Journal of the Inst. of Math. Jussieu (2007) **6**(4), 639–688 © Cambridge University Press 639
doi:10.1017/S1474748007000047 Printed in the United Kingdom

THE ALGEBRAIC THEORY OF TEMPERED REPRESENTATIONS OF p-ADIC GROUPS I. PARABOLIC INDUCTION AND RESTRICTION

PETER SCHNEIDER[1] AND ERNST-WILHELM ZINK[2]

[1] *Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany* (pschnei@math.uni-muenster.de)

[2] *Institut für reine Mathematik, Humboldt Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany* (zink@mathematik.hu-berlin.de)

(Received 1 February 2005; accepted 30 December 2005)

Dedicated to Joseph Bernstein, in admiration

Abstract In this paper we study the category of non-degenerate modules over the Harish-Chandra Schwartz algebra of a p-adic connected reductive group. We construct functors of parabolic induction and restriction and show that they are exact and both ways adjoint to each other.

Keywords: p-adic reductive group; Schwartz algebra; tempered representation

AMS 2000 *Mathematics subject classification:* Primary 22E50

Introduction

In the smooth representation theory of reductive groups G over nonarchimedean local fields from its very beginnings the parabolic induction functor played a paramount role for the construction of representations. But attention was largely restricted to admissible smooth representations. This finiteness condition was introduced by Harish-Chandra and allowed him to explore the harmonic analysis on G for the purposes of representation theory. An important example for this analytic approach to smooth representation theory is his notion of exponents (cf. [**Sil**, § 3.1]). Beginning with Jacquet and Casselman the emphasis was shifted to the more algebraic aspects. Through Jacquet's introduction of the parabolic restriction functors (often called Jacquet functors) it became possible, e.g. to characterize the analytic notions of exponents and of supercuspidal representations in an equivalent purely algebraic way. Parabolic induction and restriction are functors which through Frobenius reciprocity are adjoint to each other. But it was Bernstein who systematically freed the theory from the admissibility requirement. Most importantly he realized the existence of a second adjointness relation between parabolic induction and restriction. This seemingly formal statement in fact implies powerful finiteness properties of the category $\mathcal{M}(G)$ of all smooth G-representations.

In the Langlands classification of irreducible smooth representations as well as in many aspects of harmonic analysis (like trace formulae) Harish-Chandra's notion of tempered admissible smooth representations plays a crucial role. Having the power of Bernstein's algebraic approach in mind it seems natural to attempt an algebraic approach to tempered representations a well. This is the topic of the present paper. Since it is crucial to abandon the admissibility condition we first of all have to decide in which category we should work. The abelian category $\mathcal{M}(G)$ of smooth G-representations can equivalently be viewed as the category of non-degenerate modules over the Hecke algebra $\mathcal{H} = \mathcal{H}(G)$ of locally constant and compactly supported functions on G. This algebra $\mathcal{H}$ is contained in Harish-Chandra's Schwartz algebra $\mathcal{S} = \mathcal{S}(G)$ of uniformly locally constant and rapidly decreasing functions on G. It is known (cf. [**SSZ**, Appendix]) that on a tempered admissible representation the $\mathcal{H}$-module structure extends uniquely to an $\mathcal{S}$-module structure. It therefore seems clear that we will work in a category of $\mathcal{S}$-modules. But there is a subtle point. The algebra $\mathcal{S}$ naturally comes as a topological algebra and the $\mathcal{S}$-module structure on a tempered admissible representation is continuous in a certain sense. For this reason one might be tempted to use the category of continuous $\mathcal{S}$-modules. But this would not be a truly algebraic approach. Our point in this paper is to view $\mathcal{S}$ as an abstract algebra and to study the abelian category $\mathcal{M}^t(G)$ of all non-degenerate $\mathcal{S}$-modules. There is an obvious forgetful functor from $\mathcal{M}^t(G)$ to $\mathcal{M}(G)$.

It quickly turns out that, given any parabolic subgroup $P = MN$ of G with Levi component M and unipotent radical N, the parabolic induction functor from $\mathcal{M}(M)$ to $\mathcal{M}(G)$ directly lifts to a functor from $\mathcal{M}^t(M)$ to $\mathcal{M}^t(G)$, i.e. for any $\mathcal{S}(M)$-module its parabolic induction as an $\mathcal{H}(M)$-module carries a natural $\mathcal{S}(G)$-module structure. In particular, this lifted functor again is exact. It also is relatively formal to see that it has a left adjoint functor from $\mathcal{M}^t(G)$ to $\mathcal{M}^t(M)$ which we call the tempered parabolic restriction functor. This new restriction functor definitively is not compatible with the parabolic restriction from $\mathcal{M}(G)$ to $\mathcal{M}(M)$ under the forgetful functor. Moreover, from its construction it only appears to be right exact. Most of our paper is devoted to understanding this tempered restriction functor. In doing so we have to remember that our algebra $\mathcal{S}(G)$ is a topological algebra. This allows us to use spectral theory in order to analyse the so to speak universal case, i.e. the tempered parabolic restriction of $\mathcal{S}(G)$ itself as a left $\mathcal{S}(G)$-module. Technically Bushnell's reinterpretation in [**Bus**] of Bernstein's results in terms of localization methods will turn out to be a very useful guiding principle. We will have to work with some form of completed localization of course. As an input for our spectral analysis we use what is known about the structure of the Jacquet modules of tempered irreducible representations. As an output we obtain the exactness of our tempered parabolic restriction functor as well as an analogue of Bernstein's second adjointness.

In a second part of this paper we plan to apply these results in order to construct explicit projective generators with good additional properties for the category $\mathcal{M}^t(G)$ like Bernstein did for $\mathcal{M}(G)$. We emphasize that we do not make any use of the Plancherel isomorphism (cf. [**Wal**]) in this paper. It is conceivable that our results can actually be used to give an algebraic proof of this isomorphism.

We briefly describe now the content of the five sections of this paper. The first section is devoted to the properties of a very basic map. Whenever $P = MN$ is a parabolic subgroup of G it is well known that averaging over N gives rise to a continuous map from $\mathcal{S}(G)$ to $\mathcal{S}(M)$. In the later construction of our functors this map plays an absolutely crucial technical role. In the second section we construct the tempered parabolic induction functor $\mathrm{Ind}_P^G : \mathcal{M}^t(M) \to \mathcal{M}^t(G)$. In the third section we show that Ind_P^G has a left adjoint functor $r^t_{G,P} : \mathcal{M}^t(G) \to \mathcal{M}^t(M)$, the tempered parabolic restriction functor. We also show the formula $r^t_{G,P}(\cdot) = r^t_{G,P}(\mathcal{S}(G))\otimes_{\mathcal{S}(G)} \cdot$ which allows us to reduce the investigation of this functor to an analysis of the universal case $r^t_{G,P}(\mathcal{S}(G))$. In the central section four we express the $(\mathcal{S}(M), \mathcal{S}(G))$-bimodule $r^t_{G,P}(\mathcal{S}(G))$ in terms of topological tensor products and then use spectral theory to determine it explicitly. In the final section five we apply the acquired knowledge to obtain the exactness of the functor $r^t_{G,P}$, the second adjointness relation which says that the restriction functor $r^t_{G,\bar{P}}$ for the opposite parabolic subgroup $\bar{P}$ is right adjoint to Ind_P^G, and the compatibility of the tempered restriction functors with smooth duality.

Notation

Throughout this paper k is a locally compact nonarchimedean field with absolute value $|\cdot|_k$. Let G be the group of k-rational points of a connected reductive k-group. As usual we denote by $\mathcal{H} = \mathcal{H}(G)$ and $\mathcal{S} = \mathcal{S}(G)$ the Hecke and Schwartz algebra of G, respectively. The category of non-degenerate $\mathcal{H}$-modules, respectively $\mathcal{S}$-modules, is denoted by $\mathcal{M}(G)$, respectively $\mathcal{M}^t(G)$. We have the forgetful functor $\mathcal{M}^t(G) \to \mathcal{M}(G)$. The second category $\mathcal{M}(G)$ coincides with the category of all smooth G-representations. The multiplication in each of these two algebras as well as their action on a module always is denoted by a $*$ (for convolution). As a general convention we write the left and right translation action of a $g \in G$ on any locally constant function ϕ on G as $({}^g\phi)(h) := \phi(g^{-1}h)$ and $(\phi^g)(h) := \phi(hg^{-1})$.

For any compact open subgroup $U \subseteq G$ we let $\mathcal{H}(G, U)$, respectively $\mathcal{S}(G, U)$, denote the subalgebra of all U-bi-invariant functions in $\mathcal{H}(G)$, respectively $\mathcal{S}(G)$. Both these algebras are unital with the unit being the idempotent $\epsilon_U(g) = \mathrm{vol}_G(U)^{-1}$, respectively $\epsilon_U(g) = 0$, for $g \in U$, respectively $g \notin U$, corresponding to U. For any unital ring R we let $\mathcal{M}(R)$ be the category of left unital R-modules. The map $g \mapsto g^{-1}$ on G induces on any of the rings $\mathcal{H}(G, U)$ and $\mathcal{S}(G, U)$ a canonical anti-involution so that, for these rings, we do not have to distinguish between left and right modules.

1. The map $\psi \mapsto \psi^P$

We fix a parabolic subgroup $P \subseteq G$ together with a Levi decomposition $P = MN$ where M is the Levi subgroup and N is the unipotent radical of P. Let $\delta = \delta_P$ denote the modulus character of P (in the sense of [**B-INT**, VII.1.3]). Since there exist different conventions in the literature we recall that

$$\delta(mn) = |\det(\mathrm{ad}(m); \mathrm{Lie}(N))|_k^{-1} \quad \text{for any } m \in M,\ n \in N,$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

where $|\cdot|_k$ denotes the normalized absolute value of the field k. (To avoid confusion we warn the reader that in [**Car**] and [**Sil**] our δ is denoted by δ^{-1}.)

A crucial technical tool in this paper is the map

$$\mathcal{H}(G) \to \mathcal{H}(M),$$
$$\phi \mapsto \phi^P(m) := \delta^{-1/2}(m) \cdot \int_N \phi(mn)\,\mathrm{d}n.$$

By [**Sil**, 4.3.20] it extends continuously to a map

$$\mathcal{S}(G) \to \mathcal{S}(M),$$
$$\psi \mapsto \psi^P(m) := \delta^{-1/2}(m) \cdot \int_N \psi(mn)\,\mathrm{d}n.$$

For the convenience of the reader we begin by recalling the well known basic properties of this map.

Lemma 1.1. *$({}^{mn}\psi)^P = \delta^{-1/2}(m) \cdot {}^m(\psi^P)$ and $(\psi^{nm})^P = \delta^{1/2}(m) \cdot (\psi^P)^m$ for any $m \in M$, $n \in N$, and $\psi \in \mathcal{S}(G)$.*

Proof. Straightforward. □

In addition we now fix a special, good, maximal compact subgroup $K \subseteq G$ (this notion depends on M; cf. [**Car**, 3.5] or [**Sil**, § 0.6]). Moreover, from now on we always normalize the Haar measure on any of the unimodular groups G, K, M, N by the requirement that the intersection of the respective group with K has volume one (cf. [**Car**, 4.1]).

Lemma 1.2. *For any $\phi, \psi \in \mathcal{S}(G)$ we have*

$$\int_K (\phi^{k^{-1}})^P * ({}^k\psi)^P\,\mathrm{d}k = (\phi * \psi)^P.$$

Proof. We compute

$$\begin{aligned}
&\int_K [(\phi^{k^{-1}})^P * ({}^k\psi)^P](m')\,\mathrm{d}k \\
&\quad = \int_K \int_M (\phi^{k^{-1}})^P(m) \cdot ({}^k\psi)^P(m^{-1}m')\,\mathrm{d}m\,\mathrm{d}k \\
&\quad = \int_K \int_M \delta^{-1/2}(m) \cdot \int_N \phi(mnk)\,\mathrm{d}n \cdot \delta^{-1/2}(m^{-1}m') \cdot \int_N \psi(k^{-1}m^{-1}m'n')\,\mathrm{d}n'\,\mathrm{d}m\,\mathrm{d}k \\
&\quad = \int_K \delta^{-1/2}(m') \int_M \int_N \phi(mnk) \cdot \int_N \psi(k^{-1}m^{-1}m'n')\,\mathrm{d}n'\,\mathrm{d}n\,\mathrm{d}m\,\mathrm{d}k \\
&\quad = \int_K \delta^{-1/2}(m') \int_P \phi(pk) \int_N \psi(k^{-1}p^{-1}m'n')\,\mathrm{d}n'\,\mathrm{d}p\,\mathrm{d}k
\end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

$$= \delta^{-1/2}(m') \int_N \int_K \int_P \phi(pk)\psi((pk)^{-1}m'n')\, dp\, dk\, dn'$$
$$= \delta^{-1/2}(m') \int_N \int_G \phi(g)\psi(g^{-1}m'n')\, dg\, dn'$$
$$= \delta^{-1/2}(m') \int_N (\phi * \psi)(m'n')\, dn' = (\phi * \psi)^P(m').$$

□

Let $U \subseteq G$ be a compact open subgroup. Whenever $H \subseteq G$ is some closed subgroup we write $U_H := U \cap H$.

Lemma 1.3.

(i) *Under the map* $\psi \mapsto \psi^P$ *the algebra* $\mathcal{S}(G, U)$ *is mapped into* $\mathcal{S}(M, U_M)$.

(ii) *If* $U_P = U_M U_N$ *then* $\epsilon_U^P = \mathrm{vol}_P(U_P)\, \mathrm{vol}_G(U)^{-1} \cdot \epsilon_{U_M}$.

Proof. (i) If ψ is U-bi-invariant then the U_M-bi-invariance of ψ^P is an immediate consequence of Lemma 1.1 using that δ is trivial on U_M. (ii) This is a simple computation. □

The map $(\cdot)^P : \mathcal{H}(G) \to \mathcal{H}(M)$ is surjective: By Lemma 1.3 (ii) we have enough idempotents, which generate $\mathcal{H}(M)$ as an M-representation, in the image; but by Lemma 1.1 this image is M-invariant. We now will present an argument due to Waldspurger which shows that the extended map $(\cdot)^P : \mathcal{S}(G) \to \mathcal{S}(M)$ is surjective as well.

Let $\bar{P} \subseteq G$ denote the parabolic subgroup opposite to P with respect to M. If $\bar{N}$ is the unipotent radical of $\bar{P}$ then $\bar{P}$ has the Levi decomposition $\bar{P} = M\bar{N}$. The modulus character $\bar{\delta}$ of $\bar{P}$ satisfies $\bar{\delta} \mid M = \delta^{-1} \mid M$. In the following we fix a compact open subgroup $U \subseteq G$ which is normal in K and which is totally decomposed (with respect to a minimal Levi subgroup contained in M) in the sense of [**Bus**, § 1.1]. By [**Bus**, § 1.1, Proposition 1] we then have in particular the decomposition

$$U = U_{\bar{N}} U_M U_N.$$

It is shown in [**Bus**, § 1.2] that there exist a fundamental system of such subgroups. We also introduce the function

$$\xi_{N,U}(m) := \mathrm{vol}_N(m^{-1} U_N m U_N)$$

on M.

Lemma 1.4. *For any* $m \in M$ *we have*

$$M \cap UmU = U_M m U_M \quad \text{and} \quad N \cap m^{-1}UMU = m^{-1}U_N m U_N.$$

Proof (see also [BK, Lemma 6.10]). Let $m' = u_1 m u_2 \in M \cap UmU$ and write $u_1 = n_1 \bar{n}_1 m_1$, $u_2 = \bar{n}_2 m_2 n_2$ with $n_i \in U_N$, $\bar{n}_i \in U_{\bar{N}}$, and $m_i \in U_M$ for $i = 1, 2$. Setting $\bar{n} := \bar{n}_1 m_1 m \bar{n}_2 m^{-1} m_1^{-1}$, $x := m_1 m m_2$, and $n := n_2 m'^{-1} n_1 m'$ we have $\bar{n} x n = m'$.

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

But $\bar{n} \in \bar{N}$, $n \in N$, and $x, m' \in M$, hence $n = \bar{n} = 1$ and $x = m'$. It follows that $m' \in U_M m U_M$ which proves the first identity.

For the second one let $mn' \in UMU$ with $n' \in N$. In the same way as above (but interchanging m and m') we obtain $\bar{n}xn = m$ with $\bar{n} := \bar{n}_1 m_1 m' \bar{n}_2 m'^{-1} m_1^{-1}$, $x := m_1 m' m_2$, and $n := n_2 n'^{-1} m^{-1} n_1 m$. We must have $n = 1$ and hence $n' \in m^{-1} U_N m U_N$. □

Whenever $C \subseteq G$ is a closed U-bi-invariant subset we let $\mathcal{S}(G, U)_C$ denote the subspace in $\mathcal{S}(G, U)$ of all functions supported on C.

Lemma 1.5. *For $\psi \in \mathcal{S}(G, U)_{UMU}$ we have*

$$\psi^P(m) = \delta^{-1/2}(m)\xi_{N,U}(m)\psi(m) \quad \textit{for any } m \in M.$$

Proof. By assumption we have, for $m \in M$ and $n \in N$, that $\psi(mn) \neq 0$ if and only if $mn \in UMU$ which by Lemma 1.4 is equivalent to $n \in m^{-1}UMU \cap N = m^{-1}U_N m U_N$. Furthermore, the U-bi-invariance of ψ then implies that $\psi(mn) = \psi(m)$. Hence we may compute

$$\psi^P(m) = \delta^{-1/2}(m) \int_{m^{-1}U_N m U_N} \psi(m)\, \mathrm{d}n = \delta^{-1/2}(m)\xi_{N,U}(m)\psi(m).$$

□

The above lemma shows in particular that the restricted map

$$(\cdot)^P : \mathcal{S}(G, U)_{UMU} \to \mathcal{S}(M, U_M)$$

is injective. In order to show that it is, in fact, bijective we first need information about the growth of the function $\xi_{N,U}$. We let $\varXi_G$ and $\varXi_M$ denote the Harish-Chandra $\varXi$-function of G and M, respectively. We fix a scale function σ on G and use $\sigma \mid M$ as a scale function on M. The topology on $\mathcal{S}(G)$ and $\mathcal{S}(M)$ is given by the seminorms

$$\nu_{M,r}(\phi) = \sup_{m \in M} |\phi(m)| \varXi_M(m)^{-1}(1 + \sigma(m))^r$$

and

$$\nu_{G,s}(\psi) = \sup_{g \in G} |\psi(g)| \varXi_G(g)^{-1}(1 + \sigma(g))^s$$

with real numbers $r, s > 0$, respectively (cf. [**Sil**, Chapter 4]).

Lemma 1.6. *There is an $n_0 \in \mathbb{N}$ and a constant $c > 0$ such that*

$$\delta^{1/2}(m)\xi_{N,U}(m)^{-1}\varXi_M(m) \leqslant c\varXi_G(m)(1 + \sigma(m))^{n_0} \quad \textit{for any } m \in M.$$

Proof. According to [**Sil**, 4.1.1, 4.2.1] (or [**Wal**, I.1.(5), II.1.1]) there exist $r_G \in \mathbb{N}$ and constants $c_1, c_2 > 0$ such that

$$c_1 \leqslant \mathrm{vol}_G(KgK)^{1/2}\varXi_G(g) \leqslant c_2(1 + \sigma(g))^{r_G}$$

for any $g \in G$. A corresponding formula holds for M. Hence our assertion is implied by the inequality

$$\xi_{N,U}(m) \geqslant c \cdot \delta^{1/2}(m) \cdot \left(\frac{\mathrm{vol}_G(KmK)}{\mathrm{vol}_M((K \cap M)m(K \cap M))}\right)^{1/2}$$

or equivalently

$$\mathrm{vol}_N(m^{-1}U_N m U_N)^2 \cdot \mathrm{vol}_M((K \cap M)m(K \cap M)) \geqslant c \cdot \delta(m) \cdot \mathrm{vol}_G(KmK)$$

for any $m \in M$. Up to changing the constant c the latter is the same as

$$\mathrm{vol}_N(m^{-1}U_N m U_N)^2 \cdot \mathrm{vol}_M(U_M m U_M) \geqslant c \cdot \delta(m) \cdot \mathrm{vol}_G(UmU). \tag{1.1}$$

We have

$$\begin{aligned}
\mathrm{vol}_G(UmU) &= \mathrm{vol}_G(m^{-1}UmU) = \mathrm{vol}_G(m^{-1}Um)\,\mathrm{vol}_G(U)\,\mathrm{vol}_G(m^{-1}Um \cap U)^{-1},\\
\mathrm{vol}_M(U_M m U_M) &= \mathrm{vol}_M(m^{-1}U_M m)\,\mathrm{vol}_M(U_M)\,\mathrm{vol}_M(m^{-1}U_M m \cap U_M)^{-1},\\
\mathrm{vol}_N(m^{-1}U_N m U_N) &= \mathrm{vol}_N(m^{-1}U_N m)\,\mathrm{vol}_N(U_N)\,\mathrm{vol}_N(m^{-1}U_N m \cap U_N)^{-1}.
\end{aligned}$$

The latter formula of course has a counterpart for $\bar{N}$. Using these formulae as well as the decomposition $U = U_{\bar{N}} U_M U_N$ we see that (1.1) is equivalent to

$$\frac{\mathrm{vol}_N(m^{-1}U_N m)\,\mathrm{vol}_N(U_N)}{\mathrm{vol}_N(m^{-1}U_N m \cap U_N)} \cdot \left(\frac{\mathrm{vol}_{\bar{N}}(m^{-1}U_{\bar{N}} m)\,\mathrm{vol}_{\bar{N}}(U_{\bar{N}})}{\mathrm{vol}_{\bar{N}}(m^{-1}U_{\bar{N}} m \cap U_{\bar{N}})}\right)^{-1} \geqslant c \cdot \delta(m). \tag{1.2}$$

Since

$$\mathrm{vol}_N(m^{-1}U_N m) = \delta(m)\,\mathrm{vol}_N(U_N)$$

and

$$\mathrm{vol}_{\bar{N}}(m^{-1}U_{\bar{N}} m) = \delta_{\bar{P}}(m)\,\mathrm{vol}_{\bar{N}}(U_{\bar{N}}) = \delta^{-1}(m)\,\mathrm{vol}_{\bar{N}}(U_{\bar{N}})$$

the formula (1.2) simplifies, again allowing a change of the constant c, to

$$\frac{\mathrm{vol}_{\bar{N}}(m^{-1}U_{\bar{N}} m \cap U_{\bar{N}})}{\mathrm{vol}_N(m^{-1}U_N m \cap U_N)} \geqslant c \cdot \delta^{-1}(m). \tag{1.3}$$

We now fix a minimal Levi subgroup $M_0 \subseteq M$ of G as well as a minimal parabolic subgroup $P_0 = M_0 N_0 \subseteq P$ with unipotent radical N_0. We also fix a maximal split torus A_0 in M_0 and we let Σ denote the set of reduced roots of A_0 in $\mathrm{Lie}(G)$. Corresponding to any $\alpha \in \Sigma$ we have the root subgroup $N_\alpha \subseteq G$. We put $U_\alpha := U_{N_\alpha} = U \cap N_\alpha$ and $\delta_\alpha(a) := |\det(\mathrm{ad}(a); \mathrm{Lie}(N_\alpha))|_k^{-1}$ for any $a \in A_0$. Let $\Sigma^{M,+}$, respectively Σ^N, denote the subset of all roots $\alpha \in \Sigma$ such that $N_\alpha \subseteq M \cap P_0$, respectively $N_\alpha \subseteq N$. According to [**Cas**, Proposition 1.4.6] (or [**Wal**, I.1.(4)]) there is a finite subset $\Gamma \subseteq M$ such that

$$M = U_M \Gamma A_0 \Gamma U_M.$$

Hence the equivalent inequalities (1.1)–(1.3) need only to be shown for $m = a \in A_0$. Since U is assumed to be totally decomposed with respect to M_0 we have

$$a^{-1}U_N a \cap U_N = \prod_{\alpha\in\Sigma^N} (a^{-1}U_\alpha a \cap U_\alpha)$$

and

$$a^{-1}U_{\bar N} a \cap U_{\bar N} = \prod_{\alpha\in\Sigma^N} (a^{-1}U_{-\alpha} a \cap U_{-\alpha})$$

for any $a \in A_0$ (with an arbitrarily chosen but fixed total order on Σ^N). Hence, finally, (1.3) reduces to the inequality

$$\prod_{\alpha\in\Sigma^N} \frac{\mathrm{vol}_{N_{-\alpha}}(a^{-1}U_{-\alpha}a \cap U_{-\alpha})}{\mathrm{vol}_{N_\alpha}(a^{-1}U_\alpha a \cap U_\alpha)} \geqslant c \cdot \delta^{-1}(a) \tag{1.4}$$

for any $a \in A_0$. But there is a constant $c_3 > 0$ such that

$$[U_\alpha : (a^{-1}U_\alpha a \cap U_\alpha)] \leqslant c_3$$

if $|\alpha(a)| \leqslant 1$ and

$$[a^{-1}U_\alpha a : (a^{-1}U_\alpha a \cap U_\alpha)] \leqslant c_3$$

if $|\alpha(a)| \geqslant 1$. Using in addition that

$$\mathrm{vol}_{N_\alpha}(a^{-1}U_\alpha a) = \delta_\alpha(a)\,\mathrm{vol}_{N_\alpha}(U_\alpha)$$

it follows that the left-hand side in (1.4) can be replaced by

$$\prod_{\alpha\in\Sigma^N,\, |\alpha(a)|\neq 1} \delta_\alpha^{-1}(a).$$

Since $\delta(a) = \prod_{\alpha\in\Sigma^N} \delta_\alpha(a)$ the inequality (1.4) therefore certainly holds if

$$\delta_\alpha(a) = 1 \text{ for any } a \in A_0 \text{ and } \alpha \in \Sigma^N \text{ such that } |\alpha(a)|_k = 1.$$

But by the structure of the root subgroups N_α there are natural numbers $m(\alpha) \in \mathbb{N}$ such that

$$\delta_\alpha = |\alpha|_k^{-m(\alpha)}.$$

□

Lemma 1.7. *The restricted map*

$$(\cdot)^P : \mathcal{S}(G,U)_{UMU} \xrightarrow{\cong} \mathcal{S}(M,U_M)$$

is a linear topological isomorphism.

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

Proof. It suffices to show that the map $s^W = s^W_{N,U} : \mathcal{S}(M, U_M) \to \mathcal{S}(G, U)$ given by

$$s^W(\phi)(g) := \begin{cases} \delta^{1/2}(m)\xi_{N,U}(m)^{-1}\phi(m) & \text{if } g \in UmU,\ m \in M, \\ 0 & \text{otherwise} \end{cases}$$

is well defined and continuous. By the first formula in Lemma 1.4 the function $s^W(\phi)$ is U-bi-invariant. As a consequence of Lemma 1.6 we have

$$\nu_{G,s}(s^W(\phi)) \leqslant c\nu_{M,s+n_0}(\phi)$$

for any real number $s > 0$. □

In the limit with respect to U we obtain from Lemma 1.7 the surjectivity of the map $(\cdot)^P : \mathcal{S}(G) \to \mathcal{S}(M)$. In the rest of this section we analyse the algebraic properties of the map

$$s^W = s^W_{N,U} : \mathcal{S}(M, U_M) \to \mathcal{S}(G, U)$$

constructed in the above proof. Because of Lemma 1.3 (ii) it has the property that $\mathrm{vol}_{\bar{N}}(U_{\bar{N}})^{-1} \cdot s^W(\epsilon_{U_M}) = \epsilon_U$. It therefore is natural to introduce the renormalized map

$$s = s_{N,U} := \mathrm{vol}_{\bar{N}}(U_{\bar{N}})^{-1} \cdot s^W_{N,U}.$$

An element $m \in M$ is called *positive* for (P, U) if

$$mU_Nm^{-1} \subseteq U_N \quad \text{and} \quad m^{-1}U_{\bar{N}}m \subseteq U_{\bar{N}}.$$

The subset $M^+ = M^+_P \subseteq M$ of all elements which are positive for (P, U) clearly is multiplicatively closed and contains $K \cap M$. It follows that

$$\mathcal{H}^+(M, U_M) := \mathcal{H}(M, U_M)_{M^+} \quad \text{and} \quad \mathcal{S}^+(M, U_M) := \mathcal{S}(M, U_M)_{M^+}$$

are subalgebras of $\mathcal{H}(M, U_M)$ and $\mathcal{S}(M, U_M)$, respectively, with the same unit element.

Lemma 1.8.

(i) *The map* $s : \mathcal{S}^+(M, U_M) \to \mathcal{S}(G, U)$ *is an injective and continuous unital ring homomorphism.*

(ii) *For any* $\phi \in \mathcal{S}^+(M, U_M)$ *and* $\psi \in \mathcal{S}(G, U)$ *we have*

$$(s(\phi) * \psi)^P = \phi * \psi^P.$$

Proof. (i) Since $\xi_{N,U}(m) = \delta(m)\,\mathrm{vol}_N(U_N)$ for $m \in M^+$, we have for $\phi \in \mathcal{S}^+(M, U_M)$ the formula

$$s(\phi)(g) := \begin{cases} \mathrm{vol}_M(U_M)\,\mathrm{vol}_G(U)^{-1}\delta^{-1/2}(m)\phi(m) & \text{if } g \in UmU,\ m \in M^+, \\ 0 & \text{otherwise.} \end{cases}$$

In view of this formula it is shown in [**BK**, Corollary 6.12] that the restriction of s to the Hecke algebra $\mathcal{H}^+(M, U_M)$ is multiplicative. From this the assertion follows by continuity.

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

(ii) For any $m_0 \in M$ we compute

$$\begin{aligned}(s(\phi) * \psi)^P(m_0) &= \delta^{-1/2}(m_0) \int_N (s(\phi) * \psi)(m_0 n)\,\mathrm{d}n \\ &= \delta^{-1/2}(m_0) \int_N \int_G s(\phi)(g)\psi(g^{-1}m_0 n)\,\mathrm{d}g\,\mathrm{d}n \\ &= \delta^{-1/2}(m_0) \int_{UM^+U} \int_N s(\phi)(g)\psi(g^{-1}m_0 n)\,\mathrm{d}n\,\mathrm{d}g.\end{aligned}$$

Observe that $UM^+U = U_N M^+ U$ and that, for $g \in U_N m U$, $g = u_1 m u_2$ with $m \in M^+$, $u_1 \in U_N$, and $u_2 \in U$, we have

$$\begin{aligned}\int_N s(\phi)(g)\psi(g^{-1}m_0 n)\,\mathrm{d}n &= \frac{\mathrm{vol}_M(U_M)}{\mathrm{vol}_G(U)}\delta^{-1/2}(m)\phi(m)\int_N \psi(m^{-1}m_0 n)\,\mathrm{d}n \\ &= \frac{\mathrm{vol}_M(U_M)}{\mathrm{vol}_G(U)}\delta^{-1}(m)\delta^{1/2}(m_0)\phi(m)\psi^P(m^{-1}m_0).\end{aligned}$$

Hence

$$(s(\phi) * \psi)^P(m_0) = \sum_{m \in U_N \backslash U_N M^+ U/U} \mathrm{vol}_G(U_N m U)\frac{\mathrm{vol}_M(U_M)}{\mathrm{vol}_G(U)}\delta^{-1}(m)\phi(m)\psi^P(m^{-1}m_0).$$

For positive m we have

$$\begin{aligned}\mathrm{vol}_G(U_N m U)\,\mathrm{vol}_G(U)\delta^{-1}(m) &= [U_N m U : U]\delta^{-1}(m) \\ &= [U_N : U_N \cap mUm^{-1}]\delta^{-1}(m) \\ &= [U_N : mU_N m^{-1}]\delta^{-1}(m) \\ &= 1.\end{aligned}$$

Moreover, the first part of the proof of Lemma 1.4 shows that

$$M^+/U_M \xrightarrow{\sim} U_N \backslash U_N M^+ U/U.$$

It follows that

$$\begin{aligned}(s(\phi) * \psi)^P(m_0) &= \sum_{m \in M^+/U_M} \mathrm{vol}_M(U_M)\phi(m)\psi^P(m^{-1}m_0) \\ &= \int_{M^+} \phi(m)\psi^P(m^{-1}m_0)\,\mathrm{d}m \\ &= \int_M \phi(m)\psi^P(m^{-1}m_0)\,\mathrm{d}m \\ &= (\phi * \psi^P)(m_0).\end{aligned}$$

□

Lemma 1.8 says that the map $(\cdot)^P$ is a homomorphism of $\mathcal{S}^+(M, U_M)$-modules where the module structure of $\mathcal{S}(G, U)$ is the one induced by the map s.

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

2. Parabolic induction for $\mathcal{S}$-modules

We keep all the notation introduced in the previous section. In particular $P = MN$ is a fixed parabolic subgroup and K denotes our fixed maximal compact subgroup. Parabolic induction (in the normalized sense) is the functor

$$\mathrm{Ind}_P^G : \mathcal{M}(M) \to \mathcal{M}(G)$$

given as follows. For any smooth M-representation E one defines

$$\begin{aligned}\mathrm{Ind}_P^G(E) := {} & \text{the space of all locally constant maps } F : G \to E \text{ such that} \\ & F(gmn) = \delta^{1/2}(m) \cdot m^{-1}(F(g)) \\ & \text{for any } g \in G,\ m \in M,\ n \in N,\end{aligned}$$

and one lets G act by left translations $({}^hF)(g) := F(h^{-1}g)$.

Our aim in this section is to lift Ind_P^G to a functor $\mathcal{M}^t(M) \to \mathcal{M}^t(G)$ (which is compatible with respect to the forgetful functors). To prepare this construction we first rewrite the $\mathcal{H}(G)$-action on $\mathrm{Ind}_P^G(E)$ in terms of the $\mathcal{H}(M)$-action on E. The convolution of any $\phi \in \mathcal{H}(G)$ and any $F \in \mathrm{Ind}_P^G(E)$ can be computed as follows:

$$\begin{aligned}(\phi * F)(h) &= \int_G \phi(g)F(g^{-1}h)\,\mathrm{d}g \\ &= \int_G \phi(hg)F(g^{-1})\,\mathrm{d}g \\ &= \int_K \int_{M\times N} \phi(hmnk)F(k^{-1}n^{-1}m^{-1})\,\mathrm{d}m\,\mathrm{d}n\,\mathrm{d}k \\ &= \int_K \int_{M\times N} ({}^{h^{-1}}\phi^{k^{-1}})(mn)\delta^{-1/2}(m)m(F(k^{-1}))\,\mathrm{d}m\,\mathrm{d}n\,\mathrm{d}k \\ &= \int_K \int_M ({}^{h^{-1}}\phi^{k^{-1}})^P(m)m(F(k^{-1}))\,\mathrm{d}m\,\mathrm{d}k \\ &= \int_K ({}^{h^{-1}}\phi^{k^{-1}})^P * F(k^{-1})\,\mathrm{d}k \\ &= \int_K ({}^{h^{-1}}\phi^{k})^P * F(k)\,\mathrm{d}k. \end{aligned} \tag{2.1}$$

The last integrand only involves the $\mathcal{H}(M)$-action on E; moreover, the last integral in fact is a finite sum provided the integrand is locally constant in $k \in K$ (as it is the case above).

Suppose now that E is a non-degenerate $\mathcal{S}(M)$-module. Viewing E as a smooth M-representation we may form the parabolic induction $\mathrm{Ind}_P^G(E)$. The existence of the map $\psi \mapsto \psi^P$ on $\mathcal{S}(G)$ allows us to turn the above computation (2.1) into a definition: for any $\psi \in \mathcal{S}(G)$ and any $F \in \mathrm{Ind}_P^G(E)$ we define the map $\psi * F : G \to E$ by

$$(\psi * F)(h) := \int_K ({}^{h^{-1}}\psi^{k})^P * F(k)\,\mathrm{d}k.$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

As pointed out already the integral in fact is a finite sum. Since ψ is uniformly locally constant it is immediately clear that $\psi * F$ is locally constant. We observe that

$$({}^m\phi) * v = ({}^m(\epsilon_U * \phi)) * v = (({}^m\epsilon_U) * \phi) * v = {}^m\epsilon_U * (\phi * v) = m(\phi * v)$$

for $m \in M$, $v \in E$, and $\phi \in \mathcal{S}(M)$ where the compact open subgroup U is chosen such that ϕ is left U-invariant. Using this and Lemma 1.1 one easily checks that

$$(\psi * F)(hmn) = \delta^{1/2}(m) \cdot m^{-1}((\psi * F)(h)) \quad \text{for any } m \in M,\ n \in N.$$

So far we have seen that

$$\begin{aligned} \mathcal{S}(G) \times \operatorname{Ind}_P^G(E) &\to \operatorname{Ind}_P^G(E), \\ (\psi, F) &\mapsto \psi * F \end{aligned} \tag{2.2}$$

is a biadditive map which, because of (2.1), extends the $\mathcal{H}(G)$-module structure on $\operatorname{Ind}_P^G(E)$. To show that this is indeed an $\mathcal{S}(G)$-module structure it remains to verify that

$$\phi * (\psi * F) = (\phi * \psi) * F$$

for any $\phi, \psi \in \mathcal{S}(G)$ and $F \in \operatorname{Ind}_P^G(E)$. Using Lemma 1.2 we compute

$$\begin{aligned} (\phi * (\psi * F))(h) &= \int_K ({}^{h^{-1}}\phi^k)^P * ((\psi * F)(k))\, \mathrm{d}k \\ &= \int_K ({}^{h^{-1}}\phi^k)^P * \left(\int_K ({}^{k^{-1}}\psi^{k'})^P * F(k')\, \mathrm{d}k'\right) \mathrm{d}k \\ &= \int_K \left(\int_K ({}^{h^{-1}}\phi^k)^P * ({}^{k^{-1}}\psi^{k'})^P\, \mathrm{d}k\right) * F(k')\, \mathrm{d}k' \\ &= \int_K (({}^{h^{-1}}\phi) * (\psi^{k'}))^P * F(k')\, \mathrm{d}k' \\ &= \int_K ({}^{h^{-1}}(\phi * \psi)^{k'})^P * F(k')\, \mathrm{d}k' \\ &= ((\phi * \psi) * F)(h). \end{aligned}$$

This establishes that (2.2) is an $\mathcal{S}(G)$-module structure. It obviously is functorial in the $\mathcal{S}(M)$-module E. Hence we have constructed a functor

$$\operatorname{Ind}_P^G : \mathcal{M}^t(M) \to \mathcal{M}^t(G)$$

which after forgetting corresponds to the (normalized) parabolic induction for smooth representations. With the latter also the new functor is exact. As it stands our lifted functor seems to depend on the choice of the maximal compact subgroup K. But this choice in fact is just a matter of convenience. Recall that we have fixed Haar measures on G, M, N, and hence a left as well as a right invariant Haar measure on P. These choices determine uniquely a G-invariant functional

$$\mu_{G/P} : \operatorname{Ind}_P^G(\delta^{-1/2}) \to \mathbb{C}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

(cf. [**BZ**, 1.21]). Going back to the definition of the module structure (2.2) we consider, for any $h \in G$, $\psi \in \mathcal{S}(G)$, and $F \in \operatorname{Ind}_P^G(E)$, the map

$$\begin{aligned} \Phi_{h,\psi,F} : G &\to E, \\ g &\mapsto ({}^{h^{-1}}\psi^{g})^P * F(g). \end{aligned}$$

Using Lemma 1.1 it is easily seen that this map is locally constant and satisfies

$$\Phi_{h,\psi,F}(gp) = \delta(p) \cdot \Phi_{h,\psi,F}(g) \quad \text{for any } g \in G \text{ and } p \in P.$$

Hence

$$\Phi_{h,\psi,F} \in \operatorname{Ind}_P^G(\delta^{-1/2}) \otimes_{\mathbb{C}} E.$$

We therefore obtain the formula

$$(\psi * F)(h) = \int_{G/P} \Phi_{h,\psi,F}(g)\, \mathrm{d}\mu_{G/P}(g),$$

which shows that our lifted functor is independent of the choice of K.

It will be technically important to understand parabolic induction as a functor from $\mathcal{M}(\mathcal{S}(M, U_M))$ to $\mathcal{M}(\mathcal{S}(G, U))$ where U runs over appropriate compact open subgroups in G. In the Hecke algebra case this is described in [**Bus**] as follows. In the following we always let U, as in the previous section, be an open normal subgroup of K which is totally decomposed. We have the obvious functor

$$\begin{aligned} \mathcal{M}(\mathcal{H}(M, U_M)) &\to \mathcal{M}(\mathcal{H}(G, U)), \\ X &\mapsto \operatorname{Hom}_{\mathcal{H}^+(M,U_M)}(\mathcal{H}(G, U), X). \end{aligned}$$

Let $\mathcal{M}_U(G)$ denote the full subcategory of all V in $\mathcal{M}(G)$ such that $\mathcal{H}(G)V^U = V$. By [**Bus**, § 4.1, Corollary 1.ii and Remark 6] the diagram

$$\begin{array}{ccc} \mathcal{M}_U(G) & \xrightarrow{V \mapsto V^U} & \mathcal{M}(\mathcal{H}(G, U)) \\ {\scriptstyle \operatorname{Ind}_P^G}\uparrow & & \uparrow{\scriptstyle \operatorname{Hom}_{\mathcal{H}^+(M,U_M)}(\mathcal{H}(G,U),\cdot)} \\ \mathcal{M}_{U_M}(M) & \xrightarrow{E \mapsto E^{U_M}} & \mathcal{M}(\mathcal{H}(M, U_M)) \end{array}$$

is commutative up to the following natural isomorphism

$$\begin{aligned} \operatorname{Ind}_P^G(E)^U &\xrightarrow{\cong} \operatorname{Hom}_{\mathcal{H}^+(M,U_M)}(\mathcal{H}(G, U), E^{U_M}), \\ F &\mapsto [\phi \mapsto (\phi * F)(1)]; \end{aligned}$$

moreover, the horizontal arrows are equivalences of categories. We suppose now that E lies in $\mathcal{M}^t(M)$ and satisfies $E = \mathcal{H}(M)E^{U_M}$. Then E^{U_M} is naturally an $\mathcal{S}(M, U_M)$-module and the right-hand side of the above natural transformation is equal to

$$\operatorname{Hom}_{\mathcal{H}^+(M,U_M)}(\mathcal{H}(G, U), E^{U_M}) = \operatorname{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{S}(M, U_M) \otimes_{\mathcal{H}^+(M,U_M)} \mathcal{H}(G, U), E^{U_M}),$$

where $\mathcal{S}(M,U_M)$ is viewed as a right $\mathcal{H}^+(M,U_M)$ via the ring homomorphism s in Lemma 1.8 (i). On the other hand $\mathrm{Ind}_P^G(E)^U$ and therefore, by transport of structure, $\mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{S}(M,U_M)\otimes_{\mathcal{H}^+(M,U_M)}\mathcal{H}(G,U),E^{U_M})$ is naturally an $\mathcal{S}(G,U)$-module. We will show in the following that this latter module structure is induced by a natural right $\mathcal{S}(G,U)$-module structure on the tensor product $\mathcal{S}(M,U_M)\otimes_{\mathcal{H}^+(M,U_M)}\mathcal{H}(G,U)$.

At this point we fix an element $z\in M^+$ which is strongly (P,U)-positive and lies in the centre of M (cf. [**Bus**, 3.2]). The function $\phi_z := {}^z\epsilon_{U_M}\in\mathcal{H}^+(M,U_M)$ is supported on U_MzU_M with value $\phi_z(z)=\mathrm{vol}_M(U_M)^{-1}$. More generally, for any $i\in\mathbb{Z}$, the i-fold power $\phi_z^i := {}^{z^i}\epsilon_{U_M}\in\mathcal{H}(M,U_M)$ is supported on $U_Mz^iU_M$ with value $\phi_z^i(z^i)=\mathrm{vol}_M(U_M)^{-1}$ and, in fact, lies in the centre of $\mathcal{H}(M,U_M)$. We put $\psi_z := s(\phi_z)\in\mathcal{H}(G,U)$. Its i-fold power $\psi_z^i := s(\phi_z^i)$, for $i\in\mathbb{N}$, is supported on Uz^iU with value $\psi_z^i(z^i)=\mathrm{vol}_G(U)^{-1}\cdot\delta^{-1/2}(z^i)$.

We consider the obvious map

$$i_0:\mathcal{S}(M,U_M)\otimes_{\mathcal{H}^+(M,U_M)}\mathcal{H}(G,U)\to\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U)$$

as well as the map

$$\begin{aligned} e_0:\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U)&\to\mathcal{S}(M,U_M)\otimes_{\mathcal{H}^+(M,U_M)}\mathcal{H}(G,U),\\ \phi\otimes\psi&\mapsto\mathrm{vol}_G(U)\cdot\sum_{k\in K/U}\phi*(\psi^{k^{-1}})^P\otimes{}^k\epsilon_U, \end{aligned}$$

which is well defined by Lemma 1.8 (ii). Both maps of course are homomorphisms of left $\mathcal{S}(M,U_M)$-modules. We put

$$\mathcal{R} := \ker(e_0),$$

which is an $\mathcal{S}(M,U_M)$-submodule of $\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U)$.

Proposition 2.1. *The maps i_0 and e_0 induce isomorphisms*

$$\mathcal{S}(M,U_M)\otimes_{\mathcal{H}^+(M,U_M)}\mathcal{H}(G,U)\cong[\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U)]/\mathcal{R}$$

which are inverse to each other.

Proof. First we establish that $e_0\circ i_0$ is the identity map. This amounts to showing that, for any $\psi\in\mathcal{H}(G,U)$, the element

$$\epsilon_{U_M}\otimes\psi-\mathrm{vol}_G(U)\cdot\sum_{k\in K/U}(\psi^{k^{-1}})^P\otimes{}^k\epsilon_U \qquad (*_\psi)$$

is equal to zero in the tensor product $\mathcal{S}(M,U_M)\otimes_{\mathcal{H}^+(M,U_M)}\mathcal{H}(G,U)$. This will be achieved in three steps.

Step 1. We assume that $\psi=\epsilon_U$. The support $\mathrm{supp}(\epsilon_U^{k^{-1}})=Uk^{-1}\subseteq Pk^{-1}U$ has empty intersection with P if $k\notin K_PU$ in which case we therefore have $(\epsilon_U^{k^{-1}})^P=0$. On the other hand let $k=k_nk_m\in K_NK_M=K_P$. Then

$$(\epsilon_U^{k^{-1}})^P=(\epsilon_U^P)^{k_m^{-1}}=\frac{\mathrm{vol}_P(U_P)}{\mathrm{vol}_G(U)}\cdot\epsilon_{U_M}^{k_m^{-1}}.$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

Together we obtain

$$\begin{aligned}\mathrm{vol}_G(U)\cdot\sum_{k\in K/U}(\epsilon_U^{k^{-1}})^P\otimes{}^k\epsilon_U &= \mathrm{vol}_G(U)\cdot\sum_{k\in K_P/U_P}(\epsilon_U^{k^{-1}})^P\otimes{}^k\epsilon_U\\ &= \mathrm{vol}_P(U_P)\cdot\sum_{k\in K_P/U_P}\epsilon_{U_M}^{k_m^{-1}}\otimes{}^{k_nk_m}\epsilon_U\\ &= \frac{1}{[K_P:U_P]}\cdot\sum_{k\in K_P/U_P}\epsilon_{U_M}\otimes{}^{k_m^{-1}k_nk_m}\epsilon_U\end{aligned}$$

and hence

$$\begin{aligned}\epsilon_{U_M}\otimes\epsilon_U-\mathrm{vol}_G(U)\cdot&\sum_{k\in K/U}(\epsilon_U^{k^{-1}})^P\otimes{}^k\epsilon_U\\ &= \frac{1}{[K_P:U_P]}\cdot\sum_{k\in K_P/U_P}\epsilon_{U_M}\otimes(\epsilon_U-{}^{k_m^{-1}k_nk_m}\epsilon_U)\end{aligned}$$

(note that $k_nk_mU_P\mapsto k_m^{-1}k_nk_mU$ is a well defined map from K_P/U_P into K_NU/U). Since ϕ_z is invertible in $\mathcal{H}(M,U_M)$ we have, for any $i\in\mathbb{N}$, that

$$\begin{aligned}\epsilon_{U_M}\otimes\epsilon_U-\mathrm{vol}_G(U)\cdot&\sum_{k\in K/U}(\epsilon_U^{k^{-1}})^P\otimes{}^k\epsilon_U\\ &= \frac{1}{[K_P:U_P]}\cdot\sum_{k\in K_P/U_P}\phi_z^{-i}\otimes\psi_z^i*(\epsilon_U-{}^{k_m^{-1}k_nk_m}\epsilon_U).\end{aligned}$$

It therefore suffices to find, for any given $n\in K_N$, an $i\in\mathbb{N}$ such that

$$\psi_z^i*(\epsilon_U-{}^n\epsilon_U)=0.$$

Since z is strongly positive we do find an $i\in\mathbb{N}$ such that $z^inz^{-i}\in U_N$. Then $Uz^iUn=Uz^inU=U(z^inz^{-i})z^iU=Uz^iU$ and more precisely $(\psi_z^i)^n=\psi_z^i$.

Step 2. We assume that $\mathrm{supp}(\psi)\subseteq UM^+K$. By additivity we then may even assume that $\mathrm{supp}(\psi)\subseteq UM^+k_0U$ for some $k_0\in K$. Then $\psi_0:=\psi^{k_0^{-1}}$ has support in UM^+U and hence, by Lemma 1.7, is of the form $\psi_0=s(\phi_0)$ for a unique $\phi_0\in\mathcal{H}^+(M,U_M)$. We compute

$$\begin{aligned}\epsilon_{U_M}\otimes\psi-\mathrm{vol}_G(U)\cdot&\sum_{k\in K/U}(\psi^{k^{-1}})^P\otimes{}^k\epsilon_U\\ &=\epsilon_{U_M}\otimes s(\phi_0)^{k_0}-\mathrm{vol}_G(U)\cdot\sum_{k\in K/U}(s(\phi_0)^{k_0k^{-1}})^P\otimes{}^k\epsilon_U\\ &=\phi_0\otimes\epsilon_U^{k_0}-\mathrm{vol}_G(U)\cdot\sum_{k\in K/U}\phi_0*(\epsilon_U^{k_0k^{-1}})^P\otimes{}^k\epsilon_U\end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

$$= \phi_0 \otimes \epsilon_U^{k_0} - \mathrm{vol}_G(U) \cdot \sum_{k \in K/U} \phi_0 * (\epsilon_U^{k^{-1}})^P \otimes {}^k\epsilon_U^{k_0}$$
$$= \phi_0 * \left(\epsilon_{U_M} \otimes \epsilon_U - \mathrm{vol}_G(U) \cdot \sum_{k \in K/U} (\epsilon_U^{k^{-1}})^P \otimes {}^k\epsilon_U \right) * \epsilon_U^{k_0}$$
$$= 0,$$

where for the second identity we have used Lemma 1.8 (ii).

Step 3. We now let $\psi \in \mathcal{H}(G,U)$ be a general element. First we claim that

$$Y := \{\psi \in \mathcal{H}(G,U) : \mathrm{supp}(\psi) \subseteq UM^+K\}$$

is an $\mathcal{H}^+(M,U_M)$-submodule of $\mathcal{H}(G,U)$, and that any element in the quotient module $\mathcal{H}(G,U)/Y$ is annihilated by some power of ϕ_z. The first part of this assertion is immediate from

$$UM^+UM^+K = UM^+U_{\bar{N}}M^+K = UM^+K.$$

For the second part we recall that N is the union of its compact open subgroups and that M is the increasing union of the $z^{-i}M^+$ for $i \geqslant 0$. We therefore find an $i \geqslant 0$ and a compact open subgroup $N_0 \subseteq N$ such that

$$\mathrm{supp}(\psi) \subseteq N_0 z^{-i} M^+ K.$$

Choosing i large enough we may further assume that $z^i N_0 z^{-i} \subseteq U_N$. We then have

$$\begin{aligned} \mathrm{supp}(\psi_z^i * \psi) &\subseteq U z^i U \cdot \mathrm{supp}(\psi) \subseteq U z^i \cdot \mathrm{supp}(\psi) \\ &\subseteq U z^i N_0 z^{-i} M^+ K \subseteq U M^+ K. \end{aligned}$$

This establishes the claim.

Given a $\psi \in \mathcal{H}(G,U)$ we now find an $i \geqslant 0$ such that, by Step 2, we have

$$\epsilon_{U_M} \otimes \psi_z^i * \psi = \mathrm{vol}_G(U) \cdot \sum_{k \in K/U} (\psi_z^i * \psi^{k^{-1}})^P \otimes {}^k\epsilon_U.$$

The left-hand side is equal to $\phi_z^i * (\epsilon_{U_M} \otimes \psi)$, and the right-hand side, by Lemma 1.8 (ii), to

$$\phi_z^i * \left(\mathrm{vol}_G(U) \cdot \sum_{k \in K/U} (\psi^{k^{-1}})^P \otimes {}^k\epsilon_U \right).$$

Since ϕ_z is invertible in $\mathcal{H}(M,U_M)$ it follows that

$$\epsilon_{U_M} \otimes \psi = \mathrm{vol}_G(U) \cdot \sum_{k \in K/U} (\psi^{k^{-1}})^P \otimes {}^k\epsilon_U.$$

This finishes the proof of the identity

$$e_0 \circ i_0 = \mathrm{id}.$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

In particular, the map i_0 is injective. Moreover, $e := i_0 \circ e_0$ is a projector on $\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)$. We obtain that

$$\mathcal{R} = \ker(e_0) = \ker(e) = \mathrm{im}(\mathrm{id} - e)$$

and that

$$\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U) = \mathrm{im}(i_0) \oplus \mathcal{R}.$$

□

The tensor product $\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)$ of course is a bimodule with $\mathcal{S}(M, U_M)$ acting from the left and $\mathcal{S}(G, U)$ acting from the right. Using Lemma 1.2 one easily checks that the map $e = i_0 \circ e_0$ satisfies

$$e(\phi \otimes \psi_1 * \psi_2) = e(e(\phi \otimes \psi_1) * \psi_2).$$

It follows that $\mathcal{R} = \mathrm{im}(\mathrm{id} - e) = \ker(e)$ is a right $\mathcal{S}(G, U)$-submodule of the tensor product $\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)$. Hence $[\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)]/\mathcal{R}$ is a bimodule quotient of $\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)$. Using Proposition 2.1 we may rewrite, for any E in $\mathcal{M}^t(M)$, our earlier natural isomorphism as the isomorphism

$$\begin{aligned} \mathrm{Ind}_P^G(E)^U &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{S}(M,U_M)}([\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)]/\mathcal{R}, E^{U_M}), \\ F &\mapsto A_F(\phi \otimes \psi) = \phi * ((\psi * F)(1)). \end{aligned}$$

This visibly is an isomorphism of $\mathcal{S}(G, U)$-modules where the $\mathcal{S}(G, U)$-action on the right-hand side is the one coming from the right $\mathcal{S}(G, U)$-module structure of the first entry. To write this fact, similarly as before, as a commutative diagram of functors we introduce the full subcategory $\mathcal{M}_U^t(G)$ of all V in $\mathcal{M}^t(G)$ such that $\mathcal{S}(G)V^U = V$.

Lemma 2.2. *Let V be in $\mathcal{M}^t(G)$; we then have*

(i) *V lies in $\mathcal{M}_U^t(G)$ if and only if, as an $\mathcal{H}(G)$-module, it lies in $\mathcal{M}_U(G)$;*

(ii) *V decomposes as an $\mathcal{S}(G)$-module into a direct sum $V = V_0 \oplus V_1$ such that V_0 lies in $\mathcal{M}_U^t(G)$ and $V_1^U = 0$.*

Proof. Let

$$\mathcal{M}(G) = \prod_{\Omega} \mathcal{M}_{\Omega}(G)$$

be the decomposition of the category $\mathcal{M}(G)$ into its Bernstein components (cf. [**BD**]). Suppose that V lies in $\mathcal{M}_U^t(G)$. We consider the corresponding decomposition $V = \bigoplus_{\Omega} V(\Omega)$ of V as an $\mathcal{H}(G)$-module into its Bernstein components $V(\Omega)$. As explained in [**SSZ**, p. 166] this decomposition is in fact an $\mathcal{S}(G)$-module decomposition. Since $\epsilon_U V = \sum_{\Omega} \epsilon_U V(\Omega)$ our assumption that $V = \mathcal{S}(G)V^U$ implies that $V(\Omega) = \mathcal{S}(G)\epsilon_U V(\Omega)$ for any Ω. According to [**Bus**, § 1.4, Proposition 3] there are finitely many Bernstein components $\Omega_1, \ldots, \Omega_r$ such that $\mathcal{M}_U(G) = \mathcal{M}_{\Omega_1}(G) \times \cdots \times \mathcal{M}_{\Omega_r}(G)$. It follows that $\epsilon_U V(\Omega) = 0$ for any $\Omega \neq \Omega_i$. Hence $V = V(\Omega_1) \oplus \cdots \oplus V(\Omega_r)$ lies in $\mathcal{M}_U(G)$. For a general V in $\mathcal{M}^t(G)$ the decomposition in (ii) is given by $V_0 := V(\Omega_1) \oplus \cdots \oplus V(\Omega_r)$ and $V_1 := \bigoplus_{\Omega \neq \Omega_i} V(\Omega)$. □

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

Lemma 2.3. *The functor*

$$\begin{aligned} \mathcal{M}_U^t(G) &\xrightarrow{\sim} \mathcal{M}(\mathcal{S}(G,U)),\\ V &\mapsto V^U \end{aligned}$$

is an equivalence of categories with a quasi-inverse functor being given by

$$X \mapsto \mathcal{S}(G) * \epsilon_U \otimes_{\mathcal{S}(G,U)} X.$$

Proof. The category $\mathcal{M}_U(G)$, being a finite product of Bernstein components, is closed under the passage to $\mathcal{H}(G)$-module subquotients. Lemma 2.2 (i) then implies that the category $\mathcal{M}_U^t(G)$ is closed under the passage to $\mathcal{S}(G)$-module subquotients. The asserted equivalence of categories is a formal consequence of this fact (cf. the proof of Proposition 3.3 in [**BK**]). □

Proposition 2.4. *The diagram*

$$\begin{array}{ccc} \mathcal{M}_U^t(G) & \xrightarrow{V \mapsto V^U} & \mathcal{M}(\mathcal{S}(G,U)) \\ {\scriptstyle \mathrm{Ind}_P^G}\uparrow & & \uparrow{\scriptstyle \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U,\cdot)} \\ \mathcal{M}_{U_M}^t(M) & \xrightarrow{E \mapsto E^{U_M}} & \mathcal{M}(\mathcal{S}(M,U_M)) \end{array}$$

is commutative up to the natural isomorphism $F \mapsto A_F$, where

$$\mathcal{B}_U := [\mathcal{S}(M,U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G,U)]/\mathcal{R};$$

moreover, the horizontal arrows in the diagram are equivalences of categories.

Proof. We first of all point out that, by [**Bus**, § 1.1, Proposition 1], the compact open subgroup $U_M \subseteq M$ has properties exactly analogous to the properties which we assumed to hold for $U \subseteq G$. By [**Bus**, § 1.7] the functor Ind_P^G maps $\mathcal{M}_{U_M}(M)$ into $\mathcal{M}_U(G)$. It therefore follows from Lemma 2.2 that this functor also maps $\mathcal{M}_{U_M}^t(M)$ into $\mathcal{M}_U^t(G)$. The asserted commutativity was established before Lemma 2.2. Finally, Lemma 2.3 says that the horizontal arrows are equivalences of categories. □

3. Parabolic restriction for $\mathcal{S}$-modules

The parabolic induction functor for smooth representations has a left adjoint—the Jacquet functor—which is given by

$$\begin{aligned} r_{G,P} : \mathcal{M}(G) &\to \mathcal{M}(M),\\ V &\mapsto V_N := V/V(N), \end{aligned}$$

where $V(N) \subseteq V$ denotes the vector subspace generated by all $nv - v$ for $n \in N$ and $v \in V$ (cf. [**Car**, 2.2]). If

$$\begin{aligned} V &\to V_N,\\ v &\mapsto \bar{v} \end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

denotes the canonical projection map then M acts on V_N by $m\bar{v} = \delta^{1/2}(m) \cdot \overline{mv}$. It is known that this functor does not respect tempered admissible representations. To construct a left adjoint for our parabolic $\mathcal{S}$-module induction therefore requires a modification of the above functor even on the level of the underlying smooth representations; in other words we cannot expect compatibility with the forgetful functors.

For any V in $\mathcal{M}(G)$ there is a natural $\mathcal{H}$-module homomorphism

$$\begin{aligned} V &\to \mathcal{S} \otimes_{\mathcal{H}} V, \\ v &\mapsto \epsilon_U \otimes v, \end{aligned}$$

where ϵ_U is chosen in such a way that U fixes the vector v. We first want to show that the projection map $V \to V_N$ naturally extends to a map $\mathcal{S}(G) \otimes_{\mathcal{H}(G)} V \to \mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N$. We compute

$$\begin{aligned} \phi * v = \int_G \phi(g) gv \, \mathrm{d}g &= \int_K \int_M \int_N \phi(mnk) mnkv \, \mathrm{d}m \, \mathrm{d}n \, \mathrm{d}k \\ &\equiv \int_K \int_M \int_N \phi(mnk) mkv \, \mathrm{d}m \, \mathrm{d}n \, \mathrm{d}k \mod V(N) \\ &= \int_K \int_M (\phi^{k^{-1}})^P(m) \delta^{1/2}(m) \overline{mkv} \, \mathrm{d}m \, \mathrm{d}k \\ &= \int_K (\phi^{k^{-1}})^P * \overline{kv} \, \mathrm{d}k \end{aligned}$$

and hence

$$\overline{\phi * v} = \int_K (\phi^k)^P * \overline{k^{-1}v} \, \mathrm{d}k \tag{3.1}$$

for any $\phi \in \mathcal{H}(G)$ and $v \in V$. We now put

$$\begin{aligned} J := J_P : \mathcal{S}(G) \otimes_{\mathcal{H}(G)} V &\to \mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N, \\ \psi \otimes v &\mapsto \int_K (\psi^k)^P \otimes \overline{k^{-1}v} \, \mathrm{d}k, \end{aligned}$$

which is well defined since, using Lemma 1.2, we have

$$\begin{aligned} J(\psi * \phi \otimes v) &= \int_K ((\psi * \phi)^k)^P \otimes \overline{k^{-1}v} \, \mathrm{d}k \\ &= \int_K \int_K (\psi^{k'})^P * ({}^{k'^{-1}}\phi^k)^P \, \mathrm{d}k' \otimes \overline{k^{-1}v} \, \mathrm{d}k \\ &= \int_K (\psi^{k'})^P \otimes \int_K ({}^{k'^{-1}}\phi^k)^P * \overline{k^{-1}v} \, \mathrm{d}k \, \mathrm{d}k' \\ &\overset{(3.1)}{=} \int_K (\psi^{k'})^P \otimes \overline{({}^{k'^{-1}}\phi) * v} \, \mathrm{d}k' \\ &= \int_K (\psi^{k'})^P \otimes \overline{k'^{-1}(\phi * v)} \, \mathrm{d}k' \\ &= J(\psi \otimes \phi * v). \end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

We check that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{pr}} & V_N \\ \downarrow & & \downarrow \\ \mathcal{S}(G) \otimes_{\mathcal{H}(G)} V & \xrightarrow{J} & \mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N \end{array}$$

is indeed commutative. Given a vector $v \in V$ and a compact open subgroup $U \subseteq G$ which fixes v let $U' \subseteq U_M$ be a compact open subgroup of M which fixes $((\epsilon_U)^k)^P \in \mathcal{H}(M)$ for any $k \in K$ (in fact, these are only finitely many different elements); then

$$\begin{aligned} J(\epsilon_U \otimes v) &= \int_K ((\epsilon_U)^k)^P \otimes \overline{k^{-1}v}\,\mathrm{d}k \\ &= \epsilon_{U'} \otimes \int_K ((\epsilon_U)^k)^P * \overline{k^{-1}v}\,\mathrm{d}k \\ &\overset{(3.1)}{=} \epsilon_{U'} \otimes \overline{\epsilon_U * v} = \epsilon_{U'} \otimes \bar{v}. \end{aligned}$$

For any V in $\mathcal{M}^t(G)$ we let $m_V : \mathcal{S}(G) \otimes_{\mathcal{H}(G)} V \to V$ denote the obvious multiplication map. We now define our parabolic restriction functor by

$$\begin{aligned} r^t_{G,P} : \mathcal{M}^t(G) &\to \mathcal{M}^t(M), \\ V &\mapsto (\mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N)/(\mathcal{S}(M) \cdot J(\ker m_V)). \end{aligned}$$

Proposition 3.1. *The functor $r^t_{G,P}$ is left adjoint to the functor Ind_P^G constructed in § 2.*

Proof. We begin by observing that we have the chain of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}(G)}(\mathcal{S}(G) \otimes_{\mathcal{H}(G)} V, \mathrm{Ind}_P^G(E)) &= \mathrm{Hom}_{\mathcal{H}(G)}(V, \mathrm{Ind}_P^G(E)) \\ &= \mathrm{Hom}_{\mathcal{H}(M)}(V_N, E) \\ &= \mathrm{Hom}_{\mathcal{S}(M)}(\mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N, E) \end{aligned} \tag{3.2}$$

for any non-degenerate $\mathcal{S}(M)$-module E. More explicitly, if

$$A : \mathcal{S}(G) \otimes_{\mathcal{H}(G)} V \to \mathrm{Ind}_P^G(E) \quad \text{and} \quad B : \mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N \to E$$

correspond to each other under the above natural isomorphisms then we have

$$B(\varphi \otimes \bar{v}) = \varphi * (A(\epsilon_U \otimes v)(1)).$$

We obtain

$$\begin{aligned} B(J(\psi \otimes v)) &= B\biggl(\int_K (\psi^k)^P \otimes \overline{k^{-1}v}\,\mathrm{d}k\biggr) \\ &= \int_K B((\psi^k)^P \otimes \overline{k^{-1}v})\,\mathrm{d}k \\ &= \int_K (\psi^k)^P * A(\epsilon_U \otimes k^{-1}v)(1)\,\mathrm{d}k \end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

$$\begin{aligned} &= \int_K (\psi^k)^P * A(\epsilon_U \otimes v)(k)\,\mathrm{d}k \\ &= (\psi * A(\epsilon_U \otimes v))(1) = A(\psi \otimes v)(1), \end{aligned}$$

where the fourth identity holds by the definition of the $\mathcal{S}(G)$-module structure on $\mathrm{Ind}_P^G(E)$. This says that the diagram

$$\begin{array}{ccc} \mathcal{S}(G) \otimes_{\mathcal{H}(G)} V & \xrightarrow{\ A\ } & \mathrm{Ind}_P^G(E) \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1(F) := F(1)} \\ \mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N & \xrightarrow{\ B\ } & E \end{array} \tag{3.3}$$

is commutative. It follows that, for a given $\mathcal{S}(G)$-submodule $W \subseteq \mathcal{S}(G) \otimes_{\mathcal{H}(G)} V$, we have

$$A(W) = 0 \quad \text{if and only if } B(\mathcal{S}(M) \cdot J(W)) = 0.$$

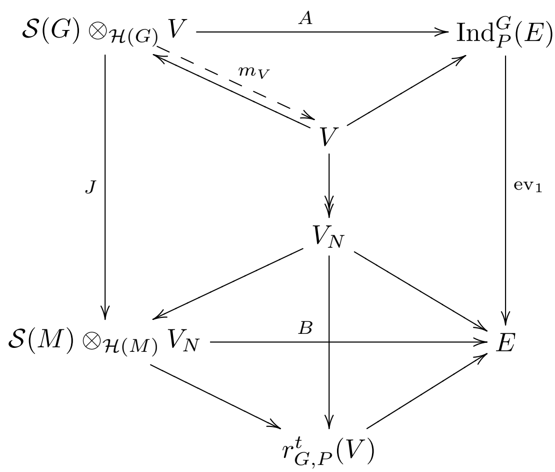
(For the reverse implication observe that $A(w) = 0$ if and only if $A(gw)(1) = 0$ for any $g \in G$.)

Let now V be a non-degenerate $\mathcal{S}(G)$-module and apply the above discussion to $W := \ker(m_V)$. In this situation it follows that the natural isomorphisms (3.2) induce, because of (3.3), a natural isomorphism

$$\mathrm{Hom}_{\mathcal{S}(G)}(V, \mathrm{Ind}_P^G(E)) = \mathrm{Hom}_{\mathcal{S}(M)}((\mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N)/(\mathcal{S}(M) \cdot J(\ker m_V)), E).$$

□

All the information obtained so far can best be displayed in one large commutative diagram (the broken arrow only makes commutative the parallelogram of which it is the top):



By the uniqueness of adjoints, with Ind_P^G also the functor $r_{G,P}^t$ is independent of the choice of the maximal compact subgroup K. As an immediate consequence of (3.2) we

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

have

$$r^t_{G,P}(\mathcal{S}(G) \otimes_{\mathcal{H}(G)} V) = \mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N,$$

i.e. that the diagram of functors

$$\begin{array}{ccc} \mathcal{M}(G) & \xrightarrow{\mathcal{S}(G)\otimes_{\mathcal{H}(G)}\cdot} & \mathcal{M}^t(G) \\ \downarrow{\scriptstyle r_{G,P}} & & \downarrow{\scriptstyle r^t_{G,P}} \\ \mathcal{M}(M) & \xrightarrow{\mathcal{S}(M)\otimes_{\mathcal{H}(M)}\cdot} & \mathcal{M}^t(M) \end{array}$$

is commutative. We also remark that it is a formal consequence of having a right adjoint that the functor $r^t_{G,P}$ is right exact.

The kernel of m_V is $\mathbb{C}$-linearly generated by elements of the form $\psi \otimes v - \epsilon_U \otimes \psi * v$ with $v \in V$, $\psi \in \mathcal{S}(G)$, and $U \subseteq G$ a compact open subgroup such that $\epsilon_U * \psi = \psi$. It follows that $r^t_{G,P}(V)$ is nothing else than the cokernel of the map

$$\mathcal{S}(M) \otimes_{\mathbb{C}} \mathcal{S}(G) \otimes_{\mathcal{H}(G)} V \to \mathcal{S}(M) \otimes_{\mathcal{H}(M)} V_N,$$
$$\phi \otimes \psi \otimes v \mapsto \phi \otimes \overline{\psi * v} - \int_K \phi * (\psi^k)^P \otimes \overline{k^{-1}v}\, \mathrm{d}k.$$

This evidently is a map of left $\mathcal{S}(M)$-modules. But in case $V = \mathcal{S}(G)$ it also is a map of right $\mathcal{S}(G)$-modules since the projection map $\mathcal{S}(G) \twoheadrightarrow \mathcal{S}(G)_N$ is. We therefore see that this map for arbitrary V arises from the map for $V = \mathcal{S}(G)$ by the tensor product $\cdot \otimes_{\mathcal{S}(G)} V$. It also follows that $r^t_{G,P}(\mathcal{S}(G))$ is an $(\mathcal{S}(M), \mathcal{S}(G))$-bimodule. Hence we have the following fact.

Proposition 3.2. *For any $V \in \mathcal{M}^t(G)$ we have $r^t_{G,P}(V) = r^t_{G,P}(\mathcal{S}(G)) \otimes_{\mathcal{S}(G)} V$.*

As in the previous section we want to understand parabolic restriction also as a functor from $\mathcal{M}(\mathcal{S}(G,U))$ to $\mathcal{M}(\mathcal{S}(M,U_M))$. We again fix an open normal subgroup U of K which is totally decomposed. We also recall the bimodule

$$\mathcal{B}_U = [\mathcal{S}(M,U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G,U)]/\mathcal{R}.$$

It is clear that the functor

$$\mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \cdot) : \mathcal{M}(\mathcal{S}(M,U_M)) \to \mathcal{M}(\mathcal{S}(G,U))$$

has the left adjoint

$$\mathcal{B}_U \otimes_{\mathcal{S}(G,U)} \cdot : \mathcal{M}(\mathcal{S}(G,U)) \to \mathcal{M}(\mathcal{S}(M,U_M)).$$

Proposition 3.3. *The diagram*

$$\begin{array}{ccc} \mathcal{M}^t(G) & \xrightarrow{V \mapsto V^U} & \mathcal{M}(\mathcal{S}(G,U)) \\ \downarrow{\scriptstyle r^t_{G,P}} & & \downarrow{\scriptstyle \mathcal{B}_U \otimes_{\mathcal{S}(G,U)} \cdot} \\ \mathcal{M}^t(M) & \xrightarrow{E \mapsto E^{U_M}} & \mathcal{M}(\mathcal{S}(M,U_M)) \end{array}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

is commutative (up to a natural isomorphism which is made explicit after Corollary 3.5). In particular, the surjection

$$\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U)\twoheadrightarrow\mathcal{B}_U$$

of $(\mathcal{S}(M,U_M),\mathcal{S}(G,U))$*-bimodules induces a surjection*

$$\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}V^U\twoheadrightarrow\mathcal{B}_U\otimes_{\mathcal{S}(G,U)}V^U\cong r^t_{G,P}(V)^{U_M}$$

of left $\mathcal{S}(M,U_M)$*-modules.*

Proof. We first of all claim that the functor $r^t_{G,P}$ maps the category $\mathcal{M}^t_U(G)$ into $\mathcal{M}^t_{U_M}(M)$. If V lies in $\mathcal{M}^t_U(G)$ then, by Lemma 2.2 (i), as an $\mathcal{H}(G)$-module it lies in $\mathcal{M}_U(G)$. Hence V_N lies in $\mathcal{M}_{U_M}(M)$ by [**Bus**, § 1.7], i.e. we have that $V_N=\mathcal{H}(M)V_N^{U_M}$. It easily follows that $r^t_{G,P}(V)=\mathcal{S}(M)r^t_{G,P}(V)^{U_M}$, i.e. that $r^t_{G,P}(V)$ lies in $\mathcal{M}^t_{U_M}(M)$. By the uniqueness of adjoints Proposition 2.4 therefore implies the commutativity of the diagram

$$\begin{array}{ccc}\mathcal{M}^t_U(G) & \xrightarrow{V\mapsto V^U} & \mathcal{M}(\mathcal{S}(G,U))\\ {\scriptstyle r^t_{G,P}}\downarrow & & \downarrow{\scriptstyle \mathcal{B}_U\otimes_{\mathcal{S}(G,U)}\cdot}\\ \mathcal{M}^t_{U_M}(M) & \xrightarrow{E\mapsto E^{U_M}} & \mathcal{M}(\mathcal{S}(M,U_M))\end{array}$$

Because of Lemma 2.2 (ii) it remains to show that for any V in $\mathcal{M}^t(G)$ with $V^U=0$ we have $r^t_{G,P}(V)^{U_M}=0$. By [**BD**, 3.5.2] we certainly have $V_N^{U_M}=0$. In terms of Bernstein components this means the following. Similarly as in the proof of Lemma 2.2, if $\mathcal{M}(M)=\prod_{\Omega'}\mathcal{M}_{\Omega'}(M)$ is the Bernstein decomposition of the category $\mathcal{M}(M)$, then there are finitely many components $\Omega'_1,\dots,\Omega'_s$ such that $\mathcal{M}_{U_M}(M)=\mathcal{M}_{\Omega'_1}(M)\times\cdots\times\mathcal{M}_{\Omega'_s}(M)$. We see that V_N has to lie in $\prod_{\Omega'\neq\Omega'_j}\mathcal{M}_{\Omega'}(M)$. The subsequent lemma (applied to M) implies that then also $\mathcal{S}(M)\otimes_{\mathcal{H}(M)}V_N$ and *a fortiori* its quotient $r^t_{G,P}(V)$, as $\mathcal{H}(M)$-modules, lie in $\prod_{\Omega'\neq\Omega'_j}\mathcal{M}_{\Omega'}(M)$ and therefore have no non-zero U_M-invariant vectors. □

We remark that it will follow from Theorem 4.18 later on that even the natural map

$$V^U\to r^t_{G,P}(V)^{U_M}$$

is surjective.

Lemma 3.4. *Suppose that* V *lies in the Bernstein component* $\mathcal{M}_{\Omega_0}(G)$ *of* $\mathcal{M}(G)$*; then* $\mathcal{S}(G)\otimes_{\mathcal{H}(G)}V$*, as an* $\mathcal{H}(G)$*-module, lies in* $\mathcal{M}_{\Omega_0}(G)$ *as well.*

Proof. As we have recalled earlier the Bernstein decomposition of $\mathcal{S}(G)\otimes_{\mathcal{H}(G)}V=\prod_\Omega(\mathcal{S}(G)\otimes_{\mathcal{H}(G)}V)(\Omega)$ is a decomposition into $\mathcal{S}(G)$-modules. By assumption V is entirely contained in $(\mathcal{S}(G)\otimes_{\mathcal{H}(G)}V)(\Omega_0)$. On the other hand, V obviously generates $\mathcal{S}(G)\otimes_{\mathcal{H}(G)}V$ as an $\mathcal{S}(G)$-module. Hence we must have

$$\mathcal{S}(G)\otimes_{\mathcal{H}(G)}V=(\mathcal{S}(G)\otimes_{\mathcal{H}(G)}V)(\Omega_0).$$

□

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

Corollary 3.5. *With $\mathcal{B} := r^t_{G,P}(\mathcal{S}(G))$ we have*

(i) $\mathcal{B}_U \cong \epsilon_{U_M}\mathcal{B}\epsilon_U$ *as $(\mathcal{S}(M,U_M),\mathcal{S}(G,U))$-bimodules;*

(ii) $\mathcal{B} \cong \varinjlim_U \mathcal{B}_U$ *as $(\mathcal{S}(M),\mathcal{S}(G))$-bimodules;*

(iii) *the functor $\mathrm{Ind}^G_P : \mathcal{M}^t(M) \to \mathcal{M}^t(G)$ is naturally isomorphic to the functor*

$$E \mapsto \mathrm{Hom}^{\infty}_{\mathcal{S}(M)}(\mathcal{B},E) := \mathcal{S}(G) * \mathrm{Hom}_{\mathcal{S}(M)}(\mathcal{B},E).$$

Proof. (i) By the functoriality of $r^t_{G,P}$ we have $\mathcal{B}\epsilon_U = r^t_{G,P}(\mathcal{S}(G/U))$ where $\mathcal{S}(G/U)$ denotes the left $\mathcal{S}(G)$-submodule of all right U-invariant functions in $\mathcal{S}(G)$. On the other hand, Proposition 3.3 implies that

$$\epsilon_{U_M} r^t_{G,P}(\mathcal{S}(G/U)) = r^t_{G,P}(\mathcal{S}(G/U))^{U_M} = \mathcal{B}_U.$$

(ii) This follows immediately from (i).

(iii) Using Proposition 3.2 we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{S}(G)}(V,\mathrm{Ind}^G_P(E)) &= \mathrm{Hom}_{\mathcal{S}(M)}(r^t_{G,P}(V),E)\\ &= \mathrm{Hom}_{\mathcal{S}(M)}(\mathcal{B}\otimes_{\mathcal{S}(G)} V,E)\\ &= \mathrm{Hom}_{\mathcal{S}(G)}(V,\mathrm{Hom}_{\mathcal{S}(M)}(\mathcal{B},E))\\ &= \mathrm{Hom}_{\mathcal{S}(G)}(V,\mathrm{Hom}^{\infty}_{\mathcal{S}(M)}(\mathcal{B},E)).\end{aligned}$$

□

We remark that by going through the formulae one can see that the natural isomorphism in Proposition 3.3 is explicitly given by

$$\begin{aligned}\mathcal{B}_U \otimes_{\mathcal{S}(G,U)} V^U &\to r^t_{G,P}(V)^{U_M} = r^t_{G,P}(\mathcal{S}(G))^{U_M} \otimes_{\mathcal{S}(G)} V,\\ (\phi\otimes\psi)\otimes v &\mapsto (\phi\otimes\bar{\psi})\otimes v,\end{aligned}$$

where $\bar{\psi} \in \mathcal{S}(G)_N$ denotes the image of $\psi \in \mathcal{S}(G,U)$ under projection.

Lemma 3.6. *Let $U' \subseteq U$ be another compact open subgroup with the same properties as U; we then have*

$$\mathcal{B}_{U'}\epsilon_U \cong \mathcal{S}(M,U'_M)\otimes_{\mathcal{S}(M,U_M)} \mathcal{B}_U$$

as $(\mathcal{S}(M,U'_M),\mathcal{S}(G,U))$-bimodules.

Proof. Quite generally, as a consequence of Lemma 2.3 applied to U as well as U' one obtains the natural isomorphism

$$X = \mathcal{S}(M,U'_M)\otimes_{\mathcal{S}(M,U_M)} \epsilon_{U_M} X$$

for any $\mathcal{S}(M,U'_M)$-module X which is generated by $\epsilon_{U_M}X$. Since $\mathcal{S}(G/U)$ lies in $\mathcal{M}^t_U(G)$ we know from the proof of Proposition 3.3 that $r^t_{G,P}(\mathcal{S}(G/U))$ lies in $\mathcal{M}^t_{U_M}(M) \subseteq \mathcal{M}^t_{U'_M}(M)$. Using Lemma 2.3 again we deduce that $\mathcal{B}_{U'}\epsilon_U \cong r^t_{G,P}(\mathcal{S}(G/U))^{U'_M}$ as an $\mathcal{S}(M,U'_M)$-module is generated by $r^t_{G,P}(\mathcal{S}(G/U))^{U_M} \cong \mathcal{B}_U$. Our assertion therefore is a special case of the initial observation. □

4. The bimodules $\mathcal{B}_U$

Keeping all our notation we will make in this section a detailed investigation of the structure of the $(\mathcal{S}(M,U_M),\mathcal{S}(G,U))$-bimodule

$$\mathcal{B}_U := [\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U)]/\mathcal{R}$$

with $\mathcal{R}=\operatorname{im}(\mathrm{id}-e)=\ker(e)$ for the projector

$$\begin{aligned} e:\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U) &\to \mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U),\\ \phi\otimes\psi &\mapsto \operatorname{vol}_G(U)\cdot\sum_{k\in K/U}\phi*(\psi^{k^{-1}})^P\otimes {}^k\epsilon_U. \end{aligned}$$

First we study $\mathcal{B}_U$ as a left $\mathcal{S}(M,U_M)$-module. As a consequence of Lemma 1.8 (ii) we have, for any $k\in K/U$, the well defined $\mathcal{S}(M,U_M)$-module homomorphism

$$\begin{aligned} \mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U) &\to \mathcal{S}(M,U_M),\\ \phi\otimes\psi &\mapsto \phi*(\psi^{k^{-1}})^P. \end{aligned}$$

By Lemma 1.2 it is zero on $\mathcal{R}$. Hence we may combine these maps into an $\mathcal{S}(M,U_M)$-module homomorphism

$$\begin{aligned} \mathcal{F}:(\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U))/\mathcal{R} &\to \mathcal{S}(M,U_M)^{[K:U]},\\ \phi\otimes\psi &\mapsto \operatorname{vol}_G(U)\cdot(\phi*(\psi^{k^{-1}})^P)_k \end{aligned}$$

into the finitely generated free $\mathcal{S}(M,U_M)$-module $\mathcal{S}(M,U_M)^{[K:U]}$ of rank equal to $[K:U]$. In the reverse direction we have the homomorphism

$$\begin{aligned} \mathcal{S}(M,U_M)^{[K:U]} &\to (\mathcal{S}(M,U_M)\otimes_{\mathcal{S}^+(M,U_M)}\mathcal{S}(G,U))/\mathcal{R},\\ (\phi_k)_k &\mapsto \sum_{k\in K/U}\phi_k\otimes {}^k\epsilon_U+\mathcal{R}. \end{aligned}$$

Using that

$$\epsilon_{U_M}\otimes\psi\equiv\operatorname{vol}_G(U)\cdot\sum_{k\in K/U}(\psi^{k^{-1}})^P\otimes {}^k\epsilon_U \bmod \mathcal{R}$$

we see that $\mathcal{F}$ is a section of this latter map. We in particular obtain the following fact.

Lemma 4.1. *$\mathcal{B}_U$ is a finitely generated projective left $\mathcal{S}(M,U_M)$-module.*

Using once more Lemma 1.2 one can show that the image of $\mathcal{F}$ is characterized by the relations

$$\phi_k=\operatorname{vol}_G(U)\cdot\sum_{k'\in K/U}\phi_{k'}*({}^{k'}\epsilon_U^{k^{-1}})^P$$

for $k\in K/U$. We also remark that

$$\begin{aligned} \mathcal{S}(G,U) &\to M_{[K:U]}(\mathcal{S}(M,U_M)),\\ \psi &\mapsto M(\psi):=\operatorname{vol}_G(U)\cdot(({}^{k'}\psi^{k^{-1}})^P)_{k',k} \end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

is a non-unital ring homomorphism into the algebra of $(r \times r)$-matrices over $\mathcal{S}(M, U_M)$ with $r := [K : U]$. The right multiplication of a $\psi \in \mathcal{S}(G, U)$ on $\mathcal{B}_U$ corresponds under the map $\mathcal{F}$ to the right multiplication of the vector $\mathcal{F}(\cdot)$ by the matrix $M(\psi)$.

Before we begin the much more crucial study of $\mathcal{B}_U$ as a right $\mathcal{S}(G, U)$-module we explain how to identify in a rather conceptual way $\mathcal{B}_U$ as a topological tensor product. All three algebras $\mathcal{S}(G, U)$, $\mathcal{S}(M, U_M)$, and $\mathcal{S}^+(M, U_M)$ are Fréchet algebras, $\mathcal{S}^+(M, U_M)$ is a closed subalgebra of $\mathcal{S}(M, U_M)$, and the algebra homomorphism s is continuous (Lemma 1.8 (i)). The (completed) projective tensor product $\mathcal{S}(M, U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)$ is formed in the following way. First we form the usual completed projective tensor product of complex Fréchet spaces $\mathcal{S}(M, U_M) \hat{\otimes}_{\mathbb{C}} \mathcal{S}(G, U)$ (cf. [**Sch**, III.6.1-3]), and then we consider the continuous linear map:

$$\begin{aligned} \mathcal{S}(M, U_M) \otimes_{\mathbb{C}} \mathcal{S}^+(M, U_M) \otimes_{\mathbb{C}} \mathcal{S}(G, U) &\to \mathcal{S}(M, U_M) \hat{\otimes}_{\mathbb{C}} \mathcal{S}(G, U), \\ \phi \otimes \phi^+ \otimes \psi &\mapsto (\phi * \phi^+) \otimes \psi - \phi \otimes (s(\phi^+) * \psi). \end{aligned}$$

The completed tensor product $\mathcal{S}(M, U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)$ is defined to be the quotient of $\mathcal{S}(M, U_M) \hat{\otimes}_{\mathbb{C}} \mathcal{S}(G, U)$ by the closure of the image of this map. Due to the open mapping theorem this quotient is naturally a Fréchet space again. The natural map

$$\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U) \to \mathcal{S}(M, U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)$$

has dense image. To compute this projective tensor product we use the map $\mathcal{F}$ above. Its image, being characterized by the above finitely many explicit relations, is closed in the Fréchet space $\mathcal{S}(M, U_M)^{[K:U]}$. On the other hand, viewing the map $\mathcal{F}$ as a map on $\mathcal{S}(M, U_M) \otimes_{\mathbb{C}} \mathcal{S}(G, U)$ it extends by continuity to the completion $\mathcal{S}(M, U_M) \hat{\otimes}_{\mathbb{C}} \mathcal{S}(G, U)$ and factorizes through the quotient $\mathcal{S}(M, U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U)$. Hence we have the commutative diagram

$$\begin{array}{ccc} \mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U) & \longrightarrow & \mathcal{S}(M, U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U) \\ \downarrow & & \downarrow \\ (\mathcal{S}(M, U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U))/\mathcal{R} & \xrightarrow[\cong]{\mathcal{F}} & \mathrm{im}(\mathcal{F}) \end{array}$$

Next we look at the continuous linear map

$$\begin{aligned} \mathcal{S}(G, U) &\to \mathcal{S}(M, U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G, U), \\ \psi &\mapsto \epsilon_{U_M} \otimes \psi - \mathrm{vol}_G(U) \cdot \sum_{k \in K/U} (\psi^{k^{-1}})^P \otimes {}^k\epsilon_U \end{aligned}$$

and claim that it is, in fact, the zero map. But in Step 3 of the proof of Proposition 2.1 we have seen that this map vanishes on the dense subspace $\mathcal{H}(G, U)$. This means that the horizontal map in the above diagram is zero on $\mathcal{R}$ and hence that we have the commutative triangle

$$\begin{array}{ccc} \mathcal{B}_U & \longrightarrow & \mathcal{S}(M,U_M)\,\hat{\otimes}_{\mathcal{S}^+(M,U_M)}\,\mathcal{S}(G,U) \\ {\scriptstyle \mathcal{F}}\searrow{\scriptstyle \cong} & & \swarrow \\ & \operatorname{im}(\mathcal{F}) & \end{array}$$

The inverse of the map $\mathcal{F}$ viewed as a map

$$\operatorname{im}(\mathcal{F}) \to \mathcal{S}(M,U_M)\,\hat{\otimes}_{\mathcal{S}^+(M,U_M)}\,\mathcal{S}(G,U),$$
$$(\phi_k)_k \mapsto \sum_{k\in K/U} \phi_k \otimes {}^k\epsilon_U$$

is visibly continuous. It follows that all three maps in the above triangle must be bijections.

Proposition 4.2. $\mathcal{B}_U \xrightarrow{\cong} \mathcal{S}(M,U_M)\,\hat{\otimes}_{\mathcal{S}^+(M,U_M)}\,\mathcal{S}(G,U)$ *as bimodules.*

It is technically important that this completed tensor product can be simplified further at the expense of choosing a strongly (P,U)-positive element $z \in M^+$ in the centre of M as in § 2. In addition to the notation introduced there we need $Z := z^{\mathbb{Z}}$ and $Z^+ := z^{\mathbb{N}_0} = Z \cap M^+$. In $\mathcal{S}(M,U_M)$ we have the closed (central) subalgebras

$$\mathcal{S}_z := \{\phi \in \mathcal{S}(M,U_M) : \operatorname{supp}(\phi) \subseteq U_M Z U_M\}$$

and

$$\mathcal{S}_z^+ := \{\phi \in \mathcal{S}(M,U_M) : \operatorname{supp}(\phi) \subseteq U_M Z^+ U_M\} = \mathcal{S}_z \cap \mathcal{S}^+(M,U_M).$$

Lemma 4.3. $\mathcal{S}_z\,\hat{\otimes}_{\mathcal{S}_z^+}\,\mathcal{S}^+(M,U_M) \cong \mathcal{S}(M,U_M)$.

Proof. We recall that M is the increasing union of the subsets $z^i M^+$, and consider first the case that the intersection $\bigcap_{i\in\mathbb{Z}} z^i M^+$ is non-empty and hence contains some element. Then there exists, for any $i \in \mathbb{Z}$, an element $m_i \in M^+$ such that $m = z^i m_i$. In particular $z^{-1} = m(m_{-1})^{-1} = m_i z^i (m_{-1})^{-1}$. By choosing i large enough so that $z^i(m_{-1})^{-1} \in M^+$ we see that $z^{-1} \in M^+$ and consequently that $M^+ = M$. In this case our assertion therefore is trivial. We henceforth assume that $\bigcap_{i\in\mathbb{Z}} z^i M^+ = \emptyset$. As a space M then decomposes into the open and closed subsets $z^i M^+ \setminus z^{i+1} M^+$ for $i \in \mathbb{Z}$ which are U_M-bi-invariant. Correspondingly we introduce the closed subspaces

$$\mathcal{S}_i^+ := \{\phi \in \mathcal{S}(M,U_M) : \operatorname{supp}(\phi) \subseteq z^i M^+ \setminus z^{i+1} M^+\}$$

of $\mathcal{S}(M,U_M)$. Acting by z is a topological isomorphism

$$\mathcal{S}_i^+ \underset{z\cdot}{\xrightarrow{\cong}} \mathcal{S}_{i+1}^+.$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

We claim that the map

$$\mu : \mathcal{S}_z \,\hat{\otimes}_{\mathbb{C}}\, \mathcal{S}_0^+ \to \mathcal{S}(M, U_M),$$
$$\varphi \otimes \phi \mapsto \varphi * \phi$$

is a topological isomorphism. The left-hand side contains as a dense subspace the algebraic tensor product $\mathcal{H}_z \otimes_{\mathbb{C}} \mathcal{S}_0^+$ where $\mathcal{H}_z$ denotes the subalgebra of all those functions in $\mathcal{S}_z$ which have compact support. The right-hand side contains as a dense subspace the algebraic direct sum $\bigoplus_{i\in\mathbb{Z}} \mathcal{S}_i^+$. Obviously, the map μ restricts to a linear isomorphism between these two subspaces. We therefore have to show that it identifies the corresponding subspace topologies. The Fréchet space structure on $\mathcal{S}(M, U_M)$ is given by the norms

$$\nu_{M,r}(\phi) = \sup_{m\in M} |\phi(m)| \varXi_M(m)^{-1}(1 + \sigma(m))^r$$

for $r > 0$. Hence on $\mathcal{H}_z \otimes_{\mathbb{C}} \mathcal{S}_0^+$ it is given by the tensor product norms $\nu_{M,r} \otimes \nu_{M,r}$. As μ clearly is continuous it suffices to find, for any given $r > 0$, an $s > 0$ and a constant $C > 0$ such that

$$\nu_{M,s}(\mu(x)) \geqslant C \cdot (\nu_{M,r} \otimes \nu_{M,r})(x)$$

holds true for any $x \in \mathcal{H}_z \otimes_{\mathbb{C}} \mathcal{S}_0^+$. Since the functions $\phi_z^i := {}^{z^i}\epsilon_{U_M}$ for $i \in \mathbb{Z}$ constitute a basis of $\mathcal{H}_z$ any element $x \in \mathcal{H}_z \otimes_{\mathbb{C}} \mathcal{S}_0^+$ can be written in a unique way as a finite sum

$$x = \sum_i \phi_z^i \otimes \phi_i.$$

Its image under the map μ then is

$$\mu(x) = \sum_i {}^{z^i}\phi_i.$$

Since the summands in this expression have pairwise disjoint support we obtain

$$\begin{aligned}\nu_{M,s}(\mu(x)) &= \sup_i \nu_{M,r}({}^{z^i}\phi_i)\\ &= \sup_i \sup_{m\in M} |\phi_i(z^{-i}m)| \varXi_M(m)^{-1}(1 + \sigma(m))^s\\ &= \sup_i \sup_{m\in M} |\phi_i(m)| \varXi_M(z^i m)^{-1}(1 + \sigma(z^i m))^s.\end{aligned}$$

From the very definition of the function $\varXi_M$ one has $\varXi_M(y) = 1$ and $\varXi_M(ym) = \varXi_M(m)$ for any $m \in M$ and any element y in the centre of M. Hence

$$\nu_{M,s}(\mu(x)) = \sup_i \sup_{m\in M} |\phi_i(m)| \varXi_M(m)^{-1}(1 + \sigma(z^i m))^s.$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

For $\nu_{M,r} \otimes \nu_{M,r}(x)$ on the other hand we certainly have

$$\begin{aligned}\nu_{M,r} \otimes \nu_{M,r}(x) &\leqslant \sum_i \nu_{M,r}(\phi_z^i)\nu_{M,r}(\phi_i) \\ &= \sum_i \mathrm{vol}_M(U_M)^{-1}(1+\sigma(z^i))^r \sup_m |\phi_i(m)| \Xi_M(m)^{-1}(1+\sigma(m))^r \\ &= \mathrm{vol}_M(U_M)^{-1} \cdot \sum_i (1+\sigma(z^i))^r \sup_m |\phi_i(m)| \Xi_M(m)^{-1}(1+\sigma(m))^r.\end{aligned}$$

As a consequence of [**Sil**, 4.2.5] there is an $r_0 > 0$ such that $C' := \mathrm{vol}_M(U_M)^{-1} \cdot \sum_i (1+\sigma(z^i))^{-r_0} < \infty$. Hence

$$\nu_{M,r} \otimes \nu_{M,r}(x) \leqslant C' \cdot \sup_i \sup_m |\phi_i(m)| \Xi_M(m)^{-1}(1+\sigma(m))^r(1+\sigma(z^i))^{r+r_0}.$$

According to (a special case of) [**Vi2**, Proposition 3.1.2] there is an $r_1 > 0$ and, for any $r > 0$, a constant $C_r > 0$ such that

$$(1+\sigma(y))^r(1+\sigma(m))^r \leqslant C_r \cdot (1+\sigma(ym))^{2r+r_1}$$

for any $m \in M$ and any y in the centre of M. It follows that

$$\begin{aligned}\nu_{M,r} \otimes \nu_{M,r}(x) &\leqslant C' C_{r+r_0} \cdot \sup_i \sup_m |\phi_i(m)| \Xi_M(m)^{-1}(1+\sigma(z^i m))^{2r+2r_0+r_1} \\ &= C' C_{r+r_0} \cdot \nu_{M,2r+2r_0+r_1}(\mu(x)),\end{aligned}$$

which is what we wanted to establish.

In the same way we obtain that $\mathcal{S}_z^+ \hat{\otimes}_{\mathbb{C}} \mathcal{S}_0^+ \xrightarrow{\cong} \mathcal{S}^+(M,U_M)$. We now conclude that

$$\mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}^+(M,U_M) \cong \mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} (\mathcal{S}_z^+ \hat{\otimes}_{\mathbb{C}} \mathcal{S}_0^+) \cong \mathcal{S}_z \hat{\otimes}_{\mathbb{C}} \mathcal{S}_0^+ \cong \mathcal{S}(M,U_M).$$

□

Lemma 4.4. $\mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}(G,U) \xrightarrow{\cong} \mathcal{S}(M,U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G,U)$.

Proof. Using the previous lemma we have

$$\begin{aligned}\mathcal{S}(M,U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G,U) &\cong (\mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}^+(M,U_M)) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G,U) \\ &\cong \mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}(G,U).\end{aligned}$$

□

Corollary 4.5. $\mathcal{B}_U \cong \mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}(G,U)$ *as right* $\mathcal{S}(G,U)$*-modules.*

The structure of the tensor product $\mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}(G,U)$ is closely related to the spectral theory of the left multiplication operator

$$L_z : \mathcal{S}(G,U) \xrightarrow{\psi_z *\cdot} \mathcal{S}(G,U)$$

on the Fréchet space $\mathcal{S}(G,U)$. We recall that the Waelbroeck spectrum $\sigma(L_z) := \mathbb{C} \cup \{\infty\} \setminus \rho(L_z)$ of the operator L_z is the complement in $\mathbb{C} \cup \{\infty\}$ of the set $\rho(L_z)$ of all $\lambda \in \mathbb{C} \cup \{\infty\}$ for which there is an open neighbourhood U_λ such that

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

(1) $L_z - \mu \cdot \mathrm{id}$ is invertible for any $\mu \in U_\lambda \cap \mathbb{C}$, and

(2) $\{(L_z - \mu \cdot \mathrm{id})^{-1} : \mu \in U_\lambda \cap \mathbb{C}\}$ is a bounded subset of the locally convex vector space $\mathcal{L}_{\mathrm{b}}(\mathcal{S}(G, U))$.

Here we denote by $\mathcal{L}_{\mathrm{b}}(X)$ the locally convex vector space of all continuous linear endomorphisms of a Fréchet space X equipped with the strong topology.

On the other hand, the naive spectrum of the element ψ_z in the Fréchet algebra $\mathcal{S}(G, U)$ is defined to be the set

$$\sigma(\psi_z) := \{\lambda \in \mathbb{C} : \psi_z - \lambda \notin \mathcal{S}(G, U)^\times\}.$$

Due to certain simplifying features of the algebra $\mathcal{S}(G, U)$ listed in the subsequent lemma these two sets turn out to coincide.

Lemma 4.6.

(i) *Every simple unital (left) $\mathcal{S}(G, U)$-module is finite dimensional over $\mathbb{C}$.*

(ii) *$\mathcal{S}(G, U)^\times$ is open in $\mathcal{S}(G, U)$.*

(iii) *The map $\mathcal{S}(G, U) \xrightarrow{\cdot^{-1}} \mathcal{S}(G, U)$ of passing to the inverse is continuous.*

(iv) *Given any $\psi \in \mathcal{S}(G, U)$ there is a constant $c > 0$ such that $\psi - \lambda \in \mathcal{S}(G, U)^\times$ for any $\lambda \in \mathbb{C}$ with $|\lambda| > c$, and $\lim_{\lambda \to \infty} (\psi - \lambda)^{-1} = 0$.*

Proof. (i) We have seen in Lemma 2.3 that the functor

$$\begin{gathered} \mathcal{M}_U^t(G) \xrightarrow{\sim} \mathcal{M}(\mathcal{S}(G, U)), \\ V \mapsto V^U \end{gathered}$$

is an equivalence of categories. Hence it suffices to show that $\dim_{\mathbb{C}} V^U < \infty$ for any simple non-degenerate $\mathcal{S}(G)$-module V. But according to [**SSZ**, Appendix, Proposition 3] any such V in particular is an irreducible and hence admissible smooth G-representation. The assertions (ii) and (iii) are shown in [**Vi1**, Theorem 29.3]. (iv) Taking [**Vi1**, Propositions 13 and 18] into account this is a version of [**Vi1**, Lemma 16] with the same proof. □

Lemma 4.7. $\sigma(L_z) = \sigma(\psi_z)$.

Proof. Suppose that the operator $L_z - \mu \cdot \mathrm{id}$ on $\mathcal{S}(G, U)$ is invertible. We then find a $\phi \in \mathcal{S}(G, U)$ such that $(\psi_z - \mu) * \phi = 1$. As a consequence of Lemma 4.6 (i) the element $(\phi * (\psi_z - \mu)) - 1$ is contained in every maximal left ideal. It follows that $\mathcal{S}(G, U) * (\psi_z - \mu) = \mathcal{S}(G, U)$, i.e. that $\psi_z - \mu$ also has a left inverse and therefore is a unit in $\mathcal{S}(G, U)$. This shows that $\sigma(\psi_z) = \mathbb{C} \setminus \rho_{\mathrm{naive}}(L_z)$ where

$$\rho_{\mathrm{naive}}(L_z) := \{\lambda \in \mathbb{C} : L_z - \lambda \cdot \mathrm{id} \text{ is invertible}\}.$$

We obviously have $\rho(L_z) \cap \mathbb{C} \subseteq \rho_{\mathrm{naive}}(L_z)$. On the other hand, as a consequence of Lemma 4.6 (ii), the set $\rho_{\mathrm{naive}}(L_z)$, as the preimage of $\mathcal{S}(G,U)^\times$ under the continuous map

$$\begin{aligned} \mathbb{C} &\to \mathcal{S}(G,U), \\ \lambda &\mapsto \psi_z - \lambda, \end{aligned}$$

is open in $\mathbb{C}$. Furthermore, by Lemma 4.6 (iii) and [**B-TVS**, III.31, Proposition 6], the composed map

$$\begin{aligned} \rho_{\mathrm{naive}}(L_z) \to \mathcal{S}(G,U)^\times &\xrightarrow{\cdot^{-1}} \mathcal{S}(G,U)^\times \xrightarrow{*} \mathcal{L}_{\mathrm{b}}(\mathcal{S}(G,U)), \\ \lambda \mapsto \psi_z - \lambda& \end{aligned}$$

is continuous. Finally, Lemma 4.6 (iv) implies that $\rho_{\mathrm{naive}}(L_z) \cup \{\infty\}$ is open in $\mathbb{C} \cup \{\infty\}$ and that this composed map extends by zero continuously to a map from $\rho_{\mathrm{naive}}(L_z) \cup \{\infty\}$ to $\mathcal{L}_{\mathrm{b}}(\mathcal{S}(G,U))$. On a compact neighbourhood of any point in $\rho_{\mathrm{naive}}(L_z) \cup \{\infty\}$ the image under this map therefore is compact and hence bounded in $\mathcal{L}_{\mathrm{b}}(\mathcal{S}(G,U))$. We conclude that $\rho(L_z) = \rho_{\mathrm{naive}}(L_z) \cup \{\infty\}$. □

Since the Waelbroeck spectrum by construction is compact it follows that $\sigma(\psi_z)$ is a compact subset of the complex plane. We let $\mathcal{O}(\sigma(\psi_z))$ denote the topological algebra of germs of holomorphic functions on $\sigma(\psi_z)$.

Proposition 4.8.

(i) *There is a unique continuous unital algebra homomorphism*

$$\begin{aligned} \mathcal{O}(\sigma(\psi_z)) &\to \mathcal{L}_{\mathrm{b}}(\mathcal{S}(G,U)), \\ f &\mapsto f(L_z) \end{aligned}$$

such that $\iota(L_z) = L_z$ where $\iota : \sigma(\psi_z) \xrightarrow{\subseteq} \mathbb{C}$; every map in the image of this homomorphism is an $(\mathcal{S}^+(M,U_M), \mathcal{S}(G,U))$-bimodule endomorphism of $\mathcal{S}(G,U)$.

(ii) *If $\sigma(\psi_z) = A_1 \dot\cup \cdots \dot\cup A_m$ is a disjoint decomposition into closed subsets then there is a corresponding $(\mathcal{S}^+(M,U_M), \mathcal{S}(G,U))$-bimodule decomposition $\mathcal{S}(G,U) = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_m$ such that $\sigma(L_z \mid \mathcal{S}_j) = A_j$ for any $1 \leqslant j \leqslant m$.*

Proof. This is, in view of Lemma 4.7, a special case of [**Vas**, III.3.10] or [**EP**, 2.5.7] for (i) and [**Vas**, III.3.11] for (ii). We point out that in (ii) the decomposition of $\sigma(\psi_z)$ gives rise, through the characteristic functions of the sets A_j, to a decomposition $1 = e_1 + \cdots + e_m$ of the unit element in $\mathcal{O}(\sigma(\psi_z))$ into a sum of idempotents e_j; then $\mathcal{S}_j = e_j(L_z)(\mathcal{S}(G,U))$. The unicity is a consequence of Runge's theorem (cf. [**Con**, III.8.1]). □

We now will construct a particular such decomposition of the spectrum $\sigma(\psi_z)$. Given a linear operator T on a finite-dimensional complex vector space X we let $E(T;X)$ denote the set of its eigenvalues. We also need the notation $|A| := \{|\mu| : \mu \in A\}$ for any subset $A \subseteq \mathbb{C}$.

Lemma 4.9. $\sigma(\psi_z) = \bigcup_V E(\psi_z; V^U) \subseteq \{0\} \cup \bigcup_V E(z; V_N^{U_M})$ *where V runs over all irreducible tempered representations in $\mathcal{M}_U(G)$.*

Proof. From Lemma 4.6 (i) and its proof we know that the V^U in question constitute all the simple unital $\mathcal{S}(G,U)$-modules and that these all are finite-dimensional vector spaces. If λ is an eigenvalue of ψ_z on some V^U then $\psi_z - \lambda$ cannot be a unit in $\mathcal{S}(G,U)$ and hence $\lambda \in \sigma(\psi_z)$. Vice versa, if $\psi_z - \lambda$ is not a unit in $\mathcal{S}(G,U)$ then, by an argument symmetric to the one given at the beginning of the proof of Lemma 4.7, this element cannot have a left inverse which means it is contained in a maximal left ideal. But then ψ_z has an eigenvector with eigenvalue λ on the corresponding simple quotient module. This establishes the first equality in the assertion. Next we consider, for each individual V, the projection map

$$V^U \to V_N^{U_M}.$$

The action of ψ_z on V^U corresponds under this map to the action of the group element z on $V_N^{U_M}$. Moreover, from [**Bus**, § 3.4, Theorem 1] we know that the map is surjective and that ψ_z is nilpotent on its kernel. It follows that $E(\psi_z; V^U) \subseteq \{0\} \cup E(z; V_N^{U_M})$. ☐

We may (cf. the proof of [**BK**, Lemma 6.14]) and always will assume in the following that our strongly positive element z lies in the maximal split torus Z_M in the centre of M.

Proposition 4.10. $|\sigma(\psi_z)|$ *is finite.*

Proof. By the previous lemma it suffices to show that the set $\bigcup_V |E(z; V_N^{U_M})|$, where V runs over all irreducible tempered representations in $\mathcal{M}_U(G)$, is finite. Since $\mathcal{M}_U(G)$ by [**Bus**, § 1.5, Proposition 3.i] is a finite product of Bernstein components it suffices to let V run over all irreducible tempered representations in a fixed Bernstein component $\mathcal{M}_\Omega(G)$. This means that there is a parabolic subgroup $Q_0 \subseteq G$ with Levi component L_0 and a supercuspidal representation τ_0 of L_0 such that any irreducible V' in $\mathcal{M}_\Omega(G)$ is a subquotient of a parabolic induction $\mathrm{Ind}_{Q_0}^G(\chi\tau_0)$ for some unramified character χ of L_0. On the other hand, any irreducible tempered V'' has a discrete support (cf. [**Wal**, III.4.1]) which similarly means that there is a parabolic subgroup $Q \subseteq G$ with Levi component L and a smooth discrete series representation τ of L such that V'' is a subquotient of the parabolic induction $\mathrm{Ind}_Q^G(\tau)$. For our tempered V in $\mathcal{M}_\Omega(G)$ we may assume that $L \supseteq L_0$ and that τ is a subquotient of $\mathrm{Ind}_{Q_0 \cap L}^L(\zeta\tau_0)$ for some unramified character ζ of L_0. Applying [**Sil**, 5.4.5.1] to these finitely many groups L (which contain L_0) we see that there are, up to unitary unramified twist, only finitely many possibilities for the occurring characters ζ. This reduces our assertion to the claim that the set

$$\bigcup_{\chi_1} |E(z; \mathrm{Ind}_{Q_0}^G(\chi_1\tau_0)_N^{U_M})|,$$

where χ_1 runs over all unitary unramified characters of L_0, is finite. By the geometric lemma (cf. [**Cas**, Theorem 6.3.5]) the Jacquet module $\mathrm{Ind}_{Q_0}^G(\chi_1\tau_0)_N$, as an M-representation, has a filtration whose subquotients are of the form

$$\mathrm{Ind}_{wQ_0w^{-1}\cap M}^M({}^w(\chi_1\tau_0))$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

where w runs over finitely many appropriate group elements such that $wL_0w^{-1} \subseteq M$. If χ_0 denotes the central character of τ_0 then $z \in Z_M \subseteq Z_{wL_0w^{-1}}$ acts on $\mathrm{Ind}^M_{wQ_0w^{-1}\cap M}({}^w(\chi_1\tau_0))$ through multiplication by the scalar

$$\delta^{-1/2}_{wQ_0w^{-1}\cap M}(z)\chi_0(w^{-1}zw)\chi_1(w^{-1}zw).$$

We therefore obtain that the set under consideration is equal to the finite set of numbers $\delta^{-1/2}_{wQ_0w^{-1}\cap M}(z)|\chi_0(w^{-1}zw)|$. ☐

This last result says that there are finitely many real numbers $0 \leqslant R_1 < \cdots < R_m$ such that the spectrum

$$\sigma(\psi_z) = \sigma_{R_1}(\psi_z) \,\dot{\cup}\, \cdots \,\dot{\cup}\, \sigma_{R_m}(\psi_z)$$

decomposes into the finitely many closed subsets $\sigma_{R_j}(\psi_z) := \{\lambda \in \sigma(\psi_z) : |\lambda| = R_j\}$. Let

$$\mathcal{S}(G,U) = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_m$$

be the corresponding (Proposition 4.8 (ii)) decomposition of the $(\mathcal{S}^+(M,U_M),\mathcal{S}(G,U))$-bimodule $\mathcal{S}(G,U)$.

Lemma 4.11.

(i) $R_m = 1$.

(ii) *If* $M \neq G$ *then* $\sigma_1(\psi_z) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *is the full unit circle.*

Proof. (i) The inequality $R_m \leqslant 1$ is an immediate consequence of the classical criterion for the temperedness of an admissible smooth representation (cf. [**Wal**, III.2.2]). This argument can be expanded to give the equality. But for the latter there is another quite simple observation which avoids representation theory. According to [**Vi1**, Proposition 28] the algebra $\mathcal{S}(G,U)$ is contained in the reduced C^*-algebra $C^*_r(G,U)$ which is defined to be the completion of $\mathcal{H}(G,U)$ in the operator norm on $L^2(U\backslash G/U)$. In addition, [**Vi1**, Propositions 13 and 18] say that $\mathcal{S}(G,U)^\times = \mathcal{S}(G,U) \cap C^*_r(G,U)^\times$. We therefore may alternatively show that the spectral radius of ψ_z as an element of $C^*_r(G,U)$ is greater than or equal to 1. But a simple straightforward computation shows that every power ψ_z^i has the same L^2-norm equal to $\mathrm{vol}_G(U)^{-1}$ and hence has an operator norm greater than or equal to 1 on $L^2(U\backslash G/U)$.

(ii) Let V be an irreducible tempered representation in $\mathcal{M}_U(G)$ such that $E(z;V_N^{U_M})$ contains a λ with $|\lambda| = 1$. For the action of Z_M the Jacquet module V_N decomposes into generalized eigenspaces for finitely many characters $\chi_1,\ldots,\chi_t$. After renumbering we may assume that precisely the characters $\chi_1,\ldots,\chi_s$ for some $0 \leqslant s \leqslant t$ are unitary. Let E denote the direct sum of the generalized eigenspaces corresponding to these unitary characters. By [**Wal**, III.3.1] this E is an admissible tempered representation of M. The classical criterion for temperedness (cf. [**Wal**, III.2.2]) implies that any $|\chi_i|$ is of the form $|\chi_i| = \prod_\alpha |\alpha|_k^{c_\alpha(i)}$ with real numbers $c_\alpha(i) \geqslant 0$ and α running over the roots which

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

are positive and simple for N. By [**BK**, Lemma 6.14] we must have $|\alpha|_k(z) < 1$ since z is strongly (P,U)-positive. It follows that $|\chi_i(z)| < 1$ for any $s < i \leqslant t$. This means that $\lambda \in E(z; E^{U_M})$. Now let χ be any unitary unramified character of M. Then $V_1 := \mathrm{Ind}_P^G(\chi \otimes E)$ is an admissible tempered representation of finite length in $\mathcal{M}_U(G)$ such that $\chi \otimes E$ is a quotient of its Jacquet module $(V_1)_N$. It follows that $\chi(z)\lambda \in E(z; (V_1)_N^{U_M})$ and a fortiori that $\chi(z)\lambda \in E(z; (V_2)_N^{U_M})$ for some irreducible tempered subquotient V_2 of V_1. We obtain $\chi(z)\lambda \in \sigma(\psi_z)$. If $M \neq G$ then $\delta_P(z)$ is a positive real number not equal to 1. Hence the values at z of the unramified characters of M of the form δ_P^{ia}, with $a \in \mathbb{R}$ arbitrary, realize the full unit circle. □

Before we apply these results to the tensor product $\mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}(G,U)$ we want to recognize the Fréchet algebras $\mathcal{S}_z \supseteq \mathcal{S}_z^+$ in more familiar terms. The Schwartz algebra $\mathcal{S}(\mathbb{Z})$ of the discrete group $\mathbb{Z}$ is the algebra of all functions $\phi : \mathbb{Z} \to \mathbb{C}$ which satisfy

$$\nu_k(\phi) := \sup_{i \in \mathbb{Z}} |\phi(i)|(1+|i|)^k < \infty$$

for any $k \in \mathbb{N}$; its Fréchet topology is defined by the norms ν_k.

Lemma 4.12. *The map*

$$\mathcal{S}_z \xrightarrow{\cong} \mathcal{S}(\mathbb{Z}),$$
$$\phi \mapsto [i \mapsto \mathrm{vol}_M(U_M)\phi(z^i)]$$

is an isomorphism of Fréchet algebras.

Proof. Recall that the norms on the left-hand side come by restriction from the norms

$$\nu_{M,r}(\phi) = \sup_{m \in M} |\phi(m)| \varXi_M(m)^{-1}(1+\sigma(m))^r.$$

Since $\varXi_M$ is trivial on the centre of M and since σ can be chosen in such a way that it satisfies $\sigma(y^i) = |i|\sigma(y)$ for any y in the centre of M we obtain

$$\nu_{M,r}(\phi) = \sup_{i \in \mathbb{Z}} |\phi(z^i)|(1+|i|\sigma(z))^r$$

for any $\phi \in \mathcal{S}_z$. □

Let $\boldsymbol{D}$ denote the closed unit ball in the complex plane, $\boldsymbol{D}^0$ its interior, and $\boldsymbol{T}$ the unit circle. The Fourier transform

$$\mathcal{S}(\mathbb{Z}) \xrightarrow{\cong} C^\infty(\boldsymbol{T}),$$
$$\phi \mapsto \sum_{i \in \mathbb{Z}} \phi(i)\xi^i$$

is an isomorphism of Fréchet algebras (cf. [**Tre**, Theorem 51.3]). Under the composed isomorphism

$$\mathcal{S}_z \cong \mathcal{S}(\mathbb{Z}) \cong C^\infty(\boldsymbol{T})$$

the subalgebra $\mathcal{S}_z^+$ in the left-hand side corresponds to the subalgebra $\tilde{\mathcal{O}}(\boldsymbol{D})$ of all continuous functions on $\boldsymbol{D}$ whose restriction to $\boldsymbol{T}$, respectively to $\boldsymbol{D}^0$, lies in $C^\infty(\boldsymbol{T})$, respectively is holomorphic.

Lemma 4.13. *$\mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}_j \cong C^\infty(\boldsymbol{T}) \hat{\otimes}_{\tilde{\mathcal{O}}(\boldsymbol{D})} \mathcal{S}_j = 0$ for any $j \neq m$.*

Proof. From Lemma 4.11 (i) we know that $R_j < 1$ for $j \neq m$. Let U be an open annulus which contains the unit circle and whose inner radius is bigger than R_j. From [**EP**, p. 135] we then have

$$\mathcal{O}(U) \hat{\otimes}_{\mathcal{O}(\mathbb{C})} \mathcal{S}_j = 0.$$

To avoid confusion we point out that our completed tensor product is the one in [**EP**] made Hausdorff. Here the ring $\mathcal{O}(\mathbb{C})$ of entire holomorphic functions acts on $\mathcal{S}_j$ through the unique continuous homomorphism of algebras $\mathcal{O}(\mathbb{C}) \to \mathcal{L}_\mathrm{b}(\mathcal{S}_j)$ which sends the complex variable ξ to the operator L_z. In particular this arrow can be obtained as the composite

$$\mathcal{O}(\mathbb{C}) \to \tilde{\mathcal{O}}(\boldsymbol{D}) \to \mathcal{L}_\mathrm{b}(\mathcal{S}_j),$$

where the left arrow is the obvious restriction map and the right arrow comes from the $\mathcal{S}_z^+$-module structure of $\mathcal{S}_j$. We see that $C^\infty(\boldsymbol{T}) \hat{\otimes}_{\tilde{\mathcal{O}}(\boldsymbol{D})} \mathcal{S}_j$ is a quotient of

$$0 = C^\infty(\boldsymbol{T}) \hat{\otimes}_{\mathcal{O}(U)} (\mathcal{O}(U) \hat{\otimes}_{\mathcal{O}(\mathbb{C})} \mathcal{S}_j) = C^\infty(\boldsymbol{T}) \hat{\otimes}_{\mathcal{O}(\mathbb{C})} \mathcal{S}_j$$

and hence is zero. □

So far we have seen that

$$\mathcal{B}_U \cong \mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}(G,U) \cong \mathcal{S}_z \hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}_m$$

as right $\mathcal{S}(G,U)$-modules. To deal with the summand $\mathcal{S}_m$ we need to extend the holomorphic functional calculus from Proposition 4.8.

Proposition 4.14. *There is a unique continuous unital algebra homomorphism from $C^\infty(\boldsymbol{T})$ into $\mathcal{L}_\mathrm{b}(\mathcal{S}_m)$ which sends the complex variable ξ to $L_z \mid \mathcal{S}_m$.*

The proof requires some preparation. In particular we need to recall a few basic facts from C^*-algebra theory. As in the proof of Lemma 4.11 we consider the reduced C^*-algebra $\mathcal{A} := C_r^*(G,U)$. We let $\hat{\mathcal{A}}$ denote the space (of isomorphism classes) of simple unital $\mathcal{A}$-modules equipped with the Jacobson topology [**Dix**, § 3.1]. One has the following properties.

- The obvious restriction map induces a bijection between $\hat{\mathcal{A}}$ and the set of isomorphism classes of simple unital $\mathcal{S}(G,U)$-modules [**SSZ**, pp. 206, 207]. In particular, by Lemma 4.6 (i), any module in $\hat{\mathcal{A}}$ is finite dimensional.

- Any module X in $\hat{\mathcal{A}}$ carries an (up to scalar multiples) unique inner product for which X becomes a unitary representation of $\mathcal{A}$ [**Dix**, 2.9.6]. In particular, the finite-dimensional space $\mathrm{End}_\mathbb{C}(X)$ carries a distinguished operator norm $\|\cdot\|_X$.

- Let $\|\cdot\|$ denote the unique C^*-algebra norm on $\mathcal{A}$; then [**Dix**, 2.7.3]

$$\|a\| = \sup_{X \in \hat{\mathcal{A}}} \|a\|_X \quad \text{for any } a \in \mathcal{A}.$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

- For any $a \in \mathcal{A}$ the function

$$\hat{\mathcal{A}} \to \mathbb{R},$$
$$X \mapsto \|a\|_X$$

is lower semi-continuous [**Dix**, 3.3.2].

The last two properties imply that given any dense $\mathcal{D} \subseteq \hat{\mathcal{A}}$ we have

$$\|a\| = \sup_{X \in \mathcal{D}} \|a\|_X \quad \text{for any } a \in \mathcal{A}.$$

This can equivalently be expressed by saying that the natural map

$$(\mathcal{A}, \|\|) \to \prod_{X \in \mathcal{D}}^{*} (\mathrm{End}_{\mathbb{C}}(X), \|\cdot\|_X)$$
$$a \mapsto (X \xrightarrow{a\cdot} X)_X,$$

where $\prod^*$ denotes the direct product in the sense of Banach spaces is a closed isometric embedding.

In order to describe the specific set $\mathcal{D}$ to which we will apply this observation let us first consider a single module X in $\hat{\mathcal{A}}$. This module is of the form $X = V^U$ for some irreducible tempered representation V in $\mathcal{M}_U(G)$ which is unique up to isomorphism. As recalled in the proof of Lemma 4.9 the projection map

$$V^U \to V_N^{U_M}$$

is surjective, ψ_z is nilpotent on its kernel, and the action of ψ_z on V^U corresponds to the action of z on $V_N^{U_M}$.

On the other hand, due to the presence of the unit element $\epsilon_U \in \mathcal{S}(G, U)$ the map in Proposition 4.8 (i) gives rise to the unique continuous unital algebra homomorphism

$$\mathcal{O}(\sigma(\psi_z)) \to \mathcal{S}(G, U),$$
$$f \mapsto \varphi_f := f(L_z)(\epsilon_U)$$

such that $\varphi_\iota = \psi_z$ and

$$f(L_z)(\varphi) = \varphi_f * \varphi \quad \text{for any } \varphi \in \mathcal{S}(G, U).$$

If $e_j \in \mathcal{O}(\sigma(\psi_z))$ denotes the characteristic function of $\sigma_{R_j}(\psi_z)$ and if we put $\epsilon_j := \varphi_{e_j}$ then

$$\mathcal{S}_j = e_j(L_z)(\mathcal{S}(G, U)) = e_j(L_z)(\epsilon_U) * \mathcal{S}(G, U) = \epsilon_j * \mathcal{S}(G, U).$$

Moreover, the idempotents ϵ_j are orthogonal to each other and satisfy $\epsilon_U = \epsilon_1 + \cdots + \epsilon_m$. For our module $X = V^U$ we therefore obtain the $\mathcal{S}^+(M, U_M)$-invariant decomposition

$$X = X_1 \oplus \cdots \oplus X_m \quad \text{with } X_j := \mathcal{S}_j X = \epsilon_j X$$

and such that $E(\psi_z; X_j) \subseteq \sigma_{R_j}(\psi_z)$. It follows that either $R_1 > 0$ and the map $X \xrightarrow{\cong} V_N^{U_M}$ is bijective or $R_1 = 0$ and the restricted map $X_2 \oplus \cdots \oplus X_m \xrightarrow{\cong} V_N^{U_M}$ is bijective. In both cases the respective bijection further restricts to the bijection

$$X_m \xrightarrow{\cong} (V_N^w)^{U_M},$$

where V_N^w denotes the tempered direct summand of the Jacquet module V_N as defined in [**Wal**, III.3] (cf. the proof of Lemma 4.11 (ii)). As an admissible tempered M-representation V_N^w in a unique way is, by [**SSZ**, Appendix, Proposition 1], a non-degenerate $\mathcal{S}(M)$-module. We see that the natural $\mathcal{H}(M, U_M) = \mathcal{H}^+(M, U_M)[\phi_z^{-1}]$-module structure on $X_m = (V_N^w)^{U_M}$ extends uniquely to an $\mathcal{S}(M, U_M)$-module structure.

We now define

$$\mathcal{D} := \text{the set of all } X = V^U \text{ in } \hat{\mathcal{A}} \text{ such that } V_N^w \text{ is semisimple as a smooth } M\text{-representation.}$$

Suppose that $X = V^U$ lies in $\mathcal{D}$. Since any simple and hence any finite length semisimple $\mathcal{S}(M, U_M)$-module naturally extends to a unitary $C_r^*(M, U_M)$-representation [**SSZ**, pp. 206, 207] the $\mathcal{S}_z^+[\phi_z^{-1}]$-action on the finite-dimensional space X_m in this case therefore extends naturally to a unitary representation of the C^*-completion $C^*(\mathbb{Z})$ of $\mathcal{S}_z \cong \mathcal{S}(\mathbb{Z})$. This means we have the commutative diagram of (non-unital) algebra homomorphisms:

$$\begin{array}{ccc} \mathcal{S}_m & \xrightarrow{\subseteq} \epsilon_m * \mathcal{A} \subseteq \mathcal{A} \longrightarrow & \prod^*_{X\in\mathcal{D}} \operatorname{End}_{\mathbb{C}}(X) \\ \uparrow{\scriptstyle s_m} & & \uparrow \\ \tilde{\mathcal{O}}(\boldsymbol{D})[\xi^{-1}] & \xrightarrow{\subseteq} & C(\boldsymbol{T}) \cong C^*(\mathbb{Z}) \end{array} \tag{4.1}$$

Here $C(\boldsymbol{T})$ denotes the C^*-algebra of continuous functions on $\boldsymbol{T}$ and the isomorphism in the lower right-hand corner is another instance of the Fourier transform. The horizontal arrows are the obvious ones. The right perpendicular arrow is the direct product (cf. [**Dix**, 2.2.3]) of the composed homomorphisms $C^*(\mathbb{Z}) \to \operatorname{End}_{\mathbb{C}}(X_m) \to \operatorname{End}_{\mathbb{C}}(X)$ with the second arrow being the extension by zero. To describe the left perpendicular arrow we recall that the left $\mathcal{S}_z^+$-module structure on $\mathcal{S}(G, U)$ is given by the continuous algebra homomorphism

$$\mathcal{S}_z^+ \subseteq \mathcal{S}^+(M, U_M) \xrightarrow{s} \mathcal{S}(G, U)$$

from Lemma 1.8 (i). It maps ϕ_z to ψ_z. Since the idempotent ϵ_m commutes with the $\mathcal{S}^+(M, U_M)$-action the induced left $\mathcal{S}_z^+$-action on $\mathcal{S}_m$ is given by the (non-unital) continuous algebra homomorphism

$$\begin{aligned} \mathcal{S}_z^+ &\to \mathcal{S}_m, \\ \phi &\mapsto \epsilon_m * s(\phi) = s(\phi) * \epsilon_m. \end{aligned}$$

Its composite with the Fourier transform is denoted by

$$s_m : \tilde{\mathcal{O}}(\boldsymbol{D}) \cong \mathcal{S}_z^+ \to \mathcal{S}_m.$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

Since ϕ_z corresponds to ξ under the Fourier isomorphism we have $s_m(\xi) = \epsilon_m * \psi_z$. The map s_m and the continuous algebra homomorphism

$$\begin{aligned}\mathcal{O}(\boldsymbol{T}) &\to \mathcal{S}_m,\\ f &\mapsto \varphi_f = \epsilon_m * \varphi_f\end{aligned}$$

obviously coincide on the intersection $\tilde{\mathcal{O}}(\boldsymbol{D}) \cap \mathcal{O}(\boldsymbol{T}) = \mathcal{O}(\boldsymbol{D})$. Therefore, the map s_m extends uniquely to an algebra homomorphism

$$s_m : \tilde{\mathcal{O}}(\boldsymbol{D})[\xi^{-1}] \to \mathcal{S}_m$$

which maps ξ^{-1} to $\varphi_{\xi^{-1}}$.

Proposition 4.15. *The subset $\mathcal{D}$ is dense in $\hat{\mathcal{A}}$.*

Proof. Let V be an irreducible tempered G-representation with a non-zero U-fixed vector so that V^U lies in $\hat{\mathcal{A}}$. We have to prove that V^U lies in the closure of $\mathcal{D}$. If $V_N^w = 0$ then nothing is to show since V^U already lies in $\mathcal{D}$. We therefore assume that $V_N^w \neq 0$. Then there is a parabolic subgroup $Q \subseteq G$ with Levi component $L \subseteq M$ and a smooth discrete series representation τ of L such that V is a subquotient of the parabolic induction $\mathrm{Ind}_Q^G(\tau)$. Let $X_{nr}^1(L)$ denote the compact torus of unitary unramified characters of L. For any $\chi \in X_{nr}^1(L)$ and any irreducible constituent V' of $\mathrm{Ind}_Q^G(\chi\tau)$ the module $(V')^U$ lies in $\hat{\mathcal{A}}$. On the other hand, it is shown in the proof of [**Wal**, V.1.1] (see also [**Wal**, IV.2.2], [**Sil**, p. 272] and [**HC**, Theorem 13]) that the unitary unramified characters χ such that $\mathrm{Ind}_Q^G(\chi\tau)$ is irreducible and $\mathrm{Ind}_Q^G(\chi\tau)_N^w$ is semisimple form a dense subset of $X_{nr}^1(L)$. Our assertion therefore reduces to the following claim.

Claim. Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence in $X_{nr}^1(L)$ which converges to the trivial character and such that $V_n := \mathrm{Ind}_Q^G(\chi_n\tau)$, for any $n \in \mathbb{N}$, is irreducible; then V^U lies in the closure of the subset $\{V_n^U : n \in \mathbb{N}\}$ of the space $\hat{\mathcal{A}}$.

This is a variant of Lemma 4.4 (ii) in [**Tad**]. We fix G-invariant inner products $\langle\cdot\,,\cdot\rangle$ and $\langle\cdot\,,\cdot\rangle_{V_n}$ on $\mathrm{Ind}_Q^G(\tau)$ and any V_n, respectively. Since $\mathrm{Ind}_Q^G(\tau)$ is preunitary V can be viewed as a direct summand of $\mathrm{Ind}_Q^G(\tau)$; in particular $\langle\cdot\,,\cdot\rangle$ restricts to an inner product on V. According to [**Dix**, 3.4.10] we have to find non-zero vectors $v \in V^U$ and $v_n \in V_n^U$ such that for the corresponding positive forms

$$\ell(a) := \langle av, v\rangle \quad \text{and} \quad \ell_n(a) := \langle av_n, v_n\rangle_{V_n}$$

on $\mathcal{A}$ we have that the sequence $(\ell_n)_n$ converges to ℓ pointwise on $\mathcal{A}$. In fact, by [**Dix**, 2.7.5], pointwise convergence on $L^1(U\backslash G/U)$ is sufficient. But for $a \in L^1(G)$ we have

$$\langle av, v\rangle = \int_G a(g)\varphi_v(g)\,\mathrm{d}g \quad \text{and} \quad \langle av_n, v_n\rangle_{V_n} = \int_G a(g)\varphi_{v_n}(g)\,\mathrm{d}g$$

with the functions of positive type

$$\varphi_v(g) := \langle gv, v\rangle \quad \text{and} \quad \varphi_{v_n}(g) := \langle gv_n, v_n\rangle_{V_n}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

in $L^\infty(G)$. By [**Dix**, 13.5.2] we therefore are reduced to finding vectors $v \in V^U$ and $v_n \in V_n^U$ of length one such that the sequence $(\varphi_{v_n})_n$ of functions on G converges uniformly on compact subsets to the function φ_v. In the proof of [**Tad**, Lemma 4.4 (ii)] it is shown that for any $v \in \mathrm{Ind}_Q^G(\tau)^U$ we find vectors $v_n \in V_n^U$ such that the sequence $(\varphi_{v_n})_n$ converges to φ_v uniformly on compact subsets. The additional length one requirement can be achieved by scaling. $\square$

Beginning the proof of Proposition 4.14 formally now, we have that the upper right horizontal arrow in the diagram (4.1) is a closed isometric embedding. On the other hand the lower horizontal arrow has dense image. We therefore conclude that the right perpendicular arrow factorizes through a (non-unital) homomorphism of C^*-algebras $C(\boldsymbol{T}) \to \mathcal{A}$ with image contained in $\epsilon_m * \mathcal{A}$ which, for simplicity, we again denote by s_m, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_m & \xrightarrow{\subseteq} & \epsilon_m * \mathcal{A} \subseteq \mathcal{A} \\ \uparrow{\scriptstyle s_m} & & \uparrow{\scriptstyle s_m} \\ \tilde{\mathcal{O}}(\boldsymbol{D})[\xi^{-1}] & \xrightarrow{\subseteq} & C(\boldsymbol{T}) \end{array} \tag{4.2}$$

In particular, the element $\epsilon_m = s_m(1)$, respectively $\epsilon_m * \psi_z = s_m(\xi)$, in $\mathcal{A}$ is self-adjoint, respectively normal. Hence $\epsilon_m * \mathcal{A} * \epsilon_m$ is a C^*-subalgebra of $\mathcal{A}$, and the diagram (4.2) induces a commutative diagram of unital algebra homomorphisms:

$$\begin{array}{ccc} \epsilon_m * \mathcal{S}(G,U) * \epsilon_m & \xrightarrow{\subseteq} & \epsilon_m * \mathcal{A} * \epsilon_m \\ \uparrow{\scriptstyle s_m} & & \uparrow{\scriptstyle s_m} \\ \tilde{\mathcal{O}}(\boldsymbol{D})[\xi^{-1}] & \xrightarrow{\subseteq} & C(\boldsymbol{T}) \end{array} \tag{4.3}$$

We note that the right perpendicular arrow can also be viewed as induced by the continuous functional calculus for the normal element $\epsilon_m * \psi_z = \psi_z * \epsilon_m$ (cf. [**Con**, VIII, § 2] or [**Dix**, § 1.5]).

Lemma 4.16. *$\mathcal{S}(G,U)$ is a smooth subalgebra of the unital C^*-algebra $\mathcal{A}$ in the sense of [**BC**, Definition 6.6].*

Proof. Our scale function σ on G is K-bi-invariant and hence induces a scale on $U\backslash G/U$. We consider the densely defined self-adjoint and hence closed unbounded linear operator

$$\begin{aligned} D : L^2(U\backslash G/U) &\to L^2(U\backslash G/U), \\ \psi &\mapsto \sigma\psi \end{aligned}$$

together with the closable (cf. [**BR**, Corollary 3.2.56]) unbounded $*$-derivation

$$\begin{aligned} \delta : \mathcal{L}_{\mathrm{b}}(L^2(U\backslash G/U)) &\to \mathcal{L}_{\mathrm{b}}(L^2(U\backslash G/U)), \\ A &\mapsto i(D \circ A - A \circ D). \end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

It is shown in [**Vi1**, Theorem 29] that

$$\mathcal{S}(G,U) = \{a \in \mathcal{A} : \delta^j(a * \cdot) \text{ is bounded on } L^2(U\backslash G/U) \text{ for any } j \geqslant 0\}$$

holds true and that the Fréchet topology on $\mathcal{S}(G,U)$ can alternatively be defined by the seminorms $\|\delta^j(a*\cdot)\|$ for $j \geqslant 0$ where $\|\cdot\|$ denotes the operator norm on $\mathcal{L}_{\mathrm{b}}(L^2(U\backslash G/U))$. The

$$T^{(\ell)}(a) := \sum_{j=0}^{\ell} \frac{1}{j!} \|\delta^j(a * \cdot)\|$$

for $\ell \geqslant 0$ then form a countable family of derived seminorms of finite order [**BC**, Definition 5.1] defining the Fréchet topology on $\mathcal{S}(G,U)$. They are closable since δ is closable. □

If $\epsilon_m \neq \epsilon_U$ then the spectra of $\epsilon_m * \psi_z$ as an element in $\mathcal{A}$, respectively in $\epsilon_m * \mathcal{A} * \epsilon_m$, are related by

$$\sigma_{\mathcal{A}}(\epsilon_m * \psi_z) = \{0\} \cup \sigma_{\epsilon_m \mathcal{A} \epsilon_m}(\epsilon_m * \psi_z).$$

From the diagram (4.3) we know that $\sigma_{\epsilon_m \mathcal{A} \epsilon_m}(\epsilon_m * \psi_z) \subseteq \boldsymbol{T}$. The continuous functional calculi in both cases therefore induce the commutative diagram

$$\begin{array}{ccc} C(\boldsymbol{T}) & \xrightarrow{\ s_m\ } & \epsilon_m * \mathcal{A} * \epsilon_m \\ {\scriptstyle \text{extension by } 0}\downarrow & & \downarrow{\scriptstyle \subseteq} \\ C(\{0\} \cup \boldsymbol{T}) & \longrightarrow & \mathcal{A} \end{array}$$

Applying Propositions 6.4 and 6.8 from [**BC**] (based on Lemma 4.16) we obtain that the lower horizontal arrow restricts to a continuous algebra homomorphism $C^\infty(\{0\} \cup \boldsymbol{T}) \to \mathcal{S}(G,U)$. Hence s_m restricts to a continuous unital algebra homomorphism $s_m : C^\infty(\boldsymbol{T}) \to \epsilon_m * \mathcal{S}(G,U) * \epsilon_m$. If $\epsilon_m = \epsilon_U$ then we may apply [**BC**] directly to s_m. We therefore have in both cases the commutative triangle of continuous algebra homomorphisms:

$$\begin{array}{ccccc} & & \mathcal{S}_m & & \\ & {\scriptstyle s_m}\nearrow & & \nwarrow{\scriptstyle s_m} & \\ \tilde{\mathcal{O}}(\boldsymbol{D})[\xi^{-1}] & & \xrightarrow{\ \subseteq\ } & & C^\infty(\boldsymbol{T}) \end{array} \tag{4.4}$$

This means that we have a (unique) continuous unital left $C^\infty(\boldsymbol{T})$-module structure on $\mathcal{S}_m$ such that ξ acts as the map L_z. By [**B-TVS**, III.31, Proposition 6] the corresponding map

$$\begin{aligned} C^\infty(\boldsymbol{T}) &\to \mathcal{L}_{\mathrm{b}}(\mathcal{S}_m), \\ f &\mapsto [\varphi \mapsto s_m(f) * \varphi] \end{aligned}$$

is continuous. This concludes the proof of Proposition 4.14.

Lemma 4.17. *The natural map*

$$\mathcal{S}_m \xrightarrow{\cong} \mathcal{S}_z \,\hat{\otimes}_{\mathcal{S}_z^+} \mathcal{S}_m$$

is an isomorphism of $(\mathcal{S}^+(M,U_M),\mathcal{S}(G,U))$*-bimodules.*

Proof. Let us denote by j the map in question. Since L_z is invertible on $\mathcal{S}_m$ the $\tilde{\mathcal{O}}(\boldsymbol{D})$-module structure on $\mathcal{S}_m$ extends uniquely to a module structure for the localization $\tilde{\mathcal{O}}(\boldsymbol{D})[\xi^{-1}]$. But $\tilde{\mathcal{O}}(\boldsymbol{D})[\xi^{-1}]$ is dense in $C^\infty(\boldsymbol{T})$. It follows that j has a dense image. On the other hand, by the previous proposition, the $\tilde{\mathcal{O}}(\boldsymbol{D})$-module structure on $\mathcal{S}_m$ even extends to a continuous $C^\infty(\boldsymbol{T})$-module structure. The corresponding scalar multiplication provides a continuous left inverse for j. Hence the image of j is closed. □

We now may establish the main result of this section.

Theorem 4.18. *The natural map of* $(\mathcal{S}^+(M,U_M),\mathcal{S}(G,U))$*-bimodules*

$$\mathcal{S}(G,U) \to \mathcal{S}(M,U_M)\,\hat{\otimes}_{\mathcal{S}^+(M,U_M)}\,\mathcal{S}(G,U) \cong \mathcal{B}_U$$

is surjective and has a unique continuous $(\mathcal{S}^+(M,U_M),\mathcal{S}(G,U))$*-bimodule splitting* ω_U^+*; in particular, as a right* $\mathcal{S}(G,U)$*-module* $\mathcal{B}_U$ *is projective of rank one.*

Proof. With the exception of the uniqueness assertion this follows from the combination of Proposition 4.2 and Lemmas 4.4, 4.13 and 4.17. The kernel of the map under consideration is equal to $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_{m-1}$. The splitting ω_U^+ is provided by the direct summand $\mathcal{S}_m$ of $\mathcal{S}(G,U)$ whose definition seems to depend on the choice of the element z. But any other continuous $\mathcal{S}^+(M,U_M)$-module splitting would give rise to a continuous $\mathcal{S}_z^+$-linear map $\mathcal{S}_m \to \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_{m-1}$, but which has to be the zero map by [**EP**, 2.5.8]. □

5. Consequences for parabolic restriction

Keeping the notation introduced earlier we now collect the fruits of our labour from the previous section.

Proposition 5.1. *The functor* $r^t_{G,P} : \mathcal{M}^t(G) \to \mathcal{M}^t(M)$ *is exact.*

Proof. We already know that $r^t_{G,P}$ is right exact. Let therefore $V_1 \to V_2$ be an injective $\mathcal{S}(G)$-module map. Since the $r^t_{G,P}(V_i)$ in particular are smooth M-representations it suffices to show that $r^t_{G,P}(V_1)^{U_M} \to r^t_{G,P}(V_2)^{U_M}$ is injective for U running over an appropriate fundamental system of compact open subgroups of G. By Proposition 3.3 this reduces to the injectivity of $\mathcal{B}_U \otimes_{\mathcal{S}(G,U)} V_1^U \to \mathcal{B}_U \otimes_{\mathcal{S}(G,U)} V_2^U$ which is a consequence of the projectivity of $\mathcal{B}_U$ as a right $\mathcal{S}(G,U)$-module (Theorem 4.18). □

Proposition 5.2. *For any* V *in* $\mathcal{M}^t(G)$ *the natural map* $V_N \to r^t_{G,P}(V)$ *is surjective and has a natural* M*-equivariant splitting.*

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

Proof. As a consequence of Proposition 3.3 and Theorem 4.18 the restricted maps $V^U \to r^t_{G,P}(V)^{U_M}$ are surjective with a natural $\mathcal{S}^+(M,U_M)$-linear splitting. Consequently, the maps

$$V_N^{U_M} \to r^t_{G,P}(V)^{U_M}$$

are surjective with a natural $\mathcal{H}^+(M,U_M)$-linear splitting. But since $\mathcal{H}(M,U_M)$ is the localization of $\mathcal{H}^+(M,U_M)$ in the element ϕ_z which acts invertibly on both sides these splittings automatically are $\mathcal{H}(M,U_M)$-linear. By passing to the limit over the U we immediately obtain the asserted surjectivity. To obtain also the splitting in the limit it remains to check that the splittings for varying U are compatible. To reduce this problem to the case of the representation $\mathcal{S}(G/U) = \mathcal{S}(G) * \epsilon_U$ we consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{S}(G/U)^U \otimes_{\mathcal{S}(G,U)} V^U & \longrightarrow & V^U \\
\downarrow & & \downarrow \\
\mathcal{S}(G/U)_N^{U_M} \otimes_{\mathcal{S}(G,U)} V^U & \longrightarrow & V_N^{U_M} \\
\downarrow & & \downarrow \\
r^t_{G,P}(\mathcal{S}(G/U))^{U_M} \otimes_{\mathcal{S}(G,U)} V^U \cong \mathcal{B}_U \otimes_{\mathcal{S}(G,U)} V^U & \longrightarrow & r^t_{G,P}(V)^{U_M}
\end{array}$$

All three horizontal maps are isomorphisms. This is obvious for the top map and follows from Proposition 3.3 for the bottom map. For the map in the middle we observe that by Lemma 2.2 (ii) we have a decomposition $V = V_0 \oplus V_1$ with V_0 in $\mathcal{M}^t_U(G)$ and $V_1^U = 0$. Furthermore, Lemma 2.3 says that $\mathcal{S}(G/U) \otimes_{\mathcal{S}(G,U)} V^U \xrightarrow{\cong} V_0$ is an isomorphism. Hence, by functoriality

$$\mathcal{S}(G/U)_N^{U_M} \otimes_{\mathcal{S}(G,U)} V^U \xrightarrow{\cong} (V_0)_N^{U_M} = V_N^{U_M}$$

is an isomorphism as well. This diagram shows that the right-hand vertical maps arise as the tensor product of the corresponding maps for the representation $\mathcal{S}(G/U)$ with the identity map on V^U. The same, by construction (cf. Theorem 4.18), holds true for our splitting of the lower right vertical map. We are therefore reduced to show that the splittings of the maps

$$\mathcal{S}(G/U)_N^{U_M} \to r^t_{G,P}(\mathcal{S}(G/U))^{U_M} \cong \mathcal{B}_U$$

are compatible if U varies. Let $U' \subseteq U$ be another compact open subgroup of the type we consider. By replacing z by an appropriate power z^j we may assume that our element z is strongly (P,U)-positive as well as strongly (P,U')-positive. According to Theorem 4.18 the splitting on the level U is provided by the summand $\mathcal{S}_m$ in the decomposition $\mathcal{S}(G/U)^U = \mathcal{S}(G,U) = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_m$. Moreover, the kernel of the projection map from $\mathcal{S}(G/U)^U$ onto $\mathcal{S}(G/U)_N^{U_M}$ either is zero or coincides with the summand $\mathcal{S}_1$ (cf. [**Bus**, § 3.4, Theorem 1]). The left multiplication by ψ_z on $\mathcal{S}(G/U)^U$ becomes the action of the group element z on $\mathcal{S}(G/U)_N^{U_M}$. Hence the Fréchet space $\mathcal{S}(G/U)_N^{U_M}$ decomposes as

$$\mathcal{S}(G/U)_N^{U_M} = \tilde{\mathcal{S}}_m \oplus \mathcal{S}_m,$$

where the spectrum of the z-action on $\mathcal{S}_m$, respectively on $\tilde{\mathcal{S}}_m$, is contained in the unit circle, respectively is contained in the open unit disk. Note that this is a decomposition as $(\mathcal{H}(M,U_M),\mathcal{S}(G,U))$-bimodules. Let

$$\mathcal{S}(G/U')_N^{U'_M} = \tilde{\mathcal{S}}'_{m'} \oplus \mathcal{S}'_{m'}$$

be the corresponding decomposition on the U'-level. The compatibility we are looking for amounts to the claim that

$$\mathcal{S}_m \subseteq \mathcal{S}'_{m'}.$$

On the other hand, we consider the decomposition

$$\mathcal{S}(G/U')_N^{U'_M} = \mathcal{S}(G/U)_N^{U_M} \oplus \mathcal{S}(G/U')_N^{U_M}(\epsilon_{U'} - \epsilon_U) \oplus (\epsilon_{U'_M} - \epsilon_{U_M})\mathcal{S}(G/U')_N^{U'_M},$$

which is z-invariant since z lies in the centre of M. Decomposing further each summand according to the spectrum of z as before we see that indeed $\mathcal{S}_m \subseteq \mathcal{S}'_{m'}$. □

Corollary 5.3. *The functor $r^t_{G,P}$ respects admissibility.*

It follows in particular that on tempered admissible representations the functor $r^t_{G,P}$ coincides with Waldspurger's construction in [**Wal**, III.3.1]. We recall that $\bar{P} = M\bar{N}$ denotes the parabolic subgroup of G opposite to P. The whole theory developed in §§ 3 and 4, of course, holds correspondingly for $\bar{P}$ instead of P. For the corresponding notation we use $M^- := (M^+)^{-1}$, $\mathcal{S}^-(M,U_M) := \mathcal{S}(M,U_M)_{M^-}$, and $\bar{\mathcal{B}}_U := \mathcal{S}(M,U_M)\,\hat{\otimes}_{\mathcal{S}^-(M,U_M)}\,\mathcal{S}(G,U)$. We also introduce the $(\mathcal{S}(G,U),\mathcal{S}(M,U_M))$-bimodule

$$\bar{\mathcal{B}}^*_U := \mathrm{Hom}_{\mathcal{S}(G,U)}(\bar{\mathcal{B}}_U, \mathcal{S}(G,U)).$$

Similarly as in Lemma 1.8 (i) we have the ring homomorphism

$$\bar{s} := s_{\bar{N},U} : \mathcal{S}^-(M,U_M) \to \mathcal{S}(G,U).$$

Lemma 5.4. $\bar{\mathcal{B}}^*_U \cong \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \mathcal{S}(M,U_M))$ *as $(\mathcal{S}(G,U),\mathcal{S}(M,U_M))$-bimodules.*

Proof. Quite generally, if f is a function on some group we let $f^{\cdot}$ denote the function defined by $f^{\cdot}(x) := f(x^{-1})$. Simple calculations show that we have

- $(\psi^{\cdot})^P = (\psi^P)^{\cdot}$ for any $\psi \in \mathcal{S}(G)$;
- $\bar{s}(\phi^{\cdot}) = s(\phi)^{\cdot}$ for any $\phi \in \mathcal{S}^+(M,U_M)$.

It then follows from Lemma 1.8 (ii) that

$$(\psi * \bar{s}(\phi))^P = \psi^P * \phi \tag{5.1}$$

holds true for any $\psi \in \mathcal{S}(G)$ and $\phi \in \mathcal{S}^-(M,U_M)$. We now may introduce (cf. the beginning of § 3) the $(\mathcal{S}(G,U),\mathcal{S}(M,U_M))$-bimodule

$$\mathcal{B}^\ell_U := [\mathcal{S}(G,U) \otimes_{\mathcal{S}^-(M,U_M)} \mathcal{S}(M,U_M)]/\mathcal{R}^\ell$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

with $\mathcal{R}^\ell := \mathrm{im}(\mathrm{id} - e^\ell) = \ker(e^\ell)$ for the projector

$$e^\ell : \mathcal{S}(G,U) \otimes_{\mathcal{S}^-(M,U_M)} \mathcal{S}(M,U_M) \to \mathcal{S}(G,U) \otimes_{\mathcal{S}^-(M,U_M)} \mathcal{S}(M,U_M), \\ \psi \otimes \phi \mapsto \mathrm{vol}_G(U) \cdot \sum_{k \in K/U} \epsilon_U^{k^{-1}} \otimes ({}^k\psi)^P * \phi.$$

On the other hand, our map $(\cdot)^P : \mathcal{S}(G,U) \to \mathcal{S}(M,U_M)$ (cf. Lemma 1.3 (i)) is continuous and a homomorphism of left $\mathcal{S}^+(M,U_M)$-modules (cf. Lemma 1.8 (ii)). Hence it defines the homomorphism of left $\mathcal{S}(M,U_M)$-modules

$$\pi_U^+ : \mathcal{B}_U \cong \mathcal{S}(M,U_M) \hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G,U) \to \mathcal{S}(M,U_M), \\ \phi \otimes \psi \mapsto \phi * \psi^P$$

(cf. Proposition 4.2). It gives rise to the homomorphism of $(\mathcal{S}(G,U), \mathcal{S}(M,U_M))$-bimodules

$$\Pi : \mathcal{S}(G,U) \otimes_{\mathbb{C}} \mathcal{S}(M,U_M) \to \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \mathcal{S}(M,U_M)), \\ \psi_0 \otimes \phi_0 \mapsto [\phi \otimes \psi \mapsto \phi * (\psi * \psi_0)^P * \phi_0].$$

It follows from (5.1) that Π factorizes over $\mathcal{S}(G,U) \otimes_{\mathcal{S}^-(M,U_M)} \mathcal{S}(M,U_M)$. Moreover, using Lemma 1.2 one checks that

$$\Pi \circ (\mathrm{id} - e^\ell) = 0.$$

Hence Π induces a homomorphism of $(\mathcal{S}(G,U), \mathcal{S}(M,U_M))$-bimodules

$$\Pi : \mathcal{B}_U^\ell \to \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \mathcal{S}(M,U_M)).$$

As a first step towards our assertion we claim that this map is an isomorphism. For that purpose we recall from § 4 that we have, at least as left $\mathcal{S}(M,U_M)$-modules, the embedding

$$\mathcal{F} : (\mathcal{S}(M,U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G,U))/\mathcal{R} \to \mathcal{S}(M,U_M)^{[K:U]}, \\ \phi \otimes \psi \mapsto \mathrm{vol}_G(U) \cdot (\phi * (\psi^{k^{-1}})^P)_k,$$

which is a section of the homomorphism

$$\Sigma : \mathcal{S}(M,U_M)^{[K:U]} \to (\mathcal{S}(M,U_M) \otimes_{\mathcal{S}^+(M,U_M)} \mathcal{S}(G,U))/\mathcal{R}, \\ (\phi_k)_k \mapsto \sum_{k \in K/U} \phi_k \otimes {}^k\epsilon_U + \mathcal{R}.$$

In a completely analogous way we obtain the maps

$$\mathcal{F}^\ell : (\mathcal{S}(G,U) \otimes_{\mathcal{S}^-(M,U_M)} \mathcal{S}(M,U_M))/\mathcal{R}^\ell \to \mathcal{S}(M,U_M)^{[K:U]}, \\ \psi \otimes \phi \mapsto \mathrm{vol}_G(U) \cdot (({}^k\psi)^P * \phi)_k$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

and

$$\Sigma^\ell : \mathcal{S}(M, U_M)^{[K:U]} \to (\mathcal{S}(G, U) \otimes_{\mathcal{S}^-(M,U_M)} \mathcal{S}(M, U_M))/\mathcal{R}^\ell,$$
$$(\phi_k)_k \mapsto \sum_{k \in K/U} \epsilon_U^{k^{-1}} \otimes \phi_k + \mathcal{R}^\ell$$

such that $\Sigma^\ell \circ \mathcal{F}^\ell = \mathrm{id}$. One easily checks that the diagrams

$$\begin{array}{ccc} \mathcal{B}_U^\ell & \xrightarrow{\ \Pi\ } & \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \mathcal{S}(M, U_M)) \\ {\scriptstyle \mathcal{F}^\ell}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\Sigma, \mathcal{S}(M,U_M))} \\ \mathcal{S}(M, U_M)^{[K:U]} & \xrightarrow{\ \Phi\ } & \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{S}(M, U_M)^{[K:U]}, \mathcal{S}(M, U_M)) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{B}_U^\ell & \xrightarrow{\ \Pi\ } & \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \mathcal{S}(M, U_M)) \\ {\scriptstyle \Sigma^\ell}\uparrow & & \uparrow{\scriptstyle \mathrm{Hom}(\mathcal{F}, \mathcal{S}(M,U_M))} \\ \mathcal{S}(M, U_M)^{[K:U]} & \xrightarrow{\ \Phi\ } & \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{S}(M, U_M)^{[K:U]}, \mathcal{S}(M, U_M)) \end{array}$$

are commutative where Φ is the obvious isomorphism

$$\Phi : \mathcal{S}(M, U_M)^{[K:U]} \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{S}(M, U_M)^{[K:U]}, \mathcal{S}(M, U_M)),$$
$$(\phi'_k)_k \mapsto \Big[(\phi_k)_k \mapsto \mathrm{vol}_G(U)^{-1} \cdot \sum_{k \in K/U} \phi_k * \phi'_k\Big].$$

This establishes our first claim. For our assertion we now have to construct in a second step a natural isomorphism of $(\mathcal{S}(G, U), \mathcal{S}(M, U_M))$-bimodules

$$\bar{\mathcal{B}}_U^* \cong \mathcal{B}_U^\ell.$$

As a general observation we first of all point out that the whole discussion in § 4, being based solely on the spectral properties of the element ψ_z in $\mathcal{S}(G, U)$, is left-right symmetric, i.e. the consideration of the right multiplication operator $R_z : \mathcal{S}(G, U) \xrightarrow{\cdot * \psi_z} \mathcal{S}(G, U)$ leads to corresponding results. In particular, parallel to Theorem 4.18 we have the isomorphism of $(\mathcal{S}(G, U), \mathcal{S}^+(M, U_M))$-bimodules

$$\mathcal{S}(G, U) * \epsilon_m \xrightarrow{\cong} \mathcal{S}(G, U) \,\hat{\otimes}_{\mathcal{S}^+(M,U_M)} \mathcal{S}(M, U_M).$$

If we apply this observation to $\bar{P}$ instead of P we obtain the chain of natural isomorphisms

$$\begin{aligned} \bar{\mathcal{B}}_U^* &= \mathrm{Hom}_{\mathcal{S}(G,U)}(\bar{\mathcal{B}}_U, \mathcal{S}(G, U)) \\ &= \mathrm{Hom}_{\mathcal{S}(G,U)}(\mathcal{S}(M, U_M) \,\hat{\otimes}_{\mathcal{S}^-(M,U_M)} \mathcal{S}(G, U), \mathcal{S}(G, U)) \\ &\cong \mathrm{Hom}_{\mathcal{S}(G,U)}(\tilde{\epsilon} * \mathcal{S}(G, U), \mathcal{S}(G, U)) \\ &\cong \mathcal{S}(G, U) * \tilde{\epsilon} \\ &\cong \mathcal{S}(G, U) \,\hat{\otimes}_{\mathcal{S}^-(M,U_M)} \mathcal{S}(M, U_M) \\ &\cong \mathcal{B}_U^\ell, \end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

where $\tilde{\epsilon} \in \mathcal{S}(G,U)$ is an appropriate idempotent and where the last isomorphism is a variant of Proposition 4.2. At first this is only a $(\mathcal{S}(G,U), \mathcal{S}^-(M,U_M))$-bimodule isomorphism. But by Theorem 4.18, both $\bar{\mathcal{B}}_U^*$ and $\mathcal{B}_U^\ell$ are projective left $\mathcal{S}(G,U)$-modules of rank one. Hence the above composed isomorphism is continuous and even a homeomorphism. Since the right $\mathcal{S}(M,U_M)$-action on both modules is continuous and since $\mathcal{S}^-(M,U_M)[\phi_{z^{-1}}^{-1}]$ is dense in $\mathcal{S}(M,U_M)$ by Lemma 4.3 the isomorphism necessarily also is $\mathcal{S}(M,U_M)$-equivariant. $\square$

We point out that the above proof shows that the isomorphism in Lemma 5.4 is the unique isomorphism of left $\mathcal{S}(G,U)$-modules which maps the element $\omega_U^- \in \bar{\mathcal{B}}_U^*$ given by the section in the $\bar{P}$-version of Theorem 4.18 to $\pi_U^+ \in \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \mathcal{S}(M,U_M))$.

Theorem 5.5. *The functor $r_{G,\bar{P}}^t$ is right adjoint to Ind_P^G.*

Proof. For any E in $\mathcal{M}^t(M)$ and any V in $\mathcal{M}^t(G)$ we have to establish a natural isomorphism

$$\mathrm{Hom}_{\mathcal{S}(G)}(\mathrm{Ind}_P^G(E), V) \cong \mathrm{Hom}_{\mathcal{S}(M)}(E, r_{G,\bar{P}}^t(V)).$$

Because of

$$\mathrm{Hom}_{\mathcal{S}(G)}(\mathrm{Ind}_P^G(E), V) = \varprojlim_U \mathrm{Hom}_{\mathcal{S}(G,U)}(\mathrm{Ind}_P^G(E)^U, V^U)$$

and

$$\mathrm{Hom}_{\mathcal{S}(M)}(E, r_{G,\bar{P}}^t(V)) = \varprojlim_U \mathrm{Hom}_{\mathcal{S}(M,U_M)}(E^{U_M}, r_{G,\bar{P}}^t(V)^{U_M})$$

we may do this by finding natural isomorphisms on each U-level in a compatible way. Since the functor Ind_P^G visibly commutes with filtered inductive limits we may assume in addition that E is a finitely generated $\mathcal{S}(M)$-module. This means that E lies in $\mathcal{M}_{U_M}^t(M)$ provided U is small enough which we will assume in the following. Using Proposition 2.4, Lemma 4.1, and Lemma 5.4 in lines 2, 3 and 4, respectively, we compute

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{S}(G,U)}(\mathrm{Ind}_P^G(E)^U, V^U)\\
&\qquad = \mathrm{Hom}_{\mathcal{S}(G,U)}(\mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, E^{U_M}), V^U)\\
&\qquad = \mathrm{Hom}_{\mathcal{S}(G,U)}(\mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \mathcal{S}(M,U_M)) \otimes_{\mathcal{S}(M,U_M)} E^{U_M}, V^U)\\
&\qquad = \mathrm{Hom}_{\mathcal{S}(G,U)}(\bar{\mathcal{B}}_U^* \otimes_{\mathcal{S}(M,U_M)} E^{U_M}, V^U).
\end{aligned}$$

Since $\bar{\mathcal{B}}_U$, by Theorem 4.18, is projective of rank one as an $\mathcal{S}(G,U)$-module we have the canonical isomorphism

$$\bar{\mathcal{B}}_U \otimes_{\mathcal{S}(G,U)} \cdot \cong \mathrm{Hom}_{\mathcal{S}(G,U)}(\bar{\mathcal{B}}_U^*, \cdot)$$

of functors from $\mathcal{M}(\mathcal{S}(G,U))$ to $\mathcal{M}(\mathcal{S}(M,U_M))$. It follows that the functor $\bar{\mathcal{B}}_U \otimes_{\mathcal{S}(G,U)} \cdot$ has the left adjoint functor $\bar{\mathcal{B}}_U^* \otimes_{\mathcal{S}(M,U_M)} \cdot$. Using also Proposition 3.3 we therefore may continue our computation by

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{S}(G,U)}(\mathrm{Ind}_P^G(E)^U, V^U) &= \mathrm{Hom}_{\mathcal{S}(M,U_M)}(E^{U_M}, \bar{\mathcal{B}}_U \otimes_{\mathcal{S}(G,U)} V^U)\\
&= \mathrm{Hom}_{\mathcal{S}(M,U_M)}(E^{U_M}, r_{G,\bar{P}}^t(V)^{U_M}).
\end{aligned}$$

If now $U' \subseteq U$ is another compact open subgroup with the same properties as U then, leaving the details of the computation to the reader, we observe the following compatibilities along the above chain of identifications. The inclusion map

$$\mathrm{Ind}_P^G(E)^U \subseteq \mathrm{Ind}_P^G(E)^{U'}$$

corresponds to the composed map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, E^{U_M}) & & \\ \Big\downarrow{\scriptstyle \mathrm{Hom}(\mathcal{B}_U,\subseteq)} & & \\ \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, E^{U'_M}) & \xrightarrow{=} & \mathrm{Hom}_{\mathcal{S}(M,U'_M)}(\mathcal{B}_{U'}\epsilon_U, E^{U'_M}) \\ & & \Big\downarrow{\scriptstyle \mathrm{Hom}(\cdot\epsilon_U, E^{U'_M})} \\ & & \mathrm{Hom}_{\mathcal{S}(M,U'_M)}(\mathcal{B}_{U'}, E^{U'_M}) \end{array}$$

where the horizontal identity in the middle comes from Lemma 3.6. Viewed as

$$\begin{array}{c} \mathrm{Hom}_{\mathcal{S}(M,U_M)}(\mathcal{B}_U, \mathcal{S}(M,U_M)) \otimes_{\mathcal{S}(M,U_M)} E^{U_M} \\ \Big\downarrow \\ \mathrm{Hom}_{\mathcal{S}(M,U'_M)}(\mathcal{B}_{U'}, \mathcal{S}(M,U'_M)) \otimes_{\mathcal{S}(M,U'_M)} E^{U'_M} \end{array}$$

this latter composite map is the tensor product of the inclusion on the second factor and the unique $\mathcal{S}(G,U)$-module homomorphism on the first factor which maps π_U^+ to $\epsilon_U \pi_{U'}^+ \epsilon_{U_M} = \epsilon_U \pi_{U'}^+$. It corresponds to the tensor product map

$$\bar{\mathcal{B}}_U^* \otimes_{\mathcal{S}(M,U_M)} E^{U_M} \to \bar{\mathcal{B}}_{U'}^* \otimes_{\mathcal{S}(M,U'_M)} E^{U'_M}$$

with the inclusion on the second factor and the unique $\mathcal{S}(G,U)$-module homomorphism on the first factor which maps ω_U^- to $\epsilon_U \omega_{U'}^- \epsilon_{U_M} = \epsilon_U \omega_{U'}^-$. It remains to see that the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{S}(G,U')}(\bar{\mathcal{B}}_{U'}^*, V^{U'}) & \xleftarrow{\cong} & \bar{\mathcal{B}}_{U'} \otimes_{\mathcal{S}(G,U')} V^{U'} & \xrightarrow{\cong} & r_{G,\bar{P}}^t(V)^{U'_M} \\ \Big\downarrow & & & & \Big\downarrow{\scriptstyle \epsilon_{U_M}*} \\ \mathrm{Hom}_{\mathcal{S}(G,U)}(\bar{\mathcal{B}}_U^*, V^U) & \xleftarrow{\cong} & \bar{\mathcal{B}}_U \otimes_{\mathcal{S}(G,U)} V^U & \xrightarrow{\cong} & r_{G,\bar{P}}^t(V)^{U_M} \end{array}$$

is commutative where the left vertical arrow is the map induced by the above map $\bar{\mathcal{B}}_U^* \to \bar{\mathcal{B}}_{U'}^*$. The horizontal composite arrow is given by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}(G,U)}(\bar{\mathcal{B}}_U^*, V^U) &\to r_{G,\bar{P}}^t(V)^{U_M}, \\ A &\mapsto \text{image of } A(\omega_U^-) \end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

and similarly for U'. The commutativity we are looking for reduces therefore to the identity

$$\text{image of } (\epsilon_U * A(\omega_{U'}^-)) = \epsilon_{U_M} * (\text{image of } A(\omega_{U'}^-))$$

for any $A \in \mathrm{Hom}_{\mathcal{S}(G,U')}(\bar{\mathcal{B}}_{U'}^*, V^{U'})$. For this it suffices that $A(\omega_{U'}^-)$ is U_N-invariant, hence that $\omega_{U'}^-$ is U_N-invariant. The latter is shown in [**Bus**, § 3.6, Lemma 5] (applied to $V = \mathcal{S}(G/U')$). □

Corollary 5.6. *The functor* $\mathrm{Ind}_P^G : \mathcal{M}^t(M) \to \mathcal{M}^t(G)$ *respects projective objects.*

Proof. Respecting projective objects is a formal consequence of having an exact right adjoint. □

It is an immediate consequence of Lemma 4.1 that the functor $r_{G,P}^t$ respects finite generation. A variant of this fact is a formal consequence of the above theorem.

Corollary 5.7. *The functors* $\mathrm{Ind}_P^G : \mathcal{M}^t(M) \to \mathcal{M}^t(G)$ *and* $r_{G,P}^t$ *respect objects of finite presentation.*

Proof. We recall that an object X in an abelian category with exact inductive limits $\mathcal{X}$ is called of finite presentation if the functor $\mathrm{Hom}_{\mathcal{X}}(X, \cdot)$ commutes with inductive limits. Any functor which has a right adjoint which itself also has a right adjoint and therefore commutes with inductive limits respects objects of finite presentation. As a consequence of Proposition 3.1 and Theorem 5.5 the two functors in the assertion have this property. □

For any V in $\mathcal{M}(G)$ we denote, as usual, by $\tilde{V}$ the smooth dual of V, i.e. $\tilde{V} = \varinjlim_U \mathrm{Hom}_{\mathbb{C}}(V^U, \mathbb{C})$. Due to the anti-involution of the algebra $\mathcal{S}(G)$ induced by $g \mapsto g^{-1}$, if V lies in $\mathcal{M}^t(G)$ then so does $\tilde{V}$ in a natural way.

Proposition 5.8. *For any V in $\mathcal{M}^t(G)$ we have a natural isomorphism*

$$r_{G,P}^t(V)^\sim \cong r_{G,\bar{P}}^t(\tilde{V})$$

in $\mathcal{M}^t(M)$.

Proof. Using Proposition 3.3, Theorem 4.18 and the notation and results established in the course of the proof of Lemma 5.4 we compute

$$\begin{aligned}
(r_{G,P}^t(V)^{U_M})^\sim &\cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{B}_U \otimes_{\mathcal{S}(G,U)} V^U, \mathbb{C}) \\
&= \mathrm{Hom}_{\mathcal{S}(G,U)}(\mathcal{B}_U, \mathrm{Hom}_{\mathbb{C}}(V^U, \mathbb{C})) \\
&\cong \mathrm{Hom}_{\mathbb{C}}(V^U, \mathbb{C}) \otimes_{\mathcal{S}(G,U)} \mathrm{Hom}_{\mathcal{S}(G,U)}(\mathcal{B}_U, \mathcal{S}(G,U)) \\
&= \tilde{V}^U \otimes_{\mathcal{S}(G,U)} \mathcal{B}_U^* \\
&\cong \tilde{V}^U \otimes_{\mathcal{S}(G,U)} \bar{\mathcal{B}}_U^\ell
\end{aligned}$$

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

as right $\mathcal{S}(M, U_M)$-modules. Rewriting this, by using the anti-involution $g \mapsto g^{-1}$, in terms of left modules gives the natural isomorphism

$$(r^t_{G,P}(V)^{U_M})^{\sim} \cong \bar{\mathcal{B}}_U \otimes_{\mathcal{S}(G,U)} \tilde{V}^U \cong r^t_{G,\bar{P}}(\tilde{V})^{U_M}.$$

The assertion follows by passing to the direct limit with respect to U; the necessary compatibilities are seen as in the proof of Theorem 5.5. □

Acknowledgements. We are very grateful to J.-L. Waldspurger for providing us with a proof of the surjectivity of the map considered in § 1. This certainly is an important fact for our whole investigation. We also thank V. Heiermann for several inspiring discussions and J. Cuntz for patiently answering questions from P.S. about spectral theory.

References

[BD] J. Bernstein (rédigé par P. Deligne), Le 'center' de Bernstein, in *Représentations des groupes réductifs sur un corps local*, pp. 1–32 (Hermann, Paris, 1984).

[BZ] J. Bernstein and A. V. Zelevinskii, Representations of the group $GL(n, F)$ where F is a non-archimedean local field, *Russ. Math. Surv.* **31**(3) (1976), 1–68.

[BC] B. Blackadar and J. Cuntz, Differential Banach algebra norms and smooth subalgebras of C^*- algebras, *J. Operat. Theory* **26** (1991), 255–282.

[B-INT] N. Bourbaki, *Intégration*, Chapters 7, 8 (Hermann, Paris, 1963).

[B-TVS] N. Bourbaki, *Topological vector spaces* (Springer, 1987).

[BR] O. Bratelli and D. R. Robinson, *Operator algebras and quantum statistical mechanics*, Volume I (Springer, 1979).

[Bus] C. J. Bushnell, Representations of reductive p-adic groups: localization of Hecke algebras and applications, *J. Lond. Math. Soc.* **63** (2001), 364–386.

[BK] C. J. Bushnell and P. C. Kutzko, Smooth representations of reductive p-adic groups: structure theory via types, *Proc. Lond. Math. Soc.* **77** (1998), 582–634.

[Car] P. Cartier, Representations of $\mathfrak{p}$-adic groups: a survey, in *Automorphic forms, representations and L-functions*, Proceedings of Symposia in Pure Mathematics, Volume 33(1), pp. 111–155 (American Mathematical Society, Providence, RI, 1979).

[Cas] W. Casselman, Introduction to the theory of admissible representations of $\wp$-adic reductive groups, preprint.

[Con] J. B. Conway, *A course in functional analysis*, 2nd edn (Springer, 1990).

[Dix] J. Dixmier, *Les C^*-algèbres et leurs représentations* (Gauthier-Villars, Paris, 1964).

[EP] J. Eschmeier and M. Putinar, *Spectral decompositions and analytic sheaves* (Clarendon, Oxford, 1996).

[HC] Harish-Chandra, Harmonic analysis on reductive p-adic groups, in *Harmonic analysis on homogeneous spaces* (ed. C. Moore), Proceedings of Symposia in Pure Mathematics, Volume 26, pp. 167–192 (American Mathematical Society, Providence, RI, 1973).

[Sch] H. H. Schaefer, *Topological vector spaces* (Springer, 1971).

[SSZ] P. Schneider, U. Stuhler and E.-W. Zink, K-types for the tempered components of a p-adic general linear group, With an appendix: the definition of the tempered category, *J. Reine Angew. Math.* **517** (1999), 161–208.

[Sil] A. Silberger, *Introduction to harmonic analysis on reductive p-adic groups* (Princeton University Press, Princeton, NJ, 1979).

[Tad] M. Tadic, Geometry of dual spaces of reductive groups (non-archimedean case), *J. Analyse Math.* **51** (1988), 139–181.

[Tre] F. Treves, *Topological vector spaces, distributions and kernels* (Academic, 1967).

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press

[Vas] F.-H. Vasilescu, *Analytic functional calculus and spectral decompositions* (Reidel, Dordrecht, 1982).

[Vi1] M.-F. Vigneras, On formal dimensions for reductive p-adic groups, in *Festschrift in honor of I. I. Piatetski-Shapiro* (ed. S. Gelbart, R. Howe and P. Sarnak), Part I, pp. 225–266, Israel Mathematical Conference Proceedings, Volume 2 (Weizmann Scientific Press, Jerusalem, 1990).

[Vi2] M.-F. Vigneras, Homologie cyclique, principe de Selberg et pseudo-coefficient, *Invent. Math.* **116** (1994), 651–676.

[Wal] J.-L. Waldspurger, La formule de Plancherel pour les groupes p-adiques, d'aprés Harish-Chandra, *J. Inst. Math. Jussieu* **2** (2003), 235–333.

https://doi.org/10.1017/S1474748007000047 Published online by Cambridge University Press