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GEOMETRY OF KOTTWITZ–VIEHMANN VARIETIES

JINGREN CHI

Department of Mathematics, University of Maryland, College Park, MD 20742, USA (chijingren@gmail.com)

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Abstract We study basic geometric properties of Kottwitz–Viehmann varieties, which are certain generalizations of affine Springer fibers that encode orbital integrals of spherical Hecke functions. Based on the previous work of A. Bouthier and the author, we show that these varieties are equidimensional and give a precise formula for their dimension. Also we give a conjectural description of their number of irreducible components in terms of certain weight multiplicities of the Langlands dual group and we prove the conjecture in the case of unramified conjugacy class.

Keywords: affine Grassmannian; Vinberg monoid; spherical Hecke algebra; Hitchin–Frenkel–Ngô fibration

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Contents

1. Introduction

1.1. Background and motivation

In this article, we study certain analogue of affine Springer fibers that we call *Kottwitz–Viehmann varieties* whose underlying set is defined as

$$X_\gamma^\lambda = \{g \in G(F)/G(\mathcal{O}) | g^{-1}\gamma g \in G(\mathcal{O})\varpi^\lambda G(\mathcal{O})\}$$

where

- G is a connected reductive algebraic group over a field k (no assumption on k at the moment);
- $F = k((\varpi))$ is the field of Laurent series with coefficients in k and $\mathcal{O} = k[[\varpi]]$ is the ring of power series;
- $\gamma \in G(F)$ is a regular semisimple element;
- $\lambda : \mathbb{G}_m \to T$ is a cocharacter of a maximal torus T of G and

$$\varpi^\lambda := \lambda(\varpi) \in G(F).$$

Also we will consider a closely related set $X_\gamma^{\leqslant\lambda}$ defined by Replacing the double coset $G(\mathcal{O})\varpi^\lambda G(\mathcal{O})$ in the definition of X_γ^λ by the union

$$\overline{G(\mathcal{O})\varpi^\lambda G(\mathcal{O})} = \bigcup_{\mu\leqslant\lambda} G(\mathcal{O})\varpi^\mu G(\mathcal{O}).$$

These sets were first studied by Kottwitz and Viehmann in [20]. More general versions of them (replacing $G(\mathcal{O})$ by parahoric subgroups of $G(F)$) have also been studied by Lusztig in [21]. When k is a finite field, they arise naturally in the study of orbital integrals of functions in the spherical Hecke algebra $\mathcal{H}(G(F), G(\mathcal{O}))$ consisting of $G(\mathcal{O})$-biinvariant locally constant functions with compact support on $G(F)$.

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It turns out that X_γ^λ can be realized as the set of k-rational points of some algebraic variety over k. We view them as group analogue of affine Springer fibers for Lie algebras studied by Kazhdan and Lusztig in [19]:

$$X_\gamma = \{g \in G(F)/G(\mathcal{O}) | \mathrm{ad}(g)^{-1}\gamma \in \mathfrak{g}(\mathcal{O})\}.$$

Here $\mathfrak{g}$ is the Lie algebra of G, $\gamma \in \mathfrak{g}(F)$ is a regular semisimple element and 'ad' denotes the adjoint action of G on $\mathfrak{g}$.

Basic geometric properties of these affine Springer fibers X_γ have been well understood through the works of Kazhdan and Lusztig [19], Bezrukavnikov [2], Ngô [23]. A key ingredient in their approach is the symmetry on X_γ arising from the centralizer $G_\gamma(F)$. More precisely, the group $G_\gamma(F)$ has a dense open orbit X_γ^{reg} (the 'regular locus') so that geometric properties of X_γ^λ such as dimensions and number of irreducible components can be studied via the commutative algebraic group $G_\gamma(F)$ (more precisely certain finite-dimensional quotient P_γ of the infinite-dimensional loop group $G_\gamma(F)$).

We would like to generalize these methods to study the Kottwitz–Viehmann varieties X_γ^λ. Similar to Lie algebra case, the (connected) centralizer $G_\gamma^0(F)$ acts naturally on X_γ^λ and we consider the open orbits $X_\gamma^{\lambda,\mathrm{reg}}$ (the 'regular locus'). However, there are the following notable differences from the Lie algebra situation:

- In general, the action of $G_\gamma^0(F)$ on $X_\gamma^{\lambda,\mathrm{reg}}$ is not transitive.
- A more serious problem is that in general the 'regular locus' $X_\gamma^{\lambda,\mathrm{reg}}$ is not dense in X_γ and there might be irreducible components disjoint from $X_\gamma^{\lambda,\mathrm{reg}}$.

Thus X_γ^λ may have more irreducible components than $X_\gamma^{\lambda,\mathrm{reg}}$. This makes it more difficult to reduce geometric properties of X_γ^λ to the commutative group $G_\gamma^0(F)$.

1.2. Main results

Our first goal is to prove a dimension formula of X_γ^λ.

Theorem 1.2.1. *Assume that k is algebraically closed and its characteristic does not divide the order of Weyl group of G. Then X_γ^λ and $X_\gamma^{\leqslant\lambda}$ defined set-theoretically as above are k-schemes locally of finite type, equidimensional with dimension*

$$\dim X_\gamma^\lambda = \dim X_\gamma^{\leqslant\lambda} = \langle\rho,\lambda\rangle + \tfrac{1}{2}(d(\gamma) - c(\gamma))$$

where

- *ρ is half sum of the positive roots for G;*
- *$d(\gamma)$ is the discriminant valuation of γ (cf. Definition 3.1.2);*
- *$c(\gamma) = \mathrm{rank}(G) - \mathrm{rank}_F(G_\gamma)$, the difference between the dimension of the maximal torus of G and the dimension of the maximal F-split subtorus of the centralizer G_γ.*

In [3] and [6], this theorem is proved when G is semisimple and simply connected. In this article, we prove it for any split connected reductive group.

As in the Lie algebra case, there are two major steps. First we prove the dimension formula for the regular open subset, this step generalizes the method of Kazhdan–Lusztig in [19]. The second step is to show that

$$\dim X_\gamma^{\lambda,\mathrm{reg}} = \dim X_\gamma^\lambda.$$

For this the argument of Kazhdan–Lusztig in [19] does not generalize, since otherwise it would imply that the complement of the regular open subset has strictly smaller dimension (see [23, Proposition 3.7.1]), which in our situation may not be true due to the possible existence of irregular components. In general, actually most components of X_γ^λ will be irregular, see Remark 3.9.14. Instead, we bypass this difficulty by studying the global analogue of Kottwitz–Viehmann varieties, the Hitchin–Frenkel–Ngô fibration. Similar ideas occurred previously in [6].

This major difference from Lie algebra case leads us naturally to the question of determining the number of irreducible components of X_γ^λ, which is our second goal. We will formulate a conjecture on the number of irreducible components of X_γ^λ and prove the conjecture in the case where γ is an unramified (or split) conjugacy class. One formulation of the conjecture involves the Newton point $\nu_\gamma \in (X_*(T)\otimes\mathbb{Q})^+$ of γ, which is an element in the dominant rational coweight cone. By the discussion in § 3.9, if X_γ^λ is nonempty, there exists a unique *smallest* dominant *integral* coweight μ such that $\nu_\gamma \leqslant_{\mathbb{Q}} \mu$ and $\mu \leqslant \lambda$.

Conjecture (Conjecture 3.9.8). Let μ be as above. The number of $G_\gamma^0(F)$-orbits on the set of irreducible components of X_γ^λ equals to $m_{\lambda\mu}$, which is the dimension of μ-weight space in the irreducible representation V_λ of the Langlands dual group $\hat{G}$ with highest weight λ.

We remark that the isomorphism class of X_γ^λ only depends on the stable conjugacy class of γ (which is the characteristic polynomial in type A). For this reason we will give an equivalent formulation of our Conjecture using the extended Steinberg base of the Vinberg monoid. See Conjecture 3.9.8 for more details.

Also we remark that there is a similar conjecture made by Miaofen Chen and Xinwen Zhu on the irreducible components of affine Deligne–Lusztig varieties, see [13] and [34] for statements.

Theorem 1.2.2. *The Conjecture is true if $\gamma \in G(F)^{\mathrm{rs}}$ is split.*

This is proved in Corollary 3.5.3.

Remark 1.2.3. Although we restrict to equal characteristic local field, we expect that most results involving only local arguments in this paper could also be generalized to mixed characteristic Kottwitz–Viehmann varieties, which could be defined based on the work of Zhu [38]. For example, Proposition 3.1.6 on nonemptiness criterion, Corollary 3.5.3 on the dimension formula and irreducible components in the unramified case and Theorem 3.7.1 on the dimension of regular locus.

However, the dimension formula in full generality involves global argument and currently it is not clear how to generalize this to mixed characteristic case. It would be interesting to see if there is a purely local argument to prove the dimension formula.

1.3. Organization of the article

In § 2, we review certain facts needed from the theory of reductive monoids. In § 3, we prove dimension formula and the conjecture on irreducible components in the unramified case. In § 4, we review basic facts of Hitchin–Frenkel–Ngô fibration. The main result we establish in this chapter is properness of the fibration over an anisotropic open subset. In § 5, we relate Kottwitz–Viehmann varieties and Hitchin–Frenkel–Ngô fibrations and finish the proof of dimension formula for X_γ^λ.

1.4. Notations and conventions

1.4.1. Group theoretic notations. We assume throughout the article that k is an algebraically closed field. $F = k((\varpi))$ and $\mathcal{O} = k[[\varpi]]$. We let G be a (split) connected reductive group over k. Assume that either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > 0$ does not divide the order of Weyl group of G.

Denote by G_{der} the derived group of G, a semisimple group of rank r. Let G^{sc} be the simply connected cover of G_{der} and G_{ad} the adjoint group of G.

Fix a maximal torus T of G and a Borel subgroup B containing T. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots determined by $T \subset B$. Let $\check{\Lambda} := X^*(T)$ (respectively $\Lambda := X_*(T)$) be the weight (respectively coweight) lattice. Let $\check{\Lambda}^+$ (respectively Λ^+) be the set of dominant weights (respectively dominant coweights). Let W be the Weyl group of G and $S \subset W$ the set of simple reflections associated to the simple roots Δ. There is a unique longest element w_0 of W under the Bruhat order determined by S. Then w_0 is a reflection and $-w_0$ defines a bijection on the sets Δ, Λ^+ and $\check{\Lambda}^+$.

Let $\hat{G}$ be the Langlands dual group of G, viewed as a complex reductive group. For each $\lambda \in \Lambda^+$, viewed as a dominant weight for $\hat{G}$, let $V(\lambda)$ be the irreducible representation of $\hat{G}$ with highest weight λ. For any $\mu \in \Lambda^+$ with $\mu \leqslant \lambda$, let $m_{\lambda\mu}$ be the dimension of μ weight space in $V(\lambda)$.

1.4.2. Notations concerning algebraic geometry. For a scheme X over $\mathrm{Spec}\, F$, let LX be the loop space of X. More precisely, LX is the k-space that associates to any k-algebra R the set $LX(R) = X(R((t)))$.

For a scheme X over $\mathrm{Spec}\, \mathcal{O}$, let $L_n^+ X$ be its nth jet space. In other words, $L_n^+ X$ is the k-space whose set of R points is $L_n^+ X(R) = X(R[t]/t^n)$ for any k algebra R. Let $L^+ X := \varprojlim L_n^+ X$ be the arc space of X.

If X is a k-scheme, then we denote $LX := L(X \otimes_k F)$, $L_n^+ X := L_n^+(X \otimes_k \mathcal{O})$ and $L^+ X := L^+(X \otimes_k \mathcal{O})$.

For any scheme X, we denote by $\mathrm{Irr}(X)$ the set of its irreducible components.

For a scheme S and a group scheme G over S, G-torsors on S are understood in the étale topology. If E is a G-torsor on S and Y is an S-scheme on which G acts, we form the twisted product $E \wedge^G Y$, which is the quotient of $E \times_S Y$ by the anti-diagonal action of G.

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2. Review on reductive monoids

In this section we summarize some results on reductive monoids needed later. We loosely follow the exposition in [3], with several modifications and improvements. We refer the reader to [33], [27], [28] for more backgrounds on this subject.

2.1. Construction of Vinberg monoid

In this section, we assume that G is semisimple *simply connected*.

The Vinberg monoid for G is an algebraic monoid Vin_G such that the derived group of its unit group is isomorphic to G, and it is characterized by certain nice universal properties. For our purpose, we construct it in an explicit manner as follows.

Let $\omega_1, \ldots, \omega_r \in X_*(T)_+$ be the fundamental weights. For each $1 \leqslant i \leqslant r$, let $\rho_{\omega_i} : G \to \mathrm{GL}(V_{\omega_i})$ be the irreducible representation with highest weight ω_i.

We introduce the extended group $G_+ := (T \times G)/Z$ where Z, the center of G, embeds anti-diagonally in $T \times G$. Then G_+ is a reductive group with center $Z_+ = (T \times Z)/Z \cong T$ and derived group G. Let $T_+ = (T \times T)/Z$ be a maximal torus of G^+. We extend the representations ρ_{ω_i} to representations of G_+:

$$\begin{aligned} \rho_i^+ : G_+ &\longrightarrow \mathrm{GL}(V_{\omega_i}), \\ (t, g) &\longmapsto \omega_i(t)\rho_{\omega_i}(g). \end{aligned}$$

For each $1 \leqslant i \leqslant r$, we also extend the simple roots α_i to $\alpha_i^+ : G^+ \to \mathbb{G}_m$ by $\alpha_i^+(t, g) = \alpha_i(t)$. Altogether, we get the following homomorphism

$$(\alpha^+, \rho^+) : G^+ \to \mathbb{G}_m^r \times \prod_{i=1}^r \mathrm{GL}(V_{\omega_i}).$$

Definition 2.1.1. The *Vinberg monoid* of G, denoted by Vin_G, is the normalization of the closure of G_+ in the product

$$\mathbb{A}^r \times \prod_{i=1}^r \mathrm{End}(V_{\omega_i}).$$

Then Vin_G is an algebraic monoid with unit group G_+. It has a smooth dense open subvariety Vin_G^0 defined as the inverse image of the following product in Vin_G

$$\mathbb{A}^r \times \prod_{i=1}^r (\mathrm{End}(V_{\omega_i}) - \{0\}).$$

Definition 2.1.2. The *abelianization* of the monoid Vin_G is the invariant quotient

$$A_G := \mathrm{Vin}_G // (G \times G).$$

Let $\alpha : \mathrm{Vin}_G \to A_G$ be the quotient map.

Using the maps α^+ we get a canonical isomorphism $A_G \cong \mathbb{A}^r$. The adjoint torus T_{ad} embeds via the simple roots as the open subset where all the r-coordinates are nonzero.

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Note that the fibers of α over points in T_{ad} are isomorphic to G. One can construct a canonical section of the abelianization map α as follows.

Let T_{diag} be the image of the diagonal embedding $T \to T_+$. Then there is a canonical isomorphism $T_{\mathrm{diag}} \cong T_{\mathrm{ad}}$ which extends to an isomorphism $\overline{T_{\mathrm{diag}}} \cong A_G$ between the closure of T_{diag} in Vin_G and A_G. The inverse of this isomorphism defines a section of the abelianization map α, which we denote by

$$\mathfrak{s} : A_G \to \mathrm{Vin}_G. \tag{2.1}$$

The group $G_+ \times G_+$ acts by left and right multiplication on Vin_G. More precisely, for all $(x, y) \in G_+ \times G_+$ and $\gamma \in \mathrm{Vin}_G$, the action is given by $(x, y) \cdot \gamma = x\gamma y^{-1}$. The $G_+ \times G_+$-orbits on Vin_G correspond bijectively to pairs (I, J) of subsets of Δ such that no connected component (in the sense of Dynkin diagram) of the complement of J is entirely contained in I. Each orbit $O_{I,J}$ contains an idempotent $e_{I,J} \in \mathrm{Vin}_G$, defined up to conjugation. We can choose $e_{I,J} \in \overline{T_+}$, the closure of T_+ in Vin_G. Then it is well defined up to W-conjugation.

Fix such a pair (I, J). Let us describe the stabilizer of $e_{I,J}$ in $G_+ \times G_+$. Let J^c be the complement of J in Δ and J^0 be the interior of J, i.e., the elements in J that is not connected to any element of J^c in the Dynkin diagram. Let $M := (I \cap J^0) \sqcup J^c$. Let $P_+(M)$ be the corresponding standard parabolic subgroup of G_+, $P_+(M)^-$ be the opposite of $P_+(M)$ and $L_+(M)$ their common Levi subgroup. Denote by $\delta : P_+(M) \to L_+(M)$ and $\delta_- : P_+(M)^- \to L_+(M)$ the canonical projections. Also let G_M be the derived group of $L_+(M)$. The following lemma is [26, Theorem 21]:

Lemma 2.1.3. *With notations as above, the stabilizer of $e_{I,J}$ under $G_+ \times G_+$ is the subgroup of $P_+(M) \times P_+(M)^-$ consisting of pairs (g, g_-) such that*

$$\delta(g) \equiv \delta_-(g_-) \mod G_{J^c} T_{I,J},$$

where $T_{I,J}$ is a subtorus of T_+ defined as follows. Consider the subset $F_{I,J} \subset X^*(T_+)_{\mathbb{Q}}$ consisting of $(\chi, \psi) \in X^*(T_+)_{\mathbb{Q}}$ such that $\chi - \psi \in D_I$ and $\eta \in C_J$, where D_I is the convex cone spanned by the simple roots α_i for all $i \in I$ and C_J is the convex cone spanned by the fundamental weights ω_j for all $j \in J$. With these notations

$$T_{I,J} := \{t_+ \in T_+ | \Lambda(t_+) = 1, \forall \Lambda \in F_{I,J} \cap X^*(T_+)\}.$$

2.1.4. The adjoint action of G on the Vinberg monoid Vin_G is the restriction of left and right multiplication by $G \times G$ along the diagonal. In other words, for any $g \in G$ and $\gamma \in \mathrm{Vin}_G$, the adjoint action is given by $\mathrm{Ad}(g)(\gamma) := g\gamma g^{-1}$. Note that this action factors through the adjoint group G_{ad}.

For any $\gamma \in \mathrm{Vin}_G$, we let G_γ be the centralizer of γ in G, i.e., the stabilizer of γ under the adjoint action of G. If $\gamma \in G_+$ belongs to the unit group of Vin_G, we know that $\dim G_\gamma \geqslant \dim T = r$. By upper semicontinuity of stabilizer dimension (cf. [1, VI B.4, Proposition 4.1]), we see that $\dim G_\gamma \geqslant r$ for all $\gamma \in \mathrm{Vin}_G$.

Definition 2.1.5. An element $\gamma \in \mathrm{Vin}_G$ is *regular* if $\dim G_\gamma = r$ (i.e., smallest possible). Let $\mathrm{Vin}_G^{\mathrm{reg}} \subset \mathrm{Vin}_G$ be the open subset consisting of regular elements.

Definition 2.1.6. The *extended Steinberg base* is defined to be the invariant quotient $\mathfrak{C}_+ := \mathrm{Vin}_G//\mathrm{Ad}(G)$. We denote the canonical quotient map by $\chi_+ : \mathrm{Vin}_G \to \mathfrak{C}_+$.

The functions α_i^+ define a canonical map $\beta : \mathfrak{C}_+ \to A_G$ so that $\alpha = \beta \circ \chi_+$. The following result is [3, Proposition 1.7]:

Theorem 2.1.7. *The closed embedding* $\overline{T_+} \subset \mathrm{Vin}_G$ *induces an isomorphism of the invariant quotient* $\overline{T_+}/W \cong \mathfrak{C}_+$. *Moreover, the functions* α_+ *and* $\mathrm{Tr}(\rho_i^+)$ *define isomorphism*

$$\mathfrak{C}_+ \cong A_G \times \mathbb{A}^r \cong \mathbb{A}^{2r}.$$

The canonical projection $q : \overline{T_+} \to \mathfrak{C}_+$ a finite flat, generically Galois étale with Galois group W.

2.2. Adjoint orbits

We keep the assumption that G is semisimple simply connected.

Let $S = \{s_1, \ldots, s_r\}$ be the set of simple reflections in W corresponding to our choice of simple roots Δ. Let $l : W \to \mathbb{N}$ be the length function determined by S. For any subsets $J \subset S$, let W_J be the subgroup of W generated by elements in J. Let W^J (respectively ${}^J W$) be the set of minimal length representatives of the cosets W/W_J (respectively $W_J \backslash W$). For any two subsets J_1, J_2 of S, denote ${}^{J_1}W^{J_2} := {}^{J_1}W \cap W^{J_2}$.

For each $w \in W$, let $\mathrm{Supp}(w) \subset S$ be the subset consisting of those simple reflections which occurs in one (and hence every) reduced word expression of w.

Definition 2.2.1. An element $w \in W$ is called an S-*Coxeter element* if it can be written as products of simple reflections in S, each occurring precisely once. In particular, $l(w) = r$ and $\mathrm{Supp}(w) = S$. Denote by $\mathrm{Cox}(W, S)$ the set of S-Coxeter elements in W.

In general, an element $w \in W$ is called a *Coxeter element* if it is conjugate to an S-Coxeter element in W.

Let $\mathcal{N} := \chi_+^{-1}(0)$ be the *nilpotent cone* in the Vinberg monoid Vin_G. Let $\mathcal{N}^0 := \mathcal{N} \cap \mathrm{Vin}_G^0$ and $\mathcal{N}^{\mathrm{reg}} := \mathcal{N} \cap \mathrm{Vin}_G^{\mathrm{reg}}$ be the corresponding open subsets.

2.2.2. Our approach in this part follows a suggestion of Xinwen Zhu. For any subset $J \subset \Delta$, we apply Lemma 2.1.3 to determine the stabilizer of $e_{\varnothing,J}$ in $G \times G$. In the notation of *loc. cit.*, we have

$$T_J := T_{\varnothing,J} \cap T = \{t \in T \mid \omega_j(t) = 1,\ \forall j \in J\}.$$

In particular, we have $T_J \subset G_{J^c}$ and hence the stabilizer of $e_{\varnothing,J}$ in $G \times G$ consists of elements of the form (zg_1u, zg_2u_-), where $g_1, g_2 \in G_{J^c}$, $u \in U_{J^c}$, $u_- \in U_{J^c}^-$ and $z \in Z(L_{J^c})$, the center of the Levi subgroup L_{J^c} of G. Consequently we have

$$O_{\varnothing,J} \cong (G/G_{J^c}U_{J^c} \times G/G_{J^c}U_{J^c}^-)/Z(L_{J^c}),$$

where $Z(L_{J^c})$, the center of the Levi L_{J^c} acts diagonally on the product. There is a canonical map

$$\pi_{\varnothing,J} : O_{\varnothing,J} \to G/P_{J^c} \times G/P_{J^c}^-.$$

The diagonal G-orbits on the product $G/P_{J^c} \times G/P^-_{J^c}$ correspond bijectively to ${}^{J^c}W^{J^c}$. The element $w \in {}^{J^c}W^{J^c}$ corresponds to the G-orbit of $(\dot{w}, 1)$ for any representative $\dot{w}$ of w in G. We denote this G-orbit by $Y_{\varnothing,J,w}$ and let $X_{\varnothing,J,w}$ be its inverse image under $\pi_{\varnothing,J}$. Then we have

$$X_{\varnothing,J,w} = \mathrm{Ad}(G)(Z(L_{J^c})\dot{w}e_{\varnothing,J}). \tag{2.2}$$

The G-orbit $Y_{\varnothing,J,w}$ has codimension $l(w)$ in $G/P_{J^c} \times G/P^-_{J^c}$. Hence we have

$$\begin{aligned} \dim X_{\varnothing,J,w} &= 2\dim(G/P_{J^c}) - l(w) + \dim Z(L_{J^c}) \\ &= \dim G - \dim L_{J^c} - l(w) + |J|. \end{aligned} \tag{2.3}$$

Lemma 2.2.3. *$X_{\varnothing,J,w} \subset \mathcal{N}$ if and only if $J \subset \mathrm{Supp}(w)$.*

Proof. First suppose $X_{\varnothing,J,w} \subset \mathcal{N}$. Then in particular $\dot{w}e_{\varnothing,J} \in \mathcal{N}$. Recall that the idempotent $e_{\varnothing,J}$ acts as projector to highest weight space in the representation V_{ω_i} if $i \in J$ and acts by 0 if $i \notin J$. If there exists $j \in J$ but $j \notin \mathrm{Supp}(w)$, then $\rho_{\omega_j}(\dot{w})$ preserves the highest weight space in V_{ω_j} and hence $\mathrm{Tr}(\rho_{\omega_j}(\dot{w}e_{\varnothing,J})) \neq 0$, which contradicts the assumption that $\dot{w}e_{\varnothing,J} \in \mathcal{N}$.

Conversely suppose that $J \subset \mathrm{Supp}(w)$. Let $x = t\dot{w}e_{\varnothing,J}$, where $t \in Z(L_{J^c}) \subset T$. Then $\rho_{\omega_i}(x) = 0$ if $i \notin J$. If $i \in J$, so $i \in \mathrm{Supp}(w)$, then by a standard result in the root system we have $w(\omega_i) \neq \omega_i$ (see, for example [18, Lemma 3.5]). Thus we have $\mathrm{Tr}(\rho_{\omega_i}(x)) = 0$ as $t \in T$ preserve the weight spaces and $\dot{w}$ maps the highest weight space into the weight space with weight $w(\omega_i)$. Thus $x \in \mathcal{N}$. □

Corollary 2.2.4. (a) *There is a stratification of $\mathcal{N}$ into $\mathrm{Ad}(G)$-stable pieces*

$$\mathcal{N} = \bigsqcup_{J \subset \Delta} \bigsqcup_{\substack{w \in {}^{J^c}W^{J^c} \\ \mathrm{Supp}(w) \supset J}} X_{\varnothing,J,w}.$$

(b) $\mathcal{N}^0 = \bigsqcup_{\substack{w \in W \\ \mathrm{Supp}(w)=\Delta}} X_{\varnothing,\Delta,w}$.

(c) *For each $w \in \mathrm{Cox}(W,S)$ (cf. Definition 2.2.1), $X_{\varnothing,\Delta,w}$ is a single $\mathrm{Ad}(G)$-orbit and*

$$\mathcal{N}^{\mathrm{reg}} = \bigsqcup_{w \in \mathrm{Cox}(W,S)} X_{\varnothing,\Delta,w}.$$

In particular, $\mathcal{N}^{\mathrm{reg}} \subset \mathcal{N}^0$.

(d) $\dim \mathcal{N} = \dim \mathcal{N}^{\mathrm{reg}} = \dim G - r$ *and the dimension of the complement $\mathcal{N} \setminus \mathcal{N}^{\mathrm{reg}}$ is strictly less than* $\dim \mathcal{N}$.

Proof. Parts (a) and (b) are immediate from Lemma 2.2.3. For each strata $X_{\varnothing,J,w} \subset \mathcal{N}$ as in Lemma 2.2.3, we have $l(w) \geqslant |J|$ since $J \subset \mathrm{Supp}(w)$. From (2.3) we see that

$$\dim X_{\varnothing,J,w} \leqslant \dim G - \dim L_{J^c} \leqslant \dim G - r$$

and equality is reached precisely when $J = \Delta$ and $l(w) = r$. This condition means that $w \in \mathrm{Cox}(W,S)$. Hence part (d) follows from part (c).

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It remains to show that for each $w \in \mathrm{Cox}(W, S)$, $X_{\varnothing,\Delta,w}$ is a single $\mathrm{Ad}(G)$-orbit. By (2.2), we have

$$X_{\varnothing,\Delta,w} = \mathrm{Ad}(G)(Twe_{\varnothing,\Delta}).$$

So it suffices to show that for each $t \in T$, the elements $t\dot{w}e_{\varnothing,\Delta}$ and $\dot{w}e_{\varnothing,\Delta}$ are conjugate. Since w is a Coxeter element, by [31, Lemma 7.6] there exists $s \in T$ such that $t = s^{-1}\dot{w}s\dot{w}^{-1}$. This implies that $s^{-1}\dot{w}e_{\varnothing,\Delta}s = t\dot{w}e_{\varnothing,\Delta}$ since $s, t \in T$ and hence commute with $e_{\varnothing,\Delta}$. □

Remark 2.2.5. Another way to show that $X_{\varnothing,\Delta,w}$ consists of a single $\mathrm{Ad}(G)$-orbit is to show that the centralizer of $we_{\varnothing,\Delta}$ in G has dimension r, i.e., $we_{\varnothing,\Delta} \in \mathcal{N}^{\mathrm{reg}}$. Since then the $\mathrm{Ad}(G)$-orbit of $we_{\varnothing,\Delta}$ is contained in the irreducible set $X_{\varnothing,\Delta,w}$ and has the same dimension, thus equals to $X_{\varnothing,\Delta,w}$.

Corollary 2.2.6. *The morphism* $\chi_+ : \mathrm{Vin}_G \to \mathfrak{C}_+$ *is flat.*

Proof. There exists a nonempty open subset $U \subset \mathfrak{C}_+$ such that the fibers of χ_+ over U have dimension $\dim \mathrm{Vin}_G - \dim \mathfrak{C}_+ = \dim G - r$. Since χ_+ is Z_+ equivariant, U is Z_+-stable. By Corollary 2.2.4(d) we know that $0 \in U$ and hence we have $U = \mathfrak{C}_+$. By [7, 6.2.9], Vin_G is Cohen–Macaulay. Moreover, $\mathfrak{C}_+ \cong \mathbb{A}^{2r}$ is regular and hence χ_+ is flat. □

Corollary 2.2.7. $\mathrm{Vin}_G^{\mathrm{reg}} \subset \mathrm{Vin}_G^0$.

Proof. Let $U := \mathrm{Vin}_G^{\mathrm{reg}} \cap \mathrm{Vin}_G^0$ and we need to show that $U = \mathrm{Vin}_G^{\mathrm{reg}}$. Clearly U is Z_+-stable and $\mathrm{Ad}(G)$-stable. By Corollary 2.2.4(c), we have $\mathcal{N}^{\mathrm{reg}} \subset \mathcal{N}^0$ and hence $U \cap \mathcal{N} = \mathcal{N}^{\mathrm{reg}}$. Now our results follow from the following Lemma. □

Lemma 2.2.8. *Let* U *be a* Z_+*-stable and* $\mathrm{Ad}(G)$*-stable open subset of* $\mathrm{Vin}_G^{\mathrm{reg}}$*. If* $U \cap \mathcal{N} = \mathcal{N}^{\mathrm{reg}}$*, then* $U = \mathrm{Vin}_G^{\mathrm{reg}}$*.*

Proof. The following argument is extracted from [3], proof of Proposition 2.12. Let $F := \mathrm{Vin}_G \setminus U$ be the complement of U in Vin_G, which is a Z_+-stable and $\mathrm{Ad}(G)$-stable closed subset of Vin_G. Let $\chi_F : F \to \mathfrak{C}_+$ be the restriction of χ_+ to F. Consider the following set

$$V = \{x \in F \mid \dim_x \chi_F^{-1}(\chi_F(x)) \leqslant \dim G - r - 1\}.$$

By the upper-semicontinuity theorem [30, Tag 02FZ] we have that V is an open subset of F. It is clear that V is Z_+-stable. Moreover, we have $0 \in V$ by Corollary 2.2.4(d). Hence we must have $V = F$ since any Z_+-orbit in Vin_G contains 0 in its closure. This implies that for each closed point $a \in \mathfrak{C}_+$, $F \cap \chi_+^{-1}(a)$ has codimension at least 1 in $\chi_+^{-1}(a)$. But since χ_+ is flat by Corollary 2.2.6, the fiber $\chi_+^{-1}(a)$ is equidimensional and hence $U \cap \chi_+^{-1}(a)$ is dense in $\chi_+^{-1}(a)$.

Now suppose there exists $x \in \mathrm{Vin}_G^{\mathrm{reg}} \setminus U$. Then the $\mathrm{Ad}(G)$-orbit of x has dimension equal to $\dim \chi_+^{-1}(\chi_+(x))$ since x is regular. Hence the $\mathrm{Ad}(G)$-orbit of x lies in an irreducible component of $\chi_+^{-1}(\chi_+(x))$ that is disjoint from U. This contradicts with the density of U in the fiber. Thus the lemma follows. □

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Proposition 2.2.9. *The nilpotent cone $\mathcal{N}$ is connected and equidimensional. Moreover, there exist bijections*

$$\mathrm{Cox}(W, S) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{N}^{\mathrm{reg}}) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{N}^{0}) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{N})$$

which send $w \in \mathrm{Cox}(W, S)$ to the irreducible component containing $\dot{w}e_{\varnothing,\Delta}$.

Proof. Since χ_+ is flat and V_G is irreducible, the fiber $\mathcal{N} = \chi_+^{-1}(0)$ is equidimensional. Since χ_+ is the invariant quotient under a reductive group, there is a unique closed orbit in $\mathcal{N}$, namely $0 \in \mathcal{N}$. In particular, $\mathcal{N}$ is connected.

From Corollary 2.2.4, we see that $\mathcal{N}^{\mathrm{reg}}$ is dense in $\mathcal{N}^0$ and $\mathcal{N}$ and $\mathrm{Irr}(\mathcal{N}^{\mathrm{reg}})$ is in bijection with $\mathrm{Cox}(W, S)$. Hence $\mathrm{Irr}(\mathcal{N}^0)$ and $\mathrm{Irr}(\mathcal{N})$ are also in bijection with $\mathrm{Cox}(W, S)$. □

Remark 2.2.10. It is worthwhile to compare with the Lie algebra case. The fibers of the Chevalley map $\mathfrak{g} \to \mathfrak{c}$ are irreducible and $\mathfrak{g}^{\mathrm{reg}}$ is the union of the unique open G-orbit in each fiber.

2.2.11. Relation with the wonderful compactification. Here we point out some relations between our results and the work of Xuhua He in [16], see also [29].

The scheme Vin_G^0 is smooth and Z_+ acts freely on Vin_G^0. The quotient Vin_G^0/Z_+ is isomorphic to the wonderful compactification $\overline{G}_{\mathrm{ad}}$ of the adjoint group G_{ad} (c.f. [3, Proposition 2.2]). In [16], He studies certain closed subvariety $\mathcal{N}_{\mathrm{ad}} \subset \overline{G}_{\mathrm{ad}}$ consisting of elements that are represented by nilpotent matrices in all fundamental representations. Note that this is denoted by '$\mathcal{N}$' in *loc. cit.*, we add a subscript 'ad' to avoid confusion with the nilpotent cone in Vin_G. It is proved in *loc. cit.* that $\mathcal{N}_{\mathrm{ad}}$ is a disjoint union of $\mathrm{Ad}(G)$ stable pieces $Z_{J,w}$ labeled by pairs (J, w) where J is a proper subset of Δ and $w \in W$ satisfies $\mathrm{Supp}(w) = \Delta$. In terms of our notation, $Z_{\varnothing,w} = X_{\varnothing,\Delta,w}/Z_+$. We caution the reader that $\mathcal{N}^0/Z_+$ is NOT $\mathcal{N}_{\mathrm{ad}}$ but only a closed subvariety of $\mathcal{N}_{\mathrm{ad}}$. More precisely, $\mathcal{N}^0/Z_+$ is the union of the stratas $Z_{\varnothing,w}$ for $\mathrm{Supp}(w) = \Delta$. On the other hand, the G-orbits in the complement $\mathcal{N} - \mathcal{N}^0$ are not visible in the wonderful compactification.

2.2.12. Discriminant divisor. Recall that on T we have the discriminant function

$$\mathrm{Disc}(t) := \prod_{\alpha\in\Phi}(1 - \alpha(t))$$

which is W-equivariant and descends to a regular function on the Steinberg base $\mathfrak{C} := T/W$. We extend the function Disc to a function Disc_+ on $T_+ = (T \times T)/Z_G$ by

$$\mathrm{Disc}_+(t_1, t_2) := 2\rho(t_1)\mathrm{Disc}(t_2).$$

Then Disc_+ extends to a regular function on $\overline{T_+}$, which further descends to a regular function on $\mathfrak{C}_+$. The vanishing locus of Disc_+ is a principal divisor on $\mathfrak{C}_+$ which we call *extended discriminant divisor* and denote by $\mathfrak{D}_+$.

From the definition, we see that Disc_+ is an eigenfunction for the Z_+-action on $\overline{T_+}$ and $\mathfrak{C}_+$, with eigenvalue 2ρ. Hence the subschemes $\mathfrak{D}_+$ are Z_+-invariant.

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For $t_+ = (t, t^{-1}) \in T_{\text{diag}} \subset T_+$, we have

$$D_+(t_+) = 2\rho(t) \prod_{\alpha\in\Phi_+} (1-\alpha(t))(1-\alpha(t^{-1}))$$
$$= (-1)^{|\Phi_+|} \prod_{\alpha\in\Phi_+} (1-\alpha(t))^2. \tag{2.4}$$

For each $\alpha \in \Phi_+$, $D_\alpha := (1-\alpha(t))^2$ extends to a polynomial function on $\overline{T_{\text{diag}}} \cong \mathbb{A}^r$.

2.2.13. Adjoint orbits in extended Steinberg fiber. An element $\gamma \in \mathrm{Vin}_G$ is called *semisimple* if it is G-conjugate to an element in $\overline{T_+}$. Let $\mathrm{Vin}_G^{\mathrm{rs}}$ be the subset of Vin_G consisting of elements that are *both regular and semisimple*.

Lemma 2.2.14. *The centralizer of any semisimple element $\gamma \in \mathrm{Vin}_G$ in G is a Levi subgroup of G.*

Proof. We may assume that $\gamma \in \overline{T_+}$ so that $\gamma = te_{I,J}$ for some $t \in T_+$ and idempotent $e_{I,J}$.

For any $g \in G_+$, we have $g\gamma g^{-1} = \gamma$ if and only if

$$t^{-1}gte_{I,J}g^{-1} = e_{I,J}.$$

By the description of the stabilizer of $e_{I,J}$ under the action of $G_+ \times G_+$, we see that $g \in (G_+)_\gamma$ if and only if the following two conditions are satisfied:

- $(t^{-1}gt, g) \in P_M \times P_M^-$;
- $\delta(t^{-1}gt)\delta_-(g)^{-1} \in (L_{J^c})_{\mathrm{der}}T_{I,J}$.

Here $M := I \cap J^0 \sqcup J^c$. Since $t \in L_M$, the first condition implies that $g \in L_M$. Since the roots in $I \cap J^0$ and J^c are orthogonal to each other, the second condition implies that $(G_+)_\gamma$ is the subgroup of L_M generated by T_+, L_{J^c} and the centralizer of t in $L_{I\cap J^0}$. This shows that $(G_+)_\gamma$ is a Levi subgroup of G_+ and hence G_γ is a Levi subgroup of G. □

Lemma 2.2.15. *For any closed point $c \in \mathfrak{C}_+$, the fiber $\chi_+^{-1}(c)$ is connected and equidimensional of dimension* $\dim G - r$. *The open* $\mathrm{Ad}(G)$*-orbits in $\chi_+^{-1}(c)$ are precisely the regular conjugacy classes in $\chi_+^{-1}(c)$. On the other hand, there is a* unique *closed* $\mathrm{Ad}(G)$*-orbit in $\chi_+^{-1}(c)$ which is also the unique semisimple conjugacy class in $\chi_+^{-1}(c)$.*

Proof. By Corollary 2.2.6, χ_+ is flat. Hence $\chi_+^{-1}(c)$ is equidimensional of dimension $\dim G - r$. Since χ_+ is the invariant quotient by the reductive group G, there is a unique closed orbit in $\chi_+^{-1}(c)$. This closed orbit is connected since G is connected. Consequently $\chi_+^{-1}(c)$ is also connected.

The regular conjugacy classes in $\chi_+^{-1}(c)$ are locally closed subsets of the same dimension as $\chi_+^{-1}(c)$. Hence they are precisely the open $\mathrm{Ad}(G)$-orbits in $\chi_+^{-1}(c)$.

Finally by [25], closed $\mathrm{Ad}(G)$-orbits are precisely the semisimple conjugacy classes. □

Unlike the group case, there might be more than one regular conjugacy class in an extended Steinberg fiber $\chi_+^{-1}(c)$, as we see in Proposition 2.2.9 for the nilpotent cone $\mathcal{N} = \chi_+^{-1}(0)$. On the other hand, regular semisimple conjugacy classes are the only $\mathrm{Ad}(G)$ orbit in the extended Steinberg fiber they live in. We give another characterization of regular semisimple conjugacy classes using the discriminant function Disc_+. The following is a generalization of [3, 2.19].

Proposition 2.2.16. *Denote* $\overline{T_+}^{\mathrm{reg}} := \overline{T_+} \cap \mathrm{Vin}_G^{\mathrm{reg}}$. *For any* $\gamma \in \overline{T_+}$, *the following are equivalent:*

(1) $\gamma \in \overline{T_+}^{\mathrm{reg}}$;

(2) $\mathrm{Disc}_+(\gamma) \neq 0$;

(3) *The map* $q : \overline{T_+} \to \mathfrak{C}_+$ *is étale at* γ;

(4) $G_\gamma = T$.

Proof. (1) $\Rightarrow$ (2): Suppose $\gamma \in \overline{T_+}^{\mathrm{reg}}$. By Corollary 2.2.7, we have $\gamma \in \mathrm{Vin}_G^0 \cap \overline{T_+}$. After conjugation and multiplying by the center Z_+, we may assume that $\gamma \in \overline{T_{\mathrm{diag}}}$. If $\mathrm{Disc}_+(\gamma) = 0$, then there exists $\alpha \in \Phi_+$ such that $D_\alpha(\gamma) = 0$. This implies that γ lies in the closure of the diagonal embedding of $\ker(\alpha)$. Since the centralizers of elements in $\ker(\alpha)$ have dimension at least $r+1$, the same is true for G_γ by upper semicontinuity of centralizer dimension. This contradicts the assumption that γ is regular and we must have $\mathrm{Disc}_+(\gamma) \neq 0$.

(1)$\Leftrightarrow$(3)$\Leftrightarrow$(4): Since $\mathfrak{C}_+ = \overline{T_+}//W$, the finite cover $q : \overline{T_+} \to \mathfrak{C}_+$ is étale at γ if and only if the stabilizer of γ in W is trivial, which is equivalent to the fact $G_\gamma = T$ since G_γ is a standard Levi subgroup of G by the proof of Lemma 2.2.14.

(2) $\Rightarrow$ (1): Let $V \subset \overline{T_+}$ be the open subset where Disc_+ is nonzero and we need to show that $V = \overline{T_+}^{\mathrm{reg}}$. In the implication '(1) $\Rightarrow$ (2)' we proved that $\overline{T_+}^{\mathrm{reg}} \subset V$.

Consider the stratification of $\overline{T_+}$ induced by the T_{ad}-orbits on $A_G = \mathbb{A}^r$. The open strata is T_+, the unit group of $\overline{T_+}$. The codimension 1 stratas are described as follows: for each $1 \leqslant i \leqslant r$, let $\mathcal{O}_i$ be the codimension 1 strata consisting of $x \in \overline{T_+}$ such that the ith coordinate of $\alpha(x)$ vanishes and the other coordinates are nonzero. Consider the complement $F := V \setminus \overline{T_+}^{\mathrm{reg}}$, which is a closed subset of V. It is a classical fact that $F \cap T_+ = \varnothing$. Also, we have $e_{\varnothing,\Delta} \in \overline{T_+}^{\mathrm{reg}}$ by direct calculation of its centralizer. Hence $e_{\varnothing,\Delta}$ lies in the closure $\overline{\mathcal{O}_i}$ for all $1 \leqslant i \leqslant r$. This shows that the generic point of $\mathcal{O}_i$ lies in $\overline{T_+}^{\mathrm{reg}}$ for all i, which implies that F has codimension at least 2 in $\overline{T_+}$. But by the equivalence '(1)$\Leftrightarrow$(3)' we just proved and purity of branch locus (see, for example [30, Tag 0BMB]), the complement $\overline{T_+} \setminus \overline{T_+}^{\mathrm{reg}}$ is pure of codimension 1 in $\overline{T_+}$. This forces F, an open subset of $\overline{T_+} \setminus \overline{T_+}^{\mathrm{reg}}$ to be empty and hence $V = \overline{T_+}^{\mathrm{reg}}$. □

Corollary 2.2.17. $\mathrm{Vin}_G^{\mathrm{rs}} = \chi_+^{-1}(\mathfrak{C}_+ \setminus \mathfrak{D}_+)$. *Moreover,* G *acts transitively on each fiber of* χ_+ *over* $\mathfrak{C}_+ \setminus \mathfrak{D}_+$.

Proof. By Proposition 2.2.16, we have $\mathrm{Vin}_G^{\mathrm{rs}} \subset \chi_+^{-1}(\mathfrak{C}_+ \setminus \mathfrak{D}_+)$.

Let $c \in \mathfrak{C}_+ \setminus \mathfrak{D}_+$. By Lemma 2.2.15 and Proposition 2.2.16, the unique closed orbit in $\chi_+^{-1}(c)$ is also open. Hence $\chi^{-1}(c)$ is a single $\mathrm{Ad}(G)$-orbit consisting of elements that are both regular and semisimple. This proves the inverse inclusion. □

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For this reason, we denote $\mathfrak{C}_+^{\mathrm{rs}} := \mathfrak{C}_+ \setminus \mathfrak{D}_+$ and call it the *regular semisimple* open subset of $\mathfrak{C}_+$.

2.2.18. Extended Steinberg section. For each S-Coxeter element $w \in \mathrm{Cox}(W, S)$ (cf. Definition 2.2.1), each choice of representatives $\dot{s}_i \in N_G(T)$ of the simple roots s_i, Steinberg defines a section $\epsilon^w : \mathfrak{C}_G \to G$ of the adjoint quotient map $\chi_G : G \to \mathfrak{C}_G$. Moreover, it is shown that the equivalence class of ϵ^w depends neither on w nor the choices $\dot{s}_i$, see [31, 7.5 and 7.8]. Here we say that two sections ϵ, ϵ' are *equivalent* if for all $a \in \mathfrak{C}_G$, $\epsilon(a)$ and $\epsilon'(a)$ are conjugate under G.

Following [3], we extend the Steinberg sections ϵ^w to the Vinberg monoid Vin_G as follows. For each $(b, a) \in \mathfrak{C}_+ \cong \mathbb{A}^{2r}$ where $b \in A_G \cong \mathbb{A}^r$, define a map

$$\epsilon_+^w : \mathfrak{C}_+ \to \mathrm{Vin}_G$$

by $\epsilon_+^w(b, a) := \epsilon^w(a)\mathfrak{s}(b)$ where $\mathfrak{s} : A_G \to \mathrm{Vin}_G$ is the section of the abelianization map α defined in § 2.1.

Proposition 2.2.19. *The map ϵ_+^w is a section of the adjoint quotient $\chi_+ : \mathrm{Vin}_G \to \mathfrak{C}_+$. Moreover, the image of ϵ_+^w is contained in $\mathrm{Vin}_G^{\mathrm{reg}}$.*

Proof. The first statement is [3, Proposition 1.10]. The second statement is Proposition 1.16 in *loc. cit.* □

Remark 2.2.20. For each $w \in \mathrm{Cox}(W, S)$, the equivalence class of the extended section ϵ_+^w is independent of the choice of representatives $\dot{s}_i$ of the simple reflections. However, for two *different* $w, w' \in \mathrm{Cox}(W, S)$, the sections ϵ_+^w and $\epsilon_+^{w'}$ are not equivalent since, as we will see, $\epsilon_+^w(0)$ and $\epsilon_+^{w'}(0)$ are not conjugate.

Next we examine the interaction of the extended Steinberg section ϵ_+^w with the action of the central torus Z_+.

To this end, we drop the semisimple simply connected assumption and allow G to be any connected reductive group. Then the adjoint action of G_{ad} on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ induces an action of G on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ which we also denote by 'Ad'. Moreover, in the following, we will only consider the extended Steinberg base for the group G^{sc} and to ease notation, we denote it simply by $\mathfrak{C}_+$. In other words,

$$\mathfrak{C}_+ := \mathrm{Vin}_{G^{\mathrm{sc}}}//\mathrm{Ad}(G^{\mathrm{sc}}) = \mathrm{Vin}_{G^{\mathrm{sc}}}//\mathrm{Ad}(G).$$

The central torus $Z_+^{\mathrm{sc}} = T^{\mathrm{sc}}$ acts naturally on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ and $\mathfrak{C}_+$ such that the morphism $\chi_+ : \mathrm{Vin}_{G^{\mathrm{sc}}} \to \mathfrak{C}_+$ is T^{sc}-equivariant. Hence χ_+ induces a morphism between stacks

$$[\chi_+] : [\mathrm{Vin}_{G^{\mathrm{sc}}}/(\mathrm{Ad}(G) \times T^{\mathrm{sc}})] \to [\mathfrak{C}_+/T^{\mathrm{sc}}]. \tag{2.5}$$

We would like to see if ϵ_+^w induces a section $[\chi_+]$. It turns out that this is not true in general. To remedy it we consider the homomorphism $\psi : T^{\mathrm{sc}} \to G_{\mathrm{ad}}$ defined as the following composition

$$\psi : T^{\mathrm{sc}} \xrightarrow{\omega_\bullet} \mathbb{G}_m^r \xrightarrow{\mathfrak{s}} G_+^{\mathrm{sc}} \to G_{\mathrm{ad}}, \tag{2.6}$$

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where the first arrow is $\omega_\bullet := (\omega_1, \ldots, \omega_r)$, the second arrow is induced by the canonical section of the abelianization α (cf. Equation 2.1) and the third arrow is the canonical quotient morphism.

Consider the action of $T^{\text{sc}} \times T^{\text{sc}}$ on $\text{Vin}_{G^{\text{sc}}}$ where the first copy of T^{sc} acts by composing ψ with the adjoint action of G_{ad} and the second copy of T^{sc} acts as central torus. In [3, Proposition 1.11], by examining the action on weight vectors of fundamental representations, it is shown that for all $a_+ \in \mathfrak{C}_+$ and $z \in T^{\text{sc}}$ we have

$$\epsilon_+^w(z \cdot a_+) = z \cdot \mathfrak{s}(\omega_\bullet(z))\epsilon_+^w(a_+)\mathfrak{s}(\omega_\bullet(z))^{-1}.$$

This shows that ϵ_+^w is equivariant with respect to the diagonal embedding $T^{\text{sc}} \to T^{\text{sc}} \times T^{\text{sc}}$ and hence induces a morphism

$$[\mathfrak{C}_+/T^{\text{sc}}] \to [\text{Vin}_{G^{\text{sc}}}/\psi(T^{\text{sc}}) \times T^{\text{sc}}].$$

If $G = G_{\text{ad}}$, then this leads to a section $[\epsilon_+^w]$ of $[\chi_+]$. In general, let $c = |Z(G_{\text{der}})|$ be the order of the center of the derived group G_{der}. Then by extracting cth roots, we would get a lifting $\psi_{[c]} : T^{\text{sc}} \to G_{\text{der}} \subset G$ of ψ. More precisely, $\psi_{[c]}$ is defined by the following commutative diagram

$$\begin{tikzcd}
T^{\text{sc}} \arrow[r, "\psi_{[c]}"] \arrow[d, "c"] & G \arrow[d] \\
T^{\text{sc}} \arrow[r, "\psi"] & G_{\text{ad}}
\end{tikzcd}$$

where the left vertical map is raising to cth power.

The cth power map $T^{\text{sc}} \to T^{\text{sc}}$ induces a morphism between classifying stacks $\mathbb{B}T^{\text{sc}} \to \mathbb{B}T^{\text{sc}}$. Base changing $[\chi_+]$ along this map, we obtain a Cartesian diagram

$$\begin{tikzcd}
{[\text{Vin}_{G^{\text{sc}}}/(\text{Ad}(G) \times T^{\text{sc}})]_{[c]}} \arrow[r] \arrow[d, "{[\chi_+]_{[c]}}"'] & {[\text{Vin}_{G^{\text{sc}}}/(\text{Ad}(G) \times T^{\text{sc}})]} \arrow[d, "{[\chi_+]}"] \\
{[\mathfrak{C}_+/T^{\text{sc}}]_{[c]}} \arrow[r] & {[\mathfrak{C}_+/T^{\text{sc}}]}
\end{tikzcd}$$

where on the left, the T^{sc} action is the composition of the cth power map and the usual action.

Proposition 2.2.21. *The map ϵ_+^w induces a section*

$$\epsilon_{+,[c]}^w : [\mathfrak{C}_+/T^{\text{sc}}]_{[c]} \to [\text{Vin}_{G^{\text{sc}}}/(\text{Ad}(G) \times T^{\text{sc}})]_{[c]}$$

of $[\chi_+]_{[c]}$ whose image lies in the open substack

$$[\text{Vin}_{G^{\text{sc}}}^{\text{reg}}/(\text{Ad}(G) \times T^{\text{sc}})]_{[c]}.$$

Proof. By what we have discussed, ϵ_+^w induces a morphism

$$[\mathfrak{C}_+/T^{\text{sc}}]_{[c]} \to [\text{Vin}_{G^{\text{sc}}}/\psi_{[c]}(T^{\text{sc}}) \times T^{\text{sc}}]$$

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where on the right, the second copy of T^{sc} acts by composing the cth power map and the usual action. Since $\psi_{[c]}(T^{\mathrm{sc}}) \subset G$, there is a canonical morphism

$$[\mathrm{Vin}_{G^{\mathrm{sc}}}/\psi_{[c]}(T^{\mathrm{sc}}) \times T^{\mathrm{sc}}] \to [\mathrm{Vin}_{G^{\mathrm{sc}}}/(\mathrm{Ad}(G) \times T^{\mathrm{sc}})]_{[c]}.$$

Composing the two morphisms above we obtain the morphism $\epsilon^{w}_{+,[c]}$ with the desired property. □

2.3. Regular centralizer for the group

In this section, we let (G, G') be a pair of connected reductive groups equipped with an isomorphism of their derived groups $G_{\mathrm{ad}} \cong G'_{\mathrm{ad}}$. Assume moreover that the derived group of G is simply connected. Then there is a natural adjoint action of G' on G and the action factors through $G'_{\mathrm{ad}} \cong G_{\mathrm{ad}}$. Let $\mathfrak{C}_G := G//\mathrm{Ad}(G')$ be the invariant quotient. Then there is a canonical isomorphism $\mathfrak{C}_G \cong T/W$. The natural map $T \to \mathfrak{C}_G$ is finite flat and its restriction to $\mathfrak{C}_G^{\mathrm{rs}}$ is a Galois étale cover with Galois group W.

Consider the centralizer group scheme $I_{G'}$ over G defined by

$$I_{G'} := \{(g, x) \in G' \times G | \mathrm{Ad}(g)x = x\}.$$

In other words, the fiber of $I_{G'}$ over $x \in G$ is the centralizer G'_x of x in G'. Since the derived group of G' is simply connected, G'_x is connected for semisimple $x \in G$. If moreover $x \in G^{\mathrm{rs}}$ is regular semisimple, then G'_x is a maximal torus in G'. More generally, the restriction $I_{G'}|_{G^{\mathrm{reg}}}$ to the regular open subscheme G^{reg} is a smooth commutative group scheme of relative dimension $\mathbf{dim}(T)$. The following lemma is the group version of [23, Lemme 2.1.1].

Lemma 2.3.1. *There exists a unique smooth commutative group scheme $J_{G'}$ over $\mathfrak{C}_G$ such that we have a G'-equivariant isomorphism*

$$(\chi^* J_{G'})_{G^{\mathrm{reg}}} \cong I_{G'}|_{G^{\mathrm{reg}}}.$$

Moreover, this isomorphism extends uniquely to a homomorphism $\chi^* J_{G'} \to I_{G'}$.

Proof. The proof of [23, Lemme 2.1.1] generalizes verbatim to our situation. For the last statement, we use the fact that the complement of G^{reg} in G has codimension at least 2, c.f. [31]. □

Fix a maximal torus $T' \subset G'$. Consider the Weil restriction of the torus $T' \times T$ on T to $\mathfrak{C}_G$:

$$\Pi_G := \Pi_{T/\mathfrak{C}_G}(T' \times T).$$

In other words, for any $\mathfrak{C}$-scheme S, we have

$$\Pi_G(S) = \mathrm{Hom}_T(S \times_C T, T' \times T).$$

The diagonal action of W on $T' \times T$ induces an action of W on Π_G. The fixed point subscheme of Π_G^W is a closed smooth subscheme of Π_G since the characteristic of the base field does not divide the order of W.

Proposition 2.3.2. *There exists a canonical open embedding $J_{G'} \to \Pi_G^W$.*

Proof. We follow the argument for the Lie algebra case in [23, § 2.4]. First we define a morphism $J \to \Pi_G^W$. By adjunction, this is the same as giving a morphism $q^*J \to T \times T$ where $q : T \to \mathfrak{C}$ and we view $T \times T$ as a constant group scheme over T. We construct this morphism by descent along the smooth morphism $\tilde{\chi}^{\mathrm{reg}} : \widetilde{G}^{\mathrm{reg}} \to T$ which sits in the Cartesian diagram

$$\begin{tikzcd} \widetilde{G}^{\mathrm{reg}} \arrow[r, "\tilde{q}"] \arrow[d, "\tilde{\chi}"'] & G^{\mathrm{reg}} \arrow[d, "\chi"] \\ T \arrow[r, "q"] & \mathfrak{C}_G \end{tikzcd}$$

Hence it suffices to construct a G-equivariant morphism $(\tilde{\chi}^{\mathrm{reg}})^*q^*J_G \to T \times \widetilde{G}^{\mathrm{reg}}$. The upshot is that for all $x \in G$ and Borel subgroup $x \in B \subset G$, we have $I_x \subset B$ by the argument of [23, Lemme 2.4.3]. Hence when composed with the quotient $B \to T$, we obtain a map $I_x \to T$ depending on the choice of Borel B containing x. Thus we get the desired morphism $(\tilde{\chi}^{\mathrm{reg}})^*q^*J_G \cong \tilde{q}^*I_{G^{\mathrm{reg}}} \to T \times \widetilde{G}^{\mathrm{reg}}$ which is G-equivariant by construction.

To show that the morphism $J_G \to \Pi_G^W$ constructed above is an isomorphism, it suffices to show the isomorphism over an open subset of $\mathfrak{C}$ whose complement has codimension at least 2.

For each simple root $\alpha \in \Phi^+$, let T_α be the kernel of α, which is a subscheme of codimension 1 in T. Then the discriminant divisor $\mathfrak{D} \subset \mathfrak{C}$ is the union of $q(T_\alpha)$ for all simple root α. Let $T_\alpha^\circ \subset T_\alpha$ be the open subscheme consisting of points that does not lie in T_β for any $\beta \neq \alpha$. Then

$$\mathfrak{C}^{\mathrm{rs}} \cup \left(\bigcup_{\alpha \in \Phi^+} q(T_\alpha^\circ) \right)$$

is an open subset of $\mathfrak{C}$ whose complement has codimension 2. It follows from the construction that it is an isomorphism over $\mathfrak{C}^{\mathrm{rs}}$. Hence it remains to show that $J_G \to \Pi_G^W$ is an isomorphism when restricted to $q(T_\alpha^\circ)$ for each positive root α.

Let $t \in T_\alpha^\circ$ and we will show that $J \to \Pi_G^W$ is an isomorphism in an étale neighborhood of t. Let G_α be the centralizer of T_α in G and $\mathfrak{C}_{G_\alpha}$ its adjoint quotient. Then the natural morphism $\pi_\alpha : \mathfrak{C}_{G_\alpha} \to \mathfrak{C}$ is étale in a neighborhood of $q_\alpha(t)$ where $q_\alpha : T \to \mathfrak{C}_{G_\alpha}$ is the natural map. This implies that in an étale neighborhood of $q_\alpha(t)$ the group schemes $\Pi_G^W \times_{\mathfrak{C}} \mathfrak{C}_{G_\alpha}$ and $\Pi_{G_\alpha}^{s_\alpha}$ are isomorphic.

There is a natural open embedding $G^{\mathrm{reg}} \cap G_\alpha \subset G_\alpha^{\mathrm{reg}}$. Consider the open subset

$$\mathfrak{C}_\alpha^{G-\mathrm{reg}} := \chi_\alpha(G^{\mathrm{reg}} \cap G_\alpha).$$

As $t \in T_\alpha^\circ$, one can choose a unipotent element $u \in G_\alpha$ such that $tu \in G^{\mathrm{reg}} \cap G_\alpha$. In particular, $q_\alpha(t) \in \mathfrak{C}_\alpha^{G-\mathrm{reg}}$. It is clear that

$$I_{G_\alpha}|_{G^{\mathrm{reg}} \cap G_\alpha} \cong I_G|_{G^{\mathrm{reg}} \cap G_\alpha}.$$

This implies that $(\pi_\alpha^* J_G)|_{\mathfrak{C}_\alpha^{G-\mathrm{reg}}} \cong (J_{G_\alpha})|_{\mathfrak{C}_\alpha^{G-\mathrm{reg}}}$.

In summary, the base changes of J_G and Π_G^W to an étale neighborhood of $q(t)$ are isomorphic to the corresponding groups defined for the group G_α. Note that by

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assumption, G_α is of rank 1 and has semisimple derived group, thus isomorphic to the product of a torus with either GL_2 or SL_2. So we are finally reduced to the case of GL_2 and SL_2, on which the isomorphism follows by direct calculation. □

2.4. Regular centralizer for Vinberg monoid

In this section we let G be an arbitrary connected reductive group over k. Let G^{sc} be the simply connected cover of its derived group. Then there is a natural adjoint action of G on $\mathrm{Vin}_{G^{\mathrm{sc}}}$ and the action factors through G_{ad}.

Consider the centralizer group scheme $\mathcal{I}$ over $\mathrm{Vin}_{G^{\mathrm{sc}}}$ defined by

$$\mathcal{I} = \{(g, \gamma) \in G \times \mathrm{Vin}_{G^{\mathrm{sc}}} | \mathrm{Ad}(g)\gamma = \gamma\}.$$

Then $\mathcal{I}|_{\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}}$ is smooth of relative dimension r. By [25], the fibers of $\mathcal{I}$ over $\mathrm{Vin}^{\mathrm{rs}}_{G^{\mathrm{sc}}}$ are maximal tori in G. In particular, $\mathcal{I}|_{\mathrm{Vin}^{\mathrm{rs}}_{G^{\mathrm{sc}}}}$ is commutative. Hence $\mathcal{I}|_{\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}}$ is also commutative.

2.4.1. Open cover of regular locus. For each $w \in \mathrm{Cox}(W, S)$, define $\mathcal{J}^w := (\epsilon^w_+)^*\mathcal{I}$. Then $\mathcal{J}^w$ is a smooth commutative group scheme on $\mathfrak{C}_+$. The morphism

$$\begin{aligned} c_w : G \times \mathfrak{C}_+ &\longrightarrow \mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}} \\ (g, a) &\longmapsto g\epsilon^w_+(a)g^{-1} \end{aligned}$$

factors through $(G \times \mathfrak{C}_+)/\mathcal{J}^w$ and induces a quasi-finite morphism

$$\bar{c}_w : (G \times \mathfrak{C}_+)/\mathcal{J}^w \to \mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}.$$

Since $\bar{c}_w$ is an isomorphism over $G^{sc,\mathrm{reg}}_+$, it is birational. Since $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$ is normal, $\bar{c}_w$ is an open embedding by Zariski Main Theorem.

Denote by $\mathrm{Vin}^w_{G^{\mathrm{sc}}}$ the image of $\bar{c}_w$, which is an open subscheme of $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$. The union $U := \bigcup_{w\in\mathrm{Cox}(W,S)} \mathrm{Vin}^w_{G^{\mathrm{sc}}}$ is a Z^{sc}_+-stable and $\mathrm{Ad}(G)$-stable open subset of $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$. By Proposition 2.2.9, we have $U \cap \mathcal{N} = \mathcal{N}^{\mathrm{reg}}$. Applying Lemma 2.2.8 we see that $U = \mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$. In other words, the sets $\mathrm{Vin}^w_{G^{\mathrm{sc}}}$ form an open cover of $\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}$:

$$\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}} = \bigcup_{w\in\mathrm{Cox}(W,S)} \mathrm{Vin}^w_{G^{\mathrm{sc}}}. \tag{2.7}$$

We generalize Lemma 2.3.1 to $\mathrm{Vin}_{G^{\mathrm{sc}}}$:

Lemma 2.4.2. *There is a unique smooth commutative group scheme $\mathcal{J}$ over $\mathfrak{C}_+$ such that we have a G-equivariant isomorphism $(\chi^{\mathrm{reg}}_+)^*\mathcal{J} \cong \mathcal{I}|_{\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}}$. Moreover, this isomorphism extends uniquely to a homomorphism $\chi^*_+\mathcal{J} \to \mathcal{I}$.*

Proof. By the same argument as Lemma 2.3.1, for each $w \in \mathrm{Cox}(W, S)$, $\mathcal{J}^w$ is the unique commutative smooth group scheme over $\mathfrak{C}_+$ such that

$$(\chi^*_+\mathcal{J}^w)|_{\mathrm{Vin}^w_{G^{\mathrm{sc}}}} \cong \mathcal{I}|_{\mathrm{Vin}^w_{G^{\mathrm{sc}}}}.$$

Next we show that for any $w, w' \in \mathrm{Cox}(W, S)$, the group schemes $\mathcal{J}^w$ and $\mathcal{J}^{w'}$ are canonically isomorphic. It suffices to show that they are canonically isomorphic over

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certain open subset whose complement has codimension at least 2. From Lemma 2.3.1, we have the isomorphism over the open subset $\mathfrak{C}_{G_+^{\mathrm{sc}}}$. Over $\mathfrak{C}_+^{\mathrm{rs}}$, each fiber of χ_+ consists of a single $\mathrm{Ad}(G)$ orbit by Lemma 2.2.15. In other words, G acts transitively on each fiber of χ_+ over $\mathfrak{C}_+^{\mathrm{rs}}$. Hence $\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{rs}} \subset \mathrm{Vin}_{G^{\mathrm{sc}}}^{w}$ for all $w \in \mathrm{Cox}(W, S)$. Thus by uniqueness of $\mathcal{J}^w$ we see that $\mathcal{J}^w$ and $\mathcal{J}^{w'}$ are isomorphic over $\mathfrak{C}_+^{\mathrm{rs}}$.

The complement of $\mathfrak{C}_{G_+^{\mathrm{sc}}}$ is the union of the closure of codimension 1 stratas in $\mathfrak{C}_+$. Since the idempotent $e_\varnothing$ is regular semisimple and belongs each of the strata closure we see that on each strata, the regular semisimple locus is nonempty open. Hence the complement of $\mathfrak{C}_{G_+^{\mathrm{sc}}} \cup \mathfrak{C}_+^{\mathrm{rs}} \subset \mathfrak{C}_+$ has codimension at least 2.

Consequently there is a unique commutative smooth group scheme $\mathcal{J}$ over $\mathfrak{C}_+$ which comes with a unique isomorphism $(\chi_+^{\mathrm{reg}})^*\mathcal{J} \cong \mathcal{I}|_{\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}}$. We know from Lemma 2.3.1 that this isomorphism extends uniquely to a homomorphism between $\chi_+^*\mathcal{J}$ and $\mathcal{I}$ over the open subset $G_+^{\mathrm{sc}} \cup \mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}$ whose complement has codimension at least 2. Hence it extends further to the whole space $\mathrm{Vin}_{G^{\mathrm{sc}}}$. □

Proposition 2.4.3. *The classifying stack $\mathbb{B}\mathcal{J}$ acts naturally on $[\mathrm{Vin}_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)]$. The action preserves the open substacks $[\mathrm{Vin}_{G^{\mathrm{sc}}}^0/\mathrm{Ad}(G)]$, $[\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}/\mathrm{Ad}(G)]$ and $[\mathrm{Vin}_{G^{\mathrm{sc}}}^{w}/\mathrm{Ad}(G)]$ for each $w \in \mathrm{Cox}(W, S)$. Moreover, the morphism*

$$[\chi_+^w] : [\mathrm{Vin}_{G^{\mathrm{sc}}}^{w}/\mathrm{Ad}(G)] \to \mathfrak{C}_+$$

induced by χ_+ is a $\mathbb{B}\mathcal{J}$ gerbe, neutralized by the extended Steinberg section ϵ_+^w.

The proof is the same as [23, Proposition 2.2.1].

Proposition 2.4.4. *The number of irreducible components of the fibers of the map*

$$\chi_+^{\mathrm{reg}} : \mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}} \to \mathfrak{C}_+$$

is bounded above by $|\mathrm{Cox}(W, S)|$ and equality is achieved at $\mathcal{N}^{\mathrm{reg}} = (\chi_+^{\mathrm{reg}})^{-1}(0)$.

Proof. The first statement follows from (2.7). The second statement is in Proposition 2.2.9. □

Remark 2.4.5. Consequently, unless all simple factors of G^{sc} are SL_2, the action of $\mathbb{B}\mathcal{J}$ on $[\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}/\mathrm{Ad}(G)]$ is not transitive. In other words, $[\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}/\mathrm{G}]$ is not a $\mathbb{B}\mathcal{J}$-gerbe, but rather a finite union of $\mathbb{B}\mathcal{J}$ gerbes as in Proposition 2.4.3. This is different from Lie algebra situation, cf [23, Proposition 2.2.1].

2.4.6. Galois description of universal centralizer. Let $\prod_{\overline{T_+^{\mathrm{sc}}}/\mathfrak{C}_+}(T \times \overline{T_+^{\mathrm{sc}}})$ be the restriction of scalar which associates to any $\mathfrak{C}_+$-scheme S the set

$$\prod_{\overline{T_+}/\mathfrak{C}_+}(T \times \overline{T_+})(S) = \mathrm{Hom}_{\overline{T_+}}(S \times_{\mathfrak{C}_+} \overline{T_+}, T \times \overline{T_+}).$$

Then W acts diagonally on $\prod_{\overline{T_+}/\mathfrak{C}_+}(T' \times \overline{T_+})$ and we consider its fixed point subscheme

$$\mathcal{J}^1 := \left(\prod_{\overline{T_+}/\mathfrak{C}_+} T \times \overline{T_+} \right)^W .$$

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The following is proved in [5, Proposition 11].

Proposition 2.4.7. *$\mathcal{J}^1$ is a smooth commutative group scheme over $\mathfrak{C}_+$. Moreover, there exists an open embedding $\mathcal{J} \to \mathcal{J}^1$ whose restriction to $\mathfrak{C}_+^{\mathrm{rs}}$ is an isomorphism.*

2.5. Arc space of Vinberg monoid

In this section, we assume that G is semisimple and simply connected.

For each $\lambda \in X_*(T_{\mathrm{ad}})_+$, define the affine scheme Vin_G^λ over $\mathrm{Spec}\,\mathcal{O}$ by the following Cartesian diagram

$$\begin{tikzcd} \mathrm{Vin}_G^\lambda \arrow[r] \arrow[d] & \mathrm{Vin}_G \times T_{\mathrm{ad}} \arrow[d] \\ \mathrm{Spec}\,\mathcal{O} \arrow[r, "\varpi^{-w_0(\lambda)}"] & A_G \end{tikzcd}$$

where the right vertical arrow is the product of the abelianization map α_G and the natural embedding $T_{\mathrm{ad}} \hookrightarrow A_G$, the lower horizontal arrow corresponds to the point $\varpi^{-w_0(\lambda)} \in A_G(\mathcal{O})$. Replacing Vin_G by its open subscheme Vin_G^0 in the above diagram, we define an open subscheme $\mathrm{Vin}_G^{\lambda,0} \subset \mathrm{Vin}_G^\lambda$.

There is a stratification of the space of nondegenerate arcs of $A_G \supset T_{\mathrm{ad}}$ by $T_{\mathrm{ad}}(\mathcal{O})$ orbits:

$$A_G(\mathcal{O}) \cap T_{\mathrm{ad}}(F) = \bigsqcup_{\lambda \in X_*(T_{\mathrm{ad}})_+} T_{\mathrm{ad}}(\mathcal{O})\varpi^{-w_0(\lambda)}.$$

The inverse image of $T_{\mathrm{ad}}(\mathcal{O})\varpi^{-w_0(\lambda)}$ under the abelianization map is precisely $L^+\mathrm{Vin}_G^\lambda(k)$. In other words we get a stratification of the space of nondegenerate arcs of $\mathrm{Vin}_G \supset G_+$ into $G_+(\mathcal{O})$-stable pieces:

$$\mathrm{Vin}_G(\mathcal{O}) \cap G_+(F) = \bigsqcup_{\lambda \in X_*(T_{\mathrm{ad}})_+} L^+\mathrm{Vin}_G^\lambda(k).$$

Also we note that

$$L^+\mathrm{Vin}_G^{\lambda,0}(k) = L^+\mathrm{Vin}_G^\lambda(k) \cap \mathrm{Vin}_G^0(\mathcal{O}).$$

Lemma 2.5.1. *For any $g_+ \in G_+(F)$, we have $g_+ \in L^+\mathrm{Vin}_G^\lambda$ if and only if $\alpha(g_+) \in \varpi^{-w_0(\lambda)}T_{\mathrm{ad}}(\mathcal{O})$ and the image of g_+ in $G_{\mathrm{ad}}(F)$ belongs to*

$$\overline{G_{\mathrm{ad}}(\mathcal{O})\varpi^\lambda G_{\mathrm{ad}}(\mathcal{O})} = \bigcup_{\substack{\mu \in X_*(T_{\mathrm{ad}})^+ \\ \mu \leqslant \lambda}} G_{\mathrm{ad}}(\mathcal{O})\varpi^\mu G_{\mathrm{ad}}(\mathcal{O}).$$

Moreover, $g_+ \in L^+\mathrm{Vin}_G^{\lambda,0}$ if and only if $\alpha(g_+) \in \varpi^{-w_0(\lambda)}T_{\mathrm{ad}}(\mathcal{O})$ and the image of g_+ in $G_{\mathrm{ad}}(F)$ belongs to the double coset $G_{\mathrm{ad}}(\mathcal{O})\varpi^\lambda G_{\mathrm{ad}}(\mathcal{O})$.

Proof. The coweight lattice for T_+ can be expressed as

$$X_+(T_+) = \{(\lambda_1, \lambda_2) \in X_*(T_{\mathrm{ad}}) \times X_*(T_{\mathrm{ad}}) | \lambda_1 + \lambda_2 \in X_*(T)\}.$$

For $(\lambda_1, \lambda_2) \in X_*(T_+)$, we have $\varpi^{(\lambda_1,\lambda_2)} \in L^+\mathrm{Vin}_G^\lambda$ if and only if

- $\alpha(\varpi^{(\lambda_1,\lambda_2)}) \in \varpi^{-w_0(\lambda)} T_{\mathrm{ad}}(\mathcal{O})$ and
- The matrix $\rho^+_{\omega_i}(\varpi^{(\lambda_1,\lambda_2)}) \in \mathrm{End}(V_{\omega_i})$ has entries in $\mathcal{O}$ for all $1 \leqslant i \leqslant r$.

Since $\alpha(\varpi^{(\lambda_1,\lambda_2)}) = \varpi^{\lambda_1}$, the first condition means that $\lambda_1 = -w_0(\lambda)$. Then the second condition means that

$$\langle (-w_0(\lambda), \lambda_2), \chi_+\rangle \geqslant 0$$

for all $1 \leqslant i \leqslant r$ and all weights χ_+ in the G_+-representation $\rho^+_{\omega_i}$. Since the weights of the representation $\rho^+_{\omega_i}$ lie in the convex hull of the W-orbit of the highest weight (ω_i, ω_i) where W acts on the second factor, the above inequality is equivalent to

$$\langle -w_0(\lambda), \omega_i\rangle + \langle \lambda_2, w(\omega_i)\rangle = \langle (-w_0(\lambda), \lambda_2), (\omega_i, w(\omega_i))\rangle \geqslant 0$$

for all $w \in W$ and $1 \leqslant i \leqslant r$. This can be further reformulated as

$$\langle \lambda - w(\lambda_2), \omega_i\rangle \geqslant 0$$

for all $w \in W$ and $1 \leqslant i \leqslant r$.

By the discussion so far, we have

$$L^+\mathrm{Vin}^{\lambda}_G(k) \cap T_+(F) = \bigcup_{\substack{\mu\in X_*(T_{\mathrm{ad}})\\ \mu_{\mathrm{dom}}\leqslant\lambda}} \varpi^{(-w_0(\lambda),\mu)} T_+(\mathcal{O})$$

where μ_{dom} denotes the unique dominant coweight in the W-orbit of μ. As $L^{\lambda}\mathrm{Vin}_G$ is stable under the action of $G_+(\mathcal{O}) \times G_+(\mathcal{O})$, it is a union of $G_+(\mathcal{O})$ double cosets in $G_+(F)$. Thus by Cartan decomposition we get

$$L^+\mathrm{Vin}^{\lambda}_G(k) = \bigsqcup_{\substack{\mu\in X_*(T_{\mathrm{ad}})^+\\ \mu\leqslant\lambda}} G_+(\mathcal{O})\varpi^{(-w_0(\lambda),\mu)} G_+(\mathcal{O}).$$

Similarly we can get a description of $L^{\lambda}\mathrm{Vin}^0_G$. The difference is that we require furthermore that $\rho^+_{\omega_i}(\varpi^{(-w_0(\lambda),\lambda_2)})$ have nonzero reduction mod ϖ for all $1 \leqslant i \leqslant r$. Hence besides the inequality $\langle \lambda - w(\lambda_2), \omega_i\rangle \geqslant 0$ for all $w \in W$ and $1 \leqslant i \leqslant r$, we require furthermore that for each i, there exists $w \in W$ such that $\langle \lambda - w(\lambda_w), \omega_i\rangle = 0$. This condition means that λ_2 is in the W-orbit of λ and hence

$$L^+\mathrm{Vin}^{\lambda,0}_G(k) = G_+(\mathcal{O})\varpi^{(-w_0(\lambda),\lambda)} G_+(\mathcal{O}).$$

From these descriptions the lemma follows. □

Lemma 2.5.2. *Suppose*

$$n \geqslant b(\lambda) := \max_{1\leqslant i\leqslant r} \langle \lambda, \omega_i - w_0(\omega_i)\rangle.$$

Then for all $\gamma, \gamma' \in L^+\mathrm{Vin}^{\lambda}_G(k)$ *having the same image in* $\mathrm{Vin}_G(\mathcal{O}/\varpi^n\mathcal{O})$, *there exists* $g \in G_+(\mathcal{O})$ *such that* $\gamma' = \gamma g$.

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Proof. The following argument is due to Zhiwei Yun. Let $i \mapsto i^*$ be the involution on the set $\{1, \ldots, r\}$ such that $\omega_{i^*} = -w_0(\omega_i)$. For each $1 \leqslant i \leqslant r$, there is a natural pairing between V_i and V_{i^*} such that for all $x \in G_+$, $v \in V_i$ and $v^* \in V_{i^*}$, we have

$$\langle \rho_i^+(x)v, \rho_{i^*}^+(x)v^* \rangle = (\omega_i + \omega_{i^*})(\alpha(x))\langle v, v^* \rangle.$$

Thus for each $x \in G_+(F)$, under the natural pairing above, the lattice $\rho_i^+(x)V_i(\mathcal{O})$ in $V_i(F)$ is dual to the lattice

$$(\omega_i + \omega_{i^*})(\alpha(x)^{-1})\rho_{i^*}^+(g)V_{i^*}(\mathcal{O}) \subset V_{i^*}(F).$$

For $\gamma \in L^+\mathrm{Vin}_G^\lambda \subset \mathrm{Vin}_G(\mathcal{O})$, we have $\rho_i^+(\gamma)V_i(\mathcal{O}) \subset V_i(\mathcal{O})$ for all $1 \leqslant i \leqslant r$. Taking duals, we get

$$V_{i^*}(\mathcal{O}) \subset \varpi^{-\langle \lambda, \omega_i + \omega_{i^*} \rangle}\rho_{i^*}^+(\gamma)V_{i^*}(\mathcal{O}).$$

In other words, we have shown that for all $1 \leqslant i \leqslant r$,

$$\varpi^{\langle \lambda, \omega_i + \omega_{i^*} \rangle}V_i(\mathcal{O}) \subset \rho_i^+(\gamma)V_i(\mathcal{O}) \subset V_i(\mathcal{O}).$$

Thus if γ and γ' have the same image in $\mathrm{Vin}_G(\mathcal{O}/\varpi^n\mathcal{O})$ for $n \geqslant b(\lambda)$, the lattices $\rho_i^+(\gamma)V_i(\mathcal{O})$ and $\rho_i(\gamma')V_i(\mathcal{O})$ are the same and hence $\gamma' = \gamma g$ for some $g \in G_+(\mathcal{O})$. $\square$

2.6. Local lifting property of extended Steinberg map

We review certain infinitesimal lifting property of the extended Steinberg morphism χ_+ needed later in § 4.4. This is based on some result of Gabber–Ramero in [11]. Our exposition below follows the treatment in [5] and [36].

2.6.1. We start by recalling certain results in [11, § 5.4]. Let A be a ring and B an A-algebra of finite presentation. Let $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ be the natural morphism. Choose a presentation $B \cong P/J$ where $P := A[X_1, \ldots, X_N]$ and $J \subset P$ is a finitely generated ideal. Then the map f is factored as the composition of a closed embedding $i : \mathrm{Spec}\, B \hookrightarrow \mathbb{A}_A^N$ and the natural projection $p : \mathbb{A}_A^N \to \mathrm{Spec}\, A$. Define the ideal

$$H_A(P, J) := \mathrm{Ann}_P \mathrm{Ext}_B^1(\mathbb{L}_{B/A}, J/J^2) \subset P$$

where $\mathbb{L}_{B/A}$ is the cotangent complex of the morphism χ. Notice that $J \subset H_A(P, J)$.

Consider the ideal in B defined by $H_{B/A} := H_A(P, J)B = H_A(P, J)/J$. Let $\Sigma_f := \mathrm{Spec}\, B/H_{B/A}$ be the closed subscheme of $\mathrm{Spec}\, B$ defined by $H_{B/A}$. We remark that $H_{B/A}$ depends on the choice of presentation $B \cong P/J$. The following is [11, Lemma 5.4.2]:

Lemma 2.6.2. (1) *For all B-module N, $H_{B/A}$ annihilates* $\mathrm{Ext}_B^1(\mathbb{L}_{B/A}, N)$.

(2) *The complement of $\Sigma_f = \mathrm{Spec}\, B/H_{B/A}$ in $\mathrm{Spec}\, B$ is the smooth locus of the morphism $\chi : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$.*

(3) *For any A-algebra A', let $B' = B \otimes_A A'$ and $f' : \mathrm{Spec}\, B' \to \mathrm{Spec}\, A'$ be the induced morphism. Define the ideal $H_{B'/A'} \subset B'$ in the same way as above, using the presentation of B' induced from $B \cong P/J$. Then we have $H_{B/A}B' \subset H_{B'/A'}$, or in other words $\Sigma_{f'} \subset \Sigma_f \times_Z Z'$.*

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2.6.3. When $A = k[[\varpi]]$, we define the *conductor* of f to be the smallest integer h such that $\varpi^h \in H_{B/A}$. Note that the conductor depends on the presentation of B.

The next lemma is the key step in establishing the local lifting result. To state it we let R be an artin local k-algebra with maximal ideal $\mathfrak{m}$ and $I \subset R$ an ideal with $I \cdot \mathfrak{m} = 0$. Suppose $A = R[[\varpi]]$. Let $B = P/J$ be a finitely presented A-algebra and $f : \operatorname{Spec} B \to \operatorname{Spec} A$ the induced morphism as above. Let f_0 be the reduction of f mod $\mathfrak{m}$ and $h \geqslant 0$ the conductor of f_0.

Lemma 2.6.4. *Suppose $n \geqslant h$ and $\sigma : \operatorname{Spec} A/\varpi^n I \to \operatorname{Spec} B$ is a morphism such that the composition $f \circ \sigma$ is the natural embedding $\operatorname{Spec} A/\varpi^n I \hookrightarrow \operatorname{Spec} A$. Then there exists a section $\tilde{\sigma} : \operatorname{Spec} A \to \operatorname{Spec} B$ of f such that the restriction of $\tilde{\sigma}$ to $\operatorname{Spec} A/\varpi^{n-h} I$ coincides with σ.*

Proof. The obstruction of extending σ to $\operatorname{Spec} A$ is an element $\omega \in \operatorname{Ext}^1_B(\mathbb{L}_{B/A}, \varpi^n I)$ where we view $\varpi^n I$ as a B-module via the map $\sigma^* : B \to A/\varpi^n I$. By the definition of conductor h, we have $\varpi^h \in H_{B/A} + \mathfrak{m}B$. By Lemma 2.6.2(1) and the assumption that $\mathfrak{m} \cdot I = 0$ we see that $\varpi^h \omega = 0$. This implies that the image of ω in $\operatorname{Ext}^1_B(\mathbb{L}_{B/A}, \varpi^{n-h} I)$ vanishes by noticing that the multiplication map $\varpi^h : \varpi^n I \to \varpi^n I$ can be factored as the composition of the natural embedding $\varpi^n I \hookrightarrow \varpi^{n-h} I$ and an isomorphism $\varpi^{n-h} I \cong \varpi^n I$. Hence we get the desired lifting of the restriction of σ to $\operatorname{Spec} A/\varpi^{n-h} I$. □

2.6.5. We apply the general discussion above to the situation where $\operatorname{Spec} A = \mathfrak{C}_+$, $\operatorname{Spec} B = \mathrm{Vin}_G$ and $f = \chi_+$. Choose a presentation $B = P/J$ where $P = A[X_1, \ldots, X_N]$ and $J \subset P$ is a finitely generated ideal as above. Recall that we have the discriminant divisor $\mathfrak{D}_+ \subset \mathfrak{C}_+$ defined by the extended discriminant function Disc_+. By Corollary 2.2.17, χ_+ is smooth over $\mathfrak{C}_+ - \mathfrak{D}_+$. Hence $\chi_+(\Sigma_{\chi_+})$ is contained in $\mathfrak{D}_+$ set-theoretically by Lemma 2.6.2(2).

Since $\mathfrak{D}_+$ is a principal divisor, there exists a positive integer m_0 (depending on the presentation $B \cong P/J$) such that $\chi_+(\Sigma_{\chi_+}) \subset m_0 \mathfrak{D}_+$ scheme-theoretically. To state the main result in this section, we consider an artin local k-algebra R with maximal ideal $\mathfrak{m}$ and let $I \subset R$ be an ideal such that $I \cdot \mathfrak{m} = 0$.

Proposition 2.6.6. *Let $\delta \in \mathrm{Vin}_G(R[[\varpi]])$ and $a_0 \in \mathfrak{C}_+(k[[\varpi]])$ be the reduction mod $\mathfrak{m}$ of $\chi_+(\delta)$. Let $d := \mathrm{val}(a_0^* \mathfrak{D}_+)$ be the discriminant valuation of a_0 (suppose that d is a finite number). Then for any integer $N \geqslant m_0 d$ and any $a \in \mathfrak{C}_+(R[[\varpi]])$ such that $a \equiv \chi_+(\delta)$ mod $\varpi^N I$, there exists $\gamma \in \mathrm{Vin}_G(R[[\varpi]])$ such that $\chi_+(\gamma) = a$ and $\gamma \equiv \delta \mod \varpi^{N - m_0 d} I$.*

Proof. Consider the following diagram in which the right square is Cartesian

$$\begin{tikzcd} & V_a \arrow[r] \arrow[d, "\chi_a"] & \mathrm{Vin}_G \arrow[d, "\chi_+"] \\ \operatorname{Spec}(R[[\varpi]]/\varpi^N I) \arrow[r] \arrow[ur, "\delta"] & \operatorname{Spec} R[[\varpi]] \arrow[r, "a"] & \mathfrak{C}_+ \end{tikzcd}$$

Also, let $\chi_{a_0} : V_{a_0} \to \operatorname{Spec} k[[\varpi]]$ be the reduction mod $\mathfrak{m}$ of χ_a. Let h be the conductor of χ_{a_0}. By Lemma 2.6.2(3), we have $\Sigma_{\chi_{a_0}} \subset V_{a_0} \cap \Sigma_{\chi_+}$. Since $\chi_+(\Sigma_{\chi_+}) \subset m_0 \mathfrak{D}_+$, we have $h \leqslant m_0 d$. By Lemma 2.6.4, there exists a section γ of χ_a such that the restriction of γ

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to Spec $R[[\varpi]]/\varpi^{N-m_0d}$ coincides with δ. Thus the element in $\mathrm{Vin}_G(R[[\varpi]])$ determined by γ is the lifting we want. □

3. Kottwitz–Viehmann varieties

We fix a connected reductive group G. Let $T \subset G$ be a maximal torus and $\lambda \in X_*(T)_+$ a dominant coweight. Let $\gamma \in G^{\mathrm{rs}}(F)$ be a regular semisimple element.

We study the following sets associated to the pair (γ, λ), which we both refer to as *Kottwitz–Viehmann varieties*:

$$X_\gamma^\lambda = \{g \in G(F)/G(\mathcal{O}) | \mathrm{Ad}(g)^{-1}(\gamma) \in G(\mathcal{O})\varpi^\lambda G(\mathcal{O})\}$$
$$X_\gamma^{\leqslant\lambda} = \{g \in G(F)/G(\mathcal{O}) | \mathrm{Ad}(g)^{-1}(\gamma) \in \overline{G(\mathcal{O})\varpi^\lambda G(\mathcal{O})}\}$$

3.1. Nonemptiness

The first immediate question is when the sets $X_\gamma^\lambda, X_\gamma^{\leqslant\lambda}$ are nonempty. To answer this we need to recall the notion of Newton points and Kottwitz map.

3.1.1. Newton Points. Following [20, § 4], for each $\gamma \in G(F)^{\mathrm{rs}}$, one associates a rational dominant coweight $\nu_\gamma \in X_*(T)^+_{\mathbb{Q}}$, called the *Newton point of* γ. Let us recall its definition. There is an integer $e \geqslant 1$ such that γ is $G(F_e)$ conjugate to an element in $T(F_e)$. Consider the W-orbit of its image in $T(F_e)/T(\mathcal{O}_e) = \frac{1}{e}X_*(T)$. The unique dominant element in this W-orbit is the Newton point ν_γ. Here F_e is the totally ramified extension of F of degree e and $\mathcal{O}_e \subset F_e$ is its ring of integers.

Definition 3.1.2. The *discriminant valuation* for $\gamma \in G(F)^{\mathrm{rs}}$ is defined by

$$d(\gamma) := \mathrm{val}\det(\mathrm{Id} - \mathrm{ad}_\gamma : \mathfrak{g}(F)/\mathfrak{g}_\gamma(F) \to \mathfrak{g}(F)/\mathfrak{g}_\gamma(F))$$

where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}_\gamma$ is the centralizer of γ, i.e., the fixed locus of the adjoint action ad_γ.

Lemma 3.1.3. *Let $\gamma \in G(F)^{\mathrm{rs}}$ and $\nu_\gamma \in \Lambda^+_{\mathbb{Q}}$ its Newton point. Let $\bar{\gamma} \in T(\bar{F})^{\mathrm{rs}}$ be a $G(\bar{F})$-conjugate of γ such that $\mathrm{val}(\alpha(\gamma)) \geqslant 0$ for all positive roots α. Then we have*

$$d(\gamma) = 2\sum_{\alpha\in\Phi^+} \mathrm{val}(\alpha(\bar{\gamma}) - 1) - \langle 2\rho, \nu_\gamma\rangle$$

where we have extended the valuation on F to its separable closure $\bar{F}$.

Proof. From the definition we see that

$$d(\gamma) = \sum_{\alpha\in\Phi} \mathrm{val}(\alpha(\gamma) - 1).$$

Separate the sum over Φ according to whether $\langle\alpha, \nu_\gamma\rangle = 0$ or not, then we get

$$d(\gamma) = \sum_{\substack{\alpha\in\Phi \\ \langle\alpha,\nu_\gamma\rangle=0}} \mathrm{val}(\alpha(\bar{\gamma}) - 1) + \sum_{\substack{\alpha\in\Phi \\ \langle\alpha,\nu_\gamma\rangle<0}} \langle\alpha, \nu_\gamma\rangle. \tag{3.1}$$

By our assumption that $\mathrm{val}(\alpha(\gamma)) \geqslant 0$ for $\alpha \in \Phi^+$, the first term in (3.1) equals to

$$2 \sum_{\substack{\alpha\in\Phi^+ \\ \langle\alpha,\nu_\gamma\rangle=0}} \mathrm{val}(\alpha(\gamma)-1) = 2\sum_{\alpha\in\Phi^+} \mathrm{val}(\alpha(\gamma)-1)$$

while the second term of (3.1) equals to

$$\sum_{\alpha\in\Phi^-} \langle\alpha, \nu_\gamma\rangle = -\sum_{\alpha\in\Phi^+} \langle\alpha, \nu_\gamma\rangle = -\langle 2\rho, \nu_\gamma\rangle.$$

Hence the lemma follows. □

3.1.4. Kottwitz map. Let $\pi_1(G) := X_*(T)/X_*(T^{\mathrm{sc}})$ be the quotient of the coweight lattice by the coroot lattice and $p_G : X_*(T) \to \pi_1(G)$ be the canonical projection. Following [20], one defines a group homomorphism

$$\kappa_G : G(F) \to \pi_1(G)$$

which we refer to as Kottwitz homomorphism. Note that in *loc. cit.*, this map is denoted by w_G. As pointed out by the referee, the map κ can be understood more explicitly as follows: let $G(F)_1$ be the subgroup of $G(F)$ generated by all parahoric subgroups, then κ_G induces an isomorphism $G(F)/G(F)_1 \cong \pi_1(G)$.

Lemma 3.1.5. *Suppose that $\kappa_G(\gamma) = p_G(\lambda)$. Then there exists an element $\gamma_\lambda \in G^{\mathrm{sc}}_+(F)$ such that*

- *the image of γ_λ in $G_{\mathrm{ad}}(F)$ coincides with the image of γ in $G_{\mathrm{ad}}(F)$;*
- *$\alpha(\gamma_\lambda) = \varpi^{-w_0(\lambda_{\mathrm{ad}})} \in T_{\mathrm{ad}}(F) \cap A_{G^{\mathrm{sc}}}(\mathcal{O})$ where $\lambda_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})^+$ is the image of $\lambda \in X_*(T)^+$.*

Moreover, γ_λ is uniquely determined up to multiplication by an element in $Z_{G^{\mathrm{sc}}}(F)$.

Proof. Let $\gamma_{\mathrm{ad}} \in G_{\mathrm{ad}}(F)$ be the image of γ. Choose any $\tilde{\gamma} \in G^{\mathrm{sc}}_+(F)$ that maps to γ_{ad}. Suppose $\alpha(\tilde{\gamma}) \in \varpi^{\mu} T_{\mathrm{ad}}(\mathcal{O})$ for $\mu \in X_*(T_{\mathrm{ad}})^+$. By the assumption $\kappa_G(\gamma) = p_G(\lambda)$, we have $\lambda_{\mathrm{ad}} - \mu \in X_*(T)$. Let $\gamma_\lambda := \varpi^{\lambda_{\mathrm{ad}}-\mu}\tilde{\gamma}$ where we view $\varpi^{\lambda_{\mathrm{ad}}-\mu} \in T(F) = Z_+(F)$ as a central element in $G^{\mathrm{sc}}_+(F)$. Then we have $\alpha(\gamma_\lambda) = \varpi^{-w_0(\lambda_{\mathrm{ad}})}$ and the image of γ_λ in $G_{\mathrm{ad}}(F)$ equals to γ_{ad}.

Suppose $\gamma_\lambda, \gamma'_\lambda \in G^{\mathrm{sc}}_+(F)$ both satisfy the requirement of the Lemma. Then $\gamma'_\lambda\gamma_\lambda^{-1} \in G^{\mathrm{sc}}(F)$ and its image in $G_{\mathrm{ad}}(F)$ is the identity. Hence $\gamma'_\lambda\gamma_\lambda^{-1} \in Z_{G^{\mathrm{sc}}}(F)$. □

Now we can state the nonemptiness criteria.

Proposition 3.1.6. *The following are equivalent:*

(1) *X^λ_γ is nonempty;*

(2) *$X^{\leqslant\lambda}_\gamma$ is nonempty;*

(3) *$\kappa_G(\gamma) = p_G(\lambda)$ and $\nu_\gamma \leqslant_{\mathbb{Q}} \lambda$, i.e., $\lambda - \nu_\gamma$ is a $\mathbb{Q}$-linear combination of simple coroots with non-negative coefficients;*

(4) *$\kappa_G(\gamma) = p_G(\lambda)$ and $\chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$, where $\gamma_\lambda \in G^{\mathrm{sc}}_+(F)$ is defined in Lemma 3.1.5.*

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Proof. The implication '(1) $\Rightarrow$ (2)' is tautological. The implication '(1) $\Rightarrow$ (3)' is done in [20, Corollary 3.6].

(3) $\Rightarrow$ (4): Let F'/F be a finite extension of degree e so that γ (and hence γ_λ) is split in $G(F')$. Let $\varpi' = \varpi^{\frac{1}{e}}$ be a uniformizer of F' and $\mathcal{O}' = k[[\varpi']] \subset F'$ be the ring of integers. Then $e \cdot \nu_\gamma \in X_*(T)_+$ and γ is $G(F')$-conjugate to an element in $(\varpi')^{e\cdot\nu_\gamma} T(\mathcal{O}')$. From (3) we deduce that γ_λ is $G_+^{\mathrm{sc}}(F')$-conjugate to an element in $\mathrm{Vin}_G^{\mathrm{sc}}(\mathcal{O}')$. Therefore

$$\chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O}') \cap \mathfrak{C}_+(F) = \mathfrak{C}_+(\mathcal{O}).$$

(2) $\Rightarrow$ (4): Let $g \in X_\gamma^{\leqslant\lambda}$. Then $\mathrm{Ad}(g)^{-1}(\gamma) \in G(\mathcal{O})\varpi^\mu G(\mathcal{O})$ for some $\mu \in X_*(T)_+$ with $\mu \leqslant \lambda$. Then we have $\kappa_G(\gamma) = p_G(\mu) = p_G(\lambda)$. In particular, we can define the element $\gamma_\lambda \in G_+^{\mathrm{sc}}(F)$ as in Lemma 3.1.5. Then by Lemma 2.5.1 we have

$$\mathrm{Ad}(g)^{-1}(\gamma_\lambda) \in L^\lambda \mathrm{Vin}_{G^{\mathrm{sc}}} \subset \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O}).$$

Thus $\chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$.

(4) $\Rightarrow$ (1): Let $a_+ := \chi_+(\gamma_\lambda)$. So $a \in \mathfrak{C}_+(\mathcal{O})$ by condition (4). Then for any Coxeter element $w \in \mathsf{Cox}(W, S)$ (cf. Definition 2.2.1), we have $\epsilon_+^w(a) \in \mathrm{Vin}_{G^{\mathrm{sc}}}^0(\mathcal{O})$. It remains to show that there exists $h \in G(F)$ such that $\mathrm{Ad}(h)^{-1}(\gamma_\lambda) = \epsilon_\lambda^w(a)$ (because then h defines a point in $\in X_\gamma^\lambda$). To see this, notice that the transporter from γ to $\epsilon_+^w(a)$ in G is a torsor under the torus G_{γ_λ} over F. Any such torsor is trivial since $H^1(F, G_{\gamma_\lambda})$ by a theorem of Steinberg (using the fact that the residue field k is algebraic closed). Thus the transporter has an F-point $h \in G(F)$. $\square$

3.2. Ind-scheme structure

3.2.1. First approach. We will equip the sets X_γ^λ and $X_\gamma^{\leqslant\lambda}$ with an ind-scheme structure. We present two approaches, one based on the original definition, the other using Vinberg monoid.

Let $\mathrm{Gr}_G := LG/L^+G$ be the affine Grassmannian for G, which are known to be ind-projective ind-scheme over k. The positive loop group L^+G acts by left multiplication on Gr_G. Let $(LG)_\lambda := L^+G\varpi^\lambda L^+G$ (respectively $(LG)_{\leqslant\lambda}$) be the k-scheme whose set of k-points is $G(\mathcal{O})\varpi^\lambda G(\mathcal{O})$ (respectively $\overline{G(\mathcal{O})\varpi^\lambda G(\mathcal{O})}$).

Definition 3.2.2. Let $\mathcal{X}_\gamma^\lambda$ be the k-functor which associates to any k-algebra R the set

$$\mathcal{X}_\gamma^\lambda(R) = \{g \in \mathrm{Gr}_G(R) | g^{-1}\gamma g \in (LG)_\lambda(R)\}.$$

Also, we define the k-functor $\mathcal{X}_\gamma^{\leqslant\lambda}$ by replacing $(LG)_\lambda$ with $(LG)_{\leqslant\lambda}$ in the above definition

By definition, $\mathcal{X}_\gamma^{\leqslant\lambda}$ is a closed sub-ind-scheme of Gr_G and $\mathcal{X}_\gamma^\lambda$ is an open sub-ind-scheme of $\mathcal{X}_\gamma^{\leqslant\lambda}$. Let X_γ^λ (respectively $X_\gamma^{\leqslant\lambda}$) be the reduced structure of $\mathcal{X}_\gamma^\lambda$ (respectively $\mathcal{X}_\gamma^{\leqslant\lambda}$).

3.2.3. Second approach. Now we use Vinberg monoids to define certain analogue of affine Springer fibers, which turns out to be isomorphic to Kottwitz–Viehmann varieties.

Recall that $\mathfrak{C}_+$ denotes the extended Steinberg base for $\mathrm{Vin}_{G^{\mathrm{sc}}}$. Let $a \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_+(F)^{\mathrm{rs}}$ and suppose that

$$\beta(a) \in \varpi^{-w_0(\lambda_{\mathrm{ad}})} T_{\mathrm{ad}}(\mathcal{O}) \subset A_{G^{\mathrm{sc}}}(\mathcal{O}) \cap T_{\mathrm{ad}}(F)$$

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where $\lambda_{\rm ad} \in X_*(T_{\rm ad})$ is the image of $\lambda \in X_*(T)$. Moreover, let $\gamma_+ \in G_+^{\rm sc}(F)$ be an element such that $\chi_+(\gamma_+) = a$.

Definition 3.2.4. The *generalized affine Springer fiber* $\mathcal{S}p_{G,\gamma_+}$ associates to any k-algebra R the set of isomorphism classes of pairs (h, ι) where h is the horizontal arrow in the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R[[\varpi]] \arrow[r, "h"] \arrow[dr, "a"'] & {[\mathrm{Vin}_{G^{\rm sc}}/\mathrm{Ad}(G)]} \arrow[d] \\ & \mathfrak{C}_+ \end{tikzcd}$$

and ι is an isomorphism between the restriction of h to $\operatorname{Spec} R((\varpi))$ and the composition

$$\operatorname{Spec} R((\varpi)) \xrightarrow{\gamma_+} \mathrm{Vin}_{G^{\rm sc}} \to [\mathrm{Vin}_{G^{\rm sc}}/\mathrm{Ad}(G)].$$

Also, we define k-functors $\mathcal{S}p^0_{G,\gamma_+}$ (respectively $\mathcal{S}p^{\rm reg}_{G,\gamma_+}$, $\mathcal{S}p^w_{G,\gamma_+}$) by replacing $\mathrm{Vin}_{G^{\rm sc}}$ with $\mathrm{Vin}^0_{G^{\rm sc}}$ (respectively $\mathrm{Vin}^{\rm reg}_{G^{\rm sc}}$, $\mathrm{Vin}^w_{G^{\rm sc}}$, c.f. § 2.4.1).

By definition $\mathcal{S}p_{G,\gamma_+}$ is a closed sub-ind-scheme of Gr_G and $\mathcal{S}p^w_{G,\gamma_+} \subset \mathcal{S}p^{\rm reg}_{G,\gamma_+} \subset \mathcal{S}p^0_{G,\gamma_+}$ are its open sub-ind-schemes. We let Sp_{G,γ_+} (respectively $\mathrm{Sp}^0_{G,\gamma_+}$, $\mathrm{Sp}^{\rm reg}_{G,\gamma_+}$, $\mathrm{Sp}^w_{G,\gamma_+}$) be the reduced structures of $\mathcal{S}p_{G,\gamma_+}$ (respectively $\mathcal{S}p^0_{G,\gamma_+}$, $\mathcal{S}p^{\rm reg}_{G,\gamma_+}$, $\mathcal{S}p^w_{G,\gamma_+}$). Concretely, the k-points of $\mathrm{Sp}^0_{G,\gamma_+}$ (respectively $\mathrm{Sp}^{\rm reg}_{G,\gamma_+}$, $\mathrm{Sp}^w_{G,\gamma_+}$) consist of $g \in \mathrm{Sp}_{G,\gamma_+}$ such that $g^{-1}\gamma_\lambda g \in \mathrm{Vin}^0_{G^{\rm sc}}(\mathcal{O})$ (respectively $\mathrm{Vin}^{\rm reg}_{G^{\rm sc}}(\mathcal{O})$, $\mathrm{Vin}^w_{G^{\rm sc}}(\mathcal{O})$).

The isomorphism classes of Sp_{G,γ_+} (respectively $\mathrm{Sp}^0_{G,\gamma_+}$, $\mathrm{Sp}^{\rm reg}_{G,\gamma_+}$, $\mathrm{Sp}^w_{G,\gamma_+}$) only depend on $a = \chi_+(\gamma_+)$, so we will also denote them by $\mathrm{Sp}_{G,a}$ (respectively $\mathrm{Sp}^0_{G,a}$, $\mathrm{Sp}^{\rm reg}_{G,a}$, $\mathrm{Sp}^w_{G,a}$). If the group G is clear from the context, we will drop it in the notation.

Next we relate the two definitions given above. Let (γ, λ) be as in the beginning of this chapter. Suppose that the ind-scheme X^λ_γ is nonempty. Then by Proposition 3.1.6 we have $\kappa_G(\gamma) = p_G(\lambda)$ and $a := \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$ where $\gamma_\lambda \in G_+^{\rm sc}(F)$ is defined in Lemma 3.1.5. It is not hard to see that

$$X^\lambda_\gamma \cong \mathrm{Sp}^0_a \quad \text{and} \quad X^{\leqslant\lambda}_\gamma \cong \mathrm{Sp}_a.$$

Conversely, let $a \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_+(F)^{\rm rs}$ and suppose that

$$\beta(a) \in \varpi^{-w_0(\lambda)} T_{\rm ad}(\mathcal{O}) \subset A_{G^{\rm sc}}(\mathcal{O}) \cap T_{\rm ad}(F)$$

for some $\lambda \in X_*(T_{\rm ad})_+$. Let $\gamma^w_a \in G_{\rm ad}(F)$ be the image of $\epsilon^w_+(a) \in G_+(F) \cap \mathrm{Vin}^0_{G^{\rm sc}}(\mathcal{O})$ under the natural quotient $G_+(F) \to G_{\rm ad}(F)$. Then we have

$$\mathrm{Sp}_a \cong X^{\leqslant\lambda}_{\gamma^w_a} \quad \text{and} \quad \mathrm{Sp}^0_a \cong X^\lambda_{\gamma^w_a}.$$

Note that the isomorphism class of $X^{\leqslant\lambda}_{\gamma^w_a}$ and $X^\lambda_{\gamma^w_a}$ does not depend on the choice of $w \in \mathrm{Cox}(W, S)$.

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3.3. Symmetries

Assume X^λ_γ is nonempty. Then by Proposition 3.1.6 we have $\kappa_G(\gamma) = p_G(\lambda)$ and

$$a = \chi_+(\gamma_\lambda) \in \mathfrak{C}^{\mathrm{rs}}_{G^{\mathrm{sc}}_+}(F) \cap \mathfrak{C}_+(\mathcal{O}).$$

Let J_a be the commutative group scheme over $\operatorname{Spec}\mathcal{O}$ obtained by pulling back $\mathcal{J}$ along a : $\operatorname{Spec}\mathcal{O} \to \mathfrak{C}_+$. Since a is generically regular semisimple, there is a canonical isomorphism $LJ_a \cong LG^0_\gamma$ which allows us to identify the positive loop group L^+J_a as a subgroup of LG^0_γ. Consider the quotient group

$$P_a := LJ_a/L^+J_a \cong LG^0_\gamma/L^+J_a.$$

In other words, $\mathcal{P}_a$ is the affine Grassmannian of J_a classifying isomorphism classes of J_a-torsors on $\operatorname{Spec}\mathcal{O}$ with a trivialization of its restriction to $\operatorname{Spec} F$.

The loop group LG^0_γ acts naturally on X^λ_γ and this action factors through P_a. Using the isomorphism $X^\lambda_\gamma \cong \mathrm{Sp}^0_a$, the P_a action is induced by the $\mathbb{B}\mathcal{J}$ action on $[V^0_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)]$ in Proposition 2.4.3. Moreover, P_a preserve the open subspaces $\mathrm{Sp}^{\mathrm{reg}}_a$ and Sp^w_a for each $w \in \mathrm{Cox}(W, S)$.

Proposition 3.3.1. *For each $w \in \mathrm{Cox}(W, S)$, Sp^w_a is a torsor under P_a.*

Proof. This is a consequence of 2.4.3. □

Remark 3.3.2. Unlike the Lie algebra case, $\mathrm{Sp}^{\mathrm{reg}}_a$ may not be a P_a-torsor in general. See the discussion in § 3.9.10.

Let R_a be the finite free $\mathcal{O}$-algebra defined by the Cartesian diagram

$$\begin{tikzcd} \widetilde{X}_a := \operatorname{Spec} R_a \arrow[r] \arrow[d] & \overline{T_+} \arrow[d] \\ \operatorname{Spec}\mathcal{O} \arrow[r, "a"] & \mathfrak{C}_+ \end{tikzcd} \tag{3.2}$$

Let $R^\flat_a$ be the normalization of R_a and $\widetilde{X}^\flat_a := \operatorname{Spec} R^\flat_a$. Then W acts naturally on the $\mathcal{O}$-algebras R_a and $R^\flat_a$.

Let $J^\flat_a$ be the finite type Neron model of J_a. Hence $J^\flat_a$ is a smooth commutative group scheme over $\mathcal{O}$ such that $J^\flat_a(F) = J_a(F) = G^0_\gamma(F)$ and $J^\flat_a(\mathcal{O})$ is the maximal bounded subgroup of $G^0_\gamma(F)$.

Lemma 3.3.3. *There is a canonical isomorphism*

$$J^\flat_a \cong \left(\prod_{R^\flat_a/\mathcal{O}} T \times \widetilde{X}^\flat_a\right)^W$$

Proof. The proof is the same as [23, Proposition 3.8.2]. □

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Corollary 3.3.4. $\mathrm{Lie}(P_a) = (\mathfrak{t} \otimes_k (R_a^\flat/R_a))^W$.

Proof. The quotient $L^+J_a^\flat/L^+J_a$ is an open subgroup of P_a. Hence we have isomorphism of $\mathcal{O}$ modules

$$\mathrm{Lie} P_a \cong \mathrm{Lie}(L^+J_a^\flat)/\mathrm{Lie}(L^+J_a).$$

On the other hand, by 2.4.7, we have

$$\mathrm{Lie} L^+J_a = (\mathfrak{t} \otimes_k R_a)^W$$

and by 3.3.3,

$$\mathrm{Lie} L^+J_a^\flat = (\mathfrak{t} \otimes_k R_a^\flat)^W.$$

Hence the Corollary follows. □

3.4. Admissible subsets of loop spaces

In this section, we closely follow [12, § 5].

Let M be a standard Levi subgroup of G and $P = MN$ the standard parabolic subgroup where N is the unipotent radical of P. Let $Z(M)^0$ be the neutral component of the center of M. Then $Z(M)^0$ is a subtorus of T. Let Φ_N be the set of roots of $Z(M)^0$ acting on N and $\Phi_N^\vee$ the corresponding set of coroots. For each $\alpha \in \Phi_N$, let N_α be the corresponding root subgroup. Then each N_α is isomorphic to a product of several copies of $\mathbb{G}_a$ and is preserved by the adjoint action of M. Denote by δ_N the half sum of elements in $\Delta_N^\vee$.

For each $\alpha \in \Delta_N$, denote $\mathrm{ht}_N(\alpha) := \langle \delta_N, \alpha \rangle$. Let $l = \max_{\alpha \in \Phi_N} \mathrm{ht}_N(\alpha)$. For each $1 \leqslant i \leqslant l$, let $N[i]$ be the subgroup of N generated by root groups N_α with $\mathrm{ht}_N(\alpha) \geqslant i$. Also we denote $N[l+1] = 1$. Then $N[1] = N$ and for each $1 \leqslant i \leqslant s+1$, $N[i]$ is a normal subgroup of N and the successive quotients $N\langle i \rangle := N[i]/N[i+1]$ are commutative groups isomorphic to products of some copies of $\mathbb{G}_a$. Let LN and L^+N be the loop space and arc space of N. For each integer $n \geqslant 0$, let $N_n := \ker(L^+N \to L_n^+N)$. Then $\{N_n\}_{n \geqslant 0}$ form a decreasing sequence of compact open subgroups of LN.

For each $\gamma \in M(F) \cap G(F)^{\mathrm{rs}}$, consider the map

$$\begin{aligned} f_\gamma : LN &\longrightarrow LN \\ u &\longmapsto u^{-1}\gamma u \gamma^{-1}. \end{aligned} \tag{3.3}$$

Then f_γ preserves the root subgroups N_α and hence each normal subgroup $N[i]$. In particular, f_γ induces morphism $f_\gamma[i] : LN[i] \to LN[i]$ and $f_\gamma\langle i \rangle : LN\langle i \rangle \to LN\langle i \rangle$.

For each $1 \leqslant i \leqslant l$, denote $r_i := \mathrm{val}\det(f_\gamma\langle i \rangle)$. Note that there is a M-equivariant isomorphism $N\langle i \rangle \cong \mathrm{Lie} N\langle i \rangle$ from which we see that

$$r_i = \mathrm{val}\det(\mathrm{ad}_\gamma : \mathrm{Lie} N\langle i \rangle(F) \to \mathrm{Lie} N\langle i \rangle(F)).$$

Consider the following invariant of γ:

$$r_N(\gamma) := \mathrm{val}\det(\mathrm{ad}_\gamma : \mathrm{Lie} N(F) \to \mathrm{Lie} N(F)). \tag{3.4}$$

Then we also have $r_N(\gamma) = \sum_{i=1}^{l} r_i$.

Now assume that $\gamma \in M(F)_+$, we have $f_\gamma(U_n) \subset U_n$ for all $n \geqslant 0$.

Let $f_{0,\gamma} : L^+N \to L^+N$ be the restriction of f_γ to the arc space L^+N.

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Lemma 3.4.1. *For any $1 \leqslant i \leqslant l+1$ and any positive integer n such that $n \geqslant \sum_{j=i}^{l+1} r_j$ we have $N[i]_n \subset f_\gamma(L^+N[i])$.*

Proof. We prove by descending induction on i. The case $i = l+1$ is trivial since $N[l+1] = 1$. Assume the statement is true for $i+1$. Let $x \in N[i]_n$. To show that $x \in f_\gamma(L^+N[i])$ it suffices to find $u \in N[i](\mathcal{O})$ with $x * u = 1$, for then $f_\gamma(u^{-1}) = x$.

Let $x_i \in N\langle i\rangle_n$ be the image of x. Since $\operatorname{val}\det(f_\gamma\langle i\rangle) = r_i$, we have

$$\varpi^{r_i} N\langle i\rangle(\mathcal{O}) \subset f_\gamma\langle i\rangle(N\langle i\rangle(\mathcal{O})).$$

Hence there exists $u_i \in N[i]_{n-r_i}$ such that $x_i * u_i = 1$ in $N\langle i\rangle(\mathcal{O})$ and hence $x * u_i \in N[i+1]_{n-r_i}$. By induction hypothesis, there exists $v \in N[i+1](\mathcal{O})$ such that $(x * u_i) * v = 1$. Then $u = u_i v$ satisfies $x * u = 1$. □

A subset of L^+N is *admissible* if it is the pre-image of a locally closed subset of L_n^+N for some n. A subset Z of LN is *admissible* if it is conjugate under $G(F)$ to an admissible subset of L^+N.

Lemma 3.4.2. *Let V be an admissible subset of L^+N. Let $n \geqslant r_N(\gamma)$ be a positive integer such that V is right invariant under N_n. Suppose moreover that $V \subset f_{0,\gamma}(L^+N)$. Then the set $f_{0,\gamma}^{-1}(V)$ is admissible and right invariant under N_n. Moreover, $f_{0,\gamma}$ induces a smooth surjective map*

$$f_{0,\gamma}^{-1}(V)/N_n \to V/N_n$$

whose fibers are isomorphic to $\mathbb{A}^{r_N(\gamma)}$.

Proof. Let $\bar{f}_{0,\gamma} : L_n^+N \to L_n^+N$ be the map induced by $f_{0,\gamma}$. Since V is right invariant under N_n, a straightforward calculation shows that $f_{0,\gamma}^{-1}(V)$ is also right invariant under N_n. Denote $\overline{V} := V/U_n$. Then we have $f_{0,\gamma}^{-1}(V)/U_n = \bar{f}_{0,\gamma}^{-1}(\overline{V})$, a locally closed subset of L_n^+N. In particular, $f_{0,\gamma}^{-1}(V)$ is admissible. Since $V \subset f_{0,\gamma}(L^+N)$, the induced map $\bar{f}_{0,\gamma}^{-1}(\overline{V}) \to \overline{V}$ is surjective and it remains to show that it is smooth with fibers isomorphic to $\mathbb{A}^{r(\gamma)}$.

Denote $H := L_n^+N$, $H[i] := L_n^+(N[i])$ and $H\langle i\rangle := L_n^+(N\langle i\rangle)$. Then for each $1 \leqslant i \leqslant l+1$, $H[i]$ is a normal subgroup of H and $H[i]/H[i+1] \cong H\langle i\rangle$. For each $1 \leqslant j \leqslant n$, we define a normal subgroup $H_j := \ker(H \to L_j^+N)$ of H; and similarly we define normal subgroups $H[i]_j$ (respectively $H\langle i\rangle_j = \varpi^j H\langle i\rangle$) of $H[i]$ (respectively $H\langle i\rangle$).

Consider the right action of H on itself defined by $v * u := u^{-1} v \gamma u \gamma^{-1}$ for $u, v \in H(k) = N(\mathcal{O}/\varpi^n\mathcal{O})$. Then $\bar{f}_{0,\gamma}(u) = 1 * u$ and hence $\bar{f}_0$ is the orbit map at 1 of the H-action. In particular, all fibers of $\bar{f}_{0,\gamma}$ are isomorphic to the stabilizer $S := \bar{f}_{0,\gamma}^{-1}(1)$.

Now we take a closer look at the structure of the stabilizer S. First note that the action $*$ induces actions of $H[i]$ and $H\langle i\rangle$ on themselves. Let $S[i]$ (respectively $S\langle i\rangle$) be the stabilizer of 1 under the $H[i]$ (respectively $H\langle i\rangle$) action.

We claim that for all i, the canonical homomorphism $S[i] \to S\langle i\rangle$ is surjective. Let $s \in S\langle i\rangle$ and choose a representative $h \in H[i]$ of s. Since

$$S\langle i\rangle = \ker(\bar{f}_{0,\gamma}\langle i\rangle) \subset \varpi^{n-r_i} H\langle i\rangle$$

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we have $h \in H[i]_{n-r_i}$ and $1 * h \in H[i]_{n-r_i} \cap H[i+1] = H[i+1]_{n-r_i}$. By assumption $n - r_i \geqslant \sum_{j=i+1}^{l+1} r_j$, then we can apply Lemma 3.4.1 to obtain an element $h' \in H[i+1]$ such that $1 * (hh') = 1$. Thus $hh' \in S[i]$ maps to $s \in S\langle i \rangle$ and the claim follows.

The kernel of the surjective homomorphism $S[i] \to S\langle i \rangle$ is $S[i] \cap H[i+1] = S[i+1]$. Moreover, we have

$$S\langle i \rangle \cong (f_0\langle i \rangle)^{-1}(\varpi^n N\langle i \rangle)/\varpi^n N\langle i \rangle \cong \mathbb{A}^{r_i}$$

From this we see that $S \cong \mathbb{A}^{r_N(\gamma)}$ as a scheme. □

The proof of the following lemma is inspired by [20, Lemma 3.8].

Lemma 3.4.3. *For any $n \geqslant r_N(\gamma)$, we have $f_\gamma^{-1}(N_n) \subset N_{n-r_N(\gamma)}$.*

Proof. Let $u \in N(F)$ with $f_\gamma(u) \in N_n$. We will show by induction that

$$u \in N[i](F) \cdot N_{n-\sum_{j<i} r_j}.$$

The case $i = 1$ says $u \in N[1](F) = N(F)$ which is clear and the case $i = s + 1$ gives the lemma since $\sum_{i=1}^{s} r_i = r_N(\gamma)$ and $N[s+1] = 1$.

It remains to finish the induction step. By induction hypothesis we have $u = u_i v$ with $u_i \in N[i](F)$ and $v \in N_{n-\sum_{j<i} r_j}$. By assumption,

$$f_\gamma(u) = f_\gamma(u_i v) = v^{-1} \cdot u_i^{-1} \gamma u_i \gamma^{-1} \cdot \gamma v \gamma^{-1} \in N_n$$

from which it follows that

$$u_i^{-1} \gamma u_i \gamma^{-1} \in N[i](F) \cap v \cdot N_n \cdot (\gamma v^{-1} \gamma^{-1}) \subset N[i]_{n-\sum_{j<i} r_j}.$$

Let $\bar{u}_i \in N\langle i \rangle$ be the image of u_i. Then we have

$$f_\gamma\langle i \rangle(\bar{u}_i) \in N\langle i \rangle_{n-\sum_{j<i} r_j}.$$

Since $\operatorname{val}\det(f_\gamma\langle i \rangle) = r_i$, we get that $\bar{u}_i \in N\langle i \rangle_{n-\sum_{j<i+1} r_j}$ and hence

$$u = u_i v \in N[i+1](F) \cdot N_{n-\sum_{j<i+1} r_j}.$$

This finishes the induction step. □

Proposition 3.4.4. *Let Z be an admissible subset of the loop space LN. Then $f_\gamma^{-1}(Z)$ is admissible and there exists a positive integer m such that for all $n \geqslant m$, $f_\gamma^{-1}(Z)$ and Z are right invariant under the group N_n and the map*

$$f_\gamma^{-1}(Z)/N_n \to Z/N_n$$

induced by f_γ is smooth surjective whose geometric fibers are irreducible of dimension $r_N(\gamma)$.

Proof. Let $n_0 \geqslant r(\gamma)$ be a positive integer. Choose a coweight $\mu_0 \in X_*(Z(M)^0)$ such that

$$Z^{\mu_0} := \operatorname{Ad}(\varpi^{\mu_0})(Z) \subset N_{n_0}.$$

Then by Lemma 3.4.3 we have

$$f_\gamma^{-1}(Z^{\mu_0}) \subset f_\gamma^{-1}(N_{n_0}) \subset N_{n_0 - r(\gamma)} \subset L^+N.$$

Hence in particular

$$\mathrm{Ad}(\varpi^{\mu_0})(f_\gamma^{-1}(Z)) = f_\gamma^{-1}(Z^{\mu_0}) = f_0^{-1}(Z^{\mu_0}).$$

Moreover, since Z^{μ_0} is an admissible subset of L^+N, $f_0^{-1}(Z^{\mu_0})$ is an admissible subset of L^+N by Lemma 3.4.2. This shows that $f_\gamma^{-1}(Z)$ is admissible.

Let $n_1 > n_0$ be a positive integer such that Z^{μ_0} and $f_\gamma^{-1}(Z^{\mu_0})$ are invariant under right multiplication by N_{n_1}. For all $n \geqslant n_1$, since the map f_γ commutes with conjugation by ϖ^{μ_0}, Z and $f_\gamma^{-1}(Z)$ are right invariant under the group $N_n^{-\mu_0} := \varpi^{-\mu_0} N_n \varpi^{\mu_0}$. Then we get the following commutative diagram

$$\begin{tikzcd} f_\gamma^{-1}(Z)/N_n^{-\mu_0} \arrow[r] \arrow[d, "\simeq"] & Z/N_n^{-\mu_0} \arrow[d, "\simeq"] \\ f_\gamma^{-1}(Z^{\mu_0})/N_n \arrow[r] & Z^{\mu_0}/N_n \end{tikzcd}$$

where the horizontal arrows are induced by f_γ and the vertical arrows are isomorphisms induced by $\mathrm{Ad}(\varpi^{\mu_0})$.

By Lemma 3.4.1, $Z^{\mu_0} \subset N_{n_0} \subset f_\gamma(L^+N)$. Therefore we can apply Lemma 3.4.2 to conclude that the lower horizontal map is surjective smooth whose fibers are isomorphic to $\mathbb{A}^{r_N(\gamma)}$. Hence the same is true for the upper horizontal map.

Let m be a positive integer such that for all $n \geqslant m$, $N_n \supset N_{n'}^{-\mu_0}$ for some $n' \geqslant n_1$. Consider the following diagram

$$\begin{tikzcd} f_\gamma^{-1}(Z)/N_{n'}^{-\mu_0} \arrow[r] \arrow[d] & Z/N_{n'}^{-\mu_0} \arrow[d] \\ f_\gamma^{-1}(Z)/N_n \arrow[r] & Z/N_n \end{tikzcd}$$

The two vertical maps are smooth surjective with fibers isomorphic to the irreducible scheme $U_n/U_{n'}^{-\mu_0}$ and the upper horizontal map is smooth surjective with fibers isomorphic to $\mathbb{A}^{r_N(\gamma)}$ as we have just seen. Hence the lower horizontal map is smooth surjective with irreducible fibers of dimension $r_N(\gamma)$. $\square$

3.5. The case of unramified conjugacy class

In this section, we assume that $\gamma \in G(F)^{\mathrm{rs}}$ is an *unramified* regular semisimple element. Since the residue field k is algebraically closed, after conjugation we may assume that $\gamma \in \varpi^\mu T(\mathcal{O}) \cap G^{\mathrm{rs}}(F)$, where $\mu = \nu_\gamma \in X_*(T)_+$ is the Newton points of γ. In this case, we have $G_\gamma^0 = T$. By Lemma 3.1.3 the discriminant valuation for γ is

$$d(\gamma) = 2 \sum_{\alpha \in \Phi^+} \mathrm{val}(\alpha(\gamma) - 1) - \langle 2\rho, \mu \rangle.$$

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We will apply the results in the previous section to the case $N = U$ is a maximal unipotent subgroup. In this case, the corresponding invariant for γ is

$$r(\gamma) := r_U(\gamma) = \sum_{\alpha\in\Phi^+} \mathrm{val}(\alpha(\gamma) - 1) = \frac{1}{2}d(\gamma) + \langle\rho, \mu\rangle. \tag{3.5}$$

Fix a dominant coweight $\lambda \in \Lambda_+$ such that $\mu \leqslant \lambda$. By Proposition 3.1.6, this implies that X_γ^λ is nonempty.

3.5.1. Relation with MV cycles. Let Y_γ^λ be the locally closed sub-ind-scheme of X_γ^λ whose set of k-points is

$$Y_\gamma^\lambda(k) = \{u \in U(F)/U(\mathcal{O}) | \mathrm{Ad}(u)^{-1}\gamma \in G(\mathcal{O})\varpi^\lambda G(\mathcal{O})\}.$$

To understand the structure of Y_γ^λ, we use the map $f_\gamma : LU \to LU$ (cf. (3.3)). In the following, we denote $K := L^+G$. Then we have

$$Y_\gamma^\lambda = (f_\gamma^{-1}(K\varpi^\lambda K\varpi^{-\mu} \cap LU)/L^+U.$$

Recall the Mirkovic–Vilonen cycles in the affine Grassmannian:

$$S_\mu \cap \mathrm{Gr}_\lambda = (LU\varpi^\mu K \cap K\varpi^\lambda K)/K.$$

From this description we get an isomorphism

$$\begin{aligned} (LU \cap K\varpi^\lambda K\varpi^{-\mu})/\varpi^\mu L^+U\varpi^{-\mu} &\longrightarrow S_\mu \cap \mathrm{Gr}_\lambda \\ u &\longmapsto u\varpi^\mu. \end{aligned} \tag{3.6}$$

In summary, we have the following diagram

$$\begin{tikzcd} f_\gamma^{-1}(K\varpi^\lambda K\varpi^{-\mu} \cap LU) \arrow[r, "f_\gamma"] \arrow[d] & K\varpi^\lambda K\varpi^{-\mu} \cap LU \arrow[d] \\ Y_\gamma^\lambda & S_\mu \cap \mathrm{Gr}_\lambda \end{tikzcd}$$

where the left vertical arrow is an L^+U-torsor and the right vertical arrow is a torsor under the group $\varpi^\mu L^+U\varpi^{-\mu}$.

Theorem 3.5.2. *Y_γ^λ is an equidimensional quasi-projective variety of dimension $\langle\rho, \lambda\rangle + \frac{1}{2}d(\gamma)$, where $d(\gamma)$ is the discriminant valuation, cf. Definition 3.1.2. Moreover, the number of irreducible components of Y_γ^λ equals to $m_{\lambda\mu}$, the dimension of μ-weight space in the irreducible representation V_λ of $\hat{G}$ with highest weight λ.*

Proof. Apply Proposition 3.4.4 to the admissible subset $Z = K\varpi^\lambda K\varpi^{-\mu} \cap LU$ of LU, we see that there exists a large enough positive integer n such that in the following diagram

$$\begin{tikzcd} f_\gamma^{-1}(K\varpi^\lambda K\varpi^{-\mu} \cap LU)/U_n \arrow[r, "\bar{f}_\gamma"] \arrow[d] & (K\varpi^\lambda K\varpi^{-\mu} \cap LU)/U_n \arrow[d] \\ Y_\gamma^\lambda & S_\mu \cap \mathrm{Gr}_\lambda \end{tikzcd}$$

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(1) All schemes are of finite type;

(2) the map $\bar{f}_\gamma$ induced by f_γ is smooth surjective whose geometric fibers are irreducible of dimension $r(\gamma)$, where we recall that $r(\gamma)$ is defined in (3.5);

(3) U_n is contained in $\varpi^\mu L^+U\varpi^{-\mu}$, hence also L^+U;

(4) the left vertical map is smooth surjective with fibers isomorphic to the irreducible scheme L^+U/U_n;

(5) the right vertical map is smooth with fibers isomorphic to the irreducible scheme $\varpi^\mu L^+U\varpi^{-\mu}/U_n$.

Since Y_γ^λ is of finite type, it is a locally closed subscheme of a closed Schubert variety. In particular, Y_γ^λ is quasi-projective since closed Schubert varieties are projective.

Recall that the MV cycle $S_\mu \cap \mathrm{Gr}_\lambda$ is equidimensional of dimension $\langle \rho, \lambda+\mu\rangle$. Hence by (2)–(5) we see that Y_γ^λ is equidimensional of dimension

$$\begin{aligned}\dim Y_\gamma^\lambda &= \dim(S_\mu \cap \mathrm{Gr}_\lambda) + \dim \varpi^\mu U(\mathcal{O})\varpi^{-\mu}/U_n^{-\lambda_0} + r(\gamma) - \dim U(\mathcal{O})/U_n^{-\lambda_0} \\ &= \langle \rho, \lambda+\mu\rangle - \langle 2\rho, \mu\rangle + r(\gamma) = \langle \rho, \lambda\rangle + \tfrac{1}{2}d(\gamma). \end{aligned} \tag{3.7}$$

Moreover, by [30, Tag 037A] the three maps in the diagram above induce a canonical bijection between the set of irreducible components

$$\mathrm{Irr}(Y_\gamma^\lambda) \xrightarrow{\sim} \mathrm{Irr}(S_\mu \cap \mathrm{Gr}_\lambda).$$

Hence the number of irreducible components of Y_γ^λ equals the number of irreducible components of the MV cycle $S_\mu \cap \mathrm{Gr}_\lambda$, which is known to be $m_{\lambda\mu}$. □

Corollary 3.5.3. *Suppose $\gamma \in G(F)^{\mathrm{rs}}$ is unramified (i.e., split) and $\nu_\gamma = \mu \in X_*(T)_+$, then X_γ^λ is a scheme locally of finite type, equidimensional of dimension*

$$\dim X_\gamma^\lambda = \langle \rho, \lambda\rangle + \tfrac{1}{2}d(\gamma).$$

Moreover, the number of $G_\gamma^0(F)$-orbits on its set of irreducible component $\mathrm{Irr}(X_\gamma^\lambda)$ *equals to $m_{\lambda\mu}$.*

Proof. There is a natural morphism

$$\begin{aligned} Y_\gamma^\lambda \times X_*(T) &\longrightarrow X_\gamma^\lambda \\ (u, \nu) &\longmapsto u\varpi^\nu \end{aligned}$$

which induces bijection on k-points and a stratification of X_γ^λ such that each strata is isomorphic to Y_γ^λ. Thus X_γ^λ is a scheme locally of finite type and the assertions about equidimensionality and dimension formula follow from the corresponding statements for Y_γ^λ.

The LG_γ^0 action on the set $\mathrm{Irr}(X_\gamma^\lambda)$ factors through $\pi_0(LG_\gamma^0) = X_*(T)$ and hence LG_γ-orbits on $\mathrm{Irr}(X_\gamma^\lambda)$ correspond bijectively to the set $\mathrm{Irr}(Y_\gamma^\lambda)$. Thus the number of orbits equals to the weight multiplicity $m_{\lambda\mu}$. □

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3.6. Finiteness of Kottwitz–Viehmann varieties

In this section, we let $\gamma \in G(F)^{\mathrm{rs}}$ be any regular semisimple element and $\lambda \in \Lambda^+$. Assume without loss of generality that X_γ^λ is nonempty and $\det(\gamma) = \det(\varpi^\lambda)$. Then we get an element $\gamma_\lambda \in \mathrm{Vin}_G^\lambda(F)$ as in Lemma 3.1.5. Moreover, the Newton point of γ satisfies $\nu_\gamma \leqslant_{\mathbb{Q}} \lambda$ and $\chi(\gamma) \in \mathfrak{C}_{\leqslant\lambda}$ by Proposition 3.1.6.

We show in this section that X_γ^λ, a priori an ind-scheme, is actually a scheme locally of finite type. This has already been proved for unramified conjugacy classes in Corollary 3.5.3. It remains to reduce the general case to the unramified case. This reduction step is completely analogous to the Lie algebra case. For the reader's convenience, we include the details, following the exposition in [35, § 2.5]. See also [3].

Let F'/F be a finite extension of degree e so that γ splits over F'. Let $\varpi' = \varpi^{1/e} \in F'$ be a uniformizer and $\mathcal{O}' = k[[\varpi']]$ the ring of integers in F'. Let σ be a generator of the cyclic group $\mathrm{Gal}(F'/F)$.

Choose $h \in G(F')$ such that $\mathrm{Ad}(h)G_\gamma^0 = T$. Then $h\sigma(h)^{-1} \in N_G(T)(F')$ and we let $w \in W$ be its image.

Consider the embedding

$$\begin{aligned} \iota_\gamma : \Lambda := X_*(T) &\longrightarrow G_\gamma(F') \\ \mu &\longmapsto \mathrm{Ad}(h)^{-1}\varpi^\mu. \end{aligned}$$

Let $\Lambda_\gamma := \iota_\gamma^{-1}(G_\gamma(F))$. It follows immediately that $\Lambda_\gamma \subset \Lambda^w$ where Λ^w is the fixed point set of w on Λ. Moreover, Λ_γ can be identified with the coweight lattice of the maximal F-split subtorus of G_γ. In particular, $(\Lambda_\gamma)_{\mathbb{Q}} = (\Lambda^w)_{\mathbb{Q}}$ so that $\Lambda_\gamma \subset \Lambda^w$ is a subgroup of finite index.

Proposition 3.6.1. *There exists a closed subscheme $Z \subset X_\gamma^\lambda$ which is projective over k such that $X_\gamma^\lambda = \bigcup_{\ell\in\Lambda_\gamma} \ell\cdot Z$. Here $\ell \in \Lambda_\gamma$ acts on X_γ^λ via the embedding ι_γ.*

Proof. We rephrase the argument in [35, § 2.5.7]. Let $\widetilde{X}_\gamma^{e\lambda}$ be the generalized affine Springer fiber of coweight $e\lambda$ for γ in $\mathrm{Gr}_{G_{F'}}$, the affine Grassmannian of $G_{F'}$. Then σ acts naturally on $\widetilde{X}_\gamma^{e\lambda}$ and the fixed points sub-ind-scheme $(\widetilde{X}_\gamma^{e\lambda})^\sigma$ contains X_γ^λ (but they are not equal in general). Let $\gamma' = h\gamma h^{-1} \in T(F')$ and $\widetilde{X}_{\gamma'}^{e\lambda}$ be the corresponding generalized affine Springer fiber in $\mathrm{Gr}_{G_{F'}}$. Then

$$\widetilde{X}_{\gamma'}^{e\lambda} = h\cdot\widetilde{X}_\gamma^{e\lambda}.$$

By Theorem 3.5.2, there is a locally closed subscheme $\widetilde{Y}_{\gamma'}^{e\lambda}$ of $\widetilde{X}_{\gamma'}^{e\lambda}$ such that

$$\widetilde{X}_{\gamma'}^{e\lambda} = \bigcup_{\ell\in\Lambda} \ell\cdot Y_{\gamma'}^{e\lambda}.$$

Let $\widetilde{Z}$ be the closure of $h^{-1}\widetilde{Y}_{\gamma'}^{e\lambda}$ in $\widetilde{X}_\gamma^{e\lambda}$. Then $\widetilde{Z}$ is projective over k and $\widetilde{X}_\gamma^{e\lambda} = \bigcup_{\ell\in\Lambda} \ell\cdot\widetilde{Z}$.

Recall that $w \in W$ is represented by $h\sigma(h)^{-1}$. One can check that $\sigma(\widetilde{Z}) = \widetilde{Z}$ and more generally $\sigma(\ell\cdot\widetilde{Z}) = w(\ell)\cdot\widetilde{Z}$ for all $\ell \in \Lambda$. Consequently,

$$(\widetilde{X}_\gamma^{e\lambda})^\sigma = \bigcup_{\ell\in\Lambda^w} \ell\cdot\widetilde{Z} = \bigcup_{\ell\in\Lambda_\gamma} \ell\cdot(C\cdot\widetilde{Z}),$$

where $C \subset \Lambda^w$ is a finite set of representatives of the quotient Λ^w/Λ_γ. Hence, $C \cdot \widetilde{Z}$ is a finite type scheme.

Finally let $Z := (C \cdot \widetilde{Z}) \cap X_\gamma^\lambda$. Then Z is a finite type subscheme of X_γ^λ. Hence Z is projective over k and $X_\gamma^\lambda = \bigcup_{\ell \in \Lambda_\gamma} \ell \cdot Z$. □

As a consequence, we immediately get the following.

Theorem 3.6.2. *The ind-scheme X_γ^λ is a finite-dimensional k-scheme, locally of finite type. Moreover, the lattice Λ_γ acts freely on X_γ^λ and the quotient $X_\gamma^\lambda/\Lambda_\gamma$ is representable by a proper algebraic space over k.*

3.7. Dimension of the regular locus

Recall that the regular locus $X_\gamma^{\lambda,\mathrm{reg}}$ is an open subscheme of X_γ^λ on which the action of $P_a = LG_\gamma^0/L^+J_a$ is free (but not necessarily transitive).

Theorem 3.7.1.
$$\dim P_a = \dim X_\gamma^{\lambda,\mathrm{reg}} = \langle \rho, \lambda \rangle + \frac{d(\gamma) - c(\gamma)}{2}$$
where

- $d(\gamma) := \mathrm{val}(\det(\mathrm{Id} - \mathrm{ad}(\gamma) : \mathfrak{g}(F)/\mathfrak{g}_\gamma(F) \to \mathfrak{g}(F)/\mathfrak{g}_\gamma(F)))$.
- $c(\gamma) := \mathrm{rank}(G) - \mathrm{rank}_F G_\gamma$, *where* $\mathrm{rank}_F G_\gamma$ *is the dimension of the maximal F-split subtorus of G_γ.*

Moreover, $X_\gamma^{\lambda,\mathrm{reg}}$ is equidimensional.

Proof. The first equality follows from the fact that the P_a-orbits in $X_\gamma^{\lambda,\mathrm{reg}}$ are open and the action is free.

When γ is unramified (hence split as k is algebraically closed), the second equality follows from Corollary 3.5.3. It remains to reduce to this case. The argument is similar to that of Bezrukavnikov's in Lie algebra case, cf. [2], which we reformulate using the Galois description of universal centralizer.

Let A be the finite free $\mathcal{O}$-algebra defined by the Cartesian diagram (3.2) and $A^\flat$ the normalization of A. Then W acts naturally on the $\mathcal{O}$-algebras A and $A^\flat$ and by 3.3.4, we get
$$\dim \mathcal{P}_a = \dim_k(\mathfrak{t} \otimes_k (A^\flat/A))^W.$$
Let $\tilde{F}/F$ be a ramified extension of degree e, with ring of integers $\tilde{\mathcal{O}} = k[[\varpi^{\frac{1}{e}}]]$, such that γ is split over $\tilde{F}$. Let σ be a generator of the cyclic group $\Gamma := \mathrm{Gal}(F'/F)$. Let $\tilde{A} := A \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ and $\tilde{A}^\flat$ its normalization. We remark that $\tilde{A}^\flat$ is not the same as $A^\flat \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ in general. Let $\tilde{\mathcal{P}}_a = LG_{\gamma,F'}/L^+J_{a,F'}$. Then by the dimension formula in split case, we have
$$\dim_k(\mathfrak{t} \otimes_k \tilde{A}^\flat/\tilde{A})^W = \dim \tilde{\mathcal{P}}_a = \langle \rho, e\lambda \rangle + \tfrac{1}{2} e \cdot d(\gamma).$$
As γ split over $\tilde{\mathcal{O}}$, we have
$$\tilde{A}^\flat \cong \tilde{\mathcal{O}}[W] := \tilde{\mathcal{O}} \otimes_k k[W]$$

as W-module. Here W acts on $\tilde{\mathcal{O}}[W]$ via right regular representation. Moreover, there exists an element $w_\gamma \in W$ of order e such that under the above isomorphism, the natural action of $\sigma \in \Gamma$ on $\tilde{A}^\flat$ becomes $\sigma \otimes l_{w_\gamma}$ where l_{w_γ} denotes the left regular action of w_γ on $k[W]$. In particular, the action of W and Γ commutes with each other. With these considerations, we obtain an isomorphism

$$(\mathfrak{t} \otimes_k \tilde{A}^\flat)^W \cong \mathfrak{t} \otimes_k \tilde{\mathcal{O}}$$

which intertwines the action of $\sigma \in \Gamma$ on the left hand side with the action of $w \otimes \sigma$ on the right hand side.

Moreover, we have an equality

$$(\mathfrak{t} \otimes_k \tilde{A}^\flat)^\Gamma = \mathfrak{t} \otimes_k A^\flat$$

which remains true after taking W-invariants since the Γ action commutes with W action. In particular, we have

$$M := (\mathfrak{t} \otimes_k \tilde{\mathcal{O}})^\Gamma = (\mathfrak{t} \otimes_k A^\flat)^W.$$

Moreover, it is clear from the definition of W action that

$$(\mathfrak{t} \otimes_k \tilde{A})^W = (\mathfrak{t} \otimes_k A)^W \otimes_{\mathcal{O}} \tilde{\mathcal{O}}.$$

Thus we get

$$\begin{aligned} \dim \mathcal{P}_a &= \dim_k (\mathfrak{t} \otimes_k A^\flat / A)^W = \frac{1}{e} \dim_k (\mathfrak{t} \otimes_k (A^\flat/A) \otimes_{\mathcal{O}} \tilde{\mathcal{O}})^W = \frac{1}{e} \dim_k \left(\frac{M \otimes_{\mathcal{O}} \tilde{\mathcal{O}}}{(\mathfrak{t} \otimes_k \tilde{A})^W} \right) \\ &= \langle \rho, \lambda \rangle + \frac{1}{2} d(\gamma) - \frac{1}{e} \dim_k \left(\frac{\mathfrak{t} \otimes_k \tilde{\mathcal{O}}}{M \otimes_{\mathcal{O}} \tilde{\mathcal{O}}} \right). \end{aligned} \tag{3.8}$$

Since the element $w_\gamma \in W$ has order e, its eigenvalues are eth roots of unit. Let ζ be a primitive eth root of unit and $\mathfrak{t}(i)$ the subspace of $\mathfrak{t}$ on which w_γ acts via the scalar ζ^i. In particular, $\mathfrak{t}(0) = \mathfrak{t}^{w_\gamma}$ is the w_γ invariant subspace. Then we have

$$M := (\mathfrak{t} \otimes_k \tilde{\mathcal{O}})^\Gamma = \bigoplus_{i=0}^{e-1} \mathfrak{t}(i) \otimes_k \varpi^{\frac{e-i}{e}}.$$

The existence of a W-invariant nondegenerate symmetric bilinear form on $\mathfrak{t}$ guarantees that $\dim_k \mathfrak{t}(i) = \dim_k \mathfrak{t}(e-i)$, from this we obtain that

$$\dim_k \left(\frac{\mathfrak{t} \otimes_k \tilde{\mathcal{O}}}{M \otimes_{\mathcal{O}} \tilde{\mathcal{O}}} \right) = e(\dim_k \mathfrak{t} - \dim_k \mathfrak{t}^{w_\gamma}) = e \cdot c(\gamma).$$

Combined with (3.8), we obtain

$$\dim \mathcal{P}_a = \langle \rho, \lambda \rangle + \tfrac{1}{2}(d(\gamma) - c(\gamma)).$$

Finally, $X_\gamma^{\lambda,\mathrm{reg}}$ is equidimensional since it is a finite union of $\mathcal{P}_a$-torsors. □

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3.7.2. Some zero-dimensional generalized affine Springer fibers. Suppose X_γ^λ is nonempty. Then there exists $\gamma_\lambda \in G_+^{\mathrm{sc}}$ satisfying the conclusion of Lemma 3.1.5. Let $a := \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}^{\mathrm{rs}}_{G_+^{\mathrm{sc}}}(F)$. Recall the extended discriminant divisor $\mathfrak{D}_+ \subset \mathfrak{C}_+$ defined in § 2.2.12. We define the *extended discriminant valuation* to be

$$d_+(a) := \mathrm{val}(a^*\mathfrak{D}_+) \in \mathbb{Z}.$$

From equation (3.1) we get

$$\begin{aligned} d_+(a) &= 2 \cdot \mathrm{val}(\rho(\alpha(\gamma_\lambda))) + d(\gamma) \\ &= \langle 2\rho, \lambda\rangle + d(\gamma) \\ &= \sum_{\substack{\alpha\in\Phi \\ \langle \alpha, \nu_\gamma\rangle = 0}} \mathrm{val}(\alpha(\gamma) - 1) + \langle 2\rho, \lambda - \nu_\gamma\rangle. \end{aligned} \tag{3.9}$$

Proposition 3.7.3. *Suppose $d_+(a) = 0$. Then γ is split and* $\dim X_\gamma^\lambda = 0$*. Moreover, $X_\gamma^\lambda = X_\gamma^{\lambda,\mathrm{reg}}$ and it is a torsor under P_a.*

Proof. The assumption $d_+(a) = 0$ implies that $a \in \mathfrak{C}_+^{\mathrm{rs}}(\mathcal{O})$. Let $\widetilde{X_a} = \mathrm{Spec}\, R_a$ be defined by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{X_a} & \longrightarrow & \overline{T_+} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathcal{O} & \longrightarrow & \mathfrak{C}_+ \end{array}$$

By Proposition 2.2.16, $\widetilde{X_a}$ is an étale cover $\mathrm{Spec}\,\mathcal{O}$ which must be trivial since the residue field k is algebraically closed. Then we see that $\gamma_\lambda \in T_+(F)$ is split and hence $\gamma \in T(F)$ is split.

Since X_γ^λ is nonempty, we have $\nu_\gamma \leqslant_{\mathbb{Q}} \lambda$ and hence the terms on the right hand side of (3.9) are non-negative. In particular, $d_+(a) = 0$ implies that $\nu_\gamma = \lambda$. Thus the proposition follows from Corollary 3.5.3. □

3.8. The case of central coweight

In this section, we deal with the case where $\lambda \in X_*(T)_+$ is a central coweight, i.e., $\langle \lambda, \alpha\rangle = 0$ for all roots α. Then $\lambda \in X_*(Z^0)$ where Z^0 is the maximal torus in the center of G. Consequently, we have $X_\gamma^\lambda \cong X^0_{\varpi^{-\lambda}\gamma}$. Hence the essential case is when $\lambda = 0$ and the corresponding Kottwitz–Viehmann variety becomes

$$X_\gamma := \{g \in G(F)/G(\mathcal{O}) | \mathrm{Ad}(g)^{-1}\gamma \in G(\mathcal{O})\}.$$

We first do some routine reductions. Let $P = MN$ be a standard parabolic subgroup with standard Levi M and unipotent radical N. For $\gamma \in M(\mathcal{O}) \cap G^{\mathrm{rs}}(F) \subset M(\mathcal{O}) \cap M^{\mathrm{rs}}(F)$, we consider the Kottwitz–Viehmann variety X_γ (respectively X_γ^M) defined for the groups G (respectively M). We have the discriminant valuation $d(\gamma)$ (respectively $d_M(\gamma)$) defined for G (respectively M). The two discriminant valuations are related by

$$d(\gamma) = d_M(\gamma) + 2r_N(\gamma) \tag{3.10}$$

where $r_N(\gamma)$ is defined in (3.4).

Proposition 3.8.1. *With notation as above, we have*

$$\dim X_\gamma = \dim X_\gamma^M + \frac{d_G(\gamma) - d_M(\gamma)}{2}.$$

Proof. Let $P = MN$ be the standard parabolic subgroup with Levi factor is M and unipotent radical N. The connected components of Gr_M and Gr_P both correspond bijectively to $\pi_1(M)$, the quotient of $X_*(T)$ by the coroot lattice of M. The canonical map $\mathrm{Gr}_P \to \mathrm{Gr}_G$ induces bijection on k-points by generalized Iwasawa decomposition. For each $\lambda \in \pi_1(M)$, let $X_{\gamma,\lambda}$ be the intersection of X_γ and the connected component of Gr_P corresponding to λ. Similarly, let $X_{\gamma,\lambda}^M$ be the intersection of X_γ^M with the connected component of Gr_M corresponding to λ. Then there is a canonical morphism

$$p_\gamma^M : X_{\gamma,\lambda} \to X_{\gamma,\lambda}^M.$$

It suffices to show that the fibers of this map have dimension $r_N(\gamma)$.

Let $h \in X_{\gamma,\lambda}^M$. Then $\gamma_h := h^{-1}\gamma h \in M(\mathcal{O})$ and we consider the fiber $Y_h := (p_\gamma^M)^{-1}(h)$. Its set of k points is

$$Y_h(k) = \{u \in N(F)/N(\mathcal{O}) | u^{-1}\gamma_h u \in G(\mathcal{O})\}.$$

In other words, we have

$$Y_h = f_{\gamma_h}^{-1}(N(\mathcal{O}))/N(\mathcal{O})$$

where $f_{\gamma_h} : N(F) \to N(F)$ is defined by $f_\gamma(u) = u^{-1}\gamma_h u \gamma_h^{-1}$. Apply Proposition 3.4.4 to the admissible set $Z = N(\mathcal{O})$ we see that Y_h is an irreducible affine space of dimension

$$\dim Y_h = r_N(\gamma_h) = r_N(\gamma)$$

and hence we conclude by (3.10). □

Corollary 3.8.2. *Let $\lambda \in X_*(T)$ be a central coweight and $\gamma \in G(F)^{\mathrm{rs}}$. Then*

$$\dim X_\gamma^\lambda = \tfrac{1}{2}(d_\gamma - c_\gamma).$$

Moreover, $X_\gamma^{\lambda,\mathrm{reg}}$ is a torsor under P_a and the dimension of the complement of $P_a = X_\gamma^{\lambda,\mathrm{reg}}$ in X_γ^λ is strictly smaller than the dimension of X_γ^λ.

Proof. We first assume that $\gamma \in G(\mathcal{O})$ is topologically unipotent mod center. In other words, the reduction mod ϖ of γ is unipotent mod center. After multiplying by an element in $Z(\mathcal{O})$, we may assume that $\gamma \in G^{\mathrm{sc}}(\mathcal{O})$ is topologically unipotent. Then when $G = G^{\mathrm{sc}}$ the argument of [19, § 4] and [23, Proposition 3.7.1] generalizes verbatim to our situation and proves $\dim X_\gamma^{\mathrm{reg}} = \dim X_\gamma$ and the complement of X_γ^{reg} has strictly smaller dimension. In particular, the dimension formula follows in this situation. More generally, we argue as in [32, Lemma 4.1] to reduce to the case $G = G^{\mathrm{sc}}$.

It remains to reduce to the case where γ is topologically unipotent mod center. After multiplying $\gamma \in G(\mathcal{O})$ by an element in $Z(\mathcal{O})$ we may assume that $\gamma \in G^{\mathrm{sc}}(\mathcal{O})$. Then G_γ is a maximal torus in G and $\gamma \in G_\gamma(F) \cap G(\mathcal{O})$. Let S be the maximal split subtorus in the centralizer G_γ. After conjugation we may assume that $S \subset T$. Let $M = C_G(S)$

be the centralizer of S in G. Then M is a standard Levi subgroup of G and $\gamma \in M(\mathcal{O})$. Let $a_M := \chi_M(\gamma) \in \mathfrak{C}_M(\mathcal{O})$. Then the pullback of T along $a_M : \operatorname{Spec} \mathcal{O} \to \mathfrak{C}_M$ is a totally ramified cover of $\operatorname{Spec} \mathcal{O}$ and we deduce that γ is topologically unipotent mod center in $M(\mathcal{O})$. Thus the result follows from the case already proved and Proposition 3.8.1. $\square$

We highlight the following special case.

Corollary 3.8.3. *Let $\lambda \in X_*(T)$ be a central coweight and $\gamma \in G(F)^{\mathrm{rs}}$. If $d_\gamma \leqslant 1$, then $X_\gamma^\lambda = X_\gamma^{\lambda,\mathrm{reg}} = P_a$ and they are zero-dimensional.*

Proof. By Corollary 3.8.2, if $d_\gamma \leqslant 1$, we must have $\dim X_\gamma^\lambda = 0$. Moreover, the complement of $X_\gamma^{\lambda,\mathrm{reg}}$ in X_γ^λ has strictly smaller dimension, hence must be empty. $\square$

3.9. Irreducible components

3.9.1. Stratification on dominant coweight cone. Let $\Lambda := X_*(T)$ and $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\mathsf{D} \subset \Lambda_{\mathbb{Q}}$ be the positive coroot cone. In other words, D consists of $\mathbb{Q}$-linear combinations of simple coroots with non-negative coefficients.

For $\lambda \in \Lambda^+$, we define the *dominant coweight polytope* to be:

$$\mathsf{P}_\lambda := \Lambda_{\mathbb{Q}}^+ \cap \operatorname{Conv}(W \cdot \lambda) = \Lambda_{\mathbb{Q}}^+ \cap (\lambda - \mathsf{D}),$$

where $\operatorname{Conv}(W \cdot \lambda)$ denotes the convex hull of the W-orbit of λ.

Lemma 3.9.2. *For each $\lambda_1, \lambda_2 \in \Lambda^+$ with $\kappa_G(\lambda_1) = \kappa_G(\lambda_2)$, there exists a unique $\mu \in \Lambda^+$ such that $\mu \leqslant \lambda_1, \mu \leqslant \lambda_2$ and*

$$(\lambda_1 - \mathsf{D}) \cap (\lambda_2 - \mathsf{D}) = \mu - \mathsf{D}.$$

In particular, we have $\mathsf{P}_{\lambda_1} \cap \mathsf{P}_{\lambda_2} = \mathsf{P}_\mu$.

Proof. Since $\kappa_G(\lambda_1) = \kappa_G(\lambda_2)$, the difference $\lambda_1 - \lambda_2$ lies in the coroot lattice. There exists a partition of the set of simple coroots $\Delta^\vee = \Delta_1^\vee \sqcup \Delta_2^\vee$ such that

$$\lambda_1 - \lambda_2 = \beta_1 - \beta_2,$$

where β_i is a non-negative integral linear combinations of simple coroots in $\Delta_i^\vee$ for $i \in \{1, 2\}$. Let $\Delta = \Delta_1 \sqcup \Delta_2$ be the corresponding partition of the set of simple roots. Consider the coweight $\mu := \lambda_1 - \beta_1 = \lambda_2 - \beta_2$. Then clearly $\mu \leqslant \lambda_1$ and $\mu \leqslant \lambda_2$.

We claim that $\mu \in \Lambda^+$. Take any simple root $\alpha \in \Delta_1$. Since β_2 is a positive linear combination of coroots in Δ_2, we have $\langle \alpha, \beta_2 \rangle \leqslant 0$ and hence $\langle \mu, \alpha \rangle = \langle \lambda_2 - \beta_2, \alpha \rangle \geqslant 0$. Similarly, using $\mu = \lambda_1 - \beta_1$, we see that for all $\alpha \in \Delta_2$, $\langle \mu, \alpha \rangle \geqslant 0$. Thus we conclude that $\mu \in \Lambda_+$.

It is clear that $\mu - \mathsf{D} \subset (\lambda_1 - \mathsf{D}) \cap (\lambda_2 - \mathsf{D})$. Now we prove the reverse inclusion. Let $\nu \in (\lambda_1 - \mathsf{D}) \cap (\lambda_2 - \mathsf{D})$. Then for $i \in \{1, 2\}$, $\lambda_i - \nu \in \mathsf{D}$ is a non-negative $\mathbb{Q}$-linear combination of simple coroots and we need to show that $\mu - \nu \in \mathsf{D}$. For any fundamental weight ω, there exists $i \in \{1, 2\}$ so that ω is orthogonal to all coroots in $\Delta_i^\vee$. Without loss of generality assume $i = 1$, then we have

$$\langle \mu - \nu, \omega \rangle = \langle \lambda_1 - \beta_1 - \nu, \omega \rangle = \langle \lambda_1 - \nu, \omega \rangle \geqslant 0.$$

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This means that $\nu \leqslant_{\mathbb{Q}} \mu$, or $\nu \in (\mu - \mathsf{D})$. Therefore, we have shown that

$$\mu - \mathsf{D} = (\lambda_1 - \mathsf{D}) \cap (\lambda_2 - \mathsf{D}).$$

Finally, taking intersection with $\Lambda^+_{\mathbb{Q}}$, we get $\mathsf{P}_{\lambda_1} \cap \mathsf{P}_{\lambda_2} = \mathsf{P}_{\mu}$. □

For each $\lambda \in \Lambda^+$, define

$$\mathsf{P}^{\circ}_{\lambda} := \mathsf{P}_{\lambda} - \bigcup_{\substack{\mu \in \Lambda^+, \\ \mu < \lambda}} \mathsf{P}_{\mu}. \tag{3.11}$$

Corollary 3.9.3. *For any $\lambda_1, \lambda_2 \in \Lambda_+$ with $\lambda_1 \neq \lambda_2$, we have $\mathsf{P}^{\circ}_{\lambda_1} \cap \mathsf{P}^{\circ}_{\lambda_2} = \varnothing$. In particular, we get a well-defined stratification*

$$\{\nu \in \Lambda^+_{\mathbb{Q}} | p_{G,\mathbb{Q}}(\nu) \in X_*(G_{\mathrm{ab}}) \subset \pi_1(G)_{\mathbb{Q}}\} = \bigsqcup_{\lambda \in \Lambda^+} \mathsf{P}^{\circ}_{\lambda}.$$

Proof. If $\det(\varpi^{\lambda_1}) \neq \det(\varpi^{\lambda_2})$, it is clear that P_{λ_1} and P_{λ_2} are disjoint. Suppose

$$\det(\varpi^{\lambda_1}) \neq \det(\varpi^{\lambda_2}).$$

Then by Lemma 3.9.2, there exists $\mu \in \Lambda^+$ such that $\mu \leqslant \lambda_1, \mu \leqslant \lambda_2$ and

$$\mathsf{P}^{\circ}_{\lambda_1} \cap \mathsf{P}^{\circ}_{\lambda_2} \subset \mathsf{P}_{\lambda_1} \cap \mathsf{P}_{\lambda_2} = \mathsf{P}_{\mu}.$$

But by (3.11), we have

$$\mathsf{P}_{\mu} \cap \mathsf{P}^{\circ}_{\lambda_i} = \varnothing$$

since $\mu \leqslant \lambda_i$ for $i \in \{1, 2\}$. Therefore $\mathsf{P}^{\circ}_{\lambda_1} \cap \mathsf{P}^{\circ}_{\lambda_2} = \varnothing$. □

3.9.4. Stratification on extended Steinberg base. To give an alternative formulation of the conjecture on irreducible components, we introduce a stratification on $\mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_{G^{\mathrm{sc}}_+}(F)$.

Recall that $\mathfrak{C}_+ \cong A_{G^{\mathrm{sc}}} \times \mathbb{A}^r$. Consider the strata

$$\mathfrak{C}^{\lambda}_+ := \varpi^{-w_0(\lambda_{\mathrm{ad}})} T_{\mathrm{ad}}(\mathcal{O}) \times \mathcal{O}^r \subset \mathfrak{C}_+(\mathcal{O})$$

where $\lambda_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})_+$ is the image of λ.

For each $\mu \in \Lambda^+$ such that $\mu \leqslant \lambda$, we have an embedding

$$i_{\mu\lambda} : \mathfrak{C}^{\mu}_+ \hookrightarrow \mathfrak{C}^{\lambda}_+$$

defined by the formula

$$\begin{aligned} &i_{\mu\lambda}(a_1, \dots, a_r, b_1, \dots, b_r) = \\ &\quad (\varpi^{\langle -w_0(\lambda-\mu), \alpha_1\rangle} a_1, \dots, \varpi^{\langle -w_0(\lambda-\mu), \alpha_r\rangle} a_r, \varpi^{\langle -w_0(\lambda-\mu), \omega_1\rangle}) b_1, \dots, \varpi^{\langle -w_0(\lambda-\mu), \omega_r\rangle} b_r). \end{aligned} \tag{3.12}$$

Note that we need to choose a uniformiser to define the embedding $i_{\mu\lambda}$ but its image does not depend on this choice.

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Proposition 3.9.5. *For any $\lambda, \mu_1, \mu_2 \in \Lambda^+$ with $\mu_1 \leqslant \lambda$ and $\mu_2 \leqslant \lambda$, there exists a unique $\mu_3 \in \Lambda^+$ such that $\mu_3 \leqslant \mu_1, \mu_3 \leqslant \mu_2$ and*

$$i_{\mu_1\lambda}(\mathfrak{C}_+^{\mu_1}) \cap i_{\mu_2\lambda}(\mathfrak{C}_+^{\mu_2}) = i_{\mu_3\lambda}(\mathfrak{C}_+^{\mu_3}).$$

Proof. By Lemma 3.9.2, there exists a unique $\mu_3 \in \Lambda^+$ such that

$$(\mu_1 - \mathsf{D}) \cap (\mu_2 - \mathsf{D}) = \mu_3 - \mathsf{D}. \tag{3.13}$$

To prove the proposition, it suffices to show that

$$i_{\mu_1\lambda}(\mathfrak{C}_+^{\mu_1}) \cap i_{\mu_2\lambda}(\mathfrak{C}_+^{\mu_2}) \subset i_{\mu_3\lambda}(\mathfrak{C}_+^{\mu_3}).$$

Let ι be the involution on the set $\{1, \dots, r\}$ such that $\omega_{\iota(i)} = -w_0(\omega_i)$ for all $1 \leqslant i \leqslant r$. For each $c = (c_1, \dots, c_r) \in \mathcal{O}^r$, let $a_i := \mathrm{val}(c_{\iota(i)})$.

Suppose that $\varpi^{(-w_0(\lambda_{\mathrm{ad}}),c)} \in i_{\mu_1\lambda}(\mathfrak{C}_+^{\mu_1}) \cap i_{\mu_2\lambda}(\mathfrak{C}_+^{\mu_2})$, then we get

$$a_i \geqslant \langle \lambda - \mu_1, \omega_i \rangle \text{ and } a_i \geqslant \langle \lambda - \mu_2, \omega_i \rangle \quad \text{for all } 1 \leqslant i \leqslant r \tag{3.14}$$

and we need to show that $a_i \geqslant \langle \lambda - \mu_3, \omega_i \rangle$ for all $1 \leqslant i \leqslant r$.

Let $\mu_1' := \sum_{i=1}^r \langle \mu_1, \omega_i \rangle \alpha_i^\vee$ and define μ_2', μ_3'. Then we have $\mu_1', \mu_2', \mu_3' \in \Lambda_0^+$. Consider the coweight $\nu := \sum_{i=1}^r (\langle \lambda, \omega_i \rangle - a_i)\alpha_i^\vee \in \Lambda_0$. By (3.13) and (3.14) we have

$$\nu \in (\mu_1' - \mathsf{D}) \cap (\mu_2' - \mathsf{D}) = \mu_3' - \mathsf{D}.$$

This implies that

$$\langle \lambda, \omega_i \rangle - a_i = \langle \nu, \omega_i \rangle \leqslant \langle \mu_3', \omega_i \rangle = \langle \mu_3, \omega_i \rangle$$

which is what we want. □

For any $\lambda, \mu \in \Lambda^+$ with $\mu \leqslant \lambda$, define

$$\mathfrak{C}_+^{\lambda\mu} := i_{\mu\lambda}(\mathfrak{C}_+^{\mu}) - \bigcup_{\substack{\nu \in \Lambda_+ \\ \nu < \mu}} i_{\nu\lambda}(\mathfrak{C}_+^{\nu}). \tag{3.15}$$

Corollary 3.9.6. *For any $\lambda, \mu_1, \mu_2 \in \Lambda^+$ with $\mu_1 \neq \lambda$ and $\mu_2 \leqslant \lambda$, we have*

$$\mathfrak{C}_+^{\lambda\mu_1} \cap \mathfrak{C}_+^{\lambda\mu_2} = \varnothing.$$

In particular, we get well-defined stratifications

$$\mathfrak{C}_+^{\lambda} = \bigsqcup_{\substack{\mu \in \Lambda_+ \\ \mu \leqslant \lambda}} \mathfrak{C}_+^{\lambda\mu}, \quad \mathfrak{C}_{G_+^{\mathrm{sc}}}(F) \cap \mathfrak{C}_+(\mathcal{O}) = \bigsqcup_{\substack{\lambda, \mu \in \Lambda_+ \\ \mu \leqslant \lambda}} \mathfrak{C}_+^{\lambda\mu}.$$

Proof. The argument is similar to the proof of Corollary 3.9.3, using Proposition 3.9.5 instead of Lemma 3.9.2. □

The following lemma relates the stratas (3.15) to the stratas (3.11).

Lemma 3.9.7. *For any $\lambda \in \Lambda^+$ and $\gamma \in G(F)^{\mathrm{rs}}$ with $\nu_\gamma \leqslant_{\mathbb{Q}} \lambda$, there exists a* unique *dominant integral coweight $\mu \in \Lambda_+$ with $\mu \leqslant \lambda$ that satisfies any (hence all) of the following equivalent conditions:*

(1) *$\mu \in \Lambda_+$ is a minimal dominant integral coweight such that $\nu_\gamma \leqslant_{\mathbb{Q}} \mu$;*

(2) *$\nu_\gamma \in \mathsf{P}_\mu^\circ$, cf. (3.11);*

(3) *$\chi_+(\gamma_\lambda) \in \mathfrak{C}_+^{\lambda\mu}$, cf. (3.15).*

Proof. The equivalence between (1) and (2) follows from the definition of P_μ. The equivalence of (1) and (3) follows from Proposition 3.1.6.

Finally, the uniqueness of μ follows from Lemma 3.9.2 or Proposition 3.9.5. □

Now we state our conjecture on irreducible components of X_γ^λ.

Conjecture 3.9.8. Let $\lambda \in \Lambda^+$ and $\gamma \in G(F)^{\mathrm{rs}}$ with $\nu_\gamma \leqslant_{\mathbb{Q}} \lambda$. Let $\mu \in \Lambda_+$ be the 'best integral approximation' of ν_γ, i.e., the unique dominant coweight that satisfies the equivalent conditions in Lemma 3.9.7. Then the number of $G_\gamma^0(F)$-orbits on $\mathrm{Irr}(X_\gamma^\lambda)$ equals to the weight multiplicity $m_{\lambda\mu}$.

By Corollary 3.5.3, this conjecture is true when γ is an unramified conjugacy class.

Remark 3.9.9. For irreducible components of affine Deligne–Lusztig varieties, there is a similar conjecture made by Miaofen Chen and Xinwen Zhu, see the discussion in [13] and [34]. Their conjecture has been proved independently by Nie [24] and Zhou–Zhu [37]. A remark is in order for readers who are interested in comparing the two conjectures. In the setting of affine Deligne–Lusztig varieties, one also approximates Newton points of twisted conjugacy classes by integral coweight. However, the 'best integral approximation' as defined in [13] is the largest integral coweight dominated by the Newton point, whereas in the formulation of our Conjecture 3.9.8, we use the smallest integral coweight dominating the Newton point. Simple examples suggest that these two integral approximations are very likely in the same Weyl group orbit, so we expect the two weight multiplicities to be equal.

3.9.10. Components of the regular locus. Now we examine the number of $G_\gamma^0(F)$ orbits on $\mathrm{Irr}(X_\gamma^{\lambda,\mathrm{reg}})$.

Theorem 3.9.11. *Let $\lambda \in \Lambda^+$ and $\gamma \in G(F)^{\mathrm{rs}}$ with $\nu_\gamma \leqslant_{\mathbb{Q}} \lambda$. Let $\mu \in \Lambda^+$ be the 'best integral approximation' of the Newton point ν_γ as in Lemma 3.9.7. Then we have an inequality*

$$|\{G_\gamma^0(F) \text{ orbits on } X_\gamma^{\lambda,\mathrm{reg}}\}| \leqslant |\mathrm{Cox}(W,S)|,$$

where $\mathrm{Cox}(W,S)$ is the set of S-Coxeter elements defined in Definition 2.2.1. Moreover, when λ lies in the interior of the Weyl chamber and $\lambda-\mu$ lies in the interior of the positive coroot cone, the equality is achieved.

Proof. The $G_\gamma^0(F)$-orbits on $\mathrm{Irr}(X_\gamma^{\lambda,\mathrm{reg}})$ correspond bijectively to $G_\gamma^0(F)$ orbits on $X_\gamma^{\lambda,\mathrm{reg}}$, which are precisely the P_a-orbits of maximal dimension on $\mathrm{Sp}_a^0 \cong X_\gamma^\lambda$. We know from Proposition 3.3.1 that these are the varieties $X_\gamma^{\lambda,w} = \mathrm{Sp}_a^w$ for $w \in \mathrm{Cox(W,S)}$.

However, for two different $w, w' \in \mathrm{Cox}(W,S)$, $X_\gamma^{\lambda,w}$ and $X_\gamma^{\lambda,w'}$ might coincide. For example, in the case $\lambda = 0$ and $\gamma \in G(\mathcal{O})$, all $X_\gamma^{\lambda,w}$ coincide (hence equal to $X_\gamma^{\lambda,\mathrm{reg}}$).

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So in this particular case, $X_\gamma^{\lambda,\mathrm{reg}}$ is the unique $\mathcal{P}_a$-orbit of maximal dimension. In general, we know from (2.7) that the number of $G_\gamma^0(F)$ orbits in $X_\gamma^{\lambda,\mathrm{reg}}$ is bounded above by the Cardinality of $\mathrm{Cox}(W,S)$.

It remains to show the last statement. Suppose λ lies in the interior of the Weyl chamber and $\lambda-\mu$ lies in the interior of the dominant coroot cone. Consider the following Cartesian diagram

$$\begin{tikzcd} \chi_+^{-1}(a) \arrow[r] \arrow[d] & \mathrm{Vin}_{G^{\mathrm{sc}}} \arrow[d, "\chi_+"] \\ \mathrm{Spec}\,\mathcal{O} \arrow[r, "a"] & \mathfrak{C}_+ \end{tikzcd}$$

For $g\in G(F)$ such that $gG(\mathcal{O})\in X_\gamma^{\lambda,\mathrm{reg}}$, let $\overline{\mathrm{Ad}(g)^{-1}\gamma}$ be the reduction mod ϖ of $\mathrm{Ad}(g)^{-1}\gamma\in \mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}(\mathcal{O})$. The condition that λ lies in the interior of the Weyl chamber means that $\langle\lambda,\alpha_i\rangle>0$ for all simple roots α_i. Hence the special fiber of $\chi_+^{-1}(a)$ lies in the asymptotic semigroup $\mathrm{As}(G^{\mathrm{sc}}):=\alpha^{-1}(0)$ and in particular $\overline{\mathrm{Ad}(g)^{-1}\gamma}\in \mathrm{As}(G^{\mathrm{sc}})\cap \mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}$.

Furthermore, the assumption that $\lambda-\mu$ lies in the interior of the positive coroot cone implies that $\langle\lambda-\mu,\omega_i\rangle>0$ for all fundamental weight ω_i. Therefore, the reduction mod ϖ of a equals to 0 and the special fiber of $\chi_+^{-1}(a)$ is the nilpotent cone $\mathcal{N}$. In particular, we get $\overline{\mathrm{Ad}(g)^{-1}\gamma}\in\mathcal{N}^{\mathrm{reg}}$.

Consequently, there is a bijection between $G_\gamma^0(F)$ orbits on $X_\gamma^{\lambda,\mathrm{reg}}$ and G orbits on $\mathcal{N}^{\mathrm{reg}}$, the latter of which corresponds bijectively to $\mathrm{Cox}(W,S)$ by Proposition 2.2.9. $\square$

As an immediate consequence, we mention the following purely combinatorial result, which might be of independent interest.

Corollary 3.9.12. *Let $\lambda\geqslant\mu$ be dominant weights of a complex reductive group G. Suppose that λ lies in the interior of the Weyl chamber and $\lambda-\mu$ lies in the interior of the positive root cone (the 'wide cone'). Then we have the following lower bound for the weight multiplicity*

$$m_{\lambda\mu}\geqslant|\mathrm{Cox}(W,S)|,$$

where the set $\mathrm{Cox}(W,S)$ is defined in § 2.2.1.

Proof. We consider the dual group $G^\vee$ of G over k. Then $\lambda\geqslant\mu$ are dominant coweights for $G^\vee$. Let $T^\vee\subset G^\vee$ be a maximal torus and $\gamma\in\varpi^\mu T^\vee(\mathcal{O})\cap G^\vee(F)^{\mathrm{rs}}$. Then the generalized affine Springer fiber X_γ^λ is nonempty and by Corollary 3.5.3, the number of $G_\gamma^{\vee,0}(F)$-orbits on $\mathrm{Irr}(X_\gamma^\lambda)$ equals to $m_{\lambda\mu}$. On the other hand, by Theorem 3.9.11, the number of $G_\gamma^{\vee,0}(F)$-orbits on $\mathrm{Irr}(X_\gamma^{\lambda,\mathrm{reg}})$ equals to $|\mathrm{Cox}(W,S)|$, hence the inequality. $\square$

Remark 3.9.13. If G_{ad} is simple of rank r, then $|\mathrm{Cox}(W,S)|=2^{r-1}$. Indeed, an S-Coxeter element is uniquely determined by requiring for any pair of noncommuting simple reflections, which one occurs first in the reduced expression. This means that each edge of the Dynkin diagram gives two possibilities. There are $r-1$ edges since the Dynkin diagram is a connected tree with r nodes. Hence there are 2^{r-1} possibilities. In general,

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if the simple factors of G_{ad} have rank $r_1, \dots, r_m$, then

$$|\mathrm{Cox}(W, S)| = \prod_{i=1}^{m} 2^{r_i - 1}.$$

We expect that there should be a more straightforward proof of Corollary 3.9.12.

Remark 3.9.14. In general, the weight multiplicity $m_{\lambda\mu}$ will increase with λ, while the right hand side in Corollary 3.9.12 is a fixed constant independent of λ, μ. Thus in general there will be much more irreducible components in X_γ^λ than the regular open subvariety $X_\gamma^{\lambda,\mathrm{reg}}$.

Remark 3.9.15. The idea of using Vinberg monoid to explore the geometry of Kottwitz–Viehmann varieties is similar in spirit to the work of He [17], where he relates the geometry of certain affine Deligne–Lusztig varieties in the affine flag variety to the wonderful compactification.

4. The Hitchin–Frenkel–Ngô fibration

In this section, we study the Hitchin–Frenkel–Ngô fibrations, viewed on the one hand as the global analogue of Kottwitz–Viehmann varieties. These are certain group analogues of Hitchin fibrations, first introduced in [10] and later studied in more detail in [5] and [4].

Throughout this section, we let X be a projective smooth curve of genus g over k and G a connected reductive group over k.

4.1. First definitions

Let $\mathcal{L}$ be a $Z_+^{\mathrm{sc}} = T^{\mathrm{sc}}$ torsor on X. Then we can twist the schemes $\mathrm{Vin}_{G^{\mathrm{sc}}}$ (respectively $\mathfrak{C}_+$, $A_{G^{\mathrm{sc}}}$) by $\mathcal{L}$ to form corresponding affine spaces $\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}$ (respectively $\mathfrak{C}_+^{\mathcal{L}}$, $A_{G^{\mathrm{sc}}}^{\mathcal{L}}$) over X.

Definition 4.1.1. The *Hitchin–Frenkel–Ngô moduli stack* associated to the T^{sc}-torsor $\mathcal{L}$ is the mapping stack

$$\mathcal{M}_{\mathcal{L}} := \mathrm{Hom}(X, [\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}/\mathrm{Ad}(G)]).$$

In other words, $\mathcal{M}_{\mathcal{L}}$ classifies pairs $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a G-torsor on X and φ is a section of the twisted product $\mathcal{E} \wedge^G \mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}$ where G acts on $\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}$ by adjoint action, and the action factors through G_{ad} (cf. § 1.4.2 for the notation). We refer to such pairs $(\mathcal{E}, \varphi)$ as *Higgs–Vinberg pairs.*

Replacing $\mathrm{Vin}_{G^{\mathrm{sc}}}$ by $\mathrm{Vin}_{G^{\mathrm{sc}}}^0$ (respectively $\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathrm{reg}}$) in the definition of $\mathcal{M}_{\mathcal{L}}$, we define open substacks $\mathcal{M}_{\mathcal{L}}^0$ (respectively $\mathcal{M}_{\mathcal{L}}^{\mathrm{reg}} \subset \mathcal{M}_{\mathcal{L}}$). Also, we define

$$\mathcal{A}_{\mathcal{L}} := \mathrm{Hom}_X(X, \mathfrak{C}_+^{\mathcal{L}}), \quad \mathcal{B}_{\mathcal{L}} := \mathrm{Hom}_X(X, A_{G^{\mathrm{sc}}}^{\mathcal{L}})$$

as the space of sections of the affine space $\mathfrak{C}_+^{\mathcal{L}}$ (respectively $A_{G^{\mathrm{sc}}}^{\mathcal{L}}$) over X. More concretely, we can describe $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{B}_{\mathcal{L}}$ as follows.

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For each $\omega \in X^*(T)$, let $\omega(\mathcal{L})$ be the invertible sheaf on X defined by pushing $\mathcal{L}$ along the morphism $\omega : T \to \mathbb{G}_m$. Then we have

$$\mathcal{B}_{\mathcal{L}} = H^0(X, A_{G^{\mathrm{sc}}}^{\mathcal{L}}) = \bigoplus_{i=1}^{r} H^0(X, \alpha_i(\mathcal{L}))$$

and

$$\mathcal{A}_{\mathcal{L}} = \mathcal{B}_{\mathcal{L}} \oplus \bigoplus_{i=1}^{r} H^0(X, \omega_i(\mathcal{L})).$$

Definition 4.1.2. The *Hitchin–Frenkel–Ngô* fibration is the morphism

$$h_{\mathcal{L}} : \mathcal{M}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$$

induced by $\chi_+ : \mathrm{Vin}_{G^{\mathrm{sc}}} \to \mathfrak{C}_+$.

Let $\beta_{\mathcal{L}} : \mathcal{A}_{\mathcal{L}} \to \mathcal{B}_{\mathcal{L}}$ be the natural projection and $\alpha_{\mathcal{L}} := \beta_{\mathcal{L}} \circ h_{\mathcal{L}} : \mathcal{M}_{\mathcal{L}} \to \mathcal{B}_{\mathcal{L}}$ be the map induced by $\alpha : \mathrm{Vin}_{G^{\mathrm{sc}}} \to A_{G^{\mathrm{sc}}}$. We call the fibers of $\alpha_{\mathcal{L}}$ *restricted Hitchin–Frenkel–Ngô moduli stack*.

4.1.3. Each point $b \in \mathcal{B}_{\mathcal{L}}$ can be written as $b = (b_1, \ldots, b_r)$, where $b_i \in H^0(X, \alpha_i(\mathcal{L}))$. Let $\mathcal{B}_{\mathcal{L}}^{\circ} \subset \mathcal{B}_{\mathcal{L}}$ be the open subset consisting of those b such that b_i is nonzero for all i. To each point $b \in \mathcal{B}_{\mathcal{L}}^{\circ}$, we can associate an $X_*(T_{\mathrm{ad}})_+$-valued divisor λ_b on X defined by

$$\lambda_b := \sum_{i=1}^{r} \check{\omega}_i D(b_i),$$

where $D(b_i)$ is the effective divisor on X associated to b_i and $\check{\omega}_i$ is the ith fundamental coweight. For any $a \in \mathcal{A}_{\mathcal{L}}$ with $\beta_{\mathcal{L}}(a_+) \in \mathcal{B}_{\mathcal{L}}^{\circ}$, we denote $\lambda_a := \lambda_{\beta_{\mathcal{L}}(a)}$.

Definition 4.1.4. The *generically regular semisimple locus* $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$ is the open subset of $\mathcal{A}_{\mathcal{L}}$ consisting of sections $a : X \to \mathfrak{C}_+^{\mathcal{L}}$ such that $\beta_{\mathcal{L}}(a) \in \mathcal{B}_{\mathcal{L}}^{\circ}$ and $a(X)$ generically lies in the open subset $\mathfrak{C}_+^{\mathrm{rs},\mathcal{L}} = \mathfrak{C}_+^{\mathcal{L}} - \mathfrak{D}_+^{\mathcal{L}}$.

4.1.5. Global Steinberg section. Let $c = |Z(G_{\mathrm{der}})|$ be the order of the center of the derived group of G. Suppose there exists a T^{sc}-torsor $\mathcal{L}'$ such that $\mathcal{L} \cong (\mathcal{L}')^{\otimes c}$. By definition, there is a canonical map $[\mathrm{ev}]_{\mathcal{L}} : \mathcal{A}_{\mathcal{L}} \times X \to [\mathfrak{C}_+/T^{\mathrm{sc}}]$ making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{A}_{\mathcal{L}} \times X \arrow[r, "{[\mathrm{ev}]_{\mathcal{L}}}"] \arrow[d] & {[\mathfrak{C}_+/T^{\mathrm{sc}}]} \arrow[d] \\
X \arrow[r, "\mathcal{L}"] & \mathbb{B}T^{\mathrm{sc}}
\end{tikzcd}$$

Here, the left arrow is the projection to X and the bottom arrow corresponds to the T^{sc}-torsor $\mathcal{L}$.

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The choice of cth root $\mathcal{L}'$ of $\mathcal{L}$ defines a morphism $[\mathrm{ev}]_{\mathcal{L}'} : \mathcal{A}_{\mathcal{L}} \times X \to [\mathfrak{C}_+/T^{\mathrm{sc}}]$ lifting $[\mathrm{ev}]_{\mathcal{L}}$. Then for each $w \in \mathrm{Cox}(W, S)$ (cf. Definition 2.2.1), the composition of $[\mathrm{ev}]_{\mathcal{L}'}$ and the section $\epsilon^{w}_{+,[c]}$ of $[\chi_+]_{[c]}$ (cf. Proposition 2.2.21) induces a section of $h_{\mathcal{L}}$:

$$\epsilon^{w}_{\mathcal{L}'} : \mathcal{A}_{\mathcal{L}} \to \mathcal{M}^{\mathrm{reg}}_{\mathcal{L}} \subset \mathcal{M}_{\mathcal{L}}.$$

We refer to $\epsilon^{w}_{\mathcal{L}'}$ as the *global Steinberg section*.

4.2. Symmetries of Hitchin–Frenkel–Ngô fibration

Definition 4.2.1. Let $\mathcal{P}_{\mathcal{L}}$ be the Picard stack over $\mathcal{A}_{\mathcal{L}}$ that associates to any S-point $a \in \mathcal{A}_{\mathcal{L}}(S)$ the Picard groupoid $\mathcal{P}_a$ of $\mathcal{J}_a$ torsors on $X \times S$. Here, $\mathcal{J}_a$ is the pullback of the universal centralizer $\mathcal{J}_{\mathcal{L}}$ on $\mathfrak{C}^{\mathcal{L}}_{+}$ along the map $a : X \times S \to \mathfrak{C}^{\mathcal{L}}_{+}$.

Proposition 4.2.2. *$\mathcal{P}_{\mathcal{L}}$ is a smooth Picard stack over $\mathcal{A}_{\mathcal{L}}$.*

Proof. The argument of [23, Proposition 4.3.5] generalizes *verbatim* to our situation. The point is that $\mathcal{J}_a$ is a smooth group scheme and the obstruction to deforming a $\mathcal{J}_a$-torsor lives in $H^2(X, \mathrm{Lie}(\mathcal{J}_a))$, which vanishes since X is a curve. □

The action of $\mathbb{B}\mathcal{J}$ on $[\mathrm{Vin}_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)]$ (respectively $[\mathrm{Vin}^{\mathrm{reg}}_{G^{\mathrm{sc}}}/\mathrm{Ad}(G)]$) induces action of $\mathcal{P}_{\mathcal{L}}$ on $\mathcal{M}_{\mathcal{L}}$ (respectively $\mathcal{M}^{\mathrm{reg}}_{\mathcal{L}}$).

To understand the connected components of the fibers of $\mathcal{P}_{\mathcal{L}}$, we utilize cameral covers.

Definition 4.2.3. The *cameral cover* associated to each $a \in \mathcal{A}_{\mathcal{L}}(k)$ is the finite flat cover $\pi_a : \widetilde{X}_a \to X$ defined by the following Cartesian diagram

$$\begin{tikzcd} \widetilde{X}_a \arrow[r] \arrow[d, "\pi_a"'] & \overline{T^{\mathrm{sc}}_{+}}^{\mathcal{L}} \arrow[d] \\ X \arrow[r, "a"] & \mathfrak{C}^{\mathcal{L}}_{+} \end{tikzcd}$$

For any closed point $a \in \mathcal{A}^{\heartsuit}_{\mathcal{L}}$, we define the *discriminant divisor* for a to be the effective divisor

$$\Delta_a := a^{-1}(\mathfrak{D}^{\mathcal{L}}_{+}).$$

Over the nonempty open subset $U_a := X - \Delta_a$, the cameral cover π_a is Galois étale with Galois group W. Choosing a point $\tilde{u} \in \widetilde{X}_a$ with $u := \pi_a(\tilde{u}) \in U_a$, we get a homomorphism

$$\rho_a : \pi_1(U_a, u) \to W$$

whose image is a subgroup $W_a \subset W$. Note that the conjugacy class of W_a in W is independent of the choice of base point $\tilde{u}$.

Let $\mathcal{J}^0_a \subset \mathcal{J}_a$ be the fiberwise neutral component and consider the Picard stack $\mathcal{P}'_a := \mathrm{Bun}_{\mathcal{J}^0_a}$ of $\mathcal{J}^0_a$-torsors on X. Then there is a natural homomorphism of Picard stacks $\mathcal{P}'_a \to \mathcal{P}_a$. The following Lemma is parallel to [23, Lemme 4.10.2] with exactly the same proof.

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Lemma 4.2.4. *The homomorphism $\mathcal{P}'_a \to \mathcal{P}_a$ is surjective with finite kernel. Same is true for the induced homomorphism $\pi_0(\mathcal{P}'_a) \to \pi_0(\mathcal{P}_a)$.*

Corollary 4.2.5. *$\pi_0(\mathcal{P}_a)$ is finite if and only if T^{W_a} is finite.*

Proof. By previous lemma, $\pi_0(\mathcal{P}_a)$ is finite if and only if $\pi_0(\mathcal{P}'_a)$ is finite. By [22, Corollaire 6.7], $\pi_0(\mathcal{P}'_a) = \hat{T}^{W_a}$. Since the finiteness of T^{W_a} is equivalent to the finiteness of $\hat{T}^{W_a}$, the result follows. □

Definition 4.2.6. The *anisotropic locus* is the subset $\mathcal{A}^{\mathrm{ani}}_{\mathcal{L}} \subset \mathcal{A}^{\heartsuit}_{\mathcal{L}}$ consisting of $a \in \mathcal{A}^{\heartsuit}_{\mathcal{L}}$ such that the component group $\pi_0(\mathcal{P}_a)$ is finite.

For each subset $I \subset \Delta$, we consider the invariant quotient $\overline{T^{\mathrm{sc}}_+}^{W_I}$. Then the natural morphism $\overline{T^{\mathrm{sc}}_+}^{W_I} \to \mathfrak{C}_+$ is finite and $Z^{\mathrm{sc}}_+ = T^{\mathrm{sc}}$ equivariant. Denote

$$\overline{T^{\mathrm{sc}}_+}^{W_I,\mathcal{L}} := \overline{T^{\mathrm{sc}}_+}^{W_I} \times^{Z^{\mathrm{sc}}_+} \mathcal{L}.$$

Let $\mathcal{A}^{W_I}_{\mathcal{L}} := H^0(X, \overline{T^{\mathrm{sc}}_+}^{W_I,\mathcal{L}})$ be the space of sections of the affine scheme $\overline{T^{\mathrm{sc}}_+}^{W_I,\mathcal{L}}$ over X. Consider the map

$$\nu_I : \mathcal{A}^{W_I}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$$

induced by the finite morphism $\overline{T^{\mathrm{sc}}_+}^{W_I,\mathcal{L}} \to \mathfrak{C}^{\mathcal{L}}_+$. Let $\mathcal{A}^{W_I,\heartsuit}_{\mathcal{L}} := \nu_I^{-1}(\mathcal{A}^{\heartsuit}_{\mathcal{L}})$.

Proposition 4.2.7. *Suppose G is semisimple. Then the complement of $\mathcal{A}^{\mathrm{ani}}_{\mathcal{L}}$ in $\mathcal{A}^{\heartsuit}_{\mathcal{L}}$ is a finite union*

$$\mathcal{A}^{\heartsuit}_{\mathcal{L}} \setminus \mathcal{A}^{\mathrm{ani}}_{\mathcal{L}} = \bigcup_{I \subsetneq \Delta} \nu_I(\mathcal{A}^{W_I,\heartsuit}_{\mathcal{L}}).$$

Proof. Let $a \in \mathcal{A}^{\heartsuit}_{\mathcal{L}} - \mathcal{A}^{\mathrm{ani}}_{\mathcal{L}}$. Then by Corollary 4.2.5, T^{W_a} contains a nontrivial torus S. Since G is semisimple, the centralizer of S is a *proper* Levi subgroup of G whose simple roots form a proper subset $I \subsetneq W$. Then we have $W_a \subset W_I$.

Consider the following diagram in which both squares are Cartesian:

$$\begin{tikzcd}
\widetilde{X}_a \arrow[r] \arrow[d] \arrow[rr, bend left, "\pi_a"] & Y_a \arrow[r, "\pi_a^I"'] \arrow[d] & X \arrow[d] \\
\overline{T^{\mathrm{sc}}_+}^{\mathcal{L}} \arrow[r] & \overline{T^{\mathrm{sc}}_+}^{W_M,\mathcal{L}} \arrow[r] & \mathfrak{C}^{\mathcal{L}}_+
\end{tikzcd}$$

Let $\widetilde{Y}_a \subset \widetilde{X}_a$ be the union of all irreducible components that contain a point in the W_I-orbit of $\tilde{u}$. Then the image of $\widetilde{Y}_a$ in Y_a is isomorphic to X and hence gives a section of the morphism π^I_a. In other words, there is a section $a_I : X \to \overline{T^{\mathrm{sc}}_+}^{W_M,\mathcal{L}}$ such that $\nu_I(a_I) = a$. This proves that

$$\mathcal{A}^{\heartsuit}_{\mathcal{L}} \setminus \mathcal{A}^{\mathrm{ani}}_{\mathcal{L}} \subset \bigcup_{I \subsetneq \Delta} \nu_I(\mathcal{A}^{W_I,\heartsuit}_{\mathcal{L}}).$$

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Conversely, for any $I \subsetneq \Delta$ and $a_I \in \mathcal{A}_{\mathcal{L}}^{W_I,\heartsuit}$ with $\nu_I(a_I) = a$, the morphism π_a^I in the diagram above has a section given by a_I. This implies that $W_a \subset W_I$ so that T^{W_a} is not finite. By Corollary 4.2.5 again we see that $a \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$. $\square$

Corollary 4.2.8. *Suppose G is semisimple. Then $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is an open subset of $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$. Moreover, for any $b \in \mathcal{B}_{\mathcal{L}}^{\circ}$ and any integer N with $N > \max\{2g-2, rg\}$, if $\deg \omega_i(\mathcal{L}) > N$ for all $1 \leqslant i \leqslant r$, then the complement of $\mathcal{A}_{\mathcal{L},b}^{\mathrm{ani}}$ in $\mathcal{A}_{\mathcal{L},b}^{\heartsuit}$ has codimension at least $N - rg$.*

Proof. By valuative criterion and [22, Lemme 7.3] we see that ν_I is proper. So the images $\nu_I(\mathcal{A}_{\mathcal{L}}^{W_I,\heartsuit})$ are closed subsets of $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$ and their complement $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is open. It remains to calculate the dimension of $\mathcal{A}_{\mathcal{L}}^{W_I}$.

Let $I \subsetneq \Delta$ and L_I be a corresponding Levi subgroup of G^{sc}. We label the fundamental weights $\omega_1, \ldots, \omega_r$ of G^{sc} so that $\omega_1, \ldots, \omega_s$ are fundamental weights for L_I where $s = |I| < r$. There is a natural morphism

$$q^I : \overline{T_+^{\mathrm{sc}}}^{W_I} \to A_{G^{\mathrm{sc}}} \times \mathbb{A}^s$$

given by the W_I-invariant functions $(\alpha_i, 0)$ for $1 \leqslant i \leqslant r$ and $(\omega_i, \chi_{\omega_i}^I)$ for $1 \leqslant i \leqslant s$, where $\chi_{\omega_i}^I$ is the character of the irreducible representation of L_I with highest weight ω_i. The map q^I induces a map

$$q_X^I : \mathcal{A}_{\mathcal{L}}^{W_I,\heartsuit} \to \mathcal{B}_{\mathcal{L}}^{\circ} \oplus \bigoplus_{i=1}^{s} H^0(X, \omega_i(\mathcal{L})).$$

The fibers of q^I over the open subset $T_{\mathrm{ad}} \times \mathbb{A}^s \subset A_{G^{\mathrm{sc}}} \times \mathbb{A}^s$ are isomorphic to $\mathbb{G}_m^{r-s}$. This implies that the nonempty fibers of q_X^I are $(k^\times)^{r-s}$. Hence

$$\dim \mathcal{A}_{\mathcal{L}}^{W_I,\heartsuit} \leqslant \dim \mathcal{B}_{\mathcal{L}} + \sum_{i=1}^{s} (\deg(\omega_i(\mathcal{L})) + 1 - g) + r - s.$$

Therefore, the codimension of $\mathcal{A}_{\mathcal{L},b}^{\heartsuit} - \mathcal{A}_{\mathcal{L},b}^{\mathrm{ani}}$ is bounded below by

$$\sum_{i=1}^{r} (\deg(\omega_i(\mathcal{L})) + 1 - g) - \left[\sum_{i=1}^{s} (\deg(\omega_i(\mathcal{L})) + 1 - g) + r - s \right] \geqslant N - rg. \qquad \square$$

Denote by $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}} := h_{\mathcal{L}}^{-1}(\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}})$ the anisotropic open substack. This is nonempty when G is semisimple. Also, let $\mathcal{P}_{\mathcal{L}}^{\mathrm{ani}}$ be the restriction of $\mathcal{P}_{\mathcal{L}}$ to $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$.

Proposition 4.2.9. *$\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}$ and $\mathcal{P}_{\mathcal{L}}^{\mathrm{ani}}$ are Deligne–Mumford stacks.*

Proof. Let $(\mathcal{E}, \varphi) \in \mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}(k)$ and $a = h_{\mathcal{L}}(\mathcal{E}, \varphi)$. Then the k-group $\mathrm{Aut}(\mathcal{E}, \varphi)$ classifies sections of the group scheme $\mathrm{Aut}_G(\mathcal{E})_\varphi$ over X, which is the closed subscheme of the centralizer of φ in the group scheme $\mathrm{Aut}_G(\mathcal{E})$.

Choose a geometric point $\bar{\eta}$ over the generic point η of X. Restricting the cameral cover to η along a, we obtain a homomorphism $\rho_a^\eta : \mathrm{Gal}(\bar{\eta}/\eta) \to W$. Let W_a be the image of ρ_a^η.

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Furthermore, choose a trivialization of $\mathcal{E}$ over the generic point η under which φ maps to a regular semisimple element in $T_+(k(X))$. With these choices, we get closed embeddings $\mathrm{Aut}(\mathcal{E},\varphi) \subset T^{W_a}$ and $H^0(X,\mathcal{J}_a) \subset T^{W_a}$.

Since $a \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$, T^{W_a} is finite. Since $\mathrm{char}(k)$ is coprime to the order of W, T^{W_a} is finite unramified k-group. This shows that $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}$ and $\mathcal{P}_{\mathcal{L}}^{\mathrm{ani}}$ are Deligne–Mumford stacks. $\square$

Theorem 4.2.10. *Assume that the T^{sc}-torsor $\mathcal{L}$ admits a cth root $\mathcal{L}'$. Then for any $a \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$, there is a homeomorphism of quotient stacks*

$$[\mathcal{M}_a/\mathcal{P}_a] \cong \prod_{x\in X-U_a} [\mathrm{Sp}_{a_x}/P_{a_x}]. \tag{4.1}$$

In particular, we have

$$\dim \mathcal{M}_a - \dim \mathcal{P}_a = \sum_{x\in \mathrm{Supp}(\Delta_a)} (\dim \mathrm{Sp}_{a_x} - P_{a_x}).$$

Proof. Choose a Coxeter element $w \in \mathrm{Cox}(W,S)$. The cth root $\mathcal{L}'$ of $\mathcal{L}$ induces a global Steinberg section $\epsilon^w_{\mathcal{L}'}$, in particular, a base point $\epsilon^w_{\mathcal{L}'}(a) \in \mathcal{M}_a^{\mathrm{reg}}$. Using Corollary 2.2.17, we argue as in the proof of [22, Théorème 4.6] to show that there is a morphism as (4.1) inducing equivalence of groupoids on k-points. Then the argument of [23] shows that the map (4.1) is a homeomorphism. $\square$

4.3. Properness over the anisotropic locus

Throughout this section, we assume G is semisimple so that $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is nonempty. Our goal is to show that the morphism $h_{\mathcal{L}}^{\mathrm{ani}} : \mathcal{M}_{\mathcal{L}}^{\mathrm{ani}} \to \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is proper.

4.3.1. Finiteness properties. We first show that the Hitchin–Frenkel–Ngô fibration is of finite type over the anisotropic locus. Our proof follows the argument of [8, Proposition 6.1.5].

We start with a more general situation. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional representation such that $\ker(\rho)$ is contained in the center of G. Fix a torus T and a Borel subgroup B containing T. Let $V^{(1)},\ldots,V^{(m)}$ be the irreducible constituents of V (counted with multiplicity) and $\lambda^{(1)},\ldots,\lambda^{(m)}$ be the corresponding highest weight.

For each $V^{(j)}$, we choose a basis $\{e_i^{(j)}, 1 \leqslant i \leqslant d_j\}$ (where $d_j = \dim V_j$) as follows. Each $e_i^{(j)}$ is a weight vector with weight $\lambda_i^{(j)} \in X^*(T)$. Then we can express $\lambda^{(j)} - \lambda_i^{(j)}$ as a linear combination of positive simple roots with non-negative integer coefficients and we call the sum of coefficients the *height* of $e_i^{(j)}$. The basis elements $e_i^{(j)}$ are indexed so that the height is nondecreasing with respect to i. In particular, $e_1^{(j)}$ is a highest weight vector and $e_{d_j}^{(j)}$ is a lowest weight vector in V_j.

Then under the basis $\{e_i^{(j)}, 1 \leqslant i \leqslant d_j, 1 \leqslant j \leqslant m\}$, $\rho(B)$ consists of upper triangular matrices in $\prod_j \mathrm{End}(V_j)$, which are the stabilizers of the standard flags

$$0 = L_0^{(j)} \subset L_1^{(j)} \subset \cdots \subset L_{d_j}^{(j)} = V^{(j)},$$

where $L_i^{(j)} = \mathrm{Span}(e_1^{(j)},\ldots,e_i^{(j)})$ for $1 \leqslant i \leqslant d_j$.

Let $I \subset \Delta$ be a subset of simple roots and $P_I \subset G$ the standard parabolic subgroup whose Levi factor has simple roots in I. Then there exist standard parabolic subalgebras $\mathfrak{p}_I^{(j)} \subset \mathrm{End}(V^{(j)})$ such that

$$\rho(P_I) = \rho(G) \cap \left(\bigoplus_{j=1}^{m} \mathfrak{p}_I^{(j)} \right).$$

More precisely, $\mathfrak{p}_I^{(j)}$ is the stabilizer of the partial flag in $V^{(j)}$ obtained from the standard flag by replacing $L_i^{(j)}$ with the span of $e_i^{(j)}$ and all basis vectors whose corresponding weight differs from the weight of $e_i^{(j)}$ by a linear combination of simple roots in I.

Fix a divisor D on a smooth projective curve X. Consider the following stack

$$\mathcal{M}_V := \mathrm{Hom}\left(X, \left[\left(\prod_{j=1}^{m} \mathrm{End}(V^{(j)})(D) \right) \Big/ G \right] \right),$$

where the action of G on $\prod_{j=1}^{m} \mathrm{End}(V^{(j)})$ is induced by ρ. More concretely, the moduli stack $\mathcal{M}_V$ classifies tuples $(E, \varphi_j, 1 \leqslant j \leqslant m)$, where E is a G-torsor and $\varphi_j : \rho_j E \to \rho_j E(D)$ is a meromorphic endomorphism of the vector bundle $\rho_j E := E \wedge^{(G,\rho)} V^{(j)}$.

From the definition, we have

$$\mathcal{M}_V = \mathcal{M}_1 \times_{\mathrm{Bun}_G} \mathcal{M}_2 \times_{\mathrm{Bun}_G} \cdots \times_{\mathrm{Bun}_G} \mathcal{M}_m,$$

where for each $1 \leqslant j \leqslant m$, we define

$$\mathcal{M}_j = \mathrm{Hom}(X, [(\mathrm{End}(V^{(j)})(D))/G]).$$

By [14, Satz 2.1.1] and [15, p. 253], we know that there exists a constant $C > 0$ such that for any G-torsor E on X there exists a Borel reduction E_B of E so that $\deg(E_B)$ belongs to

$$\mathsf{C} := \{H \in \Lambda_{\mathbb{Q}}, \alpha(H) \geqslant -c \ \forall \alpha \in \Delta\}. \tag{4.2}$$

Let N be a positive integer which is larger than the sum of coefficients of $\lambda^{(j)} - \lambda_i^{(j)}$ under the basis Δ for all i, j. Let d be an integer such that

$$d > \deg(D) + 2Nc. \tag{4.3}$$

For each subset $I \subset \Delta$, consider the following cone

$$\mathsf{C}_I := \{H \in \Lambda_{\mathbb{Q}}, \alpha(H) \leqslant d \ \forall \alpha \in I \text{ and } \alpha(H) \geqslant d \ \forall \alpha \in \Delta - I\}.$$

Lemma 4.3.2. *Let* $(E, \varphi_j) \in \mathcal{M}_V$ *and* E_B *a* B*-reduction of* E*. Suppose that* $\deg E_B \in \mathsf{C} \cap \mathsf{C}_I$*, then we have*

$$\varphi \in \mathfrak{p}(D) \wedge^B E_B,$$

where $\mathfrak{p} = \bigoplus_{j=1}^{m} \mathfrak{p}_I^{(j)}$.

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Proof. We can treat each factor $\mathcal{M}_j$ separately and assume that V is irreducible. It suffices to prove that under the adjoint action of φ, $E_B \wedge^B \mathfrak{b}$ is sent into $E_B \wedge^B \mathfrak{p}(D)$. Consider a filtration of $\mathrm{End}(V^{(j)})$:

$$(0) = \mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \cdots \subset \mathfrak{b}_r = \mathfrak{b} \subset \mathfrak{p} = \mathfrak{p}_s \subset \mathfrak{p}_{s-1} \subset \cdots \subset \mathfrak{p}_0 = \mathrm{End}(V^{(j)})$$

stable under adjoint action of B, with one-dimensional successive quotients.

Suppose the image of $E_B \wedge^B \mathfrak{b}$ under $\mathrm{ad}(\varphi)$ is not contained in $E_B \wedge^B \mathfrak{p}(D)$. Then there exist $0 < i \leqslant r$ and $0 \leqslant j < s$ such that $\mathrm{ad}(\varphi)$ induces a *nonzero* homomorphism of line bundles

$$E_B \wedge^B (\mathfrak{b}_i/\mathfrak{b}_{i-1}) \to E_B \wedge^B (\mathfrak{p}_j/\mathfrak{p}_{j+1})(D).$$

In particular, the degree of the source is not larger than the degree of the target. More precisely, let γ be the weight of B on $\mathfrak{b}_i/\mathfrak{b}_{i+1}$ and δ the weight of B on $\mathfrak{p}_j/\mathfrak{p}_{j+1}$. Then we have the inequality

$$\langle \deg E_B, \gamma - \delta \rangle \leqslant \deg D.$$

Note that γ is the difference between the highest weight $\lambda^{(j)}$ and certain weight of the G-representation $V^{(j)}$, hence a non-negative linear combination of simple roots with the sum of coefficients bounded by N. Since $\deg E_B \in \mathsf{C}$, we then have

$$\langle \deg E_B, \gamma \rangle \geqslant -Nc.$$

On the other hand, by definition of $\mathfrak{p} = \mathfrak{p}_I^{(j)}$, we see that $-\delta$ is a non-negative linear combination of simple roots such that the sum of coefficients is bounded by N and the coefficient of some root in $\Delta - I$ is positive. Hence because $\deg E_B \in \mathsf{C} \cap \mathsf{C}_I$, we have

$$\langle \deg E_B, -\delta \rangle \geqslant d - Nc.$$

Combining the above two inequalities, we get $d - 2Nc \leqslant \deg D$ which contradicts (4.3) and thus the lemma follows. □

Proposition 4.3.3. *The stack $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}$ is of finite type.*

Proof. The natural morphism $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}} \to \mathrm{Bun}_G$ is of finite type. For each $\nu \in X^*(T)$, the moduli stack Bun_B^ν of B-bundles on X with degree ν is of finite type. It suffices to show that there is a finite subset $S \subset X^*(T)$ such that the image of $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}$ in Bun_G is contained in the image of $\bigcup_{\nu \in S} \mathrm{Bun}_B^\nu$ in Bun_G.

By construction, $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}$ is the mapping stack of $\mathrm{Vin}_{G^{\mathrm{sc}}}^{\mathcal{L}}$, which by definition is the normalization of a closed subscheme of a product of matrix algebras (twisted by certain line bundles). In other words, there is a finite morphism $\mathcal{M}_{\mathcal{L}}^{\mathrm{ani}} \to \mathcal{M}_V$ (mostly likely a closed embedding, but we do not need this fact) where $\mathcal{M}_V$ is the mapping stack of a product of matrix algebras considered above.

Let $m = (\mathcal{E}, \varphi) \in \mathcal{M}_{\mathcal{L}}^{\mathrm{ani}}(k)$. We also view it as a k point of $\mathcal{M}_V$. Let $\mathcal{E}_B$ a B-reduction of $\mathcal{E}$ such that $\deg(\mathcal{E}_B) \in \mathsf{C}$ (see (4.2)). Let $a = h_{\mathcal{L}}(m) \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$. Suppose that $\deg(\mathcal{E}_B) \in \mathsf{C}_I$ for some *proper* subset $I \subset \Delta$. Then by Lemma 4.3.2, φ maps the generic point of the curve into the proper parabolic subgroup $P_{I,+}$ of G_+^{sc}. This implies that W_a is contained in the Weyl group of the Levi L_I and hence T^{W_a} is not finite, contradicting the fact that $a \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}(k)$. Thus we conclude that $\det \mathcal{E}_B$ lies in the intersection of C and the complement of C_I for any proper subgroup $I \subset \Delta$. This intersection is a bounded subset of $X^*(T)_{\mathbb{R}}$ and hence the set of weights $S \subset X^*(T)$ lying in the intersection is a finite set and we are done. □

4.3.4. Valuative criterion. First we have the existence part of the valuative criterion, which is true over the larger open subset $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$.

Proposition 4.3.5. *Let R be a complete discrete valuation ring with algebraically closed residue field containing k. Let K be the fraction field of R. Then for all $a \in \mathcal{A}_G^{\heartsuit}(R)$ and $m_K \in \mathcal{M}_G^{\heartsuit}(K)$ such that $h_G(m_K) = a$, there exists a finite extension K' of K and $m \in \mathcal{M}_G(R')$, where R' is the integral closure of R in K', such that*

(1) *The image of m in $\mathcal{M}_G^{\heartsuit}(K')$ is isomorphic to that of m_K;*

(2) *$h_G(m) = a$.*

Proof. The argument is the same as [8, § 8.4]. The key points are: 1. Any G-torsor extends uniquely over a codimension 2 subset; 2. the universal twisted monoid $\mathbb{V}_G$ over $\mathcal{A}_G \times X$ is affine, so that Higgs fields extend over any codimension 2 subset. □

Proposition 4.3.6. *Suppose G is semisimple. Let R be a complete discrete valuation ring with algebraically closed residue field κ containing k. Let $m, m' \in \mathcal{M}^{\mathrm{ani}}(R)$ be two elements and $m_K, m'_K \in \mathcal{M}^{\mathrm{ani}}(K)$ their base change. Suppose that the following two conditions are satisfied:*

(1) *$h(m) = h(m')$;*

(2) *there exists an isomorphism $\iota_K : m_K \to m'_K$.*

Then there exists a unique isomorphism $\iota : m \to m'$ extending ι_K.

Proof. We follow the argument in [8, § 9]. Let $m = (\mathcal{E}, \phi)$ and $m' = (\mathcal{E}', \phi')$. Consider the local ring B of the generic point of the special fiber of X_R. Then B is a discrete valuation ring whose residue field is the function field $\kappa(X)$ of X_κ and whose fraction field is the function field F of X_R.

By § 9.2 of *loc. cit.*, it suffices to extend ι_K to an isomorphism of G-torsors $\iota : \mathcal{E} \to \mathcal{E}'$ over $\operatorname{Spec} B$. As in § 9.3 of *loc. cit.*, it suffices to show that for some finite extension K', the base change $\iota_{K'}$ of ι_K extends to an isomorphism between $\mathcal{E}, \mathcal{E}'$ over $\operatorname{Spec} B'$. Here, B' is the integral closure of B in the function field F' of $X_{R'}$, where R' is the integral closure of R in K'.

To achieve this, after taking a finite extension K'/K one can assume that $\mathcal{E}, \mathcal{E}'$ are trivial over $\operatorname{Spec} B$ (since by [9, Theorem 2], they will be trivial in a Zariski open neighborhood of the generic point of the special fiber of X_R after a finite extension of K). Moreover, as in [8, Lemme 9.3.1], one can choose trivialization of $\mathcal{E}$ and $\mathcal{E}'$ over $\operatorname{Spec} B$ such that they map the 'Higgs fields' ϕ and ϕ' to some element $\gamma \in \mathrm{Vin}_G^{\mathrm{rs}}(B)$. Under these trivializations, the isomorphism ι_K is identified with an element $g \in G(F)$ such that $g^{-1}\gamma g = \gamma$. In other words, $g \in G_\gamma(F)$. Since m, m' lies in the anisotropic open substack and

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$\gamma \in \mathrm{Vin}_G^{\mathrm{rs}}(B)$, G_γ is an anisotropic torus over $\operatorname{Spec} B$ and hence $G_\gamma(B) = G_\gamma(F)$. Thus in particular, $g \in G(B)$ and the isomorphism ι_K extends. □

Theorem 4.3.7. *The morphism $h_{\mathcal{L}}^{\mathrm{ani}} : \mathcal{M}_{\mathcal{L}}^{\mathrm{ani}} \to \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ is proper.*

Proof. This follows from what have been proved in this section and the valuative criterion of properness for algebraic stacks. □

4.4. Singularities of restricted Hitchin–Frenkel–Ngô moduli stack

Later when proving equidimensionality of Kottwitz–Viehmann varieties, we will need the transversality theorem of Bouthier in [5], where it was shown that the singularities of certain open substack of restricted Hitchin–Frenkel–Ngô moduli stack are the same as some closed Schubert varieties in the affine Grassmannian. The method of Bouthier was later simplified by Yun in [36]. In [5] and [36] it is assumed that the group is simply connected but the argument works without this assumption. For the reader's convenience we review this result following [36].

4.4.1. Fix a $X_*(T_{\mathrm{ad}})_+$-valued divisor $\lambda = \sum_{i=1}^m \lambda_i x_i$ on the curve X. Then λ defines a T_{ad}-torsor $\mathcal{L}_\lambda$. We assume that $\mathcal{L}_\lambda$ can be lifted to a T^{sc}-torsor $\mathcal{L}$. Then λ can be identified with a closed point of $\mathcal{B}_{\mathcal{L}} = H^0(X, A_{G^{\mathrm{sc}}}^{\mathcal{L}})$. Let $\mathcal{M}_{\leqslant\lambda} := \alpha_{\mathcal{L}}^{-1}(\lambda)$ be the corresponding restricted Hitchin–Frenkel–Ngô moduli stack. Let $\mathcal{A}_{\leqslant\lambda} := \beta_{\mathcal{L}}^{-1}(\lambda)$ and $h_{\leqslant\lambda} : \mathcal{M}_{\leqslant\lambda} \to \mathcal{A}_{\leqslant\lambda}$ be the restricted Hitchin–Frenkel–Ngô fibration. Let $\mathcal{M}_\lambda := \mathcal{M}_{\leqslant\lambda} \cap \mathcal{M}_{\mathcal{L}}^0$ be the open substack where the Higgs–Vinberg field lands in $[\mathrm{Vin}_{G^{\mathrm{sc}}}^0 / T^{\mathrm{sc}} \times \mathrm{Ad}(G)]$.

Assume moreover that $\mathcal{L}$ admits a cth root $\mathcal{L}'$ where $c = |Z(G_{\mathrm{der}})|$. Then by the discussion in § 4.1.5, there exists global Steinberg section $\epsilon_{\mathcal{L}'}^w : \mathcal{A}_{\leqslant\lambda} \to \mathcal{M}_{\leqslant\lambda}^{\mathrm{reg}}$ for each choice of Coxeter element $w \in \mathrm{Cox}(W, S)$.

4.4.2. For each $a \in \mathcal{A}_{\leqslant\lambda}^{\heartsuit}$, we write the associated discriminant divisor as

$$\Delta(a) = \Delta(a)_{\mathrm{sing}} + \Delta(a)_{\mathrm{triv}},$$

where $\Delta(a)_{\mathrm{triv}}$ is multiplicity free and the multiplicity of $\Delta(a)_{\mathrm{sing}}$ at each point is at least 2.

Definition 4.4.3. Let $S \subset \mathrm{Supp}(\lambda)$ be a nonempty subset. The *transversal subset* $\mathcal{A}_{\leqslant\lambda}^{\flat} \subset \mathcal{A}_{\leqslant\lambda}$ consists of $a \in \mathcal{A}_{\leqslant\lambda}^{\heartsuit}$ satisfying the following two conditions

- $\mathrm{Supp}(\Delta(a)) \cap \mathrm{Supp}(\lambda) \subset S$
- For each $1 \leqslant i \leqslant r$,

$$2g - 2 + m_0(\deg \Delta(a)_{\mathrm{sing}} + |S|) + \sum_{s \in S} b(\lambda_s) < \deg \omega_i(\mathcal{L})$$

  where m_0 is the positive integer defined in the paragraph before Proposition 2.6.6 and $b(\lambda_s)$ is the non-negative integer in Lemma 2.5.2.

We call $\mathcal{M}_{\leqslant\lambda}^{\flat} := h_{\leqslant\lambda}^{-1}(\mathcal{A}_{\leqslant\lambda}^{\flat})$ the *transversal open substack* and denote

$$\mathcal{M}_\lambda^{\flat} := \mathcal{M}_{\leqslant\lambda}^{\flat} \cap \mathcal{M}_\lambda.$$

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4.4.4. Local evaluation map. For each $s \in S$, the arc space L^+G acts by left multiplication on $\mathrm{Gr}_{\leqslant\lambda_s} := \mathrm{Gr}^{G_{\mathrm{ad}}}_{\leqslant\lambda_s}$ and the action factors through L^+G_{ad}. We let N be a positive integer such that for all $s \in S$, the action of L^+G on $\mathrm{Gr}_{\leqslant\lambda_s}$ factors through the Nth jet space $L^+_N G$. Then the product group $L^+_{NS}G := \prod_{s\in S} L^+_N G$ acts naturally on $\prod_{s\in S}\mathrm{Gr}_{\leqslant\lambda_s}$ and we define the local evaluation map

$$\mathrm{ev}_{NS} : \mathcal{M}_{\leqslant\lambda} \to \left[L^+_{NS}G \Big\backslash \prod_{s\in S}\mathrm{Gr}_{\leqslant\lambda_s}\right]$$

by choosing trivializations of G-torsors on the Nth infinitesimal neighborhood of points $s \in S$. Let $\mathrm{ev}^\flat_{NS}$ be the restriction of ev_{NS} to $\mathcal{M}^\flat_{\leqslant\lambda}$. From the first condition in Definition 4.4.3, we see that for any $(\mathcal{E}, \varphi) \in \mathcal{M}^\flat_{\leqslant\lambda}$ the restriction of the Higgs–Vinberg field φ to points in $\mathsf{Supp}(\lambda) \setminus S$ lands in the open substack $[\mathrm{Vin}^{\mathrm{rs}}_{G^{\mathrm{sc}}}/T^{\mathrm{sc}} \times \mathrm{Ad}(G)]$, which is contained in $[\mathrm{Vin}^0_{G^{\mathrm{sc}}}/T^{\mathrm{sc}} \times \mathrm{Ad}(G)]$ by Corollary 2.2.7. Hence the inverse image of the open strata $[L^+_{NS}G\backslash \prod_{s\in S}\mathrm{Gr}_{\lambda_s}]$ under $\mathrm{ev}^\flat_{NS}$ is precisely $\mathcal{M}^\flat_\lambda = \mathcal{M}_{\leqslant\lambda} \cap \mathcal{M}_\lambda$.

Theorem 4.4.5 ([5],[36]). *The morphism*

$$\mathrm{ev}^\flat_{NS} : \mathcal{M}^\flat_{\leqslant\lambda} \to \left[L^+_{NS}G \Big\backslash \prod_{s\in S}\mathrm{Gr}_{\leqslant\lambda_s}\right]$$

is smooth.

The proof proceeds in several steps which occupy the rest of this section.

4.4.6. Let $\mathcal{M}^\flat_{\leqslant\lambda,NS}$ be the stack classifying triples $(\mathcal{E}, \varphi, \tau_{NS})$ where $(\mathcal{E}, \varphi)$ is a point in $\mathcal{M}^\flat_\lambda$ and τ_{NS} is a trivialization of $\mathcal{E}$ on the Nth infinitesimal neighborhoods of s for all $s \in S$. Then $\mathcal{M}_{\leqslant\lambda,NS}$ is a $L^+_{NS}G$-torsor over $\mathcal{M}_{\leqslant\lambda}$ and to prove the smoothness of $\mathrm{ev}^\flat_{NS}$, it suffices to prove the smoothness of its base change

$$\widetilde{\mathrm{ev}}^\flat_{NS} : \mathcal{M}^\flat_{\leqslant\lambda,NS} \to \prod_{s\in S}\mathrm{Gr}_{\leqslant\lambda_s}.$$

Notice that the source and target of $\tilde{\mathrm{ev}}^\flat_{NS}$ are locally of finite type. Hence it suffices to show that $\widetilde{\mathrm{ev}}^\flat_{NS}$ is formally smooth. In other words, we need to check the infinitesimal lifting property.

4.4.7. Let R be an artin local k-algebra with maximal ideal $\mathfrak{m}$ and let $I \subset R$ be an ideal with $I \cdot \mathfrak{m} = 0$. Denote $\bar{R} := R/I$. Consider a triple $(\bar{\mathcal{E}}, \bar{\varphi}, \bar{\tau}_{NS}) \in \mathcal{M}^\flat_{\lambda,NS}(\bar{R})$ whose image under the map $\widetilde{\mathrm{ev}}^\flat_{NS}$ is $([\bar{\gamma}_s])_{s\in S} \in \prod_{s\in S}\mathrm{Gr}_{\leqslant\lambda_s}(\bar{R})$. Let $([\gamma_s])_{s\in S} \in \prod_{s\in S}\mathrm{Gr}_{\leqslant\lambda_s}(R)$ be a lifting of $([\bar{\gamma}_s])_{s\in S}$. We need to find a lifting of $(\bar{\mathcal{E}}, \bar{\varphi}, \bar{\tau}_{NS})$ to a point in $\mathcal{M}_{\leqslant\lambda,NS}(R)$ whose image under $\widetilde{\mathrm{ev}}^\flat_{NS}$ coincides with $([\gamma_s])_{s\in S}$.

Extend $\bar{\tau}_{NS}$ to a trivialization $\bar{\tau}_{\infty S}$ of $\bar{\mathcal{E}}$ on $\prod_{s\in S} D_s \hat{\otimes} \bar{R}$ where D_s denotes the formal neighborhood of s in X. It suffices to construct a Higgs–Vinberg pair $(\mathcal{E}, \varphi) \in \mathcal{M}^\flat_{\leqslant\lambda}(R)$ lifting $(\bar{\mathcal{E}}, \bar{\varphi}) \in \mathcal{M}^\flat_{\leqslant\lambda}(\bar{R})$ and a trivialization $\tau_{\infty S}$ of $\mathcal{E}$ on $\prod_{s\in S} D_s \hat{\otimes} R$ lifting $\bar{\tau}_{\infty S}$.

4.4.8. Let $\bar{a} := h_{\leqslant\lambda}(\bar{\mathcal{E}}, \bar{\varphi}) \in \mathcal{A}^{\flat}_{\leqslant\lambda}(\bar{R})$ and $a_0 \in \mathcal{A}^{\flat}_{\leqslant\lambda}(k)$ its reduction mod $\mathfrak{m}$. We have the discriminant divisor $\Delta(a_0)$ associated to a_0. For each $v \in X$, let d_v be the multiplicity of $\Delta(a_0)$ at v. Let $S' := S \cup \mathrm{Supp}(\Delta(a_0)_{\mathrm{sing}})$ and $T := S' \setminus S$ so that $S' = S \sqcup T$. Note that the first condition in Definition 4.4.3 implies that $T \cap \mathrm{Supp}(\lambda) = \varnothing$.

For each $s \in S$, under the trivialization $\tau_{\infty S}$, the Taylor expansion of $\bar{\varphi}$ at s corresponds to an element $\bar{\gamma}_s \in L^+\mathrm{Vin}^{\lambda}(\bar{R})$ whose image in $\mathrm{Gr}_{\leqslant\lambda_s}(\bar{R})$ is $[\bar{\gamma}_s]$. Since the morphism $L^+\mathrm{Vin}^{\lambda} \to \mathrm{Gr}_{\leqslant\lambda_s}$ is formally smooth, there exists a lifting $\gamma_s \in L^+\mathrm{Vin}^{\lambda}(R)$ of $\bar{\gamma}_s$ whose image in $\mathrm{Gr}_{\leqslant\lambda_s}$ equals to the $[\gamma_s]$ given above. Let $a_s := \chi_+(\gamma_s) \in \mathfrak{C}_+(\hat{\mathcal{O}}_s \hat{\otimes} R)$. Then $\bar{a}_s = \bar{a} \in \mathfrak{C}_+(\hat{\mathcal{O}}_s \hat{\otimes} \bar{R})$.

For each $t \in T = S' \setminus S$, choose a trivialization $\bar{\tau}_{\infty t}$ of $\bar{\mathcal{E}}$ on $D_t \hat{\otimes} \bar{R}$, under which the Taylor expansion of φ at t corresponds to an element $\bar{\gamma}_t \in L^+G^{\mathrm{sc}}_+(\bar{R})$. We lift $\bar{\gamma}_t$ arbitrarily to an element $\gamma_t \in G^{\mathrm{sc}}_+(\hat{\mathcal{O}}_t \hat{\otimes} R)$ and let $a_t := \chi_+(\gamma_t) \in \mathfrak{C}_+(\hat{\mathcal{O}}_t \hat{\otimes} R)$. Then in particular $\bar{a}_t = \bar{a} \in \mathfrak{C}_+(\hat{\mathcal{O}}_t \hat{\otimes} \bar{R})$.

4.4.9. Consider the local evaluation map of the base space

$$\mathcal{A}_{\leqslant\lambda} \to \bigoplus_{s\in S} \mathfrak{C}^{\lambda_s}_+(\mathcal{O}_s/\varpi_s^{m_0 d_s + b(\lambda_s)}) \times \bigoplus_{t\in T} \mathfrak{C}(\mathcal{O}_t/\varpi_t^{m_0 d_t}).$$

By the inequality in Definition 4.4.3, this is a surjective linear map between k-vector spaces; hence it is smooth when viewed as morphisms between affine k-schemes. So there exists $a \in \mathcal{A}_{\leqslant\lambda}(R)$ lifting $\bar{a} \in \mathcal{A}_{\leqslant\lambda}(\bar{R})$ such that $a \equiv a_v \mod \varpi_v^{n_v}$ for all $v \in S'$.

Then for each $s \in S$, we have $a \equiv \chi_+(\gamma_s) \mod \varpi_s^{m_0 d_s + b(\lambda_s)}$. By Proposition 2.6.6 there exists $\theta_s \in \mathrm{Vin}_G(R[[\varpi_s]])$ such that $\chi_+(\theta_s) = a$ and $\theta_s \equiv \gamma_s \mod \varpi_s^{b(\lambda_s)}$. By Lemma 2.5.2, this implies that the image of θ_s in $\mathrm{Gr}_{\lambda_s}(R)$ coincides with $[\gamma_s]$.

For each $t \in T$, by Proposition 2.6.6 again, there exists $\theta_t \in G^{\mathrm{sc}}(\hat{\mathcal{O}}_t \hat{\otimes} R)$ such that $\chi_+(\theta_t) = a$.

4.4.10. For each $v \in \mathrm{Supp}(\lambda) \setminus S$, the restriction of the Higgs–Vinberg field $\bar{\varphi}$ to v lands in $[\mathrm{Vin}^{\mathrm{rs}}_G/T \times \mathrm{Ad}(G)]$ by the first condition in Definition 4.4.3. For each point v in the complement of $\mathrm{Supp}(\lambda) \cup S' = \mathrm{Supp}(\lambda) \sqcup T$, the restriction of $\bar{\varphi}$ to v lands in $[G^{\mathrm{sc,reg}}_+/T \times \mathrm{Ad}(G)]$ by Corollary 3.8.3.

Therefore the restriction of $(\bar{\mathcal{E}}, \bar{\varphi})$ to $(X - S') \otimes_k \bar{R}$ lands in the stack

$$[(\mathrm{Vin}^{\mathrm{rs}}_G \cup G^{\mathrm{sc,reg}}_+)/T \times \mathrm{Ad}(G)]$$

which is a $\mathbb{B}\mathcal{J}$ gerbe neutralized by a global Steinberg section $\epsilon^{w}_{\mathcal{L}'}$. By the same reasoning as in Proposition 4.2.2, there exists a Higgs–Vinberg pair $(\mathcal{E}', \varphi')$ over $(X - S') \otimes_k R$ together with trivializations $\tau^{\bullet}_{\infty v}$ of $\mathcal{E}'$ over the formal punctured disc at each $v \in S'$ that lifts $(\bar{\mathcal{E}}, \bar{\varphi})|_{(X-S')\otimes_k \bar{R}}$ and the restrictions of the trivializations $\tau_{\infty v}$ to the punctured disc and moreover $\chi_+(\mathcal{E}', \varphi') = a \in \mathrm{Hom}(X - S', \mathfrak{C}^{\mathcal{L}}_+)$.

4.4.11. Finally we construct the desired lifting $(\mathcal{E}, \varphi, \tau_{\infty S})$ by Beauville–Laszlo gluing lemma. For each $v \in S'$, restricting φ' to the formal punctured disc $D^{\bullet}_v \hat{\otimes} R$ and using the trivialization $\tau^{\bullet}_{\infty v}$, we obtain an element $\theta'_v \in G^{\mathrm{sc}}_+(R((\varpi_v)))$ with $\chi_+(\theta'_v) = a$. Recall that we have constructed elements $\theta_v \in L^+\mathrm{Vin}^{\lambda_v}(R) \subset G^{\mathrm{sc}}_+(R((\varpi_v)))$ with $\chi_+(\theta_v) = a$. Since $a \in \mathfrak{C}^{\mathrm{rs}}_+(R((\varpi_v)))$, the transporter $\mathrm{Isom}(\theta_v, \theta'_v)$ from θ_v to θ'_v is a torsor under the

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smooth centralizer $\mathcal{J}_a|_{D_v^\bullet \hat{\otimes} R}$. After reduction mod I, we know that $\bar{\theta}_v$ and $\bar{\theta}'_v$ come from a globally defined Higgs–Vinberg pair $(\mathcal{E}', \varphi')$. In other words, $\mathrm{Isom}(\theta_v, \theta'_v)$ has a $\bar{R}$-point. By smoothness, this $\bar{R}$-point lifts to an R-point of $\mathrm{Isom}(\theta_v, \theta'_v)$. Consequently by Beauville–Laszlo lemma, the Higgs–Vinberg pairs $(\mathcal{E}', \varphi')$ over $(X - S') \otimes_k R$ with the trivializations $\tau^\bullet_{\infty v}$ over the formal punctured disc $D_v^\bullet \hat{\otimes}_k R$ can be glued with the Higgs–Vinberg pairs $(\mathcal{E}_0, \theta_v)$ (where $\mathcal{E}_0$ is the trivial G-torsor) on the formal discs $D_v \hat{\otimes} R$ to get a Higgs–Vinberg pair $(\mathcal{E}, \varphi) \in \mathcal{M}^\flat_{\leqslant\lambda}(R)$. By construction it comes with tautological trivializations $\tau_{\infty S}$ on $\sqcup_{s\in S} D_s \hat{\otimes} R$ lifting $(\bar{\mathcal{E}}, \bar{\varphi}, \bar{\tau}_{\infty S})$ and its image under the evaluation map $\widetilde{\mathrm{ev}}^\flat_{NS}$ is the R-point $([\theta_s] = [\gamma_s])_{s\in S}$ of $\prod_{s\in S} \mathrm{Gr}_{\leqslant\lambda_s}$.

Corollary 4.4.12. *The stack $\mathcal{M}^\flat_{\leqslant\lambda}$ is Cohen–MaCaulay and its open substack $\mathcal{M}^\flat_\lambda$ is smooth.*

Proof. This follows from Theorem 4.4.5 and the fact that $\mathrm{Gr}_{\leqslant\lambda_s}$ is Cohen–MaCaulay and Gr_{λ_s} is smooth for all $s \in S$. □

5. From global to local

In this section, we finish the proof of Theorem 1.2.1.

Let $\lambda \in X_*(T)_+$ and $\gamma \in G(F)^{\mathrm{rs}}$. Suppose that $\kappa_G(\gamma) = p_G(\lambda)$ and $\nu_\gamma \leqslant_{\mathbb{Q}} \lambda$ so that the generalized affine Springer fibers X^λ_γ and $X^{\leqslant\lambda}_\gamma$ are nonempty. Let

$$a := \chi_+(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}^{\mathrm{rs}}_{G^{\mathrm{sc}}_+}(F)$$

where $\gamma_\lambda \in G^{\mathrm{sc,rs}}_+(F)$ is defined in Lemma 3.1.5. Then we have isomorphisms

$$X^{\leqslant\lambda}_\gamma \cong \mathrm{Sp}_a, \quad X^\lambda_\gamma \cong \mathrm{Sp}^0_a.$$

Moreover, the local Picard P_a acts on Sp_a and $\mathrm{Sp}^{\mathrm{reg}}_a$ is the union of open orbits.

5.1. Local constancy of Kottwitz–Viehmann varieties

This subsection is devoted to the proof of the following:

Theorem 5.1.1. *There exists an integer N such that for all $a' \in \mathfrak{C}_+(\mathcal{O}_x) \cap \mathfrak{C}^{\mathrm{rs}}_{G_+}(F_x)$ with $a' \equiv a \mod \varpi^N$, there are an isomorphism $\mathrm{Sp}_a \cong \mathrm{Sp}_{a'}$ and an isomorphism $P_a \cong P_{a'}$ compatible with the action of P_a (respectively $P_{a'}$) on Sp_a (respectively $\mathrm{Sp}_{a'}$).*

First we make some standard reductions. Notice that for any $a' \in \mathfrak{C}_+(\mathcal{O}_x) \cap \mathfrak{C}^{\mathrm{rs}}_{G_+}(F_x)$, $\mathrm{Sp}_{G,a'}$ is a union of certain connected components of $\mathrm{Sp}_{G_{\mathrm{ad}},a'}$, the latter of which is isomorphic to the quotient of $\mathrm{Sp}_{G^{\mathrm{sc}}_+,a'}$ by the coweight lattice $X_*(T^{\mathrm{sc}})$ of the central torus of G^{sc}_+. Hence we may assume that $G = G^{\mathrm{sc}}_+$ and for simplicity omit G in the notation.

Fix a Coxeter element $w \in \mathrm{Cox}(W, S)$, cf. 2.2.1. Let $\gamma_0 := \epsilon^w_+(a)$ (respectively $\gamma'_0 := \epsilon^w_+(a')$) be the extended Steinberg sections for a (respectively a'). Then we have a canonical isomorphism between group schemes over $\mathrm{Spec}\,\mathcal{O}$:

$$J_a \cong I_{\gamma_0}, \quad J_{a'} \cong I_{\gamma'_0}.$$

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Lemma 5.1.2. *For any $g \in G(F)$, we have $\mathrm{Ad}(g)^{-1}(\gamma_0) \in \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O})$ if and only if*

$$\mathrm{Ad}(g)^{-1}(\gamma_0 I_{\gamma_0}(\mathcal{O})) \subset \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O}).$$

Proof. Since $\gamma_0 \in \gamma_0 I_{\gamma_0}(\mathcal{O})$, the condition is sufficient. Now assume that

$$\gamma := \mathrm{Ad}(g)^{-1}(\gamma_0) \in \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O}).$$

Then the centralizer I_γ is a group scheme over $\mathrm{Spec}\, \mathcal{O}$. By Lemma 2.4.2, the isomorphism of F groups

$$\mathrm{Ad}(g)^{-1} : J_{a,F} = I_{\gamma_0,F} \to I_{\gamma,F}$$

extends to $\mathrm{Spec}\, \mathcal{O}$. Thus we have

$$\mathrm{Ad}(g)^{-1}(I_{\gamma_0}(\mathcal{O})) \subset I_\gamma(\mathcal{O}) \subset G(\mathcal{O})$$

from which we obtain

$$\mathrm{Ad}(g)^{-1}(\gamma_0 I_{\gamma_0}(\mathcal{O})) = \gamma \mathrm{Ad}(g)^{-1}(I_{\gamma_0}(\mathcal{O})) \subset \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O}).$$

□

Lemma 5.1.3. *Let $a, a' \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}^{\mathrm{rs}}_{G^{\mathrm{sc}}_+}(F)$ with $a \equiv a' \mod \varpi^N$. Suppose that there exists a W-equivariant isomorphism between the cameral covers $\widetilde{X}_a$ and $\widetilde{X}_{a'}$ lifting the identity modulo ϖ^N. Let $\gamma_0 := \epsilon_+^w(a)$ and $\gamma_0' = \epsilon_+^w(a')$. Then there exists $g \in G(\mathcal{O})$ such that*

$$\mathrm{Ad}(g)^{-1}(\gamma_0 I_{\gamma_0}(\mathcal{O})) = \gamma_0' I_{\gamma_0'}(\mathcal{O}).$$

Proof. We follow the argument of [23, Lemme 3.5.4]. Let

$$\widetilde{X}_a = \mathrm{Spec}\, R_a \quad \text{and} \quad \widetilde{X}_{a'} = \mathrm{Spec}\, R_{a'}$$

where $R_a, R_{a'}$ are finite flat $\mathcal{O}$-algebras. Let $F_a := R_a \otimes_{\mathcal{O}} F$ (respectively $F_{a'} := F_{a'} \otimes_{\mathcal{O}} F$) and $R_a^\flat$ (respectively $R_{a'}^\flat$) be the normalization of R_a (respectively $R_{a'}$) in F_a (respectively $F_{a'}$).

By assumption, we have

$$R_a/\varpi^N = R_{a'}/\varpi^N$$

and there exists a W-equivariant $\mathcal{O}$-isomorphism $\iota : R_a \xrightarrow{\sim} R_{a'}$ that lifts the identity modulo ϖ^N.

By Proposition 2.4.7, the isomorphism

$$\iota : R_a \cong R_{a'}$$

induces an isomorphism $\iota_I : I_{\gamma_0} \to I_{\gamma_0'}$ between group schemes over $\mathrm{Spec}\, \mathcal{O}$. Since $\gamma_0 \in I_{\gamma_0}(F)$, we have $\iota_I(\gamma_0) \in I_{\gamma_0'}(F)$. We can choose $h \in G(R_a^\flat)$ and $h' \in G(R_{a'}^\flat)$ such that on F-points, the map ι_I is given by the following composition

$$I_{\gamma_0}(F) \xrightarrow{\sim} T(F_a)^W \xrightarrow{\iota} T(F_{a'})^W \xrightarrow{\sim} I_{\gamma_0'}(F), \tag{5.1}$$

where the first map is $\mathrm{Ad}(h)$ and the third map is $\mathrm{Ad}(h')^{-1}$. In other words, $\iota_I = \mathrm{Ad}(h'^{-1}\iota(h))$ on F-points. In particular, we have

$$\chi_+(\iota_I(\gamma_0)) = \chi_+(\gamma_0) = a.$$

The assumption that ι is identity modulo ϖ^N implies that $\mathrm{Ad}(h'^{-1}\iota(h)) \equiv \mathrm{Id} \mod \varpi^N$. Thus we get

$$\iota(I_{\gamma_0}(F) \cap \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O})) \subset I_{\gamma_0'}(F) \cap \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O}).$$

In particular, we have $\iota(\gamma_0) \in I_{\gamma_0'} \cap \mathrm{Vin}_{G^{\mathrm{sc}}}(\mathcal{O})$ and moreover

$$\iota_I(\gamma_0) = \gamma_0 = \gamma_0' \text{ in } \mathrm{Vin}^w_{G^{\mathrm{sc}}}(\mathcal{O}/\varpi^N).$$

Since the map

$$G \times \mathrm{Vin}^w_G \to \mathrm{Vin}^w_G \times_{\mathfrak{C}_+} \mathrm{Vin}^w_G$$

is smooth and surjective, there exists $g \in G(\mathcal{O})$ with $g \equiv 1 \mod \varpi^N$ such that $\mathrm{Ad}(g)^{-1}(\gamma_0) = \iota_I(\gamma_0)$. Therefore

$$\mathrm{Ad}(g)^{-1}(I_{\gamma_0}) = I_{\iota(\gamma_0)} = I_{\gamma_0'}.$$

Finally by Lemma 2.5.2, we have $(\gamma_0')^{-1}\iota_I(\gamma_0) \in G(\mathcal{O}) \cap I_{\gamma_0'}(F) = I_{\gamma'}(\mathcal{O})$ which implies that $\iota_I(\gamma_0) \in \gamma_0' I_{\gamma_0'}(\mathcal{O})$ and hence we are done. □

5.2. Dimension of Kottwitz–Viehmann varieties

By Theorem 3.7.1, the dimension formula for $X^\lambda_\gamma \cong \mathrm{Sp}_a$ is reduced to the following statement which we prove in this subsection:

Theorem 5.2.1. $\dim \mathrm{Sp}_a = \dim P_a$.

If $C \subset G$ is the maximal torus in the center of G, then $\mathrm{Sp}_{G/C,a} \cong \mathrm{Sp}_{G,a}/X_*(C)$ and similar isomorphism holds for the local Picard P_a. Thus we may assume that G is semisimple.

Let X be a projective smooth curve over k and $x \in X$ a closed point. Let $\mathcal{O}_x$ be the completed local ring at x and F_x its fraction field. Choose a uniformiser ϖ_x at x so that we have $\mathcal{O}_x = k[[\varpi_x]]$ and $F_x = k((\varpi_x))$. Also we let $X' = X - \{x\}$ be the open curve.

We view $a \in \mathfrak{C}_+(\mathcal{O}_x)$ as a power series in ϖ_x with coefficients in $\mathfrak{C}_+$. Form the Cartesian diagram

$$\begin{array}{ccc} X_a & \longrightarrow & \overline{T_+} \\ {\scriptstyle \pi_a}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}\,\mathcal{O} & \xrightarrow{a} & \mathfrak{C}_+ \end{array}$$

where $X_a = \mathrm{Spec}\, R_a$ for a finite flat $\mathcal{O}$ algebra R_a. Moreover, $F_a = R_a \otimes_{\mathcal{O}} F$ is a product of finite tamely ramified extension of F of degree e by our assumption that $\mathrm{char}(k)$ is coprime to the order of Weyl group. Then $a(\varpi_x^e) \in \mathfrak{C}_+(\mathcal{O}_x) \cap \mathfrak{C}^{\mathrm{rs}}_{G^{\mathrm{sc}}_+}(F_x)$ will be a split conjugacy class.

For each $s \in k$ we define

$$a_s := a(s\varpi_x + (1-s)\varpi_x^e) \in \mathfrak{C}_+(\mathcal{O}) \cap \mathfrak{C}_{G_+}(F)^{\mathrm{rs}}. \tag{5.2}$$

Then $a_1 = a$ and $a_0 = a(\varpi_x^e)$. For each $s \neq 0$, Sp_{a_s} is isomorphic to Sp_a since a_s is obtained from $a = a_1$ by changing uniformizer.

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5.2.2. Let $N > 0$ be a positive integer such that both Sp_a and Sp_{a_0} only depend on a (respectively a_0) modulo ϖ_x^N. Then for all $s \in k$, Sp_{a_s} only depends on a_s modulo ϖ_x^N.

Now we choose a T^{sc}-torsor $\mathcal{L}$ on X trivialized on the formal neighborhood of x such that

(1) There exist a T^{sc}-torsor $\mathcal{L}'$ and an isomorphism $(\mathcal{L}')^{\otimes c} \cong \mathcal{L}$;

(2) For all $y \in X' = X - x$, choosing a trivialization of $\mathcal{L}$ on a formal neighborhood of y, the local evaluation map

$$\mathcal{A}_{\mathcal{L}} = H^0(X, \mathfrak{C}_+^{\mathcal{L}}) \to \mathfrak{C}_+(\mathcal{O}_x/\varpi_x^N) \times \mathfrak{C}_+(\mathcal{O}_y/\varpi_y^2) \tag{5.3}$$

is surjective.

By Riemann–Roch, condition 2 is satisfied if for all $1 \leqslant i \leqslant r$ we have

$$\deg(\alpha_i(\mathcal{L})) \geqslant 2g + N \quad \text{and} \quad \deg(\omega_i(\mathcal{L})) > 2g + N.$$

Recall that for each $a_+ \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$, we associate an $X_*(T_{\mathrm{ad}})_+$-valued divisor λ_{a_+} on X as in § 4.1.3.

Lemma 5.2.3. *Let $\Sigma \subset X$ be a finite subset. The subset $\mathcal{A}_{\mathcal{L}}^{\Sigma} \subset \mathcal{A}_{\mathcal{L}}^{\heartsuit}$ consisting of $a_+ \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$ such that*

$$\mathrm{Supp}(\lambda_{a_+}) \cap \mathrm{Supp}(\Delta_{a_+}) \subset \Sigma$$

is constructible.

Proof. For each $1 \leqslant i \leqslant r$, consider the closed subscheme $\mathcal{D}_i \subset \mathcal{A}_{\mathcal{L}}^{\heartsuit} \times X$ whose fiber over $a_+ \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$ is the effective divisor $D(b_i)$ where b_i is the ith coordinate of $\beta_{\mathcal{L}}(a_+)$ as above. Similarly, we have the closed subscheme $\Delta \subset \mathcal{A}_{\mathcal{L}}^{\heartsuit} \times X$ whose fiber over a_+ is the discriminant divisor Δ_{a_+}. Let

$$\mathcal{D}_i^{\Sigma} = \mathcal{D}_i \cap (\mathcal{A}_{\mathcal{L}}^{\heartsuit} \times (X - \Sigma)) \quad \text{and} \quad \Delta^{\Sigma} := \Delta \cap (\mathcal{A}_{\mathcal{L}}^{\heartsuit} \times (X - \Sigma)).$$

Then $\mathcal{D}_i^{\Sigma} \cap \Delta^{\Sigma}$ is a locally closed subset of $\mathcal{A}_{\mathcal{L}}^{\heartsuit} \times X$. By construction $\mathcal{A}_{\mathcal{L}}^{\Sigma}$ is the image of $\bigcup_{1 \leqslant i \leqslant r} (\mathcal{D}_i^{\Sigma} \cap \Delta^{\Sigma})$ in $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$, hence constructible. □

5.2.4. The one-parameter family (5.2) defines a curve C in $\mathfrak{C}_+(\mathcal{O}_x/\varpi_x^N)$. Let $L_C \subset \mathcal{A}_{\mathcal{L}}$ be the closed subset defined as the inverse image of C under the map (5.3). For all $s \in k$, let $L_{a_s} \subset \mathcal{A}_{\mathcal{L}}$ be the inverse image of a_s under the map (5.3). Since $a_s \in \mathfrak{C}_{G^{\mathrm{sc}}}^{\mathrm{rs}}(F)$ for all $s \in k$, we have $L_C \subset \mathcal{A}_{\mathcal{L}}^{\heartsuit}$.

Definition 5.2.5. Let $Z_C \subset L_C$ be the subset consisting of $a_+ \in L_C$ with $b = \beta_{\mathcal{L}}(a_+)$ such that

- $a_+ \in \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$;
- $\mathrm{Supp}(\lambda_{a_+}) \cap \mathrm{Supp}(\Delta_{a_+}) \subset \{x\}$;
- $a_+(X')$ intersects the discriminant divisor $\mathfrak{D}_+^{\mathcal{L}}$ transversally, where $X' = X - \{x\}$.

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Lemma 5.2.6. *Z_C is a constructible subset of L_C that is fiberwise dense with respect to the projection $L_C \to C$. In particular, there exists a fiberwise dense open subset U_C of L_C such that $U_C \subset Z_C$.*

Proof. First we show that Z_C is constructible. The first condition in Definition 5.2.5 defines an open subset of L_C. By Lemma 5.2.3, the set $L_C^x := L_C \cap \mathcal{A}_{\mathcal{L}}^x$ determined by the second condition in Definition 5.2.5 is a constructible subset of L_C.

Let $U \subset X' \times L_C$ be the open subset whose fiber over $a_+ \in L_C$ is the open curve $X' - \mathrm{Supp}(\lambda_{a_+})$. The local evaluation maps define a morphism

$$U \to \mathbb{T}\mathfrak{C}_+^{\mathcal{L}},$$

where $\mathbb{T}\mathfrak{C}_+^{\mathcal{L}}$ is the relative tangent bundle of $\mathfrak{C}_+^{\mathcal{L}}$ over X. Let U_1 be the inverse image of

$$\mathbb{T}\mathfrak{D}_+^{\mathcal{L},\mathrm{sm}} \cup \mathbb{T}\mathfrak{C}_+^{\mathcal{L}} \times_{\mathfrak{C}_+^{\mathcal{L}}} \mathfrak{D}_+^{\mathcal{L},\mathrm{sing}}.$$

Then the image of U_1 in L_C is a constructible subset that satisfies the third condition in Definition 5.2.5. Hence Z_C is a constructible subset of L_C.

Next we show that Z_C is fiberwise dense with respect to the map $L_C \to C$. We fix a point $a_s \in C$.

For any closed point $y \in X'$, the map

$$L_{a_s} \to \mathbb{T}\mathfrak{C}_{+,y}^{\mathcal{L}} = \mathfrak{C}_+^{\mathcal{L}} \otimes_{\mathcal{O}_y} \mathcal{O}_y/\mathfrak{m}_y^2$$

is surjective by our choice of $\mathcal{L}$.

Let $X'' := X' \setminus \mathrm{Supp}(\lambda_b)$. By the same argument as in [23, Lemme 4.7.2], we know that the subset $Z \subset L_{a_s}$ consisting of $a_+ \in L_{a_s}$ such that $a_+(X'')$ intersects $\mathfrak{D}_+^{\mathcal{L}}$ transversally is dense in L_{a_s}.

For each $y \in \mathrm{Supp}(\lambda_b) - \{x\}$, since the map $\mathrm{ev}_y : L_{a_s} \to \mathfrak{C}_{+,y}^{\mathcal{L}}$ is surjective, the subset $\Sigma_y := \mathrm{ev}_y^{-1}(\mathfrak{D}_+^{\mathcal{L}}) \subset L_{a_s}$ has codimension 1.

Finally, since L_{a_s} has codimension $2rN$ in $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$ and the complement of $\mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$ in $\mathcal{A}_{\mathcal{L}}^{\heartsuit}$ has codimension strictly larger than $2rN$, we see that

$$Z_{a_s} = \left(Z - \bigcup_{y \in \mathrm{Supp}(\lambda_b)} \Sigma_y \right) \cap \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$$

is dense in L_{a_s}. □

5.2.7. Thus we can choose a section σ of the surjective linear map (5.3) such that $C' := \sigma(C) \cap U_C$ is nonempty and contains the point $\sigma(a_0)$.

By Theorem 4.2.10, we have

$$\dim \mathcal{M}_{\sigma(a_0)} - \mathcal{P}_{\sigma(a_0)} = \sum_{v \in \mathrm{Supp}(\Delta_a) \cup \{x\}} (\dim \mathrm{Sp}_{\sigma(a_0)_v} - \dim P_{\sigma(a_0)_v}),$$

where $\sigma(a_0)_v$ denotes the image of $\sigma(a_0)$ in $\mathfrak{C}_+(\mathcal{O}_v)$.

For summands with $v \neq x$, since $\sigma(a_0) \in Z_C$ we have in particular $\lambda_{\sigma(a_0),v} = 0$ and hence by Corollary 3.8.2 $\dim \mathrm{Sp}_{\sigma(a_0)_v} = \dim P_{\sigma(a_0)_v}$. On the other hand, for the term $v = x$,

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we know that $\sigma(a_0)_x = a_0$ is split and hence by Corollary 3.5.3 $\dim \mathrm{Sp}_{a_0} = \dim P_{a_0}$. Thus the above equality simplifies to

$$\dim \mathcal{M}_{\sigma(a_0)} - \dim \mathcal{P}_{\sigma(a_0)} = 0.$$

Since $C' \subset \mathcal{A}_{\mathcal{L}}^{\mathrm{ani}}$, the restriction of the Hitchin–Frenkel–Ngô fibration to C' is proper. Hence by upper semicontinuity of fiber dimension we have for

$$\dim \mathcal{M}_{\sigma(a_s)} \leqslant \dim \mathcal{M}_{\sigma(a_0)} = \dim \mathcal{P}_{\sigma(a_0)}$$

for all $\sigma(a_s) \in C'$ with $s \neq 0$. Since $\mathcal{P}$ is smooth over $\mathcal{A}_{\mathcal{L}}$ by Proposition 4.2.2, we have $\dim \mathcal{P}_{\sigma(a_s)} = \mathcal{P}_{\sigma(a_0)}$, which forces

$$\dim \mathcal{M}_{\sigma(a_s)} = \dim \mathcal{P}_{\sigma(a_s)}.$$

Applying product formula Theorem 4.2.10 again, we get

$$0 = \dim \mathcal{M}_{\sigma(a_s)} - \dim \mathcal{P}_{\sigma(a_s)} = \sum_{v \in \mathrm{Supp}(\Delta_{a_s}) \cup \{x\}} (\dim \mathrm{Sp}_{\sigma(a_s),v} - \dim P_{\sigma(a_s),v}).$$

By similar reasoning as above, all terms on the right hand side where $v \neq x$ are zero; at $v = x$ notice that $\sigma(a_s)_x = a_s$ and then we get

$$\dim \mathrm{Sp}_{a_s} - \dim P_{a_s} = 0.$$

Since $s \neq 0$, we have $\mathrm{Sp}_{a_s} \cong \mathrm{Sp}_a$ and hence

$$\dim \mathrm{Sp}_a = \dim P_a.$$

This finishes the proof of Theorem 5.2.1 and hence the dimension formula in Theorem 1.2.1.

5.3. Equidimensionality

To finish the proof of Theorem 1.2.1 it remains to show the equidimensionality statement. Again our argument is of global nature, this time using a restricted Hitchin–Frenkel–Ngô moduli stack instead of the whole moduli stack. As in the previous subsection, we may assume that G is semisimple.

5.3.1. Recall that by Theorem 5.1.1 there exists a positive integer $N > 0$ such that the isomorphism class of Sp_a equipped with the action of P_a only depends on a modulo ϖ^N. Let X be the projective smooth curve as in the previous section. Fix two distinct closed points $x, x_0 \in X$. We consider an $X_*(T_{\mathrm{ad}})_+$-valued divisor on X of the form $\lambda[x] + \lambda_0[x_0]$, where $\lambda_0 \in X_*(T_{\mathrm{ad}})_+$ is chosen such that the following properties are satisfied:

- The T_{ad}-torsor associated to the divisor $\lambda[x] + \lambda_0[x_0]$ lifts to a T^{sc}-torsor $\mathcal{L}$ and there exists a T^{sc}-torsor $\mathcal{L}'$ together with an isomorphism $(\mathcal{L}')^{\otimes c} \cong \mathcal{L}$.
- For each $1 \leqslant i \leqslant r$, the following three inequalities are satisfied:

$$\langle \omega_i, \lambda + \lambda_0 \rangle > 2g - 2 + (N+3)r$$
$$\langle \omega_i, \lambda + \lambda_0 \rangle > 2g - 2 + m_0(d_a + 1) + b(\lambda)$$
$$\langle \omega_i, \lambda + \lambda_0 \rangle > \max\{Nr, 2g-2, rg\} + 1 + rg,$$

where $d_a = d_\gamma + \langle \rho, \lambda \rangle$ is the valuation of the extended discriminant divisor of $a = \chi(\gamma_\lambda) \in \mathfrak{C}_+(\mathcal{O})$ and the numbers m_0, $b(\lambda)$ are as in Definition 4.4.3.

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5.3.2. Let $h_{\leqslant\lambda} : \mathcal{M}_{\leqslant\lambda} \to \mathcal{A}_{\leqslant\lambda}$ be the restricted Hitchin–Frenkel–Ngô moduli stack associated to the divisor $\lambda[x] + \lambda_0[x_0]$. For simplicity we have omitted λ_0 from the notation.

We apply the result of § 4.4 to the our current situation. The set S in Definition 4.4.3 is taken to be $\{x\}$ in the current situation. Then we get open subset $\mathcal{A}^{\flat}_{\leqslant\lambda}$ and open substack $\mathcal{M}^{\flat,\mathrm{ani}}_{\leqslant\lambda}$. For some positive integer $n > 0$ large enough, there is a local evaluation map

$$\mathrm{ev}^{\flat}_{nx} : \mathcal{M}^{\flat}_{\leqslant\lambda} \to [L^+_n G\backslash \mathrm{Gr}_{\leqslant\lambda}]$$

which is smooth by Theorem 4.4.5. Moreover, the inverse image of $[L^+_n G\backslash \mathrm{Gr}_{\lambda}]$ under $\mathrm{ev}^{\flat}_{nx}$ is the open substack $\mathcal{M}^{\flat}_{\lambda}$.

Corollary 5.3.3. *The restriction of the Hitchin–Frenkel–Ngô fibration to the transversal anisotropic open substack $h^{\flat,\mathrm{ani}}_{\leqslant\lambda} : \mathcal{M}^{\flat,\mathrm{ani}}_{\leqslant\lambda} \to \mathcal{A}^{\flat,\mathrm{ani}}_{\leqslant\lambda}$ is flat.*

Proof. By product formula (Theorem 4.2.10) and Theorem 5.2.1 we have $\dim \mathcal{M}_a = \dim \mathcal{P}_a$ for each $a \in \mathcal{A}^{\flat,\mathrm{ani}}_{\leqslant\lambda}$. In particular, the fiber dimension of $h^{\flat,\mathrm{ani}}_{\leqslant\lambda}$ is constant since $\mathcal{P}$ is a smooth Deligne–Mumford stack over $\mathcal{A}^{\heartsuit}_{\mathcal{L}}$. By Corollary 4.4.12, the source $\mathcal{M}^{\flat,\mathrm{ani}}_{\leqslant\lambda}$ is Cohen–MaCaulay and hence we conclude that the morphism is flat. $\square$

Lemma 5.3.4. *There exists a point $a_+ \in \mathcal{A}^{\flat,\mathrm{ani}}_{\leqslant\lambda}$ such that*

- $x_0 \notin \mathrm{Supp}(\Delta(a_+))$,
- $\mathrm{Supp}(\Delta(a_+)_{\mathrm{sing}}) \subset \{x\}$, *in other words, $a_+(X - x)$ is transversal to the discriminant divisor.*
- $a_+ \equiv a \mod \varpi_x^N$.

Proof. The proof is similar to Lemma 5.2.6. Let $L \subset \mathcal{A}_{\leqslant\lambda}$ be the linear subspace consisting of $a_+ \in \mathcal{A}_{\leqslant\lambda}$ such that $a_+ \equiv a \mod \varpi_x^N$. Since $a \in \mathfrak{C}^{\lambda}_+(\mathcal{O})$ is generically regular semisimple, we have $L \subset \mathcal{A}^{\heartsuit}_{\leqslant\lambda}$.

The first inequality in § 5.3.1 implies that for any point $y \in X - \{x, x_0\}$, the local evaluation map

$$\mathcal{A}_{\leqslant\lambda} \to \mathfrak{C}^{\lambda}_+(\mathcal{O}_x/\varpi_x^N) \times \mathfrak{C}(\mathcal{O}_y/\varpi_y^2) \times \mathfrak{C}^{\lambda_0}_+(\mathcal{O}_{x_0}/\varpi_{x_0})$$

is surjective. By similar argument as in the proof of Lemma 5.2.6, this implies that there is dense subset $Z \subset L$ consisting of points $a_+ \in \mathcal{A}^{\heartsuit}_{\leqslant\lambda}$ such that $a_+(X - x)$ intersects the discriminant divisor transversally and $a_+(x_0)$ does not intersect with the discriminant divisor.

Then for each $a_+ \in Z$, we have $\Delta(a_+)_{\mathrm{sing}} = d_a[x]$ and hence the second inequality in § 5.3.1 ensures that $Z \subset \mathcal{A}^{\flat}_{\leqslant\lambda}$.

Finally since $Z \subset \mathcal{A}_{\leqslant\lambda}$ is a subset of codimension Nr, the third inequality in § 5.3.1 ensures that Z has nonempty intersection with $\mathcal{A}^{\mathrm{ani}}_{\leqslant\lambda}$ by Corollary 4.2.8. Then any point $a_+ \in Z \cap \mathcal{A}^{\mathrm{ani}}_{\leqslant\lambda}$ satisfies the condition we want. $\square$

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Choose $a_+ \in \mathcal{A}^{\flat,\mathrm{ani}}_{\leqslant\lambda}$ as in the Lemma above. Then by Theorem 4.2.10 and Corollary 3.8.3, there is a homeomorphism of stacks

$$[\mathcal{M}_{\leqslant\lambda,a_+}/\mathcal{P}_{a_+}] \cong [\mathrm{Sp}_a/P_a].$$

Corollary 5.3.3 implies that $\mathcal{M}_{\leqslant\lambda,a_+}$ is equidimensional. Therefore Sp_a and its open subset Sp_a^0 are also equidimensional. This finishes the proof of Theorem 1.2.1.

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