x,s)\,\mathrm{d}s$, we define the functionals $I_\lambda, J_1, J_2 : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega A_R(|\nabla u|^2)\mathrm{d}x - \lambda \int_\Omega \Psi(x,u)\,\mathrm{d}x,$$
$$J_1(u) = \frac{G_0}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \lambda \int_\Omega \Psi(x,u)\,\mathrm{d}x,$$

and

$$J_2(u) = \frac{\gamma_0}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \lambda \int_\Omega \Psi(x,u)\,\mathrm{d}x.$$

It is easy to see that

$$J_1(u) \leqslant I_\lambda(u) \leqslant J_2(u). \tag{3.1}$$

The following lemma provides the energy of a solution obtained by the Mountain Pass Theorem.

Lemma 3.1. *Assume* (H$_2$), *then there exists* $\rho_\lambda > 0$, *such that we can find explicitly* $\beta_\lambda > 0$ *such that*

$$I_\lambda(u) > \beta_\lambda,$$

for all $u \in X_{\rho_\lambda} := \{u \in H_0^1(\Omega) : \|u\|_{H_0^1(\Omega)} = \rho_\lambda\}$.

Proof. By definition of functional, we have

$$I_\lambda(u) = \frac{1}{2}\int_\Omega A_R(|\nabla u|^2)\mathrm{d}x - \lambda \int_\Omega \Psi(x,u)\,\mathrm{d}x,$$

Where, by remark 2.1 and Sobolev inequalities, we have

$$\begin{aligned} J_1(u) &\geqslant \frac{G_0}{2}\|u\|^2_{H_0^1(\Omega)} - \lambda \frac{L}{q}\|u\|^q_{L^q(\Omega)} \\ &\geqslant \left(\frac{G_0}{2} - \lambda \frac{LC}{q}\|u\|^{q-2}_{H_0^1(\Omega)} \right)\|u\|^2_{H_0^1(\Omega)}. \end{aligned}$$

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Setting $\rho_\lambda := (\frac{qG_0}{4\lambda LC})^{1/(q-2)}$, then for $u \in X_\rho$ we have

$$J_1(u) \geqslant \left(\frac{G_0}{4}\right)\left(\frac{qG_0}{4\lambda LC}\right)^{2/(q-2)} := \beta_\lambda\,.$$

Then, by (3.1), we obtain

$$I_\lambda(u) \geqslant \beta_\lambda, \quad \text{for all } u \in X_{\rho_\lambda},$$

and the lemma follows. □

LEMMA 3.2. *Assume* (H$_2$), *there exists* $u_0 \in H_0^1(\Omega)$ *such that* $\|u_0\|_{H_0^1(\Omega)} > \rho$ *and* $I_\lambda(u_0) \leqslant 0$.

Proof. Let $v \in C_c^\infty(\Omega_0)$ be a nonnegative function such that $\|v\|_{H_0^1(\Omega)} = 1$. Then, for $\zeta > 0$ we have $\psi(x, \zeta v) \geqslant 0$ and by remark 2.1, we have

$$\begin{aligned} &J_2(\zeta v)\\ &= \frac{\gamma_0\zeta^2}{2} - \lambda\int_\Omega \Psi(x,\zeta v)\,\mathrm{d}x\\ &= \frac{\gamma_0\zeta^2}{2} - \lambda\int_{\{x:\,\zeta v(x) > s_0\}} \Psi(x,\zeta v)\,\mathrm{d}x - \lambda\int_{\{x\in\Omega_0:\,\zeta v(x)\leqslant s_0\}} \Psi(x,\zeta v)\,\mathrm{d}x\\ &\leqslant \frac{\gamma_0\zeta^2}{2} - \lambda m_0 s_0^{p-q}\frac{\zeta^q}{q}\int_{\{x:\,\zeta v(x) > s_0\}} |v|^q\,\mathrm{d}x - \lambda m_0\frac{\zeta^p}{p}\int_{\{x\in\Omega_0:\,\zeta v(x)\leqslant s_0\}} |v|^p\,\mathrm{d}x\\ &\leqslant \zeta^2\left[\frac{\gamma_0}{2} - \lambda m_0\zeta^{q-2}\left(\frac{s_0^{p-q}}{q}\|v\|_q^q + \left(\frac{\zeta^{p-q}}{p} - \frac{s_0^{p-q}}{q}\right)\int_{\{x\in\Omega_0:\,\zeta v(x)\leqslant s_0\}} |v|^p\,\mathrm{d}x\right)\right] \end{aligned}$$

Taking $\zeta_0 \geqslant (\frac{\gamma_0 q}{2m_0\lambda\|v\|^q_{L^q(\Omega)}})^{\frac{1}{q-2}}$ with $(\frac{\zeta_0^{p-q}}{p} - \frac{s_0^{p-q}}{q}) \geqslant 0$, by considering $u_0 = \zeta_0 v$, we have

$$I_\lambda(u_0) \leqslant 0,$$

and by (3.1) which concludes the proof. □

REMARK 3.3. *Notice that, if we consider the function*

$$r(t) = \gamma_0\zeta_0^2\frac{t^2}{2} - \lambda m_0\zeta_0^q\|v\|^q_{L^q(\Omega_0)}\frac{t^q}{q},$$

then $J_2(\gamma(t)) \leqslant r(t_0)$, *where* ζ_0 *and* v *are from the proof of lemma above,* $\gamma(t) = t\zeta_0 v$ *and* $t_0^{q-2} = (\gamma_0\zeta_0^2/\lambda m_0\|v\|_q^q\zeta_0^p)$. *In addition, it is easy to check that*

$$\max_{0\leqslant t\leqslant 1} I_\lambda(\gamma(t)) \leqslant \max_{0\leqslant t\leqslant 1} J_2(\gamma(t)) \leqslant r(t_0) = \left(\frac{(q-2)\gamma_0}{2q}\right)\left(\frac{\gamma_0}{\lambda m_0\|v\|_q^q}\right)^{2/(q-2)}.$$

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4. Mountain Pass Solution

In this section, we obtain a critical point of I_λ by using the following standard version of the Mountain Pass Theorem.

THEOREM 4.1 (see [**21**]). *Let X be a real Banach space with dual space X^* and $J \in C^1(X, \mathbb{R})$ be a functional, satisfying the Palais–Smale condition (PS). If $u_0 \in X$ and $0 < \rho < \|u_0\|$ are such that*

$$a =: \max\{J(0), J(u_0)\} < \inf_{\|u\|=\rho} J(u) =: b\,,$$

then

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t))$$

is a critical value of J with $c \geqslant b$. Where $\Gamma = \{\gamma \in C(([0,1]), X);\, \gamma(0) = 0$ and $\gamma(1) = u_0\}$ is the set of continuous paths joining 0 and u_0.

Notice that from lemmas 3.1 and 3.2, the functional I_λ satisfies the geometry of the Mountain Pass Theorem. Now, for $\lambda > 0$ we will show that I_λ satisfies (PS). Let $\lambda > 0$ and $\{u_n\}_n \subset H_0^1(\Omega)$ such that

$$I_\lambda(u_n) \to c, \tag{4.1}$$

$$I'_\lambda(u_n) \to 0. \tag{4.2}$$

Now, our goal is to show the boundedness of the sequence $\{u_n\}$. From $p > 2$, we have that

$$\begin{aligned} o(1) + qc &= qI_\lambda(u_n) - I'(u_n)u_n \\ &= \frac{q}{2}\int_\Omega A_R(|\nabla u_n|^2) - \int_\Omega a_R(|\nabla u_n|^2)|\nabla u_n|^2 \\ &\quad + \lambda\int_\Omega (u_n\psi(x,u_n) - q\Psi(x,u_n)). \end{aligned} \tag{4.3}$$

It is not difficult to show that

$$t\psi(x,t) - q\Psi(x,t) \geqslant -M, \quad \forall\,(x,t) \in \Omega\times\mathbb{R}$$

for some $M > 0$. On the other hand, setting $\mathcal{B} = \{x \in \Omega : |\nabla u_n| \geqslant R_0\}$ and $\mathcal{C} = \{x \in \Omega : |\nabla u_n| \leqslant R_0\}$ we obtain

$$\begin{aligned} \int_\Omega \left[\frac{q}{2}A_R(|\nabla u_n|^2) - a_R(|\nabla u_n|^2)|\nabla u_n|^2\right] &= \int_{\mathcal{B}} \left[\frac{q}{2}A_R(|\nabla u_n|^2) - a_R(|\nabla u_n|^2)|\nabla u_n|^2\right] \\ &\quad + \int_{\mathcal{C}} \left[\frac{q}{2}A_R(|\nabla u_n|^2) - a_R(|\nabla u_n|^2)|\nabla u_n|^2\right]. \end{aligned}$$

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Note that the integral over $\mathcal{C}$ is bounded, then we restrict our attention on the integral over $\mathcal{B}$. Using the definition of the function ψ we have that

$$\begin{aligned}
&\int_{\mathcal{B}} \left[\frac{q}{2} A_R(|\nabla u_n|^2) - a_R(|\nabla u_n|^2)|\nabla u_n|^2\right] \\
&\quad = \int_{\mathcal{B}} \left[\frac{q}{2}\int_0^{|\nabla u_n|^2} a_R(t)\mathrm{d}t - a_R\left(\frac{R+R_0}{2}\right)|\nabla u_n|^2\right] \\
&\quad = \int_{\mathcal{B}} \left[\frac{q}{2}\int_0^{R_0} a_R(t)\mathrm{d}t + \frac{q}{2}\int_{R_0}^{|\nabla u_n|^2} a_R(t)\mathrm{d}t - a_R\left(\frac{R+R_0}{2}\right)|\nabla u_n|^2\right] \\
&\quad = \int_{\mathcal{B}} \left[\frac{q}{2}\int_0^{R_0} a_R(t)\mathrm{d}t + \frac{q}{2}(|\nabla u_n|^2 - R_0)a_R\left(\frac{R+R_0}{2}\right) - a_R\left(\frac{R+R_0}{2}\right)|\nabla u_n|^2\right] \\
&\quad = \frac{(q-2)}{2}a_R\left(\frac{R+R_0}{2}\right)\int_{\mathcal{B}}|\nabla u_n|^2 + \frac{q}{2}\int_{\mathcal{B}}\left[\int_0^{R_0} a_R(t)\mathrm{d}t - R_0 a_R\left(\frac{R+R_0}{2}\right)\right] \\
&\quad = \frac{(q-2)}{2}a_R\left(\frac{R+R_0}{2}\right)\|u_n\|^2 - \frac{(q-2)}{2}a_R\left(\frac{R+R_0}{2}\right)\int_{\mathcal{C}}|\nabla u_n|^2 \\
&\qquad + \frac{q}{2}\int_{\mathcal{B}}\left[\int_0^{R_0} a_R(t)\mathrm{d}t - R_0 a_R\left(\frac{R+R_0}{2}\right)\right]
\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\begin{aligned}
&\frac{(q-2)}{2}a_R\left(\frac{R+R_0}{2}\right)\|u_n\|^2 = o(1) + qc - \lambda\int_{\Omega}(u_n\psi(x,u_n) - q\Psi(x,u_n)) \\
&\quad - \int_{\mathcal{C}}\left[\frac{q}{2}A_R(|\nabla u_n|^2) - a_R(|\nabla u_n|^2)|\nabla u_n|^2\right] + \frac{(q-2)}{2}a_R\left(\frac{R+R_0}{2}\right)\int_{\mathcal{C}}|\nabla u_n|^2 \\
&\quad - \frac{q}{2}\int_{\mathcal{B}}\left[\int_0^{R_0} a_R(t)\mathrm{d}t - R_0 a_R\left(\frac{R+R_0}{2}\right)\right],
\end{aligned}$$

which means that $\|u_n\|$ is bounded.

5. Proof of the main result

Let u_λ be the MP-solution of the auxiliary problem. Therefore, there is $c_\lambda > 0$, critical value of I_λ such that

$$0 < \beta_\lambda \leqslant c_\lambda \leqslant r(t_0),$$

where β_λ and $r(t_0)$ are from lemma 3.1 and remark 3.3, respectively. More precisely,

$$\left(\frac{G_0}{4}\right)\left(\frac{qG_0}{4\lambda LC}\right)^{2/(q-2)} \leqslant c_\lambda \leqslant \left(\frac{(q-2)\gamma_0}{2q}\right)\left(\frac{\gamma_0}{\lambda m_0\|v\|_q^q}\right)^{2/(q-2)},$$

that is, for $\lambda > 0$

$$c_0\lambda^{-2/(q-2)} \leqslant c_\lambda \leqslant c_1\lambda^{-2/(q-2)}.$$

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On the other hand, using (H_3) we have that

$$qF(x,t) - f(x,t)t \leqslant 0 \quad \forall\,(x,t) \in \Omega \times (-s_0, s_0).$$

Then

$$\begin{aligned}
c_\lambda &= I_\lambda(u_\lambda) \\
&= \int_\Omega A_R(|\nabla u_\lambda|^2) - \lambda \int_\Omega \Psi(x,u_\lambda) \\
&= \int_\Omega A_R(|\nabla u_\lambda|^2) - \frac{\lambda}{q}\int_\Omega (q\Psi(x,u_\lambda) - \psi(x,u_\lambda)u_\lambda + \psi(x,u_\lambda)u_\lambda) \\
&= \frac{1}{q}\int_\Omega (qA_R(|\nabla u_\lambda|^2) - \lambda\psi(x,u_\lambda)u_\lambda) - \frac{\lambda}{q}\int_\Omega (q\Psi(x,u_\lambda) - \psi(x,u_\lambda)u_\lambda) \\
&= \frac{(q-1)}{q}\int_\Omega a_R(|\nabla u_\lambda|^2)|\nabla u_\lambda|^2 - \frac{1}{q}\int_\Omega \left(\int_0^{|\nabla u_\lambda|^2} t a_R'(t)\mathrm{d}t\right) \\
&\quad - \frac{\lambda}{q}\int_{\Omega\cap\{|u_\lambda|\leqslant s_0\}} (qF(x,u_\lambda) - u_\lambda f(x,u_\lambda)) \\
&\quad - \frac{\lambda}{q}\int_{\Omega\cap\{|u_\lambda|\geqslant s_0\}} (qF(x,\mathrm{sgn}(u_\lambda)s_0) - \mathrm{sgn}(u_\lambda)s_0 f(x,\mathrm{sgn}(u_\lambda)s_0)) \\
&\geqslant \frac{(q-1)}{q}\int_\Omega a_R(|\nabla u_\lambda|^2)|\nabla u_\lambda|^2 + \frac{1}{q}\int_\Omega \left(\int_0^{|\nabla u_\lambda|^2} t|a_R'(t)|\mathrm{d}t\right) \\
&\geqslant \frac{(q-1)}{q}\int_\Omega a_R(|\nabla u_\lambda|^2)|\nabla u_\lambda|^2.
\end{aligned}$$

The last expression implies

$$\|u_\lambda\|^2 = \mathcal{O}(c_\lambda), \text{ as } \lambda \to \infty.$$

Now, using lemma 2.4 we have

$$\|u\|_\infty = \mathcal{O}(\lambda^{\frac{1}{2^*-q}}\rho_\lambda^{\frac{2^*-2}{2^*-q}}) = \mathcal{O}(\lambda^{\frac{1}{2^*-q}}\left(\lambda^{\frac{-1}{q-2}}\right)^{\frac{2^*-2}{2^*-q}}) = \mathcal{O}(\lambda^{\frac{-1}{q-2}}).$$

Finally, since $|\psi(x,u)| \leqslant |u|^{q-1}$, it follows that $\|\lambda\psi\|_\infty = \mathcal{O}(\lambda^{-1/(q-2)})$ and, by lemma 2.2, $\|u_\lambda\|_{1,\alpha} = \mathcal{O}(\lambda^{-1/(q-2)})$. This means that there is $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ the MP solution verifies $\|u_\lambda\| \leqslant s_0$ and $\|\nabla u_\lambda\| \leqslant R_0$, which prove our main theorem.

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