Macroeconomic Dynamics, **22**, 2018, 779–804. Printed in the United States of America.
doi:10.1017/S1365100516000407

A GENERALIZED STEADY-STATE GROWTH THEOREM

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Is there an economic justification for why technical change is by assumption labor-augmenting in dynamic macroeconomics? The literature on the endogenous choice of capital- and labor-augmenting technical change finds that technical change is purely labor-augmenting in steady state. The present paper shows that this finding is mainly an artifact of the underlying mathematical models. To make this point, Uzawa's steady-state growth theorem is generalized to a neoclassical economy that, besides consumption and capital accumulation, uses current output to create technical progress or to manufacture intermediates. The generalized steady-state growth theorem is shown to encompass four models of endogenous capital- and labor-augmenting technical change and the typical model of the induced innovations literature of the 1960s.

Keywords: Steady-State Growth, Capital Accumulation, Uzawa's Theorem, Endogenous Direction of Technical Change

1. INTRODUCTION

Technical change is by assumption almost always labor-augmenting in dynamic macroeconomics. This observation holds true irrespective of whether technical change is treated as an exogenous or as an endogenous variable. However, is there an economic justification for this assumption?

The so-called induced innovations literature of the 1960s constitutes the first systematic attempt to address this question. The answer is given in the framework of the neoclassical growth model of Solow (1956) and Swan (1956), which is extended to allow for the endogenous choice of capital- and labor-augmenting technical change.[1] Recently, Acemoglu (2003) and Irmen and Tabaković (2015) revisit this territory.[2] These authors study the choice of capital- and labor-augmenting technical change in models with an elaborate micro-foundation reminiscent of

This paper is a revised and extended version of my CESifo Working Paper Irmen (2013). Financial support from the University of Luxembourg under the program "Agecon C - Population Aging: An Exploration of its Effect on Economic Performance and Culture" is gratefully acknowledged. I would like to thank two anonymous referees, Anastasia Litina, Amer Tabaković, Gautam Tripathi, Bertrand Wigniolle, and Benteng Zou for helpful comments.
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 1365-1005/16

the modern theory of endogenous technical change initiated by Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992).

The main finding of all these contributions is that economies may converge to a steady-state path with only labor-augmenting technical change even though capital-augmenting technical change is feasible.[3] While the literature of the 1960s and Acemoglu (2003) find this convergence to depend on an elasticity of substitution between capital and labor strictly smaller than unity, it obtains for any positive value of the elasticity of substitution in the model studied in Irmen and Taboković (2015).[4] These results may be traced back to the differential incentives that profit-maximizing firms face in these models. While the direction of technical change hinges on the relative share of capital in models of the 1960s and in Acemoglu (2003) it is linked to relative factor prices in the model of Irmen and Taboković (2015). Yet, do these findings provide a satisfactory answer to the question of why technical change should be expected to be labor-augmenting, at least in the long run?

This paper argues that the main finding of the existing literature on the endogenous choice of capital- and labor-augmenting technical change is mainly an artifact of its underlying analytical structure. To make this point, I devise a generalized steady-state growth theorem. This theorem is shown to encompass all contributions mentioned in the preceding paragraph. More precisely, the reduced form of all these models fits the generalized steady-state growth theorem. Roughly speaking, this leads to the conclusion that—by design—none of them can find something different from a steady-state path with only labor-augmenting technical change.

The generalized steady-state growth theorem of this paper complements and extends Uzawa's steady-state growth theorem [Uzawa (1961)].[5] Uzawa derived his insight with a view to neoclassical growth models that depict the process of capital accumulation in a setting void of externalities where agents interact in a system of complete, competitive markets, and technical change is exogenous. The modern theory of endogenous technical change has called for a substantial extension of this framework to capture the notion of technological knowledge and the economics of its creation. This led to multi-sector models with incomplete and (im)perfectly competitive markets that may feature intra- and inter-temporal externalities. In spite of these complications, the present paper shows that Uzawa's main insight may also apply to such settings as exemplified by the contributions of Acemoglu (2003) and Irmen and Taboković (2015).

The generalized steady-state growth theorem applies to an economy where, in addition to consumption and capital accumulation, current output is also used to generate technical progress or to manufacture intermediates. I refer to the latter resources as *aggregate intermediate expenses*. Typically, this additional element matters in an economy where technical progress is endogenous and costly. Moreover, it gives rise to the notion of *net output* defined as the difference between aggregate final-good production and aggregate intermediate expenses.

Hence, the economy under scrutiny here comprises an aggregate production function, an aggregate intermediate expenses function, a resource constraint, and an equation of motion describing the accumulation of capital. For such an economy, the generalized steady-state growth theorem characterizes steady-state paths starting in finite time. The first part of the theorem establishes that net output, aggregate output, aggregate intermediate expenses, capital, and aggregate consumption grow at the same rate. Its second part shows that technical change is purely labor-augmenting in the net output function. Moreover, the growth rate of labor-augmenting technical change is shown to coincide with the growth rate of all per-worker variables. While the proof of the first part follows directly from the analytical structure of the model, its second part is shown to rely on the assumption that both the aggregate production function and the aggregate intermediate expenses function exhibit constant returns to scale in capital and labor. This proof strategy builds on and extends Schlicht (2006)'s elegant and intuitive proof of Uzawa's original theorem.

From the perspective of the generalized steady-state growth theorem, I take a new look at the question about why steady-state technical change is labor-augmenting in the literature on the endogenous choice of capital- and labor-augmenting technical change. I argue that in steady state the reduced form of these models either involves a net output function that has constant returns in capital and labor as required by the generalized steady-state growth theorem or is consistent with Uzawa's original formulation. Therefore, in steady-state capital-augmenting technical change vanishes and labor-augmenting technical change determines the growth rate of the economy.[6]

This point is made for the one-sector model of Irmen and Taboković (2015), for the multi-sector model of Acemoglu (2003) and its extension [Acemoglu (2009, Chapter 15)] and for the typical model of the induced innovations literature of the 1960s.[7]

For the question at hand, it matters that these models differ in the way technical change is generated. In Irmen and Tabaković (2015), this requires the input of current final-good production. Therefore, the generalized steady-state growth theorem can be applied. In Acemoglu's two variants, technical change is the result of research conducted by labor. For Acemoglu (2003), this is shown to lead to an application of Uzawa's original theorem. As current final output is used up as an input in the production of intermediates in Acemoglu (2009, Chapter 15), aggregate intermediate expenses are strictly positive. Therefore, the generalized steady-state growth theorem is shown to apply. Finally, I argue that the models of the induced innovations literature lend themselves to a direct application of Uzawa's theorem.

The remainder of this paper is organized as follows. Section 2 has the statement and the proof of the generalized steady-state growth theorem. Section 2.1 gives the precise setup of the neoclassical economy under scrutiny. The generalized steady-state growth theorem appears as Theorem 1 in Section 2.2. Section 2.3 discusses important assumptions and features of it. They include the role of differing technologies affecting aggregate production and aggregate investment, the link to Uzawa's original result, the importance of capital accumulation and of constant returns. Finally, I turn to the special, yet important case of factor-augmenting technologies. Section 3 establishes the link between the generalized steady-state growth theorem and the steady-state properties of the above-mentioned models of endogenous technical change. Section 4 concludes this paper. If not indicated otherwise proofs are relegated to the appendix.

2. STATEMENT AND PROOF OF THE THEOREM

2.1. The Model

Consider a closed economy, and, without loss of generality, let time be continuous, i. e., $t \in (-\infty, +\infty)$. The production sector consists of two elements. First, there is an *aggregate production function* of the final good

$$Y(t) = \tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right], \tag{1}$$

where $\tilde{F} : \mathbf{R}^2_+ \times \mathfrak{A}_F \rightarrow \mathbf{R}_+$, $Y(t)$ is aggregate output of the final good, $K(t) > 0$ is the capital stock, $L(t) > 0$ is the labor endowment, and $\mathbf{A}_F(t) \in \mathfrak{A}_F$ represents the components of technological knowledge available at t that affect the production of the final good. Here, $\mathfrak{A}_F$ is an arbitrary set.[8]

Second, there is a function that specifies aggregate intermediate expenses. It states the amount of period-t final-good output that is used up in the same period as an input somewhere in the economy. For instance, the economy may invest contemporaneous final output to generate technical progress in its research sector or, alternatively, use it as an input in an intermediate-good industry of the production sector. In any case, the defining property of these resources is that they are neither available for consumption nor for the accumulation of capital. I refer to them as *aggregate intermediate expenses* denoted by $I(t)$. Let

$$I(t) = \tilde{I}\left[K(t), L(t), \mathbf{A}_I(t)\right], \tag{2}$$

where $\tilde{I} : \mathbf{R}^2_+ \times \mathfrak{A}_I \rightarrow \mathbf{R}_+$ is the *aggregate intermediate expenses function* and $\mathbf{A}_I(t) \in \mathfrak{A}_I$ represents components of technological knowledge available at t that affect the amount of expended final output given capital and labor. Again, $\mathfrak{A}_I$ is an arbitrary set.

The inclusion of aggregate intermediate expenses generalizes Uzawa's original setting. In most applications, the functions $\tilde{F}$ and $\tilde{I}$ will correspond to reduced-form production and investment functions of the economy under scrutiny. As such, they will reflect the technological environment and the market-clearing conditions. This justifies the assumption that these functions depend both on the capital and the labor endowment of the economy.[9] However, there is little reason why the technology applied in the production of the final good should coincide with the technology used in the economy's research or intermediate-good sector. This is why $\mathbf{A}_F(t)$ is allowed to differ from $\mathbf{A}_I(t)$.

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Suppose that both $\tilde{F}$ and $\tilde{I}$ are increasing in $K(t)$ and $L(t)$ and exhibit constant returns to scale in these arguments. Then, $V(t) = Y(t) - I(t)$ is net output, i. e.,

$$V(t) = \tilde{F}[K(t), L(t), \mathbf{A}_F(t)] - \tilde{I}[K(t), L(t), \mathbf{A}_I(t)],$$
$$\equiv \tilde{V}[K(t), L(t), \mathbf{A}_F(t), \mathbf{A}_I(t)], \tag{3}$$

where $\tilde{V} : \mathbf{R}_+^2 \times \mathfrak{A}_F \times \mathfrak{A}_I \to \mathbf{R}_+$ exhibits constant returns to scale in $K(t)$ and $L(t)$, too. Hence, net output is defined as the amount of the final good that is available for consumption and capital accumulation. Henceforth, I refer to $\tilde{V}$ as the *net output function*. Capital and aggregate consumption, $C(t)$, are measured in units of the final good. Then, at all t capital accumulates according to

$$\dot{K}(t) = V(t) - C(t) - \delta_K K(t), \quad \delta_K \in \mathbf{R}_+, \tag{4}$$

where δ_K is the instantaneous depreciation rate of capital. Finally, the evolution of the labor endowment is given by

$$L(t) = L(0)e^{g_L t}, \quad L(0) > 0, \quad g_L \in \mathbf{R}, \tag{5}$$

i. e., the instantaneous growth rate of the labor force is time-invariant and may be positive, zero, or negative.

In what follows, I denote by $g_x(t) \in \mathbf{R}$ the instantaneous growth rate of a variable $x(t)$ at t. By definition, a steady state has $g_x(t) = g_x$ for all variables featured in the model.

2.2. The Generalized Steady-State Growth Theorem

THEOREM 1. *Consider an economy described by equations (3), (4), and (5). Suppose there exists a steady-state path starting at some date $\tau < \infty$ such that $Y(t) > V(t) > C(t) > 0$ for all $t \geq \tau$. Then, the following holds:*

I. $g_V = g_Y = g_I = g_K = g_C$.

II. For any $t \geq \tau$, net output has a representation as

$$V(t) = V[K(t), A(t)L(t)],$$

where $A(t) = e^{(g_V - g_L)(t-\tau)} \in \mathbf{R}_{++}$, and

$$g = g_V - g_L,$$

is the growth rate of per-worker variables.

The main message of the generalized steady-state growth theorem is that steady-state technical change is labor-augmenting in the net output function. Moreover, the growth rate at which the technology evolves determines the growth rate of all per-worker variables. This insight comes in two steps.

Part I shows that the steady-state growth rates of net output, aggregate final-good output, aggregate intermediate expenses, capital, and aggregate consumption are the same. This follows since in steady state a strictly positive difference between two strictly positive variables satisfies that the growth rates of the minuend and the subtrahend coincide. Both, the definition of net output and of steady-state capital accumulation give rise to such differences [see equations (A.1) and (A.2) in the Proof of Theorem 1]. In the present context, this property has two implications. First, net output requires $g_V = g_Y = g_I$ since $Y(t) > I(t) > 0$. Second, steady-state capital accumulation requires $g_V = g_K = g_C$ since $V(t) > C(t) > 0$.

Part II exploits constant returns to capital and labor in the net output function in conjunction with $g_V = g_K$, the requirement of steady-state capital accumulation. Together, these properties imply steady-state labor-augmenting technical change with a growth rate equal to $g_V - g_L$. Labor-augmenting technical change at this rate assures that the first two arguments in $\tilde{V}$ of equation (3)—with respect to which $\tilde{V}$ has constant returns to scale—grow at the same rate. Accordingly, the steady state has $g_V = g_K = g + g_L$ and, in light of Part I, g is the growth rate of all per-worker variables.

Whether per-worker variables grow or shrink hinges on how $g_V = g_K$ relates to the exogenous growth rate of the labor force, g_L. If $g_V = g_K = g_L$ then $g = 0$, i. e., there is no technical change and per-worker variables remain constant over time. If $g_V = g_K > g_L$ then capital grows faster than labor and strictly positive labor-augmenting technical change makes up for the difference. Moreover, per-worker variables grow at the rate of technical change. Finally, if $g_V = g_K < g_L$ then capital grows slower than labor and labor-augmenting technical change is negative to close the gap. In this case, per-worker variables shrink at the rate of technical decline.

2.3. Discussion

Technical Change in $Y(t)$ and $I(t)$. In the economy under scrutiny here the technology may affect aggregate production and aggregate intermediate expenses in different ways, i. e., $\mathbf{A}_F(t) \neq \mathbf{A}_I(t)$. However, in light of equation (3) the generalized steady-state growth theorem implies the existence of two linear homogeneous functions, $F : \mathbf{R}^2_+ \to \mathbf{R}_+$ and $I : \mathbf{R}^2_+ \to \mathbf{R}_+$, such that

$$Y(t) = F\left[K(t), A(t)L(t)\right] \quad \text{and} \quad I(t) = I\left[K(t), A(t)L(t)\right],$$

for all $t \geq \tau$ where $A(t) = e^{(g_V - g_L)t} \in \mathbf{R}_{++}$. Hence, even though $\mathbf{A}_F(\tau) \neq \mathbf{A}_I(\tau)$ may hold, steady-state technical change must be labor-augmenting and evolve at the same pace in both the aggregate production and the aggregate intermediate expenses function.

What if $I(t) = 0$? The generalized steady-state growth theorem postulates an aggregate intermediate expenses function that takes on strictly positive values

https://doi.org/10.1017/S1365100516000407 Published online by Cambridge University Press

for all $t \geq \tau$. As shown in Section 3 below, this extension is important to understand key structural properties of some models with endogenous capital- and labor-augmenting technical change. Absent an aggregate intermediate expenses function, i. e., if $I(t) = 0$ for all $t \geq \tau$, the distinction between gross and net output vanishes. In this case, Theorem 1 and Uzawa's original theorem coincide.

What if $V(t) = C(t) > 0$? Capital accumulation is a central ingredient to the generalized steady-state growth theorem. Indeed, the assumption that $V(t) > C(t) > 0$ for all $t \geq \tau$ assures that some final output is always used to accumulate capital. If instead $V(t) = C(t) > 0$ holds, then no final output is allocated to the accumulation of capital. The following corollary highlights the necessary changes to Theorem 1.

COROLLARY 1. *Reconsider the economy described by equations (3), (4), and (5). Suppose there exists a steady-state path starting at some date $\tau < \infty$ such that $Y(t) > V(t) = C(t) > 0$ for all $t \geq \tau$. Then, the following holds:*

I. $g_V = g_Y = g_I = g_C$ *and* $g_K = -\delta_K$.

II. For any $t \geq \tau$, net output has a representation as

$$V(t) = V\left[B(t)K(t), A(t)L(t)\right],$$

where $B(t) = e^{(g_V+\delta_K)(t-\tau)} \in \mathbf{R}_{++}$ and $A(t) = e^{(g_V-g_L)(t-\tau)} \in \mathbf{R}_{++}$.
Capital per worker grows at rate $-(\delta_K + g_L)$. All remaining per-worker variables grow at rate $g = g_V - g_L$.

Corollary 1 states that steady-state capital-augmenting technical change does not necessarily disappear in a world without capital accumulation. The intuition for this comes in two steps. As to Part I, the new feature is that equation (4) and $V(t) = C(t)$ now imply $g_K = -\delta_K$ and $g_V = g_C$. Hence, the evolution of net output is decoupled from the evolution of capital whereas steady-state growth of net output still requires $g_V = g_Y = g_I$ since $Y(t) > I(t) > 0$.

Part II shows that this decoupling requires capital-augmenting technical change at rate $g_V + \delta_K$ to have "efficient capital" and "efficient labor" grow at the same rate in the net output function $\tilde{V}$ of equation (3). Capital-augmenting technical change disappears only if $g_V = -\delta_K$, the case in which net output and capital grow at the same rate.

Clearly, the growth rate of capital per worker is $-(\delta_K + g_L)$. As $g_V = g + g_L$, the growth rate of all other per-worker variables is given by g, the growth rate of labor-augmenting technical change.

Constant Returns to Capital and Labor in $\tilde{F}$ and $\tilde{I}$. Constant returns to capital and labor in the net output function (3) is key to the generalized steady-state growth theorem. However, for this to hold it is not necessary that both $\tilde{F}$ and $\tilde{I}$ share this property. In fact, Theorem 1 does not change if we allow for the aggregate production and/or the aggregate intermediate expenses function to be linear in either capital or labor. Doing so gives rise to the following six variants:

$$V(t) = \begin{cases} L(t)\tilde{\tilde{F}}(\mathbf{A}_F(t)) - \tilde{I}[K(t), L(t), \mathbf{A}_I(t)], & \text{or} \\ \tilde{F}[K(t), L(t), \mathbf{A}_F(t)] - L(t)\tilde{\tilde{I}}(\mathbf{A}_I(t)), & \text{or} \\ K(t)\tilde{\tilde{F}}(\mathbf{A}_F(t)) - \tilde{I}[K(t), L(t), \mathbf{A}_I(t)], & \text{or} \\ \tilde{F}[K(t), L(t), \mathbf{A}_F(t)] - K(t)\tilde{\tilde{I}}(\mathbf{A}_I(t)), & \text{or} \\ L(t)\tilde{\tilde{F}}(\mathbf{A}_F(t)) - K(t)\tilde{\tilde{I}}(\mathbf{A}_I(t)), & \text{or} \\ K(t)\tilde{\tilde{F}}(\mathbf{A}_F(t)) - L(t)\tilde{\tilde{I}}(\mathbf{A}_I(t)), & \end{cases} \tag{6}$$

where $\tilde{\tilde{F}} : \mathfrak{A}_F \to \mathbf{R}_{++}$ and $\tilde{\tilde{I}} : \mathfrak{A}_I \to \mathbf{R}_{++}$. Intuitively, for all these specifications Part I of Theorem 1 goes through since its proof relies only on the analytical structure of the underlying model and not on functional forms. Moreover, Part II of Theorem 1 remains valid since it relies on constant returns of the net output function, a property that all specifications of equation (6) preserve.[10]

Factor-Augmenting Technical Change. Technical change is factor-augmenting in the aggregate production function and the aggregate intermediate expenses function if and only if these aggregates can be put into the form

$$Y(t) = F[B_F(t)K(t), A_F(t)L(t)] \quad \text{and} \quad I(t) = I[B_I(t)K(t), A_I(t)L(t)],$$

where $B_j(t) \in \mathbf{R}_{++}$ and $A_j(t) \in \mathbf{R}_{++}$, $j = F, I$, represent the capital- and the labor-augmenting technology in the respective aggregate. Compared to the general specification of technical change that appears in $\tilde{F}$ and $\tilde{I}$ of equations (1) and (2), the stipulation that technical change has to be factor-augmenting is restrictive. In a sense, it assumes the "form of technical change" that results as an implication in Theorem 1. However, imposing factor-augmenting technical change also leads to additional insights as it allows for the identification of circumstances where technical change involving growth rates $g_{B_F} \neq 0$ and $g_{B_I} \neq 0$ is consistent with an overall representation of technical change as labor-augmenting. The following corollary sharpens this statement further:

COROLLARY 2. *Consider an economy comprising net output*

$$V(t) = F[B_F(t)K(t), A_F(t)L(t)] - I[B_I(t)K(t), A_I(t)L(t)], \tag{7}$$

and equations (4) and (5). Suppose there exists a steady-state path starting at some date $\tau < \infty$ such that $Y(t) > V(t) > C(t) > 0$ for all $t \geq \tau$. Then, the following holds:

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1. *If* $g_{B_F} = g_{B_I} = 0$ *then* $g = g_{A_F} = g_{A_I}$.
2. *If* $g_{B_F} = 0$ *and* $g_{B_I} \neq 0$, *then net output has the form*

$$V(t) = F\left[B_F(\tau)K(t), A_F(t)L(t)\right] - \beta_I K(t)^{\alpha_I}\left(e^{g(t-\tau)}L(t)\right)^{1-\alpha_I},$$

where $0 < \alpha_I < 1$, $\beta_I = c_I B_I(\tau)^{\alpha_I} A_I(\tau)^{1-\alpha_I} > 0$, *and*

$$g = \frac{\alpha_I g_{B_I}}{1-\alpha_I} + g_{A_I} = g_{A_F}. \tag{8}$$

3. *If* $g_{B_F} \neq 0$ *and* $g_{B_I} = 0$, *then net output has the form*

$$V(t) = \beta_F K(t)^{\alpha_F}\left(e^{g(t-\tau)}L(t)\right)^{1-\alpha_F} - I\left[B_I(\tau)K(t), A_I(t)L(t)\right],$$

where $0 < \alpha_F < 1$, $\beta_F = c_F B_F(\tau)^{\alpha_F} A_F(\tau)^{1-\alpha_F} > 0$, *and*

$$g = \frac{\alpha_F g_{B_F}}{1-\alpha_F} + g_{A_F} = g_{A_I}. \tag{9}$$

4. *If* $g_{B_F} \neq 0$ *and* $g_{B_I} \neq 0$ *then net output has the form*

$$V(t) = \beta_F K(t)^{\alpha_F}\left(e^{g(t-\tau)}L(t)\right)^{1-\alpha_F} - \beta_I K(t)^{\alpha_I}\left(e^{g(t-\tau)}L(t)\right)^{1-\alpha_I},$$

and

$$g = \frac{\alpha_I g_{B_I}}{1-\alpha_I} + g_{A_I} = \frac{\alpha_F g_{B_F}}{1-\alpha_F} + g_{A_F}. \tag{10}$$

The upshot of Corollary 2 is that steady-state technical change may involve $g_{B_F} \neq 0$ and/or $g_{B_I} \neq 0$. However, this is only permissible if the respective aggregate is Cobb–Douglas and the growth rates g_{B_F}, g_{A_F}, g_{B_I}, and g_{A_I} are aligned such that aggregate production and aggregate intermediate expenses grow at the same rate.

Claim 1 is a benchmark and immediate from Theorem 1. If $g_{B_F} = g_{B_I} = 0$, then technical change is labor-augmenting in F and I and evolves at the same rate, g, which is also equal to the growth rate of per-worker variables.

Claims 2 and 3 deal with the related cases where either $g_{B_F} = 0$ and $g_{B_I} \neq 0$ or $g_{B_F} \neq 0$ and $g_{B_I} = 0$. They show that whenever $g_{B_j} \neq 0$, $j = F, I$, the respective aggregate must be Cobb–Douglas. Under this functional form, technical change can be expressed as purely labor-augmenting at rate g.[11] It is in this sense that the distinction between capital- and labor-augmenting technical change is blurred under a Cobb–Douglas.

In steady state the growth rates of $Y(t)$ and $I(t)$ must coincide. This requires an alignment in accordance with conditions (8) and (9), respectively. Intuitively, the growth rate of "labor-augmenting technical change" in the Cobb–Douglas aggregate must coincide with the growth rate of labor-augmenting technical change in the other aggregate. A remarkable feature of both conditions is then that $g_{B_j} \neq 0$ requires $g_{A_F} \neq g_{A_I}$. Hence, one may think of capital-augmenting technical change in the Cobb–Douglas aggregate as a necessary means to fill the gap between g_{A_F} and g_{A_I}. Void of such a gap, there is no room for $B_j(t)$ to grow at a rate different from zero.

Claim 4 allows for $g_{B_I} \neq 0$ and $g_{B_F} \neq 0$. Accordingly, both aggregates must be Cobb–Douglas. Again, g, the growth rate of "labor-augmenting technical change" must be the same in both aggregates. Condition (10) states the required alignment. Unlike Claims 2 and 3, a constellation involving $g_{A_F} = g_{A_I}$ is now consistent with a steady state if g_{B_F} and g_{B_I} adjust accordingly. Moreover, observe that $g_{A_F} = g_{A_I}$ and $g_{B_I} = g_{B_F}$ imply $\alpha_F = \alpha_I$.

Remark 1. Finally, it is worth mentioning that Corollary 2 also has some bearing on the cases where either $I(t) = 0$, or $I(t) > 0$, $B_F(t) = B_I(t)$, $A_F(t) = A_I(t)$, and $I(t) = \beta F[B_F(t)K(t), A_F(t)L(t)]$, with $0 < \beta < 1$. The first of these cases concerns the scenario to which Uzawa's theorem directly applies.[12] In the second case, the aggregate intermediate expenses function has a form that coincides with the one of the aggregate production function up to a multiplicative constant. In both cases, net output will look like

$$V(t) = c_V F[B_F(t)K(t), A_F(t)L(t)], \quad c_V > 0.$$

Hence, if $g_{B_F} = 0$ then $g = g_{A_F}$. Moreover, if $g_{B_F} \neq 0$ then

$$V(t) = \beta_F K(t)^{\alpha_F} \left(e^{g(t-\tau)} L(t)\right)^{1-\alpha_F},$$

and $g = \alpha_F g_{B_F} / (1 - \alpha_F) + g_{A_F}$.

3. ENDOGENOUS CAPITAL- AND LABOR-AUGMENTING TECHNICAL CHANGE: FOUR EXAMPLES

Why is steady-state technical change purely labor-augmenting even in environments where capital-augmenting technical change is feasible? This section revisits four growth models with endogenous capital- and labor-augmenting technical change to shed light on this question. I show that the reduced form of all these models satisfies the assumptions of Theorem 1. As a consequence, steady-state technical change must have a representation as labor-augmenting. Moreover, unless Cobb–Douglas functions are involved capital-augmenting technical change vanishes in the steady state, and the economy's growth rate will be determined by labor-augmenting technical change alone.

Section 3.1 takes a new look at the competitive one-sector growth model developed in Irmen and Tabaković (2015). Here, $I(t)$, reflects aggregate productivity enhancing innovation investments and constitutes foregone output of the final good. This rightly suggests an application of Theorem 1. Section 3.2 revisits the research and development-based variety expansion model of Acemoglu (2003). Here, scientists invent new varieties of differentiated intermediate goods. Each of them is manufactured and marketed by a single intermediate-good firm. I show that this model does not feature intermediate expenses, i. e., $I(t) = 0$. This leads to the conclusion that Uzawa's original theorem characterizes the steady state. Section 3.3 studies an extension of Acemoglu (2003) that allows for a market

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size effect to play a role in the determination of the direction of technical change [Acemoglu (2009, Chapter 15.6)].[13] In this model, some current final-good output is used to produce contemporaneous intermediate goods. These resources qualify as intermediate expenses. Hence, $I(t) > 0$ and Theorem 1 is shown to inform us about how technical change looks like in the steady state. Finally, Section 3.4 deals with a typical model of the induced innovations literature of the 1960s that involves capital accumulation. I show that here the generation of technical change has no costs in terms of real resources. Hence, $I(t) = 0$ and, as for the model of Acemoglu (2003), it is Uzawa's original theorem that prescribes the mode of steady-state technical change.[14]

3.1. Example 1: The One-Sector Model of Irmen and Tabaković (2015)

The economy studied in Irmen and Tabaković (2015) has a single sector on the production side. It manufactures the final good and spends some current output to increase the productivity of capital and labor in aggregate production. Productivity enhancing outlays give rise to aggregate intermediate expenses.

Aggregate Production and Aggregate Intermediate Expenses. The aggregate production function of the final good is $Y(t) = F[M(t), N(t)]$, where $Y(t)$ is output, $M(t) > 0$ and $N(t) > 0$ denote the total amount of tasks performed by either capital or labor. The function $F : \mathbf{R}_+^2 \to \mathbf{R}_+$ has constant returns to scale and is increasing in both arguments.

Let $k(t) = 1/B(t)$ denote the amount of capital required to perform each of the $M(t)$ tasks. Similarly, let $l(t) = 1/A(t)$ denote the amount of labor necessary to perform each of the $N(t)$ tasks. Accordingly, $B(t) > 0$ and $A(t) > 0$ indicate the productivity of capital and labor in the performance of their respective tasks. As before, $K(t) > 0$ and $L(t) > 0$ denote the capital and labor endowments. Then, full employment of capital and labor implies

$$\begin{aligned} M(t)k(t) = K(t) \quad &\Rightarrow \quad M(t) = B(t)K(t), \\ N(t)l(t) = L(t) \quad &\Rightarrow \quad N(t) = A(t)L(t). \end{aligned} \tag{11}$$

Accordingly, aggregate production of the final good is equal to

$$Y(t) = F[B(t)K(t), A(t)L(t)], \tag{12}$$

and technical change represented by the evolution of $B(t)$ and $A(t)$ is capital- and labor-augmenting, respectively.

The economy may expend $M(t)i(q_B(t))$ and $N(t)i(q_A(t))$ units of contemporaneous output to increase $B(t)$ and $A(t)$ according to

$$\dot{B}(t) = B(t)(q_B(t) - \delta_B) \quad \text{and} \quad \dot{A}(t) = A(t)(q_A(t) - \delta_A). \tag{13}$$

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Here, $q_B(t) > 0$ and $q_A(t) > 0$ denote the growth rates of the respective productivity indicator gross of depreciation at rate $\delta_B > 0$ and $\delta_A > 0$, respectively. Moreover, $i : \mathbf{R}_+ \to \mathbf{R}_+$ specifies strictly positive intermediate expenditures per task.

Using equation (11), the expenses necessary to achieve strictly positive growth rates $q_A(t)$ and $q_B(t)$ amount to $A(t)L(t)i(q_A(t))$ and $B(t)K(t)i(q_B(t))$, respectively. The sum of these two outlays corresponds to what I refer to as aggregate intermediate expenses, i. e.,

$$I(t) = A(t)L(t)i(q_A(t)) + B(t)K(t)i(q_B(t)). \tag{14}$$

Net Output and the Steady State. The difference between $Y(t)$ of equation (12) and $I(t)$ of equation (14) delivers net output as

$$V(t) = F[B(t)K(t), A(t)L(t)] - A(t)L(t)i(q_A(t)) - B(t)K(t)i(q_B(t)). \tag{15}$$

PROPOSITION 1. *Consider the economy described by equations (15), (4), and (5). Suppose there exists a steady-state path starting at date $\tau < \infty$ such that $Y(t) > V(t) > C(t) > 0$ for all $t \geq \tau$. Then,*

$$V(t) = F[B(\tau)K(t), A(t)L(t)] - A(t)L(t)i(q_A(\tau)) - B(\tau)K(t)i(q_B(\tau)),$$

$g_B = 0$ *and* $g = g_A$.

The proof of Proposition 1 consists of two arguments that involve Theorem 1 and Corollary 2, respectively. I develop the proof here since it also reveals the intuition underlying Proposition 1.[15]

To see that Theorem 1 applies observe that the right-hand sides of equations (12) and (14) correspond, respectively, to the economy's aggregate production function of equation (1), $\tilde{F}[K(t), L(t), \mathbf{A}_F(t)]$, and to its aggregate intermediate expenses function of equation (2), $\tilde{I}[K(t), L(t), \mathbf{A}_I(t)]$. Accordingly, the right-hand side of equation (15) states the economy's net output function $\tilde{V}[K(t), L(t), \mathbf{A}_F(t), \mathbf{A}_I(t)]$, which indeed exhibits constant returns to scale in $K(t)$ and $L(t)$. Hence, the economy described by equations (15), (4), and (5) has all features assumed in Theorem 1. Accordingly, net output has a representation as $V(t) = V[K(t), A(t)L(t)]$ with $A(t) = e^{(g_V - g_L)(t-\tau)} \in \mathbf{R}_{++}$. Moreover, as Theorem 1 requires $\mathbf{A}_I(t) = \mathbf{A}_I(\tau)$ for all $t \geq \tau$ it must hold that $q_B(t) = q_B(\tau)$ and $q_A(t) = q_A(\tau)$ so that $i(q_B(\tau))$ and $i(q_A(\tau))$.

The second argument excludes a representation of labor-augmenting technical change involving $g_B \neq 0$. It is based on Corollary 2. To see this, define

$$B_F(t) \equiv B(t), \quad A_F(t) \equiv A(t), \quad B_I(t) \equiv B(t)i\left(q_B(\tau)\right),$$
$$A_I(t) \equiv A(t)i\left(q_A(\tau)\right).$$

Then, in steady state equation (15) may be written as

$$V(t) = F\left[B_F(t)K(t), A_F(t)L(t)\right] - \left(A_I(t)L(t) + B_I(t)K(t)\right),$$

where the term in parenthesis is $I\left[B_I(t)K(t), A_I(t)L(t)\right]$. Hence, technical change is factor-augmenting in the net output function so that Corollary 2 indeed applies. More precisely, since the aggregate intermediate expenses function cannot be Cobb–Douglas I have to refer to either Claim 1 or Claim 3. The definitions of $A_F(t)$ and $A_I(t)$ imply $g_{A_F} = g_{A_I} = g_A$. Then, equation (9) excludes $g_{B_F} \neq 0$. Moreover, the definitions of $B_F(t)$ and $B_I(t)$ imply $g_{B_F} = g_{B_I} = g_B$. Since $g_{B_F} = 0$ it follows that $g_{B_F} = g_{B_I} = g_B = 0$. Hence, $g = g_V - g_L = g_A$ for all $t \geq \tau$.

Before concluding this section, two points are worth mentioning. First, let me reemphasize that Theorem 1 excludes any form of technical change that is not labor-augmenting. For instance, in steady state no technical change is permitted that would reduce the amount of intermediate expenditures per task. This is quite restrictive since the function $i(\cdot)$ represents a technical relationship that, in principle, may change over time due to technical progress.

Second, observe that Proposition 1 does in no way rely on the accumulation equations (13). The nature of steady-state technical change is entirely determined by the structural properties of the production sector and its consistency with Theorem 1. However, the stipulated accumulation processes for $B(t)$ and $A(t)$ matter for the existence of a steady state starting at $\tau < \infty$. More precisely, they must allow for steady-state technical change with a representation as labor-augmenting. Here, this is obviously the case for $q_B(\tau) = \delta_B$. Then, the steady state has $g_B = 0$ and $g_A = q_A(\tau) - \delta_A$ for all $t \geq \tau$.

3.2. Example 2: The Multi-Sector Model of Acemoglu (2003)

Acemoglu's economy comprises four sectors on the production side. Section 3.2 sketches the relevant features. The main conclusion is that gross and net output coincide in this economy. Therefore, Uzawa's original theorem conveys the information about technical change along the steady-state path. This result is derived as Proposition 2 in Section 3.2. Throughout, I follow Acemoglu and set $L(t) = L$.

Production and Research. The first sector manufactures the final good, $Y(t)$, out of a capital-intensive intermediate good, $Y_K(t)$, and a labor-intensive intermediate good, $Y_L(t)$. The production function of this sector is[16]

$$Y(t) = F\left[Y_K(t), Y_L(t)\right]. \tag{16}$$

It exhibits constant returns to scale and is increasing in both arguments.

The second sector produces the intermediate goods $Y_K(t)$ and $Y_L(t)$ out of (other) differentiated intermediate goods. The respective production functions are of the CES-type,

$$Y_K(t) = \left[\int_0^{m(t)} \sqrt{y_k(i,t)}di\right]^2 \quad \text{and} \quad Y_L(t) = \left[\int_0^{n(t)} \sqrt{y_l(i,t)}di\right]^2. \tag{17}$$

Here, $[0, m(t)]$ and $[0, n(t)]$ denote disjoint sets of intermediate goods available at t. All intermediate goods in use at t fully depreciate afterward.

The third sector comprises single-good firms each producing one variety of the intermediate good $y_k(i,t)$ or $y_l(i,t)$. The production functions are linear, i. e.,

$$\begin{aligned} y_k(i,t) &= k(i,t) \quad \text{for all } i \in [0, m(t)], \\ y_l(i,t) &= l(i,t) \quad \text{for all } i \in [0, n(t)], \end{aligned} \tag{18}$$

where $k(i,t)$ is capital input and $l(i,t)$ the input of unskilled labor at t.

Consider a symmetric configuration of these three sectors. Then, $y_k(t) = k(t)$, $y_l(t) = l(t)$, and the factor market clearing conditions read $m(t)k(t) = K(t)$ and $n(t)l(t) = L$, respectively. Here, $K(t) > 0$ and $L > 0$ denote the endowments of capital and unskilled labor. As a consequence, $Y_K(t) = m(t)K(t)$ and $Y_L(t) = n(t)L$, so that

$$Y(t) = F\left(m(t)K(t), n(t)L\right). \tag{19}$$

Hence, increasing $m(t)$ and $n(t)$ has an interpretation as capital-, respectively, labor-augmenting technical change.

The fourth sector is the research sector. Let $S_k(t) \geq 0$ and $S_l(t) \geq 0$ denote the "number" of scientists engaged in the invention of new varieties that expand either the set $[0, m(t)]$ or the set $[0, n(t)]$. The technologies for the creation of new inventions are

$$\frac{\dot{m}(t)}{m(t)} = S_k(t) - \delta \quad \text{and} \quad \frac{\dot{n}(t)}{n(t)} = S_l(t) - \delta, \tag{20}$$

where $\delta \in \mathbf{R}_{++}$ is the obsolescence rate of existing varieties. At all t, there are S scientists in the economy, i. e., market clearing requires $S_k(t) + S_l(t) = S$.

Net Output and the Steady State. The key observation of the preceding section is that current final-good output is neither used as an input in the two vertical chains that end in the production of the final good, nor as an input in the research sector. Hence, there are no intermediate expenses. Therefore, $I(t) = 0$ and the reduced form for Y_t of equation (19) states the amount of the final good available at t for consumption and capital accumulation, i. e., $V(t) = Y(t)$.

The following proposition highlights the link between the steady-state path of the economy and Theorem 1.

PROPOSITION 2. *Consider the economy described by equations (19), (4), and (5) with $g_L = 0$. Suppose there exists a steady-state path starting at date $\tau < \infty$ such that $Y(t) > C(t) > 0$ for all $t \geq \tau$. Then, the following holds:*

1. *If $g_m \neq 0$, then net output has the form*

$$V(t) = \beta_F [K(t)]^{\alpha_F} \left[e^{g(t-\tau)}L\right]^{1-\alpha_F},$$

where

$$g = \frac{\alpha_F g_m}{1-\alpha_F} + g_n.$$

2. *If F is not Cobb–Douglas, then*

$$V(t) = F(m(\tau)K(t), n(t)L),$$

$g_m = 0$ *and* $g = g_n$.

To prove Proposition 2 start with the observation that F of equation (19) has constant returns to capital and labor. Moreover, F corresponds to the economy's aggregate production function $\tilde{F}[K(t), L(t), \mathbf{A}_F(t)]$ of equation (1). Hence, as $V(t) = Y(t)$, the economy described by equations (19), (4), and (5) with $g_L = 0$ satisfies all assumptions of Theorem 1 for $I(t) = 0$. Accordingly, steady-state technical change has a representation as labor-augmenting.

Let $B_F(t) \equiv m(t)$ and $A_F(t) \equiv n(t)$. Then, (net) output can be written as $V(t) = F[B_F(t)K(t), A_F(t)L]$. Accordingly, Section 2.3 applies and suggests the distinction between the two cases mentioned in the proposition. First, $g_m \neq 0$ can only occur if F is of the Cobb–Douglas type. Here, the parameters α_F and β_F are as introduced in Corollary 2, and $g = \alpha_F g_m/(1-\alpha_F) + g_n$ is the steady-state growth rate of the economy.

The second case follows immediately from the first. Any functional form other than the Cobb-Douglas type implies $g_m = 0$ and $g = g_n$.

Again, two closing remarks are in order. First, it is worth emphasizing that the production functions (17) and (18) of the intermediate-good sectors are time-invariant and therefore not subject to technical progress. While this is so by assumption, it is also required by Theorem 1. Formally, these functions are contained in $\mathbf{A}_F(t)$, and for $t \geq \tau$ we must have $\mathbf{A}_F(t) = \mathbf{A}_F(\tau)$. This is another instance of the restrictiveness of Theorem 1. Any mode of technical change other than labor-augmenting is forbidden in steady state.

Second, let me underline that Proposition 2 does not hinge on the presence nor on the structural properties of the research sector.[17] The nature of steady-state technical change is entirely determined by the structural properties of the production sector and its consistency with Theorem 1. Finally, recall that Theorem 1 coincides with Uzawa's original theorem for $I(t) = 0$. Hence, Uzawa's theorem directly applies to the model of Acemoglu (2003). We shall see in Section 3.4 that the same conclusion holds for the models that belong to the induced innovations literature of the 1960s.

3.3. Example 3: The Multi-Sector Model of Acemoglu (2009)

The economy studied in Acemoglu (2009, Chapter 15.6), extends the model of Acemoglu (2003) by allowing for a market size effect that affects the direction of technical change. In Section 3.3, I show how the production sector of Acemoglu (2003) is modified and that these modifications give rise to a strictly positive aggregate intermediate expenses function, i. e., $I(t) > 0$. This suggests an application of Theorem 1. Proposition 3 of Section 3.3 has the details.[18]

Production and Research. Two changes to the production sector of Section 3.2 are made. First, the production functions of the intermediates Y_K and Y_L are modified so that the market-size effect can play a role. Second, the single-good firms that manufacture one type of machines now use current final-good output as the only input. Due to the latter change, gross and net output differ. To make this more precise let us reconsider the production and research sectors of Section 3.2.

The first sector, i. e., the production of the final good, is unchanged. Hence, $Y(t) = F[Y_K(t), Y_L(t)]$ as in equation (16).

In the second sector, the production functions for $Y_K(t)$ and $Y_L(t)$ of equation (17) are replaced by

$$Y_K(t) = 2\sqrt{K(t)} \int_0^{m(t)} \sqrt{y_k(i,t)}di \quad \text{and}$$
$$Y_L(t) = 2\sqrt{L} \int_0^{n(t)} \sqrt{y_l(i,t)}di. \tag{21}$$

Here, the appearance of $K(t)$ and L is new. Hence, the disjoint sets $[0, m(t)]$ and $[0, n(t)]$ represent machines that are either complementary to capital or to labor. All machines in use at t fully depreciate afterward.

The third sector comprising the single-good firms that each produce one variety of the machines used in equation (21) is modified. The new feature concerns the production of machines that now uses current final output as the sole input. For all machines the production functions are identical and linear. Without loss of generality, assume that the required input per manufactured machine is one unit of the final good.

With these changes a symmetric configuration of the three production sectors has the following properties.

With $y_k(i,t) = y_k(t)$ and $y_l(i,t) = y_l(t)$ intermediate expenses for capital-comple- mentary machines amount to $m(t)y_k(t)$, those for labor-complementary

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machines are equal to $n(t)y_l(t)$. Since the sum of these magnitudes corresponds to aggregate intermediate expenses it holds indeed that $I(t) > 0$.

As both types of machines are produced out of current final-good output using a linear production function, we have

$$m(t)y_k(t) = \zeta_k(t)Y(t) \quad \text{and} \quad n(t)y_l(t) = \zeta_l(t)Y(t), \tag{22}$$

where $\zeta_k(t) > 0$ and $\zeta_l(t) > 0$ denote the fractions of total output used in the production of the respective machine type, and $0 < \zeta_k(t) + \zeta_l(t) < 1$. Hence, aggregate intermediate expenses may be written as

$$I(t) = [\zeta_k(t) + \zeta_l(t)]\, Y(t). \tag{23}$$

Here, the conditions on the ζs assure that $Y(t) > V(t) > 0$.

Symmetry and equation (22) imply that the output of $Y_K(t)$ and $Y_L(t)$ of (21) may be expressed as

$$Y_K(t) = 2\sqrt{\zeta_k(t)m(t)K(t)Y(t)} \quad \text{and} \quad Y_L(t) = 2\sqrt{\zeta_l(t)n(t)LY(t)}. \tag{24}$$

The substitution of these expressions into the production function of the final good (16) delivers

$$Y(t) = F\left[2\sqrt{\zeta_k(t)m(t)K(t)Y(t)}, 2\sqrt{\zeta_l(t)n(t)LY(t)}\right]. \tag{25}$$

Since F has constant returns to scale this expression may be solved for $Y(t)$. This gives the aggregate output of the final good as

$$Y(t) = 4\left(F\left[\sqrt{\zeta_k(t)m(t)K(t)}, \sqrt{\zeta_l(t)n(t)L}\right]\right)^2. \tag{26}$$

Hence, in line with equation (1), the economy's aggregate production function is

$$\tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right] = 4\left(F\left[\sqrt{\zeta_k(t)m(t)K(t)}, \sqrt{\zeta_l(t)n(t)L}\right]\right)^2, \tag{27}$$

and exhibits constant returns to scale in $K(t)$ and L.

Using equations (23) and (27) delivers the aggregate intermediate expenses function corresponding to equation (2) as

$$\tilde{I}\left[K(t), L(t), \mathbf{A}_I(t)\right] = [\zeta_k(t) + \zeta_l(t)]\, \tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right], \tag{28}$$

which has constant returns to scale in $K(t)$ and L, too.

Finally, observe that the research sector of Section 3.2 remains unchanged.

Net Output and the Steady State. The upshot of the preceding section is that $\tilde{F}$ of equation (27) has constant returns to scale in $K(t)$ and L even though it results as the solution to the fixed-point problem involved in equation (25). Then, with

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equations (27) and (28) it becomes obvious that the economy's net output function corresponding to equation (3) is

$$\tilde{V}\left[K(t), L(t), \mathbf{A}_F(t), \mathbf{A}_I(t)\right] = (1 - \zeta_k(t) - \zeta_l(t))\, \tilde{F}\left[K(t), L(t), \mathbf{A}_F(t)\right]. \tag{29}$$

The following proposition exploits this fact and provides the link between the steady-state path of the economy and Theorem 1. To simplify the notation, define $B_F(t) \equiv \zeta_k(t)m(t)$, $A_F(t) \equiv \zeta_l(t)n(t)$.

PROPOSITION 3. *Consider the economy described by equations (29), (4), and (5) with $g_L = 0$. Suppose there exists a steady-state path starting at date $\tau < \infty$ such that $Y(t) > V(t) > C(t) > 0$ for all $t \geq \tau$. Then, the following holds:*

1. *If $g_m \neq 0$, then net output has the form*

$$V(t) = (1 - \zeta_k(\tau) - \zeta_l(\tau))\, \beta_F \left[K(t)\right]^{\alpha_F} \left[e^{g(t-\tau)}L\right]^{1-\alpha_F},$$

where

$$g = \frac{\alpha_F g_{B_F}}{1 - \alpha_F} + g_{A_F}.$$

2. *If F is not Cobb–Douglas, then*

$$V(t) = (1 - \zeta_k(\tau) - \zeta_l(\tau))\, F\left(m(\tau)K(t), n(t)L\right),$$

$g_m = 0$ *and* $g = g_n$.

Despite the more complex structure of the economy, Proposition 3 looks strikingly similar to Proposition 2. In fact, the only difference is the factor of proportionality $[1 - \zeta_k(\tau) - \zeta_l(\tau)]$ reflecting the gap between gross and net output. In steady state, both ζs must be constant. To see why start with the observation that net output of equation (29) exhibits constant returns to capital and labor. Hence, the economy described by equations (29), (4), and (5) with $g_L = 0$ satisfies the assumptions of Theorem 1 with $I(t) > 0$. According to Claim 1 of the Theorem, $g_I = g_Y$. Since equation (23) must hold for all $t \geq \tau$ and $0 < \zeta_k(t) < 1$, $0 < \zeta_l(t) < 1$ neither $g_{\zeta_k} \neq 0$ nor $g_{\zeta_l} \neq 0$ is sustainable in steady state.

In accordance with Claim 2 of Theorem 1, technical change has a representation as purely labor-augmenting in steady state. In conjunction with Section 2.3, this gives rise to the two cases stated in Proposition 3. The underlying intuition mimics the one of Proposition 2.[19] Finally, observe that, mutatis mutandis, the two closing remarks of Sections 3.1 and 3.2 apply here, too.

3.4. Example 4: "Induced Innovations" and Capital Accumulation

This section revisits the so-called induced innovations literature of the 1960s. I focus on a typical model with capital accumulation that allows for endogenous

https://doi.org/10.1017/S1365100516000407 Published online by Cambridge University Press

capital- and labor-augmenting technical change [see, e. g., von Weizsäcker (1962), Drandakis and Phelps (1966), Samuelson (1966), or Funk (2002)].

There is one sector on the production side with access to the aggregate production function

$$Y(t) = F\left(B(t)K(t), A(t)L(t)\right). \tag{30}$$

The function F has constant returns to scale, is increasing in both arguments, and technical change is by assumption capital- and labor-augmenting.

The most remarkable features of this literature are related to the endogenous choice of the growth rates $g_A(t)$ and $g_B(t)$. First, there is the assumption of an innovation possibility frontier that specifies the set of feasible rates of factor augmentation as

$$\{(g_A(t), g_B(t)) \in \mathbf{R}^2_+ | g_B(t) = \gamma\left(g_A(t)\right)\}.$$

The frontier $\gamma(.)$ is time-invariant, decreasing, and strictly concave. Hence, there is a trade-off: a greater $g_A(t)$ requires a smaller $g_B(t)$ and vice versa.

Second, the growth rates $[g_A(t), g_B(t)]$ are endogenous in the sense that they maximize the instantaneous rate of technical progress at given factor shares and subject to the innovation possibility frontier.

What form of technical change will arise in the steady state starting at some date $\tau < \infty$? Since F has constant returns to scale in $K(t)$ and $L(t)$ an economy described by equations (30), (4), and (5) satisfies all assumptions of Theorem 1. Moreover, since the choice of $g_A(t)$ and $g_B(t)$ does not involve a cost in terms of resources, there are no intermediate expenses, i. e., $I(t) = 0$. Accordingly, $V(t) = Y(t)$ and Uzawa's original theorem informs us about the nature of technical change in steady state. As technical change is by definition factor-augmenting in F, we may also directly invoke Section 2.3 to characterize the steady state of the economy. This reasoning leads to the conclusion that up to few notational changes Proposition 2 applies also to the "induced innovations" economy with capital accumulation under scrutiny here.[20]

4. CONCLUDING REMARKS

Why is endogenous technical change labor-augmenting in the steady state even though capital-augmenting technical change is feasible? For the literature on induced innovations with capital accumulation of the 1960s, Solow (1970, p. ix), remarks that this theory "is set up to generate labor-augmenting technical change because that is the only kind that combines with the other standard assumptions to permit a steady state." Section 3.4 confirms this assessment: these "other standard assumptions" make this theory fit Uzawa's original theorem.

The main conclusion of the present paper is that a similar assessment holds for the modern theory of endogenous capital- and labor-augmenting technical change. This may seem puzzling since compared to the literature of the 1960s the picture of a growing economy drawn in the modern literature is far more complex. Features like a micro-founded research sector requiring resources to generate new capital- or labor-augmenting technological knowledge, a micro-founded production sector possibly operating under imperfect competition, or the presence of knowledge spill-overs do certainly not belong to the set of "the other standard assumptions" to which Solow refers.

Nevertheless, I establish that Uzawa's Theorem explains why the steady state in the model of Acemoglu (2003) has only labor-augmenting technical change. The generalized steady-state theorem devised in this paper provides the same answer for the models of Irmen and Tabaković (2015) and Acemoglu (2009, Chapter 15).

Arguably, among growth theorists there may be an intuitive feel or a folk theorem saying that Uzawa's theorem should somehow carry over to the modern literature of endogenous capital- and labor-augmenting technical change. The generalized steady-state growth theorem confirms this intuition and provides its precise analytical underpinning. The four examples discussed in Section 3 show in detail how the theorem is to be applied to the existing literature.

The main conclusion of this paper should not be interpreted as a plea to neglect or eliminate capital-augmenting technical change altogether. Indeed, the results derived in Irmen (2017) and Irmen and Tabaković (2015) suggest that both the normative and the positive implications of models with endogenous technical change crucially hinge upon whether capital-augmenting technical change is included in the analysis or not. Future research will have to elucidate this point. It will be of particular interest to study whether policy recommendations are robust if endogenous capital-augmenting technical change is added to the picture.

NOTES

1. See, e. g., von Weizsäcker (1962), Kennedy (1964), Samuelson (1965), Samuelson (1966), or Drandakis and Phelps (1966).

2. Jones (2005) develops an alternative argument for why technical change may be labor-augmenting.

3. A *steady state* is defined as a path along which *all* variables in a model grow at constant, and possibly different, exponential rates. These rates may be positive, zero, or negative.

4. Some recent estimates of the elasticity of substitution between capital and labor find values greater than unity [see, e. g., Duffy and Papageorgiou (2000), Karabarbounis and Neiman (2014), or Piketty (2014)]. In this case, the steady-state path of the typical model of the literature of the 1960s is a saddle [see, e. g., Drandakis and Phelps (1966)]. In the model of Acemoglu (2003), the steady state is unstable.

5. Quite remarkably, Uzawa calls his main result "Robinson's Theorem" [Uzawa (1961, p. 119)]. It is meant to formalize the graphical analysis of neutral inventions that appears in Robinson (1938). Robinson's Theorem shows the equivalence between labor-augmenting and Harrod-neutral technical change that, by definition, does not affect the value of the capital coefficient at a constant rate of interest [Harrod (1937)]. With this finding at hand, only a small step is needed to establish that steady-state growth requires technical change to be Harrod-neutral. It is the latter result that the literature refers to as Uzawa's steady-state growth theorem. Concise statements of it can be found in Schlicht (2006), Jones and Scrimgeour (2008), or Acemoglu (2009). As will become clear subsequently, the present paper generalizes this theorem with the introduction of *aggregate intermediate expenses* that represent resources used to generate technical progress or to manufacture intermediates.

6. In other words, along the transition toward such a steady-state technical change should become purely labor-augmenting and capital-augmenting technical change should peter out. Klump, McAdam, and Willman (2007) find this pattern of technical change empirically for the US economy over the period 1953 to 1998.

7. The reduced form of the three-sector model of endogenous capital- and labor-augmenting technical change devised in Irmen (2011) shares the relevant properties with the one-sector model of Irmen and Taboković (2015). Hence, all findings derived in Section 3.1 also apply to this three-sector model. They do also apply to the one-sector model under scrutiny in Irmen (2017).

8. In general, the specification of $\mathfrak{A}_{\mathfrak{F}}$ (and $\mathfrak{A}_{\mathfrak{I}}$ introduced below) will depend on how technological knowledge and its components are represented.

9. This assumption will be relaxed in Section 2.3 below. In any case, observe that the dependency of $\tilde{F}$ and $\tilde{I}$ on K and L does not necessarily imply that all machines and the entire labor force produce both the final and the investment good. For instance, let K_Y and K_I denote homogeneous machines allocated, respectively, to the production of the final and the investment good. Then market clearing requires $K_Y + K_I = K$. Accordingly, the equilibrium allocation of machines will hinge on the exogenous capital endowment, K, which appears as an argument in $\tilde{F}$ and $\tilde{I}$. Mutatis mutandis, the same point may be made for labor.

10. Observe that unlike Theorem 1, the character of Uzawa's original theorem drastically changes if the aggregate production function becomes either linear in labor or in capital. In the former case, $Y(t) = L(t)\tilde{\tilde{F}}(\mathbf{A}_F(t))$ and, void of capital and its accumulation, steady-state technical change has to be labor-augmenting. In the latter case, $Y(t) = K(t)\tilde{\tilde{F}}(\mathbf{A}_F(t))$, i. e., aggregate production is of the AK-type. Here, labor is not explicitly accounted for and the mere notion of labor-augmenting technical change becomes pointless.

11. In Theorem 1, I introduce g as the growth rate of per-worker variables. In a slight abuse of notation, here I also use g to denote the growth rate of "labor-augmenting" technical change in a Cobb–Douglas function. Observe that both growth rates coincide in steady state.

12. See, e. g., Barro and Sala-í-Martin (2004), 78–80, for a discussion of this case.

13. A brief discussion of this model is also contained in Section 5.3 of Acemoglu (2003).

14. All four examples are presented in continuous time. While the analysis in Irmen and Taboković (2015) is set up in discrete time the switch to continuous time is without loss of generality for my qualitative results.

15. Observe that general conditions can be given so that the steady-state path of Proposition 1 indeed exists. The same remark applies to the steady-state paths of Proposition 2, Proposition 3, and the one studied in Section 3.4.

16. Acemoglu assumes F to be a CES production function. This specification is not necessary for my purpose here (though, the elasticity of substitution plays an important role in Acemoglu's analysis). When referring to other functional forms that appear in Acemoglu (2003), I use particular values for the following parameters: $\beta = 1/2, b_k = b_l = 1$. Moreover, I set the function $\phi(s) = 1$, for $s = S_k, S_l$. These choices simplify the exposition but are without loss of generality for the qualitative results I derive.

17. However, Proposition 2 imposes severe constraints on the way the accumulation equations (20) may be specified to support a steady state where technical change has a representation as labor-augmenting. Here, full employment of all researchers and equation (20) imply in the first case of Proposition 2 that any pair (g_m, g_n) must satisfy $g_m = S_k - \delta$, $g_n = S_l - \delta$, and $S_k + S_l = S$. In the second case, it must be that $g_m = 0$ and $g_n = S - 2\delta$.

18. In what follows, I use the same parameter values as set out in note 16. Again, these choices simplify the exposition but are without loss of generality for the qualitative results I derive.

19. To justify Claim 1 of Proposition 3, one may alternatively invoke Claim 4 of Corollary 2 with $\alpha_F = \alpha_I$.

20. To be precise, we need to replace (m, g_m) and (n, g_n) by (B, g_B) and (A, g_A), respectively. Moreover, there is no reason here to keep (unskilled) labor constant. Hence, we may replace L by $L(\tau)e^{g_L(t-\tau)}$, $g_L \in \mathbf{R}$.

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APPENDIX: PROOFS

A.1. PROOF OF THEOREM 1

Observe that $Y(t) > V(t) > C(t) > 0$ implies both $Y(t) > I(t) > 0$ and $V(t) > C(t) > 0$. Moreover, without loss of generality let $\tau = 0$.

Part I. Given time-invariant growth rates, date t quantities may be expressed in terms of date 0 quantities, i. e., $V(t) = V(0)e^{g_V t}$, $Y(t) = Y(0)e^{g_Y t}$, and $I(t) = I(0)e^{g_I t}$. Then, from the definition of $V(t)$, I have for all $t \geq 0$

$$V(0)e^{g_V t} = Y(0)e^{g_Y t} - I(0)e^{g_I t}. \tag{A.1}$$

Dividing both sides by $e^{g_V t}$ gives

$$V(0) = Y(0)e^{(g_Y - g_V)t} - I(0)e^{(g_I - g_V)t}.$$

Differentiation with respect to t delivers

$$0 = (g_Y - g_V)\,Y(0)e^{(g_Y - g_V)t} - (g_I - g_V)\,I(0)e^{(g_I - g_V)t}.$$

The latter can hold for all t if any of the following conditions are satisfied: (a) $g_V = g_Y = g_I$, (b) $g_Y = g_I$ and $Y(0) = I(0)$, (c) $g_V = g_Y$ and $I(0) = 0$, and (d) $g_V = g_I$ and $Y(0) = 0$. Alternatives (b) to (d) contradict $Y(0) > I(0) > 0$. Hence, $g_V = g_Y = g_I$ must apply.

With $K(t) = K(0)e^{g_K t}$ and $C(t) = C(0)e^{g_C t}$, capital accumulation of equation (4) delivers

$$K(0)e^{g_K t}(g_K + \delta_K) = V(0)e^{g_V t} - C(0)e^{g_C t}. \tag{A.2}$$

Divide both sides by $e^{g_K t}$ and obtain

$$K(0)(g_K + \delta_K) = V(0)e^{(g_V - g_K)t} - C(0)e^{(g_C - g_K)t}.$$

Differentiation of the latter with respect to t gives

$$0 = (g_V - g_K)\,V(0)e^{(g_V - g_K)t} - (g_C - g_K)\,C(0)e^{(g_C - g_K)t}.$$

The latter can hold for all t if any of the following conditions are satisfied: (a) $g_V = g_K = g_C$, (b) $g_V = g_C$ and $V(0) = C(0)$, (c) $g_V = g_K$ and $C(0) = 0$, and (d) $g_C = g_K$ and $V(0) = 0$. Alternatives (b)–(d) contradict $V(0) > C(0) > 0$. Hence, $g_V = g_K = g_C$ must apply as claimed. This completes the proof of Part I.

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Part II. In light of equation (3), for any $t \geq 0$, net output at time 0 may be written as

$$e^{-g_V t} \cdot V(t) = \tilde{V}\left[e^{-g_K t} \cdot K(t), e^{-g_L t} \cdot L(t), \mathbf{A}_F(0), \mathbf{A}_I(0)\right].$$

Multiplying both sides by $e^{g_V t}$ and using constant returns of $\tilde{V}$ gives

$$V(t) = \tilde{V}\left[e^{(g_V - g_K)t} \cdot K(t), e^{(g_V - g_L)t} \cdot L(t), \mathbf{A}_F(0), \mathbf{A}_I(0)\right]. \quad \textbf{(A.3)}$$

From Part I, I have $g_V = g_K$, hence for any $t \geq 0$

$$V(t) = \tilde{V}\left[K(t), e^{(g_V - g_L)t} \cdot L(t), \mathbf{A}_F(0), \mathbf{A}_I(0)\right].$$

Since the latter equation is true for all $t \geq 0$ and $\tilde{V}$ is linear homogeneous in the first two arguments, there exists a linear homogeneous function $V : \mathbf{R}_+^2 \to \mathbf{R}_+$ such that

$$V(t) = V\left[K(t), e^{(g_V - g_L)t} \cdot L(t)\right] = V\left[K(t), A(t)L(t)\right],$$

with $A(t) = e^{(g_V - g_L)t} \in \mathbf{R}_{++}$.

Part I and constant returns to scale imply that net output per-worker as well as all other per-worker variables grow at rate $g = g_V - g_L$. This establishes the second part of the theorem. ■

A.2. PROOF OF COROLLARY 1

Again, without loss of generality let $\tau = 0$.

Part I. Since $Y(t) > V(t) = C(t) > 0$ we now have $Y(t) > I(t) > 0$ and $V(t) = C(t) > 0$. Therefore, the proof of $g_V = g_Y = g_I$ remains as in the proof of Theorem 1. However, $V(t) = C(t)$ and equation (4) deliver $g_V = g_C$ and $g_K = -\delta_K$, respectively. This proves Part I of Corollary 1.

Part II. The first two steps in the proof of Part II of Theorem 1 remain valid. Then, using $g_K = -\delta_K$ in equation (A.3) delivers for any $t \geq 0$

$$V(t) = \tilde{V}\left[e^{(g_V + \delta)t} \cdot K(t), e^{(g_V - g_L)t} \cdot L(t), \mathbf{A}_F(0), \mathbf{A}_I(0)\right].$$

Since the latter equation is true for all $t \geq 0$ and $\tilde{V}$ is linear homogeneous in the first two arguments, there exists a linear homogeneous function $V : \mathbf{R}_+^2 \to \mathbf{R}_+$ such that

$$V(t) = V\left[e^{(g_V + \delta_K)t} \cdot K(t), e^{(g_V - g_L)t} \cdot L(t)\right] = V\left[B(t)K(t), A(t)L(t)\right],$$

with $B(t) = e^{(g_V + \delta_K)t} \in \mathbf{R}_{++}$ and $A(t) = e^{(g_V - g_L)t} \in \mathbf{R}_{++}$. Hence, $V(t)$ is as stated in Corollary 1. Capital per worker, $K(t)/L(t)$, grows at rate $g_K - g_L = -(\delta_K + g_L)$. Moreover, since $g_V = g_Y = g_I = g_C$ all remaining per-worker variables grow at rate $g = g_V - g_L$. ■

A.3. PROOF OF COROLLARY 2

Let me introduce

$$\kappa_F(t) = \frac{A_F(t)L(t)}{B_F(t)K(t)} \quad \text{and} \quad \kappa_I(t) = \frac{A_I(t)L(t)}{B_I(t)K(t)}.$$

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Now, consider equation (7) and divide by $K(t)$. This gives

$$\frac{V(t)}{K(t)} = \frac{Y(t)}{K(t)} - \frac{I(t)}{K(t)},$$

$$= B_F(t)F[1, \kappa_F(t)] - B_I(t)I[1, \kappa_I(t)],$$

$$= B_F(t)f(\kappa_F(t)) - B_I(t)i(\kappa_I(t)),$$

where $f : \mathbf{R}_+ \to \mathbf{R}_+$ and $i : \mathbf{R}_+^2 \to \mathbf{R}_+$ are respectively defined as $f(\kappa_F(t)) \equiv F[1, \kappa_F(t)]$ and $i(\kappa_I(t)) \equiv I[1, \kappa_I(t)]$.

Observe that net output of equation (7) is a special case of the formulation given in equation (3). Therefore, Part I of Theorem 1 applies and imposes the requirement that $g_V = g_Y = g_I = g_K$, i. e., the fraction $V(t)/K(t)$ as well as $Y(t)/K(t) = B_F(t)f(\kappa_F(t))$ and $I(t)/K(t) = B_I(t)i(\kappa_I(t))$ are constant in steady state. Then, four cases may arise. They correspond to the four claims made in the corollary. I consider each in turn.

1. If $g_{B_F} = g_{B_I} = 0$ then $g_{\kappa_F} = g_{\kappa_I} = 0$. Taken together, the implication is that $g_{A_F} = g_{A_I} = g_K - g_L$. Since $g_V = g_K$, the growth rate of per-worker variables is $g_{A_F} = g_{A_I} = g_V - g_L = g$.
2. If $g_{B_F} = 0$ and $g_{B_I} \neq 0$ then $g_{\kappa_F} = 0$ whereas $g_{\kappa_I} \neq 0$. For aggregate production the implication is that $g_{A_F} = g_K - g_L$. For aggregate intermediate expenses the growth rates $g_{B_I} \neq 0$ and $g_{\kappa_I} \neq 0$ must be of opposite sign such that $B_I(t)i(\kappa_I(t))$ with $i'(\kappa_I(t)) > 0$ can remain constant over time. In other words, the time derivative of this product must vanish, i. e.,

$$\dot{B}_I(t)i(\kappa_I(t)) + B_I(t)\dot{\kappa}_I(t)i'(\kappa_I(t)) = 0, \qquad \textbf{(A.4)}$$

or, in steady state,

$$\frac{i'(\kappa_I(t))\,\kappa_I(t)}{i(\kappa_I(t))} = -\frac{g_{B_I}}{g_{\kappa_I}}.$$

Integration reveals that the solution can be written as

$$i(\kappa_I(t)) = c_I \kappa_I(t)^{1-\alpha_I},$$

where $c_I > 0$ is a constant of integration and $\alpha_I = 1 + g_{B_I}/g_{\kappa_I}$. A positive, yet declining marginal product of capital requires $0 < \alpha_I < 1$. Then, for all $t \geq \tau$

$$I(t) = c_I (B_I(t)K(t))^{\alpha_I} (A_I(t)L(t))^{1-\alpha_I} = c_I (B_I(\tau)K(t))^{\alpha_I} \left(A_I(\tau)e^{g(t-\tau)}L(t)\right)^{1-\alpha_I},$$

where $g = \alpha_I g_{B_I}/(1-\alpha_I) + g_{A_I}$. Introducing the constant β_I, $I(t)$ may be written as stated in Claim 2.

To align g_I and g_Y, express the growth rate of $I(t)$ as

$$g_I = \alpha_I \left(g_{B_I} + g_K\right) + (1-\alpha_I)\left(g_{A_I} + g_L\right).$$

Since $g_I = g_K$ the latter becomes

$$g_I = \frac{\alpha_I g_{B_I}}{1-\alpha_I} + g_{A_I} + g_L.$$

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Next, recall that $g_Y = g_K = g_{A_F} + g_L$. Then, $g_Y = g_I$ requires

$$g_{A_F} = \frac{\alpha_I g_{B_I}}{1-\alpha_I} + g_{A_I} = g.$$

Hence, any set of growth rates $\{g_{A_F}, g_{B_I}, g_{A_I}\} \in \mathbf{R}^2 \times \mathbf{R} \setminus \{0\}$ must satisfy the latter condition to be consistent with a steady state.

3. The case where $g_{B_F} \neq 0$ and $g_{B_I} = 0$ is the mirror image of the previous case. Mutatis mutandis, the proof of Claim 2 applies here, too.
4. If $g_{B_F} \neq 0$ and $g_{B_I} \neq 0$ then the proof of Cases 2 and 3 implies immediately that aggregate production and aggregate intermediate expenses may be written as

$$Y(t) = \beta_F K(t)^{\alpha_F} \left(e^{g(t-\tau)} L(t)\right)^{1-\alpha_F},$$

$$I(t) = \beta_I K(t)^{\alpha_I} \left(e^{g(t-\tau)} L(t)\right)^{1-\alpha_I},$$

where g is given by equation (10), the condition that any set of growth rates $\{g_{B_F}, g_{A_F}, g_{B_I}, g_{A_I}\} \in \mathbf{R}^2 \times \mathbf{R} \setminus \{0, 0\}$ must satisfy to be consistent with a steady state. Accordingly, net output has the form given in Claim 4.

■

A.4. PROOF OF PROPOSITIONS 1, 2, AND 3

To be found in the main text. ■

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