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Real Dimension Groups

David Handelman

Abstract. Dimension groups (not countable) that are also real ordered vector spaces can be obtained as direct limits (over directed sets) of simplicial real vector spaces (finite dimensional vector spaces with the coordinatewise ordering), but the directed set is not as interesting as one would like; for instance, it is not true that a countable-dimensional real vector space that has interpolation can be represented as such a direct limit over a countable directed set. It turns out this is the case when the group is additionally simple, and it is shown that the latter have an ordered tensor product decomposition. In an appendix, we provide a huge class of polynomial rings that, with a pointwise ordering, are shown to satisfy interpolation, extending a result outlined by Fuchs.

The EHS-theorem ([EHS, Theorem 2.2]) asserts that a partially ordered abelian group is unperforated and satisfies Riesz interpolation if and only if it is a direct limit (over a directed set) of *simplicial* abelian groups (*i.e.,* those order isomorphic to $\mathbf{Z}^n$ with the coordinatewise ordering). These are known as *dimension groups.* In particular, if such a group is countable, then it is a direct limit with index set the positive integers. We consider the corresponding questions in the context of ordered vector spaces.

Let F be a subfield of the reals (although the real case of interest occurs when $F = \mathbf{R}$), equipped with the relative ordering; F^+ will denote the set of positive real numbers in F. A *partially ordered F-vector space* will be a vector space V over F, together with its positive cone, V^+, which satisfies the following properties:

$$V^+ + V^+ \subseteq V^+; \quad V^+ - V^+ = V; \quad V^+ \cap -V^+ = \{0\}; \quad V^+ \cdot F^+ \subseteq V^+.$$

We say a partially ordered F-vector space is *F-simplicial* (or simply *simplicial*) if there exists an integer n such that it is isomorphic (as ordered F-vector spaces) to F^n (the space of columns of size n with the coordinatewise ordering acquired from F). Finally, a partially ordered abelian group (of which partially ordered F-vector spaces are special cases) G satisfies (Riesz) *interpolation* if for all pairs of pairs x_1, x_2 and y_1, y_2 of elements of G such that $x_i \leq y_j$ for all $i, j = 1, 2$, there exists z in G such that $x_i \leq z \leq y_j$ for all i, j.

We wish to emulate [EHS, Theorem 2.2], that is, to characterize the direct limits of simplicial F-vector spaces (all direct limits are over directed sets, and with positive F-linear maps). Note that such objects are already dimension groups, since unperforation is automatic for vector spaces over an ordered field (this requires inverses of nonzero positive elements be positive, which is trivially the case here). There is a theorem, but it is not quite what we expect.

The following comes as no surprise.

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Theorem 1 *Let V be a partially ordered F-vector space. Then V can be written as a direct limit (over a directed set) of F-simplicial partially ordered vector spaces if and only if V satisfies interpolation.*

Suppose however, that V satisfies the hypotheses of Theorem 1, and in addition, has countable dimension as an F-vector space. We expect a direct limit representation over a countable directed set, which is equivalent to a direct limit of the form

$$F^{n(1)} \xrightarrow{M(1)} F^{n(2)} \xrightarrow{M(2)} F^{n(3)} \xrightarrow{M(3)} \cdots, \tag{1}$$

where the maps $(M(i), i = 1, 2, 3, \dots)$ between the simplicial vector spaces are implemented by matrices all of whose entries are in F^+ (that is, the directed set can be taken to be the positive integers). Typical representation theorems of EHS-type first deal with this countably indexed version, and later extend to arbitrary direct limits when cardinality conditions are lifted. In fact, the obvious result does not hold—there exists a countable dimensional real partially ordered vector space with interpolation that cannot be represented as in (1), given in Example 8, although from Theorem 1, it is a direct limit over an uncountable directed set. To obtain a direct limit over the positive integers in the context of partially ordered vector spaces requires an additional hypothesis.

We say a partially ordered F-vector space V is *countably F^+-generated* if there exists a countable subset S of V^+ such that every element of V^+ is in the F^+-span of S; in other words, as an F^+-semigroup, V^+ is countably generated. It is not true that a countable-dimensional partially ordered $\mathbf{R}$-vector space is countably $\mathbf{R}^+$-generated, even if interpolation is thrown in (Example 8). There may be similar cardinality problems for bigger dimensions, especially if the continuum hypothesis is negated.

If V is a direct limit as in (1), then it obviously satisfies countable F^+-generation.

Theorem 2 *Let V be a countable dimensional partially ordered F-vector space. Then V is a direct limit over a countable index set of F-simplicial vector spaces if and only if V is countably F^+-generated and satisfies interpolation.*

Obvious examples of partially ordered F-vector spaces that are direct limits of simplicial ones include those of the form $G \otimes_{\mathbf{Z}} F$, where G is a dimension group, and tensor product ordering is imposed, as in [GH]. However, in these cases, the maps between the simplicial vector spaces are matrices all of whose entries are non-negative integers (and this is a useless characterization of such limits). We later show that simple partially ordered F-vector spaces admit such a decomposition, giving an alternative proof of Theorem 1 for this class.

The method of proof of Theorem 1 of course goes back to Shen's idea, which is rather simple to adapt here. However, because of the countability problem, we have a difficulty, which fortunately can be circumvented by the method used in [G, Theorem 3.17].

Lemma 3 *Let V be a partially ordered F-vector space with interpolation, let $g\colon V_0 \to V$ be an F-linear positive map from a simplicial F-vector space V_0, and let a be an element of the kernel of g. Then there exist a simplicial F-vector space V_1 together with F-linear positive maps $g_{01}\colon V_0 \to V_1$ and $h\colon V_1 \to V$ such that $g = hg_{01}$ and $g_{01}(a) = 0$.*

Proof Let $\{e_i\}_{i=1}^n$ be the standard basis for $V_0 \cong F^n$ with the usual ordering, write $a = \sum p_i e_i$ with p_i in F. Chop the index set $\{1, 2, 3, \ldots, n\}$ into three pieces, $S = \{i \mid p_i > 0\}$, $T = \{i \mid p_i < 0\}$, and $U = \{i \mid p_i = 0\}$. If either S or T is empty, it is clear what to do, and so we assume both are nonempty.

Let $e_i \mapsto x_i$ in V^+, so that $\sum_{i \in S} p_i x_i = \sum_{i \in T} |p_i| x_i$. Define $E_i = |p_i| e_i$ if $i \in S \cup T$ and e_i otherwise, so the map V_0 is given by $E_i \mapsto |p_i| x_i$ if $p_i \neq 0$ and $E_i \mapsto x_i$ otherwise. Define the elements $y_i = |p_i| x_i$ in V^+ for all i. We now have the equation

$$\sum_S y_s = \sum_T y_t.$$

Since Riesz interpolation is equivalent to Riesz decomposition, there exist y_{st} (with (s, t) running over $S \times T$) in V^+ such that for all $s \in S$, $y_s = \sum_{t \in T} y_{st}$ and for all $t \in T$, $y_t = \sum_{s \in S} y_{st}$.

Let V_1 be the simplicial F-vector space with standard basis given by $\{f_{st}\}_{S \times T} \cup \{f_u\}_{u \in U}$, and consider the assignment

$$E_s \mapsto \sum_T f_{st} \qquad E_t \mapsto \sum_S f_{st} \qquad E_u \mapsto f_u$$

This extends to an F-linear positive homomorphism $g_{01} \colon V_0 \to V_1$, since it is extendible from $e_i \mapsto \sum_j f_{ij}/|p_i|$ when $p_i \neq 0$. Define $h \colon V_1 \to V$ via $f_{st} \mapsto y_{st}$ and $f_u \mapsto y_u$.

For $s \in S$, $hg_{01}(e_s) = \sum_T y_{st}/p_s$, and this is $y_s/p_s = x_s = g(e_s)$; similarly $hg_{01}(e_t) = g(e_t)$ for $t \in T$; finally, for u in U, $hg_{01}(e_u) = h(f_u) = y_u = g(e_u)$. Hence $hg_{01} = g$. Next,

$$\begin{aligned} g_{01}(a) &= g_{01}\Big(\sum_S p_s e_s\Big) - g_{01}\Big(\sum_T |p_t| e_t\Big) \\ &= \sum_S p_s \sum_T f_{st}/p_s - \sum_T |p_t| \sum_S f_{st}/|p_t| \\ &= \sum_S \sum_T f_{st} - \sum_T \sum_S f_{st} = 0. \end{aligned}$$ ■

Lemma 4 *Let V be a partially ordered F-vector space, and let $g \colon V_0 \to V$ be an F-linear positive map from a simplicial F-vector space V_0. Then there exist a simplicial F-vector space V_1 and F-linear positive maps $g^{01} \colon V_0 \to V_1$ and $h \colon V_1 \to V$ such that $g = hg^{01}$ and $\ker g \subseteq \ker g^{01}$.*

Proof Since $\ker g$ is finite dimensional, we may iterate the previous construction. ■

Proof of Theorems 1 and 2 Now Theorem 2 follows in the usual way (index the generating set by the positive integers, *etc.*). However, to prove Theorem 1 (results of this type are easiest to prove from the countable case, but as discussed earlier, this cannot be done here), we rely on the method of proof of [G, Theorem 3.17]. Let Y be the set of all finite subsets of V^+; this is a directed set, and the method of op cit (modified

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by replacing **Z** by F), combined with Lemma 4 here shows that for we may construct simplicial F-vector spaces, G_A, together with positive F-linear maps $G_A \to G_B$ whenever $A \subseteq B$ that are compatible with the directed structure of Y, and positive F-linear maps $G_A \to V$ such that the image of G_A^+ in V^+ contains A, and such that any element of the kernel of $G_A \to V$ gets sent to zero in some G_B under the map $G_A \to G_B$. As usual this means the limit over Y is isomorphic to V, as partially ordered F-vector spaces. (Theorem 2 can also be proved from Theorem 1 by limiting Y to finite subsets of a countable generating set, and observing that this is itself countable and directed, hence order final to the positive integers.) ■

Now we consider the ostensibly stronger property of partially ordered F-vector spaces, that V factorize as $W \otimes_{\mathbf{Z}} F$ (where W is a dimension group and the tensor product ordering [GH] is imposed). Since direct limits factor through tensor products, such a decomposition automatically implies it is a direct limit of simplicial F-vector spaces. Moreover, it is routine that $\dim_F(W \otimes_{\mathbf{Z}} F) = \operatorname{rank}_{\mathbf{Z}} W$ (this has nothing to do with the ordering, and reduces to the case where W is a finite dimensional **Q**-vector space, for which it is trivial). In particular, if V admits such a factorization and is countable dimensional, then the corresponding W must be countable, so that V can be expressed as a direct limit with index set **N**. Not all countable dimensional real vector spaces with interpolation satisfy this condition (Example 8), but simple ones do.

Proposition 5 *If V is a simple partially ordered F-vector space with interpolation, then there exists a dimension group W such that V is order-isomorphic (as partially ordered F-vector spaces) to $W \otimes_{\mathbf{Z}} F$.*

Proof Pick a basis over F for V, say $B := \{x_\alpha\}$, such that at least one of the basis elements is an order unit for V, and set W to be the rational vector space spanned by B, with the relative ordering inherited from V. We note that even though traces are defined only as positive group homomorphisms to the reals, they are automatically F-linear (on an ordered F-vector space); this merely uses the fact that the rationals are order dense in F.

The trace spaces of W and V are identical, that is, the inclusion induces an affine homeomorphism on the trace spaces; restriction is one to one, since a trace is determined by its effect on W, and conversely, every trace on W extends to a trace on V, since W contains an order unit of V.

Since W is a rational vector space, it is unperforated. Suppose that w is a nonzero element of $W^+ = W \cap V^+$; then $t(w) > 0$ for all traces t of V, hence of W, and thus w is an order unit. This yields that W is simple, and moreover, that $W^+ \setminus \{0\}$ consists of the elements that are strictly positive at all traces of V. The trace space of V is a Choquet simplex $K = S(V, u)$, where u is a fixed order unit in W, and we have the representation $W \to \operatorname{Aff} K$ obtained by restricting that of V.

The range of V is dense in $\operatorname{Aff} K$, since V is a simple dimension group unequal to **Z**. Since W is a rational vector space, so is its image in $\operatorname{Aff} K$, and obviously the latter's closure contains the F-span of the image of W, in particular, the image of V. Since the latter is dense, so is the image of W in $\operatorname{Aff} K$. It follows that W is a dimension group. Next, consider the map $W \otimes F \to V$ (given by $\sum w_r \otimes_{\mathbf{Z}} \lambda_r \mapsto \sum w_r \lambda_r$). This is obviously an isomorphism of F-vector spaces (because of the construction), but we have to show it is an order isomorphism.

The positive cone of the tensor product consists of sums of terms of the form $w_r \otimes \lambda_r$, where $w_r \in W^+$ and $\lambda_r \in F^+$. These clearly go to positive elements in V. Since F has unique trace, it is easy to see that $H = W \otimes F$ has the same traces as W and thus of V, that is, determined by $w \otimes \lambda \mapsto t(w)\lambda$. So we have a vector space isomorphism $H \to V$ that is positive and induces a homeomophism on the trace spaces; moreover, H and V are simple (and unperforated), and it is immediate that this must be an order isomorphism (if $h \in H$ goes to something positive and not zero, its image must be strictly positive under all traces; from the homeomorphism, this pulls back to H, hence h is strictly positive at all traces, and thus is positive). ■

Corollary 6 *Suppose V is a countable dimensional simple partially ordered F-vector space with interpolation. Then V is order isomorphic to a direct limit of the form* (1).

Now we return to the question of whether every dimension group that is also an ordered $\mathbf{R}$-vector space can be so decomposed (as $W \otimes_{\mathbf{Z}} \mathbf{R}$). We will see below that a plausible countable dimensional real algebra that is a dimension group (and an ordered ring) cannot be so decomposed, because its positive cone is not countably $\mathbf{R}^+$-generated, so that it cannot even be represented as in (1).

We investigate necessary conditions for countable (and other) F^+-generation. Suppose that (V, u) is a partially ordered F^+-vector space with order unit, and let $\hat{}$ denote the natural homomorphism $\hat{}: V \to \operatorname{Aff} S(V, u)$. For every element g in V^+, define the *zero set* of g, $Z(g) = \{\tau \in \partial_e S(V, u) \mid \tau(g) = 0\}$. Each $Z(g)$ is of the form $\partial_e L = L \cap \partial_e S(V, u)$ for L a closed face of $S(V, u)$. Note that if V is simple, then the only zero sets are the trivial ones, $\varnothing$ and $\partial_e S(V, u)$.

Lemma 7 *Suppose the partially ordered F-vector space with order unit (V, u) has the property that V^+ is F^+-generated by a set of infinite cardinality $\aleph$.*

(a) *The number of subsets of $\partial_e S(V, u)$ of the form $Z(g)$ (with g in V^+) is at most $\aleph$.*
(b) *The number of order ideals of V having their own (relative) order unit is at most $\aleph$.*
(c) *The number of order ideals of V is at most $2^{\aleph}$.*

Proof (a) Obviously, if $g = \sum r_i g_i$ where $\{r_i\}$ is a finite set of positive real numbers and g_i are in V^+, then $Z(g) = \cap_i Z(g_i)$. Hence the zero set of an arbitrary element of V^+ is a finite intersection of zero sets of the generating set, hence the number of zero sets (of elements of V_+) is bounded above by the number of finite subsets of a set with $\aleph$ elements. Since $\aleph$ is infinite, the number of its finite subsets is again $\aleph$.

(b) If H is an order ideal with order unit h, we can write $h = \sum g_i r_i$ where $\{r_i\}$ is a finite set of positive real numbers and g_i are in V^+. Since $g_i \le h/r_i$, it follows that g_i all belong to H, and obviously determine H (the order ideal generated by the finite set $\{g_i\}$ is H). This yields an onto map from the finite subsets of the generating set to the set of order ideals with order unit, hence the result.

(c) Trivial, and useless anyway. ■

Example 8 A countable-dimensional real ordered vector space that is a dimension group which cannot be represented as a sequential direct limit of simplicial $\mathbf{R}$-vector spaces, that is, it cannot be represented as a direct limit of the form (1).

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Proof Let $R = \mathbf{R}[x]$, the polynomial ring with the pointwise ordering on the unit interval (that is, the polynomial f belongs to R^+ if and only if $f(r) \geq 0$ for all r in $[0,1]$). This is countable-dimensional, but cannot have a countable $\mathbf{R}^+$-generating set. If it did, then with $\aleph = \aleph_0$ in Lemma 7, there would only be countably many zero sets of positive elements. However, the pure trace space consists of the point evaluations from points in the unit interval, and each element of $\{(x-t)^2\}_{t\in[0,1]}$ yields a zero set consisting of the singleton, the point evalution at t; so uncountably many zero sets exist.

Now R has interpolation. This has an interesting history. It is stated without proof or references in [F1, F2], but in [F3, pp. 19–20] (I am indebted to George Elliott for finding this reference), a proof is sketched. Because the last is not readily available (even on-line), we include Fuchs' result, and extend it to cover subfields of the reals in Appendix A. ■

A Interpolation for Polynomial Rings with the Pointwise Ordering

Let $I = [\gamma, \delta]$ be a closed bounded interval with interior, in $\mathbf{R}$. Let L be a subfield of the reals, and form $R = L[x]$, the polynomial ring. Define a positive cone on the latter to be the set of elements of R that are nonnegative on I. This of course depends on L, γ, and δ, so if there is any possible ambiguity, we write $R_{L,\gamma,\delta}$. If $L = \mathbf{R}$, then the rings are order isomorphic to each other (via $x \mapsto cx + d$, $c \neq 0$) regardless of the choice of $\gamma < \delta$. At the other extreme, if $L = \mathbf{Q}$, then there are uncountably many order-isomorphism classes, as the only ring automorphisms that implement an order isomorphism are given by $x \mapsto cx + d$, $c \neq 0$, with c, d rational (we can recover some of the properties by ordered ring invariants, *e.g.,* the number of rationals in the set $\{\gamma, \delta\}$—that is, 0, 1, or 2—is equal to the number of codimension 1 order ideals in $R_{\mathbf{Q},\gamma,\delta}$).

We will show that for all choices of L, $\gamma < \delta$, the resulting ordered ring satisfies Riesz interpolation. If $L = \mathbf{R}$, Fuchs has given a proof, outlined in [F3, pp. 19–20], which uses Hermite interpolation, but really only requires the Chinese remainder theorem. On the other hand, if $L = \mathbf{Q}$, then there are many more technicalities (concerning behaviour of algebraic conjugates, etc.). We use two key steps from Fuchs' argument. If one wants a proof just for the reals, the following simplifies considerably to that outlined by Fuchs.

Recall the Chinese remainder theorem: if $\{I_s\}_S$ is a finite collection of ideals in a ring R such that for all $s \neq t \in S$, $I_s + I_t = R$, then the natural map $R \to \prod_S R/I_s$ is onto (usually stated in the equivalent form, $R/\cap_S I \to \prod_S R/I_s$ is an isomorphism).

Suppose $f_i \leq g_j$ (as functions on I), $i, j = 1, 2$ with all four polynomials in R; to find w in R such that $f_i \leq w \leq g_j$, we may assume f_2 is identically zero (by subtracting f_2 from the other terms). We may also assume that the four polynomials are distinct. In this case, the set of zeros of the equation $f_1 \vee 0 = g_1 \wedge g_2$ is finite, and we let S denote the set of these zeros that lie in I. We note that S consists of real numbers that are algebraic over L.

If α and β are roots of the same irreducible polynomial over L, we say α and β are (algebraic) *conjugates* (over L); when both are sitting inside the reals (as is the case with the elements of S), then there is a field isomorphism $\sigma\colon K := L[\alpha] \to L[\beta] \subseteq \mathbf{R}$

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fixing L pointwise, induced by $\alpha \mapsto \beta$. In the case where $L = \mathbf{R}$, this paragraph can be ignored.

Let T be a cross-section of the conjugacy classes of elements of S; in other words, T is a subset of S such that each element of S is conjugate to exactly one element of T; for technical reasons, we also choose elements of T to be in the interior of I if the conjugacy class contains such an element. Obviously, if $L = \mathbf{R}$, or more generally, if all the zeros lie in L, then $S = T$. For each $t \in T$, define the class $S_t = \{s \in S \mid s \text{ is conjugate to } t\}$, so that $\{S_t\}$ is a partition of S.

The proof is divided into three steps.

Step I *For each t in T, there exists h_t in $L[x]$ together with a relatively open neighbourhood in I, U_t, of S_t, such that* $0, f_1 \le h_t \le g_1, g_2$ *as functions on I restricted to U_t.* This is a somewhat technical (but not difficult) result (routine over $\mathbf{R}$), which will take the most time.

Step II *There exists h in $L[x]$ together with a relatively open neighbourhood in I, U, of S, such that* $0, f_1 \le h \le g_1, g_2$ *as functions on U.* This is the first step of Fuchs' argument over the reals (obtained by appealing to Hermite interpolation), and here it follows from Step I by the Chinese remainder theorem.

Step III *There exists a polynomial, nonnegative on I, f in $\boldsymbol{R}[x]$, with zeros on a finite set including S such that $-h/f, (f_1 - h/f) \le (g_i - h)/f$ on all of I, except the zeros of f; additionally, these inequalities can be interpolated by an element of $L[x]$.* This is virtually identical to Fuchs' second step and easily concludes the proof.

We now proceed to the proof of Step I. For each $t \in T$, let p_t denote the unique monic polynomial in $L[x]$ irreducible over L satisfied by t; automatically, it is also the irreducible polynomial of all s in S_t. Now we make an observation about local nonnegativity of polynomials at algebraic numbers.

Suppose α is algebraic over L with monic irreducible polynomial p in $L[x]$, and $\alpha \in I$. Let c in $L[x]$ be nonconstant and satisfy $c(\alpha) = 0$. Then there is a relative neighbourhood, U, of α, in I (that is, we have to permit relatively open sets of the form $[\gamma, \gamma + \epsilon)$ and $(\delta - \epsilon, \delta]$) such that $c|U \ge 0$ if and only if there exists an integer k and an element N of $L[x]$ with $N(\alpha) \neq 0$ such that $c = p^k N$ such that

$$\begin{cases} k \text{ is even and } N(\alpha) > 0 & \text{if } \alpha \in (\gamma, \delta) \\ N(\alpha) > 0 & \text{if } k \text{ is even and } \alpha \in \{\gamma, \delta\} \\ (p'(\alpha))^k N(\alpha) > 0 & \text{if } k \text{ is odd and } \alpha = \gamma \\ (p'(\alpha))^k N(\alpha) < 0 & \text{if } k \text{ is odd and } \alpha = \delta. \end{cases}$$

This follows easily from the fact that $p(\alpha) = 0$ implies

$$D^k(p^k N)(\alpha) = k!(p'(\alpha))^k N(\alpha).$$

In the latter two cases (k is odd, and α is an endpoint), we may obviously replace $(p'(\alpha))^k N(\alpha)$ by $p'(\alpha)N(\alpha)$.

(Technical) Lemma A.1 *For t in T, there exist an integer $m \equiv m(t)$ together with a family of nonempty open intervals in the reals, $\{(a(s), b(s))\}_{s \in S_t}$ and $\{r_j\}_{j=1}^{m-1}$ in the extension field $L[t]$ such that if h is any polynomial with coefficients from L satisfying*

(i) $h(t) = \inf\{g_1(t), g_2(t)\}$,
(ii) $h^{(j)}(t) = r_j$ *for* $1 \leq j \leq m-1$,
(iii) *for all* s *in* S_t, $h^{(m)}(s) \in (a(s), b(s))$,

then there exists a relative neighbourhood $U_t \equiv U(t,h)$ *of* S_t *such that as functions on* U_t, $f_1, 0 \leq h \leq g_1, g_2$.

Proof Case 1. $f_1(t) > 0$. Then $\inf\{g_1(t), g_2(t)\} = f_1(t) > 0$; in particular, $f_1(s) \neq 0$ for any algebraic conjugate s (*i.e.,* for $s \in S_t$). From $g_i(t) = f_1(t)$ (which holds for at least one of the g_1, g_2), we have $f_1(s) = g_i(s) \neq 0$; as $g_i \geq 0$, $g_i(s) > 0$, so that $f_1(s) > 0$, whence $g_{2-i}(s) > 0$. In particular, for all s in S_t, we have $\inf\{g_1(s), g_2(s)\} = f_1(s) > 0$, and obviously $g_i(t) = f_1(t)$ if and only if for all s in S_t, if and only if for some $s \in S_t$, we have $g_i(s) = f_1(s)$. In particular, by (i), $h(s) = f_1(s) > 0$ for all s in S_t, so there exists a neighbourhood of S_t on which h is nonnegative.

Case 1a. *Assume* $t \in (\gamma, \delta)$. Let p be the monic irreducible polynomial with coefficients from L satisfied by t (hence by all other elements of S_t). From $g_i - f_1 \geq 0$, at least one vanishing at t, we have a factorization $g_i - f_1 = p^{2m_i} M_i$ where m_i are nonnegative integers with at least one being nonzero, and M_i are in $L[x]$ with $M_i(t) > 0$. Since $g_i - f_1 \geq 0$ on all of I, this forces $M_i(s) \geq 0$ for all s in S_t; since $M_i(t) \neq 0$, we have $M_i(s) \neq 0$ for every such s, and therefore $M_i(s) > 0$ for all $s \in S_t$.

Set $m = \max\{2m_1, 2m_2\}$ and $r_j = D^j(f_1)(t)$. Without yet specifying the values of $h^{(m)}(s)$ yet, we see that $h - f_1$ has a zero of order (at least) m at t, and thus we can write it as $p^m P$ where P is a polynomial (obviously depending on our choice of h), and $h^{(m)}(s) = m!(p'(s))^m P(s) + f_1^{(m)}(s)$ for all $s \in S_t$) (since all fields here are separable, it follows that $p'(s) \neq 0$); since m is even, we see that the condition that $D^m(h - f_1)(s) > 0$ is sufficient for $h \geq f_1$ on a neighbourhood of s; this translates to $h^{(m)}(s) > f_1^{(m)}(s)$.

Thus we let $a(s) = f_1^{(m)}(s)$. The set of conditions $h^{(m)}(s) > f_1^{(m)}(s)$ (one for each s in S_t) is sufficient to guarantee $0, f_1 \leq h$ on a neighbourhood of each s in S_t. (Notice that there is no problem if one or both of the endpoints are included in S_t.)

Now consider $g_i - h = (g_i - f_1) + (f_1 - h) = p^{2m_i} M_i - p^m P$; if for one of the i, $m_i < m/2$, then dividing by p^{2m_i} we see that $M_i(s) > 0$ is sufficient for $g_i - h \geq 0$ on a neighbourhood of s, and this automatically follows from the first paragraph for every s in S_t. In addition, in this case, we must have $m_{2-i} = m/2$, so that $g_{2-i} - h = p^m(M_i - P)$; thus for $g_{2-i} - h$ to be nonnegative on a neighbourhood of s (again, since m is even) it is sufficient that $M_{2-i}(s) - P(s) > 0$. Since $P(s) = (h^m(s) - f_1^{(m)}(s))/p'(s)^m m!$, we just need to ensure that $h^{(m)}(s) < m!p'(s)^m M_{2-i}(s) + f_1^{(m)}(s)$ (since m is even, the sign of $p'(s)$ is irrelevant). In this case, we set $b(s) = m!p'(s)^m M_{2-i}(s) + f_1^{(m)}(s)$ (where i is defined by $m_i < m/2$). Now we must check that $a(s) < b(s)$, that is, $0 < m!p'(s)^m M_{2-i}(s)$, which is obvious since (again) m is even.

There remains the possibility that $m_1 = m_2 = m/2$, and the same analysis yields a choice for $b(s)$, namely $m!p'(s)^m \min\{M_1(s), M_2(s)\} + f_1^m(s)$.

Case 1b. $t = \gamma$ *or* δ *but* $|S_t| = 1$. The process for $t = \delta$ is obtained from the process for $t = \gamma$ by applying an automorphism of $L[x]$ that reverses the orientation (possibly shifting the interval at the same time, and we find that the definitions of $a(s)$

and $b(s)$ are obtained in reverse to the way they were obtained in the other subcases), so we can just assume that $t = \gamma$ and t has no conjugates in the interval $(\gamma, \delta]$. We write $g_i - f_1 = p^{k_i} M_i$ with $M_i(t) > 0$ if either k_i is even or if both k_i is odd and $p'(t) > 0$, and $M_i(t) < 0$ if both k_i is odd and $p'(t) < 0$. To simplify matters, we replace p by $-p$ if $p'(t) < 0$, that is, we can assume $p'(t) > 0$ if we do not mind losing monicity, and eliminating the last possibility, so that $M_i(t) > 0$ in any case. Set $m = \max\{k_1, k_2\}$, and $r_j = f_1^{(j)}(t)$. Then h satisfying (i) and (ii) entails $h - f_1$ has order at least m at t, and thus factors as $p^m P$. As in Case 1a, we see quickly that $h^{(m)}(t) > f_1^{(m)}(t)$ is sufficient for $h \geq f_1, 0$ on a relative neighbourhood of γ (that is, of the form $[\gamma, \gamma + \epsilon)$).

As in Case 1a, we can write $g_i - h = p^{k_i} M_i - p^m P$; now the fact that $M_i(t) > 0$ and $p'(t) > 0$ allows the same analysis as in Case 1a to show that we may choose $b(s) = m! p'(s)^m M_{2-i}(s) + f_1^{(m)}(s)$ if $k_i < k_{2-i} = m$, and

$$b(s) = m! p'(s)^m \min\{M_1(s), M_2(s)\} + f_1^m(s)$$

if $k_1 = k_2 = m$.

Case 1c. $S_t = \{\gamma, \delta\}$. In this weird case, we have that (labelling $t = \gamma$ and $s = \delta$) $g_i - f_1 \geq 0$ entails $g_i - f_1 = p^{k_i} M_i$ where if k_i is even, then $M_i(s), M_i(t) > 0$, while if k_i is odd, then $M_i(t) p'(t) > 0$ and $M_i(s) p'(s) < 0$. Let $m = \max\{k_1, k_2\}$ and again set $r_j = f_1^{(m)}(t)$, so that $h - f_1 = p^m P$.

Suppose m is even; then for nonnegativity of $h - f_1$ on a neighbourhood of S_t it is sufficient that $P(s), P(t) > 0$. Since $h_1^{(m)}(\alpha) - f_1^{(m)}(\alpha) = m! p'(\alpha))^m P(\alpha)$ for $\alpha \in S_t$, as in all the previous subcases, we can set $a(t) = f_1^{(m)}(t)$ and $a(s) = f_1^{(m)}(s)$. Continuing with even m, the same arguments as in the previous subcases give the same choice for $b(t)$ and $b(s)$.

Finally (for Case 1), suppose m is odd. We want to ensure that $p'(t)P(t) > 0$ and $p'(s)P(s) < 0$. Since S_t consists only of t and s and nothing in between, we see that they are consecutive real roots (each of multiplicity one) of the real polynomial p, hence $p'(t)p'(s) < 0$ (signs of the derivatives are opposite). By replacing p by $-p$ if necessary, we may assume $p'(t) > 0$, and thus $p'(s) < 0$. We want to ensure that $D^m(h - f_1)(t) > 0$ and $D^m(h - f_1)(s) < 0$. Set $a(t) = f_1^{(m)}(t)$ and $b(s) = f_1^{(m)}(s)$; note the appearance of $b(s)$ instead of $a(s)$. Now similar analysis with the g_i as in all the previous subcases realizes the complementary of $b(t)$ and $a(s)$.

There are no other subcases to consider, since we picked the cross-section T, so that if a conjugacy class contains an interior point of the interval, then we chose the corresponding representative in T to be an interior point.

Case 2. $f_1(t) = 0$. There exists i such that $g_i(t) = 0$, hence $g_i(s) = 0$ for all s in S_t.

Case 2a. $t \in (\gamma, \delta)$. If $g_i(t) > 0$ (necessarily $g_{2-i}(t) = 0$), then $g_i(s) \neq 0$ for all s in S_t, so from $g_i \geq 0$, we have $g_i(s) > g_{2-i}(s) = 0$, hence $g_i > g_{2-i}$ on a neighbourhood of S_t. Hence we can disregard g_i; we only have to guarantee that $g_{2-i} \geq h \geq f_1, 0$ on a neighbourhood of S_t. If the order of t as a zero of f_1 is odd, it must be at least as large as the order of t for g_{2-i}, since $g_{2-i} - f_1 \geq 0$. Dividing by a sufficiently high (but nonzero) even power of p, p^{2l} that divides both g_{2-i} and f_1, we reduce to the situation that either $f_1/p^{2l}(s) < 0$ or $g_{2-i}/p^{2l}(s) > 0$. In the former situation, set $m = 2l$ and $r_j = 0$, and verification is trivial (we can take $b(s) = \infty$ for every s for which $f_1/p^{2l}(s) < 0$), and in the latter case, if it occurs for one value of s in S_t, it occurs for all, and then we either have $g_{2-i}/p^{2l}(t) = f_1/p^{2l}(t) > 0$, which is Case 1, or $g_{2-i}/p^{2l}(t) > f_1/p^{2l}(t)$, which is not any case at all. In every single one of these possibilities, the choices for the intervals $(a(s), b(s))$ are straightforward.

If both $g_i(t) = 0$, the order of t as a zero of each g_i is even (as t is an interior point), and of course $g_i - f_1 \geq 0$. If the order of t as a zero of f_1 is odd, its order must be at least as large (and therefore more than the order at g_i). We may thus divide everything by p^2 (not affecting any of the inequalities, since p^2 is nonnegative), and continue this process as far as possible. At that point, either we reduce to Case 1, or to not both $g_i(t)$ being zero, the situation of the previous example, and again the choices for r_j and m are routine.

Case 2b. *$t = \gamma$ or δ but $|S_t| = 1$.* Reduce to $t = \gamma$ as in Case 1b. Replace p by $-p$ if necessary to ensure that $p'(\gamma) > 0$. Then $p^k \geq 0$ on I, since p has no zeros on $(\gamma, \delta]$. Hence we may proceed as in Case 2a, not worrying about the parity of the power of p.

Case 2c. *$S_t = \{\gamma, \delta\}$.* Again replace p by $-p$ if necessary to ensure that $p'(\gamma) > 0$, so that p is strictly positive on the interior of I, and thus $p|I \geq 0$, so we can again proceed as in Case 2a, dividing by a power (not worrying about the parity) to reduce to other cases. ∎

Lemma A.2 *For r a real number that is algebraic over $L \subseteq \mathbf{R}$ of degree n, set $R = L[x]$ and let K be the n-dimensional field extension $L[r]$. Let m be a positive integer. The map $\phi\colon L[x] \to K^{m+1}$, defined by $f \mapsto (f^{(j)}(r))_{j=0}^m$, is onto.*

Proof Let p be a minimal polynomial of r with respect to L, so that $\deg p = n$. Then the kernel of ϕ is exactly the ideal $J = p^{m+1}R$ of R and ϕ is L-linear; it thus induces the one to map L-linear map $\overline{\phi}\colon R/J \to K^{m+1}$. The L-dimension of the left side is $n(m+1)$, so that the left side is equidimensional (as an L-vector space) with the right side, hence $\overline{\phi}$ is an isomorphism. ∎

The following observation, when applied to $L = \mathbf{Q}$, is well known. Let K be a formally real finite dimensional extension field of L, and let $\Sigma := \{\sigma\}$ be a family of homomorphisms $K \to \mathbf{R}$ (at least one exists by formal reality) that act as the identity on L. Then the image of K under the obvious map $K \to \mathbf{R}^{\Sigma}$ is dense. We note that Σ is automatically linearly independent, so both $|\Sigma| < \infty$ and density follow immediately.

Lemma A.3 (Step I) *For each t in T, there exists h_t in $L[x]$ together with a relatively open neighbourhood in I, U_t, of S_t, such that $0, f_1 \leq h_t \leq g_1, g_2$ as functions on I restricted to U_t.*

Proof For each s in S_t, choose field isomorphisms (over L) $\sigma_s\colon L[t] \to L[s] \subseteq \mathbf{R}$ such that $\sigma_s(t) = s$, and let Σ be the set of such σ_s (this will of course vary as t varies). The map $L[t] \to \mathbf{R}^{\Sigma}$ has dense range, as we observed earlier. Hence we may find q in $L[t]$ such that for all s in S_t, we have $\sigma_s q \in (a(s), b(s))$. With $K = L[t]$, apply the

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previous lemma to the sequence $(f_1(t), r_1, \ldots, r_{m-1}, q)$. Ontoness of the map entails that there exists h_t in $L[x]$ satisfying the technical lemma, hence $0, f_1 \leq h_t \leq g_i$ on a relative neighbourhood of S_t in I. ■

Lemma A.4 (Step II) *There exists h in $L[x]$ together with a relatively open neighbourhood in I, U, of S, such that $0, f_1 \leq h \leq g_1, g_2$ as functions on U.*

Proof For each t in T, let p_t be an irreducible polynomial in $L[x]$ for t; the corresponding m_t comes from the technical lemma above. Set $J_t = p_t^{m_t}$, so that the $\{J_t\}_{t \in T}$ are pairwise comaximal. By the Chinese remainder theorem, there exists h in $L[x]$ such that for each t, $h - h_t \in J_t$. But this simply means that $D^k h(t) = D^k h_t(t)$ for $0 \leq k \leq m_t$, and since all the derivatives of h and h_t belong to $L[x]$, we also have $D^{m_t} h(s) = D^{m_t} h_t(s)$ for all s in S_t. Hence h satisfies the conditions of the technical lemma for each equivalence class, and thus $0, f_1 \leq h \leq g_1, g_2$ on a neighbourhood of $S = \cup_{t \in T} S_t$. ■

Lemma A.5 (Step III) *There exists a nonzero polynomial, nonnegative on I, f in $L[x]$, together with a finite subset S_0 of I containing S such that $-h/f, (f_1 - h)/f \leq (g_i - h)/f$ on $I \setminus S_0$ and can be interpolated by an element of $L[x]$.*

Proof Let S_0 be the union of the zero sets of $h, f_1 - h, g_i - h$ intersected with I. Obviously $S \subseteq S_0$, and none of $S_0 \setminus S$ can be conjugate to an element of T. For each conjugacy class in S_0, with representative u, let p_u be a minimal polynomial of u over L; let T_0 be a cross-section of S_0 (enlarging T). Let M be any even number exceeding the orders of all the zeros of the four polynomials, so that with $f = \prod_{T_0} p_u^M$, we have that $f|I \geq 0$, and moreover, the poles of each of $h/f, (f_1 - h)/f, (g_1 - h)/f, (g_2 - h)/f$ in I occur exactly at the points of S_0 and no others.

In a (relative) neighbourhood of $s \in S_0$, we must have $\lim_{x \to s} (g_i - h)/f = +\infty$ (the limit is two-sided if s is not a boundary point, but only one-sided if it is one of γ or δ), since $g_i - h$ is nonnegative on $I \setminus S$. Since $-h$ and $f_1 - h$ are negative on $I \setminus S_0$, it similarly follows that $\lim_{x \to s} -h/f = \lim_{x \to s} (f_1 - h)/f = -\infty$. On the other hand, we note that all of $d_0 = \sup_{x \in I} -(h/f)(x)$, $d_1 = \sup_{x \in I}((f_1 - h)/f)(x)$, $e_i = \inf_{x \in I}((g_i - h)/f)(x)$ are finite. There exists a relative neighbourhood V of S_0 such that on $V \setminus S_0$, all of $(g_i - h)/f > 1 + \max\{d_0, d_1\}$, and $-h/f, (f_1 - h)/f < \inf\{e_i\} - 1$ hold. On the remainder of I, $((g_i - h)/f)(x) > -(h/f)(x), (f_1 - h)/f)(x)$; by compactness of $I \setminus V$, there exists $\eta > 0$ such that all four differences are at least η on $I \setminus V$. Set $G_i = (g_i - h)/f \wedge (1 + \max\{d_0, d_1\})$, $E_1 = -h/f \vee (\min\{e_i\} - 1)$, and $E_2 = (f_1 - h)/f \vee (\min\{e_i\} - 1)$ so that G_i, E_i are all continuous, and $G_i - E_j > \min\{1, \eta\}$ on all of I. Let $\kappa = \min\{1, \eta\}$, and set $G = G_1 \wedge G_2$ and $E = E_1 \vee E_2$. Then $G - E > \kappa$ on I.

By the Weierstrass density theorem, there exists a *real* polynomial w such that with respect to the sup norm $\|w - (G + E)/2\| < \kappa/4$. Since L is dense in $\mathbf{R}$, there exists a polynomial W in $L[x]$ such that $\|W - w\| < \kappa/8$. As $\|G - (G + E)/2\| \geq \kappa/2$, it easily follows that $G \geq W \geq E$ on I, and W is thus the desired interpolant. ■

Theorem A.6 *The ring $R_{L,\gamma,\delta} = L[x]$ equipped with the pointwise ordering from the interval $[\gamma, \delta]$ satisfies the Riesz interpolation property.*

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Proof From Steps I–III, we can find W such that $-h/f, (f_1-h)/f \leq W \leq (g_i-h)/f$ on $I \setminus S$, with f a square. Hence $0, f_1 \leq h + Wf \leq g_i$ on $I \setminus S$, and by continuity on all of I. ∎

The use of the Chinese remainder theorem together with the obvious result on values of derivatives in Step II yields a version of Hermite interpolation over the subfield L. We have to be careful here, because if, for example, ρ is transcendental over L and f is in $L[x]$, then prescribing a value for $f(\rho)$ is either impossible or determines f (and thus all of its values) completely! So we restrict the parameters to numbers algebraic over L (and again noting that if $f^{(j)}(\alpha)$ is given, then $f^{(j)}(\beta)$ is uniquely determined whenever β is an algebraic conjugate of α). The following unexciting result is proved as Step II was.

Proposition A.7 (Hermite interpolation for real subfields) *Let L be a subfield of the reals. Suppose $\{\alpha\}_{\alpha\in Y}$ is a finite set of real numbers that are algebraic over L. Suppose that for each α, there exists $m \equiv m(\alpha)$ together with $\{r_{j,\alpha}\}$, $0 \leq j \leq m(\alpha)$ with the following properties.*

(i) *For each α, all $r_{j,\alpha}$ belong to the extension field $K_\alpha := L[\alpha]$.*
(ii) *If α is conjugate to β, and $\sigma\colon K_\alpha \to K_\beta$ is the field isomorphism fixing L induced by $\alpha \mapsto \beta$, then $m(\beta) = m(\alpha)$ and $r_{j,\beta} = \sigma(r_{j,\alpha})$ for all $0 \leq j \leq m(\alpha)$.*

Then there exists a polynomial h in $L[x]$ such that for all α in Y, for all $0 \leq j \leq m(\alpha)$, $h^{(j)}(\alpha) = r_{j,\alpha}$.

Isomorphisms among the $R_{L,\gamma,\delta}$ are easily determined. For real c, d with $c \neq 0$, let $\psi_{c,d}$ be the ring automorphism determined by $x \mapsto cx + d$. The ring automorphisms of $L[x]$ are exactly those of the form $\psi_{c,d}$ (restricted to $L[x]$), where c and d belong to L. These induce ordered ring isomorphisms $R_{L,\gamma,\delta} \to R_{L,\gamma',\delta'}$, where $\psi_{c,d}$ sends the two-element set $\{\gamma, \delta\}$ to $\{\gamma', \delta'\}$. Since the pure traces are determined, even topologically, by the ordering, any ordered ring isomorphism will have to have this property, so that the order-isomorphism classes (for L fixed) are precisely the orbits of $\{\gamma, \delta\}$ under $\psi_{c,d}$ where $c, d \in L$ (and $c \neq 0$).

It is also true that any order-isomorphism that sends 1 to 1 among these objects is a ring isomorphism, since the intersections of the kernels of the pure traces is zero.

More interesting is what happens when we change the base field. Let $L \subset K \subseteq \mathbf{R}$ be a proper field extension of L, inside K. We form the tensor product, $R_{L,\gamma,\delta} \otimes_L K$ as L-vector spaces; of course, as an L-algebra, this is just $K[x]$. We can impose the tensor product ordering, by considering the cone generated by the pure tensors of the form $f \otimes k$ where $f \in R_{L,\gamma,\delta}$ and $k \in K^+$ (ordering inherited from $\mathbf{R}$) (we would have to check that this is a proper cone, as in [GH], but since $\mathbf{R}$ is an injective K-module—as K is a field—the same argument works). Although $R_{L,\gamma,\delta}$ is by definition archimedean (for ordered abelian groups with order unit, this is equivalent to $\tau(g) \geq 0$ for all pure traces τ implies $g \geq 0$), the tensor product is not.

We simply note that the zero sets of positive elements of the tensor product are finite subsets of $[\gamma, \delta]$ that consist of L-algebraic numbers and which are relatively closed under conjugacy (that is, if α and β are algebraic conjugates and both are in the interval, then α is in a zero set of a positive element if and only if β is in the same one). In particular, if a singleton $\{\alpha\}$ arises as a zero set of a positive element, then $\alpha \in L$. So select η in $K \setminus L$; since $L\eta \setminus \{0\}$ is dense in the reals, we can find nonzero a in L such that $\alpha = a\eta$ at least one other algebraic conjugate of α belong to $[\gamma, \delta]$. The element $(x - \alpha)^2$ (strictly speaking, $x^2 \otimes 1 - 2ax \otimes \eta + a^2 \otimes \eta^2$) is in the tensor product. It is nonnegative as a function on the interval, hence on the pure traces, but it cannot be in the positive cone of the tensor product, as its zero set is the singleton $\{\alpha\}$, which is not closed under conjugation with respect to L.

What can be shown is that if $L_i \subset K_i \subseteq \mathbf{R}$ are two proper field extensions, then the ordered tensor products $R_{L_i,\gamma_i,\delta_i} \otimes_{L_i} K_i$ are isomorphic (as ordered rings) if and only if there is a field isomorphism $\sigma\colon K_1 \to K_2$ whose restriction to L_1 yields an isomorphism to L_2, for which there is a $\phi_{c,d}$ with $c, d \in L_2$ compatible with the maps, mapping the endpoints to endpoints.

Similar interpolation results apply to the trigonometric polynomial ring (another of Fuchs' examples of ordered rings satisfying interpolation). The criterion in the technical lemma applies even more generally to real analytic functions. On the other hand, when the ring has enough invertibles (for example, the ring of all functions which are real analytic on a neighbourhood of the interval), it is straightforward to show that interpolation holds, owing to a trick due to Riesz.

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Mathematics Department, University of Ottawa, Ottawa ON K1N 6N5
e-mail: dehsg@uottawa.ca