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VALUATED BUTLER GROUPS OF FINITE RANK

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Dedicated to the memory of B. H. Neumann

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Abstract

Valuated Butler groups of finite rank are investigated. The valuated B_2-groups are both epic images and pure subgroups of completely decomposable valuated groups of finite rank (Theorem 3.1). However, there are fundamental changes in the theory of Butler groups when valuations are involved. We introduce valuated B_1-groups and show that they are valuated B_2-groups. Surprisingly, valuated B_2-groups of rank greater than 1 need not be valuated B_1-groups, unless they carry a special kind valuation, see Theorem 7.1. Theorem 6.5 gives a full characterization of valuated B_1-groups of finite rank, generalizing Bican's characterization of Butler groups.

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1. Introduction

Butler groups as well as valuated groups are topics in the center of interest in abelian group theory. In this note, we wish to combine the two theories and initiate a general study of valuated Butler groups in the finite rank case.

A most important class of abelian groups, the so-called Butler groups, possesses several remarkable properties which have been investigated by a number of authors, see Arnold [1]. Finite rank Butler groups have been studied also by concentrating on free essential subgroups, furnished with the height valuation. Furthermore, in the classification of an important class of mixed abelian groups (called Warfield groups),

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free abelian groups with ordinal valuation turned out to be crucial tools. Valuated free abelian groups have been studied by several authors (see, for example, [2, 3, 8]).

Our setting is more general: rather than free abelian groups we deal with genuine Butler groups equipped with valuations of ordinal values for every prime p; their theory cannot be reduced to the study of valuated free groups, because the valuations for the whole group cannot be recaptured from a subgroup. In this note, we restrict ourselves to groups of finite rank. Consequently, we will tacitly assume throughout that all torsion-free groups in this note are of finite rank.

Recall that a finite rank torsion-free group B is called a *Butler group* if it satisfies one of the following equivalent conditions:

(1) (Butler [6]) B is a pure subgroup of a completely decomposable group (by a completely decomposable group is meant a direct sum of torsion-free groups, each isomorphic to some subgroup of the rational group $\mathbb{Q}$);
(2) (Butler [6]) B is a surjective image of a finite rank completely decomposable group;
(3) (Bican [4]) The set Π of prime numbers has a partition $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ such that, for each j, the tensor product $B \otimes \mathbb{Z}_{\Pi_j}$ (the localization of B at the set Π_j of primes) is a completely decomposable group;
(4) (Bican and Salce [5]) Balanced extensions of torsion groups by B are splitting, that is, $\mathrm{Bext}^1(B, T) = 0$ for all torsion groups T (G is a balanced extension of T by B if the subgroups of $\mathbb{Q}$ have the projective property with respect to the exact sequence $0 \to T \to G \to B \to 0$).

Each of these properties are meaningful if the groups are furnished with a valuation. But are they still equivalent? This is the question which we want to answer in this note.

Our results will show that these four conditions are no longer equivalent in the setting of valuated groups, but partial equivalences and some implications are still valid. Theorem 3.1 asserts that (1) and (2) are equivalent for valuated Butler groups, that is, the classical equivalence theorem by Butler [6] carries over to the valuated situation (just as in the case of free valuated groups); we call these groups *valuated B_2-groups*. It is more challenging to verify the equivalence of (3) and (4) in the valuated case (Theorem 6.5) ((3) and (4) have not been considered as yet for free valuated groups); these groups will be called *valuated B_1-groups*—this terminology is in accordance with the standard terminology in the theory of Butler groups. It turns out that (3) [or (4)] implies (1) [and (2)], as is shown by Theorem 6.1. Somewhat surprisingly, the converse implication fails in general (see Examples 2 and 3). However, Butler groups equipped with the height valuation, and more generally, valuated B_2-groups with gap-free valuation, are still valuated B_1-groups (compare Corollary 7.2). In Theorem 7.1 we establish a necessary and sufficient condition for a valuated B_2-group to be a valuated B_1-group.

While our discussion of valuated B_2-groups makes use of methods already established in the theory of valuated groups, the study of valuated B_1-groups requires a new approach. The key seems to be the localization process; in fact, our results are based on a careful investigation of the local behavior of valuated B_1-groups. It will be clear that the fundamental difference between valuated B_1- and B_2-groups lies in the local case: the former groups ought to be completely decomposable, but not the latter ones (Examples 2 and 3).

2. Preliminaries

We recall the definition of valuated groups. For a prime number p, by the *p-valuation* v_p of an abelian group G is meant a function from G to the class of ordinals with the symbol ∞ adjoined (which is regarded to be larger than any ordinal) such that, for all $a, b \in G$,

(i) $v_p(a) = \infty$ if $a = 0$;
(ii) $v_p(pa) \geq v_p(a)$, where strict inequality holds unless $v_p(a) = \infty$;
(iii) $v_p(na) = v_p(a)$ whenever the integer n is relatively prime to p;
(iv) $v_p(a + b) \geq \min(v_p(a), v_p(b))$.

The *valuation* v of the group G is a collection of p-valuations v_p of G, one for each prime p. Thus $v(a)$ for $a \in G$ is the sequence $v_p(a)$ of p-values of a for $p = 2, 3, 5, \ldots$.

The p-value of an element a cannot be smaller than its p-height: $v_p(a) \geq h_p(a)$, where h_p denotes the p-height. Thus, if $a \in G$ is an element of order p^k (a prime power), then for all primes $q \neq p$ necessarily $v_q(a) = h_q(a) = \infty$. Also, if a belongs to a p-divisible subgroup of G, then $v_p(a) = h_p(a) = \infty$.

In order to get more precise information about an element a of a valuated group G, one also needs to know the values $v(na)$ for each integer $n > 0$. These values are known if we are given $v_p(p^j a)$ for each prime p and for each non-negative integer j. Therefore we consider the *value-matrix* $V(a)$ of $a \in G$ which encodes all the needed value information about a. $V(a)$ is an $\omega \times \omega$ matrix whose i, j-entry is $v_p(p^j a)$ where p is the ith prime. It is easy to see that every $\omega \times \omega$-matrix with ordinal entries (∞ symbols are also admitted) is a value-matrix of some element in a suitable valuated group provided that the rows are strictly increasing (except when ∞ is reached). Value-matrices can be partially ordered pointwise.

A *morphism* ϕ between two valuated groups $G \to G'$ is a group homomorphism that does not decrease values: $v_p(a) \leq v_p(\phi a)$ for all $a \in G$ and primes p. Valuated groups with these morphisms form a category $\mathcal{V}$.

Two valuated groups A, B are called *isometric* if there is a value-preserving isomorphism between them; we then write $A \approx B$ (and preserve the sign $\cong$ for pure group isomorphism). For example, two valuated free groups of rank 1 are isometric if and only if they can be generated by elements with the same value-matrix.

If A is a subgroup of a valuated group G, then it is understood that the valuation in A is the induced valuation from G, unless stated otherwise.

We assume familiarity with the basic notions in the theory of valuated groups, in particular, with the valuation of direct sums and of factor groups. By an *exact sequence*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of valuated groups is meant an ordinary exact sequence such that α is a kernel and β is a cokernel map, that is, α is an isometry between A and $\operatorname{Im}\alpha$, and β induces an isometry between $B/\operatorname{Im}\alpha$ and C.

It should be emphasized that the category $\mathscr{V}$ of valuated groups is not abelian, it is only *pre-abelian* in the sense that it is an additive category with kernels and cokernels. We shall use the fact, as pointed out by Richman and Walker in [9], that in such a category, and hence in $\mathscr{V}$, both pull-backs and push-outs exist, enjoying the usual universal properties.

We refer to Arnold [1] for results on finite rank Butler groups (without valuation); the basic facts will be used without explicit reference.

3. Valuated B_2-groups

By a *completely decomposable valuated group* is meant a group F that is a direct sum of valuated torsion-free groups of rank 1. In the finite rank case, we have $F \approx F_1 \oplus \cdots \oplus F_n$ such that each F_i is a rank 1 torsion-free valuated group and, for each prime p, $v_p(x_1 + \cdots + x_n) = \min_i v_p(x_i)$ for $x_i \in F_i$.

Let B be a finite rank torsion-free group which carries a valuation v. We call it a *valuated B_2-group* if there is a finite rank completely decomposable valuated group $F = F_1 \oplus \cdots \oplus F_n$ (each F_i is a valuated rank 1 torsion-free group) such that B is isometric to the valuated factor group F/H for some pure subgroup H of F. Thus if $\phi : F \to B$ is the canonical map, then the elements $a_i = \phi x_i$, $(x_i \in F_i)$, generate B and valuation is given by $v(a) = \sup \min_i v(x_i)$, where the sup is taken for all possible representations of $a \in B$ as $a = \sum a_i$, $a_i \in \phi F_i$. Evidently, $B = \sum \phi F_i$ which is a valuated generation, that is, the valuation on B is induced by the valuations on the ϕF_i in the indicated way. From the definition of valuation in the image we can derive at once the formula

$$B(p, \alpha) = \sum \phi F_i(p, \alpha)$$

for each ordinal α, where we have used the notation $B(p,\alpha) = \{b \in B \mid v_p(b) \geq \alpha\}$.

The class of valuated B_2-groups is evidently closed under epimorphisms. That it is also closed under the formation of pure subgroups will follow from the next theorem.

THEOREM 3.1. *For a finite rank valuated torsion-free group A the following are equivalent*:

(i) *A is an epic image of a completely decomposable valuated torsion-free group of finite rank*;

(ii) *A is isometric to a pure subgroup of a completely decomposable valuated torsion-free group of finite rank.*

PROOF. (i) $\Rightarrow$ (ii) Let $A = \sum_{i=1}^{n} B_i$ with $n > 1$ be a sum of rank 1 valuated torsion-free pure subgroups B_i such that n is minimal for A. Define $X_i = A/B_i$, $i = 1, \ldots, n$, and consider the injection $\phi : A \to X_1 \oplus \cdots \oplus X_n$ induced by the canonical valuated maps $A \to X_i$. The X_i are valuated torsion-free groups of smaller rank with property (i), so if we show that this ϕ is value-preserving, then by induction we are done with the implication (i) $\Rightarrow$ (ii).

Assume to the contrary that there exist a prime p and an element $a \in A$ such that $v_p(a) = \alpha$, but $v_p(a + B_1, \ldots, a + B_n) > \alpha$, that is, $v_p(a + B_i) > \alpha$ for every i. Pick $b_i \in B_i$ such that $v_p(a - b_i) > \alpha$. The last inequality means that there are $c_{ij} \in B_j$ with $v_p(c_{ij}) > \alpha$ satisfying $a - b_i = \sum_{j=1}^{n} c_{ij}$. Clearly, $v_p(b_i) = \alpha$, so there are integers r_{ij}, s_{ij} such that $p \mid r_{ij}$, $p \nmid s_{ij}$ and $s_{ij}c_{ij} = r_{ij}b_j$ for all i, j. By modifying r_{ij} if necessary, we may without loss of generality assume that $s_{ij} = s$ is the same for all i, j. This leads to the equations

$$sa = \sum_{j=1}^{n} (\delta_{ij}s + r_{ij})b_j, \quad i = 1, \ldots, n,$$

where $\det \|\delta_{ij}s + r_{ij}\| \equiv s^n \pmod{p}$ with $\gcd(s, p) = 1$. Hence Cramer's rule yields that each b_j is linearly dependent on a, that is, A is of rank 1, a contradiction to $n > 1$. Repeating the above arguments with height valuation instead of the given valuation v (as is done in the theory of Butler groups), we get that the image of ϕ is pure in $X_1 \oplus \cdots \oplus X_n$.

(ii) $\Rightarrow$ (i) Assume A is a pure subgroup of the completely decomposable group $C = C_1 \oplus \cdots \oplus C_n$, where the components C_i are valuated torsion-free groups of rank 1. The theory of Butler groups tells us that if $B_1, \ldots, B_k$ are the pure subgroups of A with minimal supports in the direct sum, then they are of rank 1 and the canonical map $\psi : B = B_1 \oplus \cdots \oplus B_k \to A$ is surjective. Thus it remains to check that the induced valuation on $B/\operatorname{Ker}\psi$ coincides with the valuation in A.

Fix a prime p, and let $a \in A$ with $v_p(a) = \alpha$. We show that a has a preimage under ψ with p-value α. If a belongs to some B_j, then this is obvious. Otherwise,

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there is a $b_j \in B_j$ for some j such that $\operatorname{supp} b_j \subset \operatorname{supp} a$. Write $a = x_1 + \cdots + x_m$, $b_j = y_1 + \cdots + y_\ell$, $\ell < m$, with $x_i, y_i \in C_i$. As B_j is pure in A, $b_j \in B_j$ can be chosen so as to have $v_p(x_i) \le v_p(y_i)$ for all i with $y_i \ne 0$, but for at least one index, say i_0, we have $y_{i_0} = x_{i_0}$. Then $a - b_j \in A$ has a smaller support than a, so inducting on the size of the support, we argue that B contains an element b' such that $\psi(b') = a - b_j$ and $v_p(b') = v_p(a - b_j)$. By construction, $v_p(a - b_j) \ge \alpha$ and $v_p(b_j) \ge \alpha$, therefore

$$v_p(b' + b_j) \ge \min\{v_p(b'), v_p(b_j)\} = \min(v_p(a - b_j), v_p(b_j) \ge \alpha = v_p(a).$$

Since $\psi(b' + b_j) = a$, the reverse inequality is obvious, and our claim follows. □

The next example exhibits a finitely generated valuated group which is not a valuated B_2-group.

EXAMPLE 1. We define a valuated free group F of rank two as follows. As a group, we set $F = \langle a \rangle \oplus \langle b \rangle$. Let $\alpha_0 = 0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ be a strictly increasing sequence of ordinals — these will be the p-values of elements in F. The p-valuation v_p in F is given in terms of a transcendental p-adic unit π whose nth partial sum will be denoted by π_n. For non-negative integers k and n, set

$$v_p(p^k a) = \alpha_k, \quad v_p(p^k b) = \alpha_k, \quad v_p(p^k(a + \pi_n b)) = \alpha_{k+n}.$$

Furthermore, let $v_q(x) = \infty$ for all primes $q \ne p$ and all $x \in F$. It is straightforward to see that in this way every element of F will have a well-defined value. F cannot be embedded in a valuated free group as required by Theorem 3.1.

4. Balanced subgroups

Let A be a subgroup of the group G such that G/A is torsion-free of rank 1, and suppose that G is equipped with a valuation v. We will say that A is a (*valuated*) *balanced* subgroup in G if there exist a valuated torsion-free group X of rank 1 and an isometry $\phi : A \oplus X \to G$ with $\phi \mid A = \mathbf{1}_A$. In this case, every coset modulo A in G contains an element of maximal value-matrix.

If A is pure and not of corank one in G, then A is defined to be balanced in G if it is balanced in every pure subgroup C of G that contains A such that rank of C/A is 1. It is easily checked that balancedness is a transitive property.

An exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of valuated groups is said to be *balanced-exact* if C is torsion-free and $\operatorname{Im} \alpha$ is a balanced subgroup in B.

It is clear from the definition that a balanced-exact sequence of torsion-free valuated groups is a balanced-exact sequence of ordinary groups (if we ignore the valuations).

Standard proofs apply to verify the following two lemmas.

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LEMMA 4.1. *Let* $0 \to A \to B \xrightarrow{\alpha} C \to 0$ *be a valuated balanced-exact sequence where* C *is torsion-free. Given a* $\mathscr{V}$*-map* $\gamma : C' \to C$, *there is a pull-back diagram with balanced-exact top row:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B' & \xrightarrow{\beta} & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\alpha} & C & \longrightarrow & 0. \end{array}$$

If γ *induces an isometry between* $C'/\operatorname{Ker}\gamma$ *and* C, *then* η *induces an isometry between* $B'/\operatorname{Ker}\eta$ *and* B.

A sort of dual to this lemma is the following result (we state it for the sake of completeness, though we will not need it).

LEMMA 4.2. *Let* $0 \to A \to B \xrightarrow{\gamma} C \to 0$ *be a valuated balanced-exact sequence where* C *is torsion-free. For a* $\mathscr{V}$*-map* $\alpha : A \to A'$, *there is a push-out diagram with balanced-exact bottom row*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\gamma} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{\delta} & C & \longrightarrow & 0. \end{array}$$

If α *is an isometry of* A *with* $\operatorname{Im}\alpha$, *then* β *is an isometry of* B *with* $\operatorname{Im}\beta$.

5. Valuated B_1-groups

We now introduce the valuated analogue of what is called a B_1-group in the theory of Butler groups. (By the way, the results in this section are valid without restricting the ranks of the groups.) The definition is as follows.

A valuated torsion-free group B will be called a *valuated* B_1*-group* if every valuated balanced-exact sequence $0 \to T \to G \to B \to 0$ of valuated groups is splitting for every valuated torsion group T. Equivalently, if B has the projective property with respect to all valuated balanced-exact sequences of the mentioned type (that is, valuated extensions of torsion groups by torsion-free groups).

In view of the definition of balancedness, it is evident that all valuated rank 1 torsion-free groups are valuated B_1-groups. The same holds of course for their direct sums.

LEMMA 5.1. *Completely decomposable valuated groups, as well as their summands are valuated* B_1*-groups.*

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More generally, we can prove the following result that offers a sufficient condition for the B_1-property (compare Theorem 6.5).

If G is a valuated group, then its *localization* at a prime p is the group $G_p = G \otimes \mathbb{Z}_p$ (where $\mathbb{Z}_p$ denotes the localization of the ring $\mathbb{Z}$ at the prime p), whose valuation w is given as follows: for an element $a/r \in G_p$ ($a \in G$, $r \in \mathbb{Z}$, $\gcd(r, p) = 1$) and a prime q, $w_q(a/r) = v_q(a)$ or ∞ according as $q = p$ or $q \neq p$. G_p is a q-divisible group, so it has to carry the height-valuation for every prime $q \neq p$. We say G is *p-local* if it is identical with its own localization at p.

Similar definition of valuation applies if we localize at a set Π' of primes: the p-values are preserved for primes $p \in \Pi'$, and are reset as ∞ otherwise.

THEOREM 5.2. *A torsion-free valuated group B is a valuated B_1-group if there is a partition $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ of the set Π of prime numbers such that for each j, $j = 1, \ldots, k$, the tensor product $B \otimes \mathbb{Z}_{\Pi_j}$ is a completely decomposable valuated group.*

PROOF. Let $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ be a partition of primes such that $B \otimes \mathbb{Z}_{\Pi_h}$ is completely decomposable as a valuated group. If $0 \to T \to G \to B \to 0$ is a valuated balanced-exact sequence with T torsion, then, for every $j = 1, \ldots, k$, the sequence $0 \to T \otimes \mathbb{Z}_{\Pi_j} \to G \otimes \mathbb{Z}_{\Pi_j} \to B \otimes \mathbb{Z}_{\Pi_j} \to 0$ is also balanced-exact (under the induced valuations), so it splits. If $\phi_j : G \otimes \mathbb{Z}_{\Pi_j} \to T \otimes \mathbb{Z}_{\Pi_j}$ is a splitting map for the inclusion $T \otimes \mathbb{Z}_{\Pi_j} \to G \otimes \mathbb{Z}_{\Pi_j}$, then the composition

$$G \to \bigoplus_{j=1}^{k} (G \otimes \mathbb{Z}_{\Pi_j}) \xrightarrow{\phi} \bigoplus_{j=1}^{k} (T \otimes \mathbb{Z}_{\Pi_j}) = T$$

is a valuated splitting map for the inclusion $T \to G$, where $\phi = \phi_1 + \cdots + \phi_k$. □

We proceed with the discussion by comparing valuated B_1-groups to B_1-groups. Recall that a B_1-group is defined as a (torsion-free) group B satisfying

$$\operatorname{Bext}^1(B, T) = 0$$

for all torsion groups T, that is, all balanced extensions of torsion groups by B are splitting.

LEMMA 5.3. *A B_1-group with its height valuation is a valuated B_1-group.*

PROOF. Suppose B is a torsion-free B_1-group equipped with its height valuation, and G is a valuated extension of a valuated torsion group T by B. We have a direct sum decomposition $G = T \oplus B'$ as a group (that is, valuations are ignored for the time being) with $B' \cong B$. First, observe that B' as a subgroup of G must carry the height-valuation, because this is the minimal p-valuation. In order to see that this decomposition is a valuated decomposition, we have to show that $v_p(t, b') \leq \min(v_p(t), v_p(b'))$ for all $t \in T$, $b' \in B'$ (the reverse inequality being obvious). As the canonical homomorphism $G \to G/T$ maps (t, b') upon $b \in B$ (the element corresponding to $b' \in B'$) and the valuation in G/T must agree with the valuation in B, we clearly have $v_p(t, b') \leq v_p(b')$ for all $t \in T, b' \in B'$. In case $v_p(t) < v_p(b')$ we must have $v_p(t, b') = v_p(t)$, so also $v_p(t, b') \leq v_p(t)$. This completes the proof. □

In view of Bican [4], the preceding lemma is a special case of our Theorem 6.5 below. A sort of converse is our next result.

THEOREM 5.4. *Ignoring valuations, a valuated B_1-group is a B_1-group.*

PROOF. Assume B is a valuated B_1-group and T is any torsion group. Every extension G of T by the group B (so far no valuations) may be viewed as a valuated extension. Indeed, furnish T with the trivial valuation: $v_p(a) = \infty$ for all primes p and for all $a \in T$, keep the valuation of B, and for $g \in G$ define $v(g) = v(g + T)$, the latter value being computed in B. This way G becomes a valuated extension of T by B. Moreover, if the extension G is a balanced one, then it is a valuated balanced extension. Consequently, it splits. Hence, ignoring valuations, B is indeed a B_1-group. □

6. Valuated B_1-groups of finite rank

We now focus on finite rank valuated B_1-groups, and start with an improved version of Theorem 5.4. Though, as we shall see, the class of valuated B_2-groups is not identical with the class of valuated B_1-groups; we have an implication in one direction.

THEOREM 6.1. *A finite rank valuated B_1-group is a valuated B_2-group.*

PROOF. By virtue of Theorem 5.4, a valuated B_1-group is a B_1-group. As is well known from the theory of Butler groups, B_1-groups are B_2-groups.

In order to prove the valuated part of the claim, suppose by way of contradiction that the valuated B_1-group B is a Butler group of finite rank r, but not a valuated B_2-group, that is, it requires infinitely many pure subgroup generators to obtain its valuation, $B = \sum_{n<\omega} B_n$. Note that but finitely many of the B_n may carry their height valuation (since such B_n's do not contribute anything to the valuation, it suffices to keep only those needed to generate B as a B_2-group). Observe that if $v_p(b) > h_p(b)$ for some element $b \in B$, then also $v_p(p^k b) > h_p(p^k b)$ for all $k > 0$.

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Let A be an essential subgroup of B, and select subgroups $0 \neq K_n \subseteq A \cap B_n$ for $n < \omega$ such that there exist elements in $B_n \setminus K_n$ with higher than height-valuation whenever B_n contains such elements. Let $\phi_n : B_n \to B_n/K_n = C_n$ denote the canonical map. Clearly, A can be chosen such that B contains a maximal independent set $x_1, \ldots, x_r$, none of which is contained in A; we may moreover assume without loss of generality that $x_j \in B_j$, $j = 1, \ldots, r$.

Set $T = \bigoplus_{n<\omega} C_n$, a torsion group, and let B' be a group $\cong B$. Let $b'_n \in B'_n$ denote the element corresponding to $b_n \in B_n$ under a fixed isomorphism between B and B'. Define $G = T \oplus B'$ with the valuation induced by:

(1) T carries the height valuation;
(2) B' carries its height valuation;
(3) $v\big(\sum_{i=1}^n \phi_i(b_i), \sum_{i=1}^n b'_i\big) = v\big(\sum_{i=1}^n b_i\big)$ for all $b_n \in B_n$, $n < \omega$.

In this way, G becomes a valuated balanced-extension of T by the valuated B_1-group B. By hypothesis, this is a valuated splitting extension, so there exists a subgroup B^* in G isometric to B, such that $G = T \oplus B^*$ (valuated direct sum). Then using * to denote elements in B^* corresponding to those in B, note that changing b'_n to b^*_n requires the addition of a torsion element, say $b^*_n = b'_n + t_n$ with $t_n \in T$, where $t_n \neq 0$ for all $b_n \in B_n \setminus K_n$ for which b_n is not valuated by its height. But once $x^*_1 = b^*_1, \ldots, x^*_r = b^*_r$ have been fixed, all b^*_n are fixed by linear dependence relations. Since we have infinitely many groups C_j, we can choose an index j such that none of $t_1, \ldots, t_r$ has a coordinate in C_j. Then the linear combinations $b_j = \sum_{i=1}^r m_i x_i$ and $b^*_j = \sum_{i=1}^r m_i x^*_i$, $m_i \in \mathbb{Z}$, are contradictory, since they would imply $t_j = \sum_{i=1}^r m_i t_i$, which is clearly impossible, because t_j must have a non-zero coordinate in C_j. Hence B is indeed a valuated B_2-group. □

Now that we know that finite rank valuated B_1-groups are valuated B_2-groups, our task is to single out those valuated B_2-groups that are also valuated B_1-groups.

We start with p-local valuated groups.

Before entering into the discussion of the p-local case, we recall from the theory of Butler groups that if B is a Butler group, then its localization B_p at any prime p is a completely decomposable group: it is a direct sum of copies of $\mathbb{Z}_p$ and $\mathbb{Q}$. Thus we can write $B_p = A \oplus D$ where A is a free $\mathbb{Z}_p$-module and D is a divisible group. It should be emphasized that this is a valuated direct sum, since D is trivially valuated, so $v_q(a, d) = v_q(a)$ for all $a \in A$, $d \in D$ and all primes q.

Central to our discussion is the fact stated in the following theorem.

THEOREM 6.2. *A finite rank p-local valuated group A is a valuated B_1-group if and only if it is a completely decomposable valuated group.*

PROOF. Sufficiency is included in Lemma 5.1. To prove necessity, assume that A

is a valuated B_1-group. We know from Theorem 6.1 that it is a valuated B_2-group, so we can write $A = \sum_{i=1}^{n} B_i$ with rank 1 valuated pure groups B_i. As A is p-local, each B_i is isomorphic either to $\mathbb{Z}_p$ or to $\mathbb{Q}$.

Let $B_1, \dots, B_k$ denote those B_i's whose p-valuation is not the height valuation, and $B_{k+1}, \dots, B_n$ those which carry the height p-valuation (of course, the generator subgroups B_i isomorphic to $\mathbb{Q}$ are in the second set). Without loss of generality we may assume that none of $B_1, \dots, B_k$ can be omitted (that is, each of these contribute something to the valuation of A), and none of $B_{k+1}, \dots, B_n$ can be omitted. It is important to keep in mind that the pure subgroups B_i carry the same valuation as the one given in A.

Choose the integer m such that each of $B_1, \dots, B_k$ has elements not contained in $p^m A$ whose p-valuations exceed their p-heights (in particular, all the elements of $\left(\sum_{i=1}^{n} B_i\right) \cap p^m A$ have valuations exceeding their heights). Define T_j, $j = 1, \dots, k$ to be a cyclic p-group isomorphic to $B_j/(B_j \cap p^m A)$ with $\psi_j : B_j \to T_j$ as a fixed surjective map, and set $T = T_1 \oplus \cdots \oplus T_k$. Define the valuated group A' to have the same underlying group as A, but its valuation is induced by $B_1', \dots, B_k', B_{k+1}, \dots, B_n$, where B_j' is isomorphic to B_j, with no change in the valuations of elements in $p^m A$, but for elements not in $p^m A$ the p-valuation is just the height valuation. Define $G = T \oplus A'$ (pure group-theoretically), and furnish it with the valuation induced by:

(a) the height valuation of T;
(b) the indicated valuation of A';
(c) $v_p\left(\sum_{i=1}^{n} \psi_i(b_i), \sum_{i=1}^{n} b_i'\right) = v_p\left(\sum_{i=1}^{n} b_i\right)$ for all $b_j' \in B_j'$, $j \le n$,

where $b_j \in B_j$ and $b_j' \in B_j'$ denote corresponding elements under a fixed group isomorphism $A \cong A'$. It is readily checked that G is a valuated balanced-extension of T by A. As A was supposed to be a valuated B_1-group, there exists a valuated complement $A^* \approx A$ to T in G, $G = T \oplus A^*$ (valuated direct sum). Evidently, $A^* = \sum_{i=1}^{n} B_i^*$ with $B_i^* \approx B_i$ for each i, and none of the B_i^* can be omitted. There is a group isomorphism $\sigma_i : B_i' \to B_i^*$ (the restriction of $A' \to A$) such that corresponding elements are mapped upon the same element of B_i by the canonical map $G \to A$. For $j = 1, \dots, k$ we write $\sigma_j(b_j') - b_j' = \psi_j(b_j) + t_j$, where t_j is a torsion element such that $h_p(t_j) \ge v_p(\sigma_j(b_j')) = v_p(b_j) > h_p(\psi_j(b_j))$; the last inequality holds whenever b_j is not height-valued (compare p-values, taking (c) into consideration). By way of contradiction, suppose there is a dependence relation between $B_1', \dots, B_k'$, say $\sum_{j=1}^{k} b_j' = 0$, where $b_j' \in B_j'$. We may assume that also $b_j \notin p^m A$ and $v_p(b_j) > h_p(b_j)$ for all j, since if necessary we can increase m. Then $t_j \neq 0$, and also $\sum_{j=1}^{k} \sigma_j(b_j') = 0$, whence

$$\sum_{j=1}^{k} \psi_j(b_j) = -\sum_{j=1}^{k} t_j$$

follows. Here the terms on the left are independent and carry their height valuations, so the p-value on the left is the minimum of the p-heights of the $\psi_j(b_j)$, which is certainly less than any $v_p(t_j)$. We reached a contradiction. Consequently, $C = B_1 \oplus \cdots \oplus B_k$ must be a valuated direct sum.

Any subgroup with height valuation ought to be independent of C, because every non-zero element of $p^m C$ has valuation higher than its height. Thus, if we set $C^* = B_{k+1} + \cdots + B_n$, then we have a valuated direct sum decomposition $A = C \oplus C^*$. Here the reduced part of C^* is a free $\mathbb{Z}_p$-module with height valuation, so C^* is a completely decomposable valuated group. This completes the proof. □

We wish to record a consequence of the last proof which will be needed later on.

COROLLARY 6.3. *Let $A = \sum_{i=1}^n B_i$ be a p-local valuated B_1-group with rank 1 valuated pure subgroups B_i. There is a subset $\{B_1, \dots, B_\ell\}$ of the set $\{B_1, \dots, B_n\}$ of generator subgroups such that $A = B_1 \oplus \cdots \oplus B_\ell$ is a valuated direct sum.*

PROOF. It remains to show that C^* is a direct sum of some of the B_i. But this follows at once from Nakamaya's lemma, noting that $\mathbb{Z}_p$ is a local domain. □

The next lemma will be most helpful in reducing the global case to the local one.

LEMMA 6.4. *If B is a valuated B_1-group, then so is its localization B_p for every prime p.*

PROOF. In accordance with the above notation, we write $B_p = A \oplus D$, where A is a free $\mathbb{Z}_p$-module and D is a divisible group. As a divisible group, D is a trivially valuated completely decomposable group, so D is a valuated B_1-group. It remains to show that A is a valuated B_1-group. Since A is a q-divisible B_1-group for all primes $q \neq p$, only the balanced extensions of valuated p-groups by A need to be considered.

Let G be a valuated balanced extension of a valuated p-group T by B_p; the group B_p is now assumed to be reduced (that is, has no divisible subgroups $\neq 0$). Let σ denote the natural injection $B \to B_p$. This is a $\mathcal{V}$-map, so we can form the valuated pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \longrightarrow & T & \longrightarrow & G & \xrightarrow{\gamma} & B_p & \longrightarrow & 0. \end{array}$$

By Lemma 4.1, the top row is balanced-exact in the valuated sense, so in view of the hypothesis on B it splits. Hence there is a valuated map $\beta : B \to G$ such that $\gamma\beta = \sigma$. Consider the exact sequence

$$\mathrm{Hom}(B_p, G) \to \mathrm{Hom}(B, G) \to \mathrm{Ext}(B_p/B, G)$$

of abelian groups. Here B_p/B is a divisible torsion group whose p-component is 0, while G is q-divisible for every prime $q \neq p$. Observing that every extension of a q-divisible group by a q-group splits (see [7, page 223]), it follows that $\mathrm{Ext}(B_p/B, G) = 0$. Hence every group homomorphism from B to G extends to one from B_p to G. We conclude that β extends to a map $\alpha : B_p \to G$ which has to be a valuated map, since β was one (recall the definition of valuations in localizations). Thus $\mathrm{Im}\,\alpha \approx B_p$.

Manifestly, $T \cap \mathrm{Im}\,\beta = 0$ implies $T \cap \mathrm{Im}\,\alpha = 0$, thus $G = T \oplus \mathrm{Im}\,\alpha$ is a direct sum. To see that this is a valuated direct sum, note that for the p-valuation this holds, because β and hence α preserves p-values, while for q-values (for primes $q \neq p$) this is evident, since they are all ∞. □

It remains to characterize the valuated B_1-groups in the global case. This is our main result for valuated B_1-groups.

THEOREM 6.5. *A valuated torsion-free group B of finite rank is a valuated B_1-group if and only if there exists a partition $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ of the set Π of prime numbers such that the tensor product $B \otimes \mathbb{Z}_{\Pi_j}$, $j = 1, \ldots, k$, is a completely decomposable valuated group.*

PROOF. Theorem 5.2 takes care of the proof of sufficiency.

Turning our attention to the proof of necessity, suppose that $B = \sum_{i=1}^{n} B_i$ is a valuated B_1-group of rank r. From Lemma 6.4 we know that the localizations B_p of B at primes p are likewise B_1-groups, while Corollary 6.3 shows that B_p is the valuated direct sum of some of the localized generators B_i; of course, of exactly r of them. For every prime p, choose an r-element subset X of the set $\{B_1, \ldots, B_n\}$ of the generating groups such that the localization B_p is the valuated direct sum of the localized B_i in X, and define Π_X as the set of all primes p to which X is assigned. Let $\Pi_1, \ldots, \Pi_k$ be the list of the non-empty sets Π_X. Clearly, $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ is a partition of the set Π of primes. From the theory of torsion-free groups we know that if we consider, as we may, localizations as subgroups in the divisible hull, then we have $B = \bigcap_{j=1}^{k} (B \otimes \mathbb{Z}_{\Pi_j})$ (intersection in the divisible hull of B). Since the p-valuation of the elements of B in $B \otimes \mathbb{Z}_{\Pi_j}$ is either the same as in B or is ∞ according to whether p belongs to the class of partition corresponding to Π_j or not, this intersection yields the correct valuations of the elements of B ($v_p(a)$ in the intersection is defined as the infimum of the p-values of a in the components). That $B \otimes \mathbb{Z}_{\Pi_j}$ is a completely decomposable valuated group is evident from the definition. □

7. When valuated B_2-groups are valuated B_1-groups

Theorem 6.1 states that all valuated B_1-groups of finite rank are valuated B_2-groups. We now exhibit two examples to show that the converse implication fails in general.

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(For those familiar with the theory of Butler groups of arbitrary cardinality, this fact should be somewhat surprising, since the converse implication is expected to hold.)

Our examples are p-local valuated B_2-groups that fail to be valuated B_1-groups. Both examples are given as epimorphic images of completely decomposable valuated groups of finite rank.

EXAMPLE 2. We make the free group $F = \langle a \rangle \oplus \langle b \rangle$ with basis a, b into a valuated p-local group as follows (p is an arbitrary, but fixed prime). We assign the following p-values to $p^k a$, $p^k b$, $p^k(a+b)$, for $k = 0, 1, \dots$: $0, 2, 5, 6, 8, \dots; 0, 3, 4, 6, 9, \dots;$ and $1, 2, 4, 7, 8, \dots$, respectively (the increases are periodically $2, 3, 1; 3, 1, 2;$ and $1, 2, 3$). This is a valuated B_2-group, but not a valuated B_1-group, since the subgroups $\langle a \rangle$, $\langle b \rangle$, $\langle a+b \rangle$ are dependent, and F is not a completely decomposable valuated group (see Theorem 6.2). This F is an epic image of the valuated free group $F' = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$ where the summands are valuated in the same way as the subgroups $\langle a \rangle$, $\langle b \rangle$, $\langle a+b \rangle$ are in F.

In the second example, the generating subgroups are almost isometric.

EXAMPLE 3. Let p be a prime and A the epic image of the direct sum $B = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ modulo the pure subgroup $\langle x - y + z \rangle$ where the p-values of $p^k x$, $p^k y$, $p^k z$, for $k = 0, 1, \dots$, are $0, 3, 4, 6, 7, 8, \dots; 0, 2, 5, 6, 7, 8, \dots;$ and $1, 2, 4, 6, 7, 8, \dots$, respectively (thus, for example, $v_p(p^k x) = k + 3$ if $k \geq 3$), while the q-values at other primes q are ∞. If x^* and z^* denote the images of x and z in A, then it is easy to verify that x^* and x (similarly, z^* and z) have the same value-matrix. In order to see that $A = \langle a \rangle \oplus \langle b \rangle$ (valuated) is impossible for any choice of $a, b \in A$, note first that if this was a valuated decomposition, then one of a, b had p-value 1 and the other had p-value 0, say $v_p(a) = 1$ and $v_p(b) = 0$. It can then be seen that, for $k = 0, 1, 2$, $v_p(p^k a) = v_p(p^k z)$ and $v_p(p^k b) = v_p(p^k x)$. Furthermore, if $x^* = ka + \ell b$ and $z^* = ma + nb$ ($k, \ell, m, n \in \mathbb{Z}$), then we have $p \mid k$ and $\gcd(p, \ell) = 1 = \gcd(p, m)$. This yields that $p^2 x^* = p^2 z^* = (k + m)p^2 a + (\ell + n)p^2 b$ has p-value 4. But the element $p^2 x^* + p^2 z^* \in A$ is the image of $p^2 y \in B$, so its p-value has to be ≥ 5.

It is an obvious question: under what conditions is a valuated B_2-group a valuated B_1-group? We answer this question satisfactorily in the next theorem (all groups are of finite rank).

THEOREM 7.1. *A valuated B_2-group B of finite rank is a valuated B_1-group if and only if all of its localizations B_p at primes p are completely decomposable valuated groups.*

PROOF. By virtue of Theorem 6.5, the condition is necessary.

To prove sufficiency, write $B = \sum_{i=1}^{n} B_i$ with rank 1 pure subgroups B_i. Observe that the hypothesis implies that every localization B_p is a local B_1-group. Hence from Corollary 6.3 we deduce that for every prime p, there is a subset of $\{B_1, \dots, B_n\}$ such that B_p is the valuated direct sum of the localizations at p of the groups in this set. We now argue as in the proofs of Lemma 6.4 and Theorem 6.2 that there is a partition of the set of primes as required by this theorem. Consequently, B is a valuated B_1-group. □

By making use of this criterion, we can generalize Lemma 5.3 to valuated B_2-groups whose valuations are *gap-free* in the sense that

$$v_p(pa) = v_p(a) + 1$$

for all elements a and all primes p provided that $v_p(a) \neq \infty$.

COROLLARY 7.2. *A valuated B_2-group of finite rank with gap-free valuation is a valuated B_1-group.*

PROOF. By Theorem 7.1, it is enough to show that a p-local valuated Butler group B with gap-free valuation is completely decomposable. As the divisible part D of B is a completely decomposable summand with trivial valuation and $F = B/D$ is a finitely generated free $\mathbb{Z}_p$-module, we may restrict the proof to the case $B = F$.

Thus assume $B = \sum_{i=1}^{n} B_i$ with $B_i = \mathbb{Z}_p b_i$ for some $b_i \in B_i$. Here the b_i are of p-height 0; their values $v_p(b_i) = \alpha_i$ are assumed to satisfy $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Choose a maximal independent set $\{b_1, \dots, b_j\}$ of maximal value α_1, then extend this set to a maximal independent set $\{b_1, \dots, b_j, \dots, b_h\}$ with elements whose p-values are maximal among the non-maximal α_i, and keep going. At the end of this process we obtain an independent set $\{b_1, \dots, b_j, \dots, b_h, \dots, b_k\}$ that generates the free $\mathbb{Z}_p$-module B, since the b_i generate B/pB (argue with Nakayama's lemma), so $B = \bigoplus_{i=1}^{k} \mathbb{Z}_p b_i$. This is easily seen to be a valuated direct sum, due to the construction and the gap-free hypothesis. Thus B is a completely decomposable valuated group. □

In view of this corollary, we can claim that for valuated torsion-free groups of finite rank with gap-free valuation, conditions (1)–(4) listed in the introduction and rephrased for the valuated case are equivalent.

It is natural to raise the question whether or not pure subgroups and epimorphic images of finite rank B_1-groups are again of the same kind. The answer is in the negative: Example 3 is a counterexample. Indeed, the group A in this example is an epic image of a completely decomposable valuated p-local group B, but it is not completely decomposable, so by Theorem 6.2 it is not a valuated B_1-group. By Theorem 3.1, the group A is also a pure subgroup of a completely decomposable valuated group. Thus the class of valuated B_1-groups of finite rank is not closed either under taking epic images or under pure subgroups (but it is obviously closed under the formation of valuated direct summands).

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