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Aperiodic automorphisms of nuclear purely infinite simple C^*-algebras

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Abstract. We show that if two aperiodic automorphisms of a separable nuclear unital purely infinite simple C^*-algebra are asymptotically unitarily equivalent, then they are outer conjugate with respect to an automorphism which is isotopic to the identity automorphism. Thus, by Kirchberg and Phillips, they have the same KK-class if and only if they are outer conjugate with respect to an automorphism which is in the KK-class of the identity automorphism.

1. *Introduction*

In the theory of operator algebras, the problem of classifying group actions on an algebra has a long history as well as that of classifying algebras themselves. In the category of von Neumann algebras, the classification theory of group actions begins with the work of Dye [**9**] (for commutative algebras) and of Connes [**5**] (for non-commutative algebras) and much progress has been made so far (see, for example, [**23**]). In the category of C^*-algebras, in contrast, there are still a lot of things which should be done, although some fruitful results have also been obtained (see [**13, 15, 17, 18, 27, 42**] for compact group actions and see [**10, 12, 14, 19, 20, 29, 31, 32**] for discrete group actions). In fact many of the previous works treat group actions on a finite C^*-algebra, while the classification problem of group actions on an infinite C^*-algebra has not been studied well. In this paper we study automorphisms of purely infinite C^*-algebras. Purely infinite C^*-algebras and their automorphisms also have a connection with topological dynamical systems. For example (irreducible) topological Markov chains induce (simple) purely infinite C^*-algebras (called Cuntz–Krieger algebras) as their topological invariants [**8**] and then shift-commuting transformations of Markov chains naturally define automorphisms of induced C^*-algebras [**43**].

Recently, Kirchberg [**24**] and Phillips [**38**] classified separable nuclear unital purely infinite simple C^*-algebras using Kasparov's KK-theory [**22**]. Motivated by their remarkable success, we classify aperiodic automorphisms of such a C^*-algebra in terms of

KK-theory. Our main result says that two automorphisms have the same KK-class if and only if they are outer conjugate with respect to an automorphism which is in the KK-class of the identity automorphism.

The contents of this paper are as follows. In §2 we argue a non-commutative Rohlin property for aperiodic automorphisms of nuclear purely infinite simple C^*-algebras. This property has been known to be very important for the classification of automorphisms of operator algebras since the above-mentioned work of Connes and has been investigated by many authors [**3, 10, 12, 17, 18, 19, 20, 28, 29, 30, 31, 32, 33, 34, 35, 39**]. As to nuclear purely infinite simple C^*-algebras, a Rohlin-type theorem was first established by Kishimoto [**28**] for the Cuntz algebras O_n (with n finite) and subsequently by the author [**33**] for a certain class of algebras with trivial K_1-groups, which contains O_∞. Here we show the theorem for general nuclear purely infinite simple C^*-algebras, which was first noticed by Izumi [**21**]. In §3 we consider the classification problem of aperiodic automorphisms of a nuclear purely infinite simple C^*-algebra, up to outer conjugacy. When we show the stability of automorphisms from the Rohlin property as in [**19**], we encounter a certain difficulty relating to (almost central) unitary paths in the algebra. We deal with this problem in Theorem 7, which is the main technical part of this paper. After that we show outer conjugacy of aperiodic automorphisms which are asymptotically unitarily equivalent, following the Evans–Kishimoto intertwining argument [**12**]. At the same time we see that the automorphism constructed there is isotopic to the identity automorphism of the algebra.

Here is some notation and terminology we use throughout this paper. Let $\mathrm{Proj}(A)$ and $U(A)$ denote the set of projections and the set of unitaries of a unital C^*-algebra A, respectively. Let 1_A and id_A denote the unit and the identity automorphism of A, respectively. For $x, y \in A$ and $u \in U(A)$, define $[x, y] = xy - yx$ and $\mathrm{Ad}\, u(x) = uxu^*$. An automorphism α of a unital C^*-algebra A is called *inner* if $\alpha = \mathrm{Ad}\, u$ for some $u \in U(A)$. If there is no such u then α is called *outer*. Furthermore, if α^k is outer for any nonzero integer k then α is called *aperiodic*. A unital C^*-algebra A is called *purely infinite* if every nonzero hereditary C^*-subalgebra of A contains an infinite projection. We also use the notation of K-theory and KK-theory for operator algebras without any explanation. We refer the reader to [**2**] for the details.

2. *Rohlin-type theorem*

In this section we show a Rohlin-type theorem for automorphisms of nuclear purely infinite simple C^*-algebras. This theorem (Theorem 1) was proved first by Izumi as an application of Kishimoto's method [**28**] to the result [**25**, Proposition 3.4]. Subsequently Izumi's proof of Lemma 2 was simplified by Kishimoto. By their courtesy we combine their unpublished arguments here.

THEOREM 1. *Let A be a separable nuclear unital purely infinite simple C^*-algebra and let α be an automorphism of A. Then the following conditions are equivalent:*

(1) *α is aperiodic;*

(2) *α has the Rohlin property, that is for any positive integer M, finite subset F of A and*

$\varepsilon > 0$, *there exist projections* $e_0, \ldots, e_{M-1}, f_0, \ldots, f_M$ *in* A *such that*

$$\sum_{i=0}^{M-1} e_i + \sum_{j=0}^{M} f_j = 1,$$

$$\|e_i x - x e_i\| < \varepsilon, \quad \|f_j x - x f_j\| < \varepsilon,$$

$$\|\alpha(e_i) - e_{i+1}\| < \varepsilon, \quad \|\alpha(f_j) - f_{j+1}\| < \varepsilon$$

for $i = 0, \ldots, M-1$, $j = 0, \ldots, M$ *and all* $x \in F$, *where* e_M *and* f_{M+1} *mean* e_0 *and* f_0, *respectively.*

It is evident that (2) implies (1). We show the converse in the rest of this section. The basic idea of the proof comes from the same type of theorem for the Cuntz algebras O_n (with n finite) in [**28**], but here we employ the method of central sequence algebras which simplifies the argument a little. Let ω be a free ultrafilter on $\mathbb{N}$ (see [**40**, §3] for the definition). We refer the reader to [**25**, Notation 3.1] for our notation regarding ω-central sequence algebras. In particular, for a C^*-algebra A, we denote by A_ω the ultrapower of A and by π_ω the natural quotient mapping from the bounded sequence algebra $\ell^\infty(A)$ onto A_ω. It is shown in [**25**, Proposition 3.4] that for a C^*-algebra A, as in Theorem 1, the ω-central sequence algebra $A' \cap A_\omega$ is unital, purely infinite and simple. Assume that (1) holds, then we first show that the induced automorphism of α on $A' \cap A_\omega$ is also aperiodic.

LEMMA 2. $\alpha_\omega \restriction (A' \cap A_\omega)$ *is aperiodic.*

Proof. It is sufficient to show that $\alpha_\omega \restriction (A' \cap A_\omega)$ is outer when α is outer. For an outer automorphism α of A we claim that $\alpha_\omega \restriction (A' \cap A_\omega) \neq \mathrm{id}$. By [**26**, Theorem 2.1 and Remark 2.2] we have an irreducible representation π of A such that π is disjoint from $\pi \circ \alpha$. That is, if we denote by $c(\pi)$ the central cover of π then $c(\pi)\alpha(c(\pi)) = 0$. Take a unit vector ξ in the Hilbert space associated with the universal representation of A such that $c(\pi)\xi = \xi$, then $\|(\alpha(c(\pi)) - c(\pi))\xi\| = 1$. On the other hand, by using [**1**, Lemma 1.1], we can approximate $c(\pi)$ strongly by a bounded net $(x_\lambda \mid \lambda \in \Lambda)$ in A such that

$$\lim \|(c(\pi) - x_\lambda)a - a(c(\pi) - x_\lambda)\| = \lim \|x_\lambda a - a x_\lambda\| = 0 \quad (a \in A).$$

Therefore, for a sequence $(u_n \mid n \in \mathbb{N})$ in $U(A)$, whose linear span is dense in A, we find $\lambda(n) \in \Lambda$ $(n \in \mathbb{N})$ such that

$$\|x_{\lambda(n)} u_i - u_i x_{\lambda(n)}\| < \frac{1}{n} \quad (i = 1, \ldots, n),$$

$$\|(\alpha(x_{\lambda(n)}) - x_{\lambda(n)})\xi\| > \tfrac{1}{2}.$$

This means that we have $x = \pi_\omega(x_{\lambda(n)} \mid n \in \mathbb{N}) \in A' \cap A_\omega$ with $\|\alpha_\omega(x) - x\| \geq \frac{1}{2}$, which implies that $\alpha_\omega \restriction (A' \cap A_\omega) \neq \mathrm{id}$. Hence we can conclude the outerness of $\alpha_\omega \restriction (A' \cap A_\omega)$ following the method in [**5**, Proposition 2.1.2]. Indeed, for any $y = \pi_\omega(y_n \mid n \in \mathbb{N}) \in A_\omega$, we find $L_n \in \omega$ $(n \in \mathbb{N})$ such that

$$\|[x_{\lambda(k)}, y_n]\| < \frac{1}{n}, \quad \|[x_{\lambda(k)}, u_i]\| < \frac{1}{n} \quad (k \in L_n, i = 1, \ldots, n).$$

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Take $k_n \in L_n$ satisfying $k_n < k_{n+1}$ and define $x' = \pi_\omega(x_{\lambda(k_n)} \mid n \in \mathbb{N})$. Then

$$\|[x', y]\| = \lim_{n\to\omega} \|[x_{\lambda(k_n)}, y_n]\| = 0,$$
$$\|[x', u_i]\| = \lim_{n\to\omega} \|[x_{\lambda(k_n)}, u_i]\| = 0,$$
$$\|\alpha_\omega(x') - x'\| = \lim_{n\to\omega} \|x_{\lambda(k_n)} - x_{\lambda(k_n)}\| \geq \tfrac{1}{2}.$$

Since y is arbitrary, it follows that $\alpha_\omega \restriction (A' \cap A_\omega)$ can not be inner. We complete the proof. □

Combining Lemma 2 with the fact that $A' \cap A_\omega$ is purely infinite and simple, we can prove the following two lemmas and conclude Theorem 1 in an analogous manner as in [**28, 33**].

LEMMA 3. *For any $K \in \mathbb{N}$, $g \in K_0(A' \cap A_\omega)$ and nonzero projection p in $A' \cap A_\omega$, there exists a nonzero projection e in $p(A' \cap A_\omega)p$ such that*

$$e\alpha_\omega^k(e) = 0 \quad (k = 1, \ldots, K),$$
$$[e] = g \quad \textit{in } K_0(A' \cap A_\omega).$$

Proof. Let K, g and p be given as above. We first claim that, for any $\varepsilon > 0$, there is a nonzero projection $e' \in p(A' \cap A_\omega)p$ satisfying

$$\|e'\alpha_\omega^k(e')\| < \varepsilon \quad (k = 1, \ldots, K).$$

Indeed, by the previous lemma and [**26**, Lemma 1.1], we have for any $k \in \mathbb{Z} \setminus \{0\}$

$$\inf\{\|q{\alpha_\omega}^k(q)\| \mid q \in \mathrm{Proj}(p(A' \cap A_\omega)p) \setminus \{0\}\} = 0.$$

Hence we find a nonzero projection $e_1 \in p(A' \cap A_\omega)p$ such that $\|e_1\alpha_\omega(e_1)\| < \varepsilon$. Since $A' \cap A_\omega$ is purely infinite simple, we further find a nonzero projection $e_2 \in e_1(A' \cap A_\omega)e_1$ such that $\|e_2\alpha_\omega^2(e_2)\| < \varepsilon$. Repeating this process we can reach a desired projection e'.

To show the lemma, let $(u_n \mid n \in \mathbb{N})$ be a sequence in $U(A)$ whose linear span is dense in A and let $p = \pi_\omega(p_n \mid n \in \mathbb{N})$. By a standard argument we may assume that p_n is a nonzero projection. Using the preceding paragraph, we have a nonzero projection $e_m = \pi_\omega(e_{m,n} \mid n \in \mathbb{N}) \in A' \cap A_\omega$ and $L_m \in \omega$ for each $m \in \mathbb{N}$ such that

$$0 \lneq e_{m,n} \leq p_n,$$
$$\|e_{m,n}\alpha^k(e_{m,n})\| < \frac{1}{m} \quad (k = 1, \ldots, K),$$
$$\|[e_{m,n}, u_i]\| < \frac{1}{m} \quad (i = 1, \ldots, m)$$

for all $n \in L_m$ and that $L_m \supsetneq L_{m+1}$. Define a projection $e'' = \pi_\omega(e''_n \mid n \in \mathbb{N})$ by

$$e''_n = \begin{cases} e_{m,n} & (n \in L_m \setminus L_{m+1}) \\ 0 & (n \in \mathbb{N} \setminus L_1). \end{cases}$$

Then e'' is a nonzero projection in $A' \cap A_\omega$ satisfying

$$e''\alpha_\omega^k(e'') = 0 \quad (k = 1, \ldots, K).$$

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Since $A' \cap A_\omega$ is purely infinite simple, we further find a nonzero projection e in $A' \cap A_\omega$ such that

$$e \le e'', \quad [e] = g.$$

This completes the proof. $\square$

LEMMA 4. *For any $M \in \mathbb{N}$ there exist nonzero mutually orthogonal projections $e_0, \ldots, e_{M-1}$ in $A' \cap A_\omega$ such that*

$$\alpha_\omega(e_i) = e_{i+1} \quad (i = 0, \ldots, M-1),$$

where e_M means e_0.

Proof. Temporarily fix a positive integer m. By Lemma 3 we have a nonzero projection $e \in A' \cap A_\omega$ such that

$$e\alpha_\omega^k(e) = 0 \quad (k = 1, \ldots, mM),$$
$$[e] = 0 \quad \text{in } K_0(A' \cap A_\omega).$$

Since $[\alpha_\omega(e)] = 0$, there is a partial isometry $w \in A' \cap A_\omega$ such that $w^*w = e$ and $ww^* = \alpha_\omega(e)$. Define $e_{i,j} \in A' \cap A_\omega$ $(i, j = 0, \ldots, mM)$ as follows:

$$e_{i,j} = \begin{cases} \alpha_\omega^i(e) & (i = j) \\ \alpha_\omega^{i-1}(w)\alpha_\omega^{i-2}(w)\ldots\alpha_\omega^j(w) & (i > j) \\ \alpha_\omega^i(w)^*\alpha_\omega^{i+1}(w)^*\ldots\alpha_\omega^{j-1}(w)^* & (i < j). \end{cases}$$

Then we can easily verify that $(e_{i,j} \mid i, j = 0, \ldots, mM)$ is a system of matrix units satisfying

$$\alpha_\omega(e_{i,j}) = e_{i+1,j+1} \quad (i, j = 0, \ldots, mM-1).$$

Further, define $f_i \in A' \cap A_\omega$ $(i = 0, \ldots, M-1)$ by the formula:

$$f_i = \frac{1}{m}\sum_{k,l=0}^{m-1} e_{i+kM,i+lM}.$$

Then we also verify that

$$\alpha_\omega(f_i) = f_{i+1} \quad (i = 0, \ldots, M-2)$$

and that

$$\begin{aligned}\alpha_\omega(f_{M-1}) - f_0 &= \frac{1}{m}\sum_{k,l=0}^{m-1}(e_{(k+1)M,(l+1)M} - e_{kM,lM}) \\ &= \frac{1}{m}\sum_{l=0}^{m-1}(e_{mM,(l+1)M} + e_{(l+1)M,mM} - e_{0,lM} - e_{lM,0}).\end{aligned}$$

The last formula implies an estimate $\|\alpha_\omega(f_{M-1}) - f_0\| \le 4m^{-1/2}$. Since m is arbitrary, we can take desired projections e_i $(i = 0, \ldots, M-1)$ from $A' \cap A_\omega$ as in the proof of the preceding lemma. $\square$

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Proof of Theorem 1. Let M, F and ε be given as in the statement of the theorem. Let m be a positive integer which we make very large later. By Lemma 4 there exist nonzero mutually orthogonal projections $e_0, \dots, e_{mM-1} \in A' \cap A_\omega$ such that

$$\alpha_\omega(e_i) = e_{i+1} \quad (i = 0, \dots, mM-1),$$

where e_{mM} means e_0. Set $e = \sum_{i=0}^{mM-1} e_i$. If $e = 1$ then we are done, so we assume that $e \neq 1$. Let n be a positive integer which we also make very large later. Using Lemma 3 we find a nonzero projection $f \in e_0(A' \cap A_\omega)e_0$ such that

$$f\alpha_\omega^{mMk}(f) = 0 \quad (k = 1, \dots, n-1),$$
$$[f] = [1-e] \quad \text{in } K_0(A' \cap A_\omega).$$

Thus we have a partial isometry $v \in A' \cap A_\omega$ such that $v^*v = 1 - e$ and $vv^* = f$. Define

$$w = n^{-1/2} \sum_{k=0}^{n-1} \alpha_\omega^{mMk}(v).$$

Then we easily verify that w is a partial isometry satisfying

$$w^*w = 1 - e, \ ww^* \leq e_0$$

and that

$$\|\alpha_\omega^{mM}(w) - w\| = \|n^{-1/2}(\alpha_\omega^{mMn}(v) - v)\| \leq 2n^{-1/2}.$$

Let D be the C^*-subalgebra of $A' \cap A_\omega$ generated by

$$\{w, \alpha_\omega(ww^*), \dots, \alpha_\omega^{mM-1}(ww^*)\},$$

then D is isomorphic to M_{mM+1} (the $(mM+1)$-by-$(mM+1)$ matrices) with the unit

$$w^*w + ww^* + \alpha_\omega(ww^*) + \cdots + \alpha_\omega^{mM-1}(ww^*).$$

Here, by considering the preceding estimate, $\alpha \restriction D$ is almost conjugate (i.e. up to conjugacy, very close in norm) to $\mathrm{Ad}(1 \oplus S(mM))$ when n is large enough, where $S(k)$ denotes the k-by-k matrix

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}.$$

Moreover, since the eigenvalues of the above unitary is uniformly distributed (see [**29**, §5]), the above automorphism of D is almost conjugate (when m is large enough) to

$$\mathrm{Ad}\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \otimes S(M) \oplus \begin{bmatrix} \eta_1 & & \\ & \ddots & \\ & & \eta_l \end{bmatrix} \otimes S(M+1)\right)$$

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for some $k, l \in \mathbb{N}$ and $\lambda_i, \eta_j \in \mathbb{C}$. Therefore, we can take mutually orthogonal projections $f_0, \ldots, f_{M-1}, g_0, \ldots, g_M$ from D such that

$$\sum_{i=0}^{M-1} f_i + \sum_{j=0}^{M} g_j = 1_D,$$
$$\|\alpha_\omega(f_i) - f_{i+1}\| < \frac{\varepsilon}{2} \quad (i = 0, \ldots, M-1),$$
$$\|\alpha_\omega(g_j) - g_{j+1}\| < \frac{\varepsilon}{2} \quad (j = 0, \ldots, M).$$

where $f_M \equiv f_0$, $g_{M+1} \equiv g_0$. Define

$$f_i' = f_i + \sum_{k=0}^{m-1} (e_{i+Mk} - \alpha_\omega^{i+Mk}(ww^*)) \quad (i = 0, \ldots, M-1).$$

Then f_i' is also a projection in $A' \cap A_\omega$ and $f_0', \ldots, f_{M-1}', g_0 \ldots, g_M$ satisfy that

$$\sum_{i=0}^{M-1} f_i' + \sum_{j=0}^{M} g_j = 1_A,$$
$$\|\alpha_\omega(f_i') - f_{i+1}'\| < \varepsilon \quad (i = 0, \ldots, M-1),$$

where $f_M' \equiv f_0'$. Using a standard method for these projections in $A' \cap A_\omega$, we can find projections in A, which ensure the Rohlin property of α. We complete the proof. □

3. *Outer conjugacy*

We first state the main theorem in this section. For automorphisms α and β of a unital C^*-algebra A, α is said to be *asymptotically unitarily equivalent* to β if there exists a continuous mapping u from $[0, \infty)$ into $U(A)$ such that $\alpha = \lim_{t\to\infty} \mathrm{Ad}\, u(t) \circ \beta$ (pointwise). α is said to be *outer conjugate* to β if there exist a unitary u in A and an automorphism γ of A such that $\alpha = \mathrm{Ad}\, u \circ \gamma \circ \beta \circ \gamma^{-1}$.

THEOREM 5. *Let A be a separable nuclear unital purely infinite simple C^*-algebra and let α and β be aperiodic automorphisms of A. If α is asymptotically unitarily equivalent to β then α is outer conjugate to β with respect to an automorphism γ which is isotopic to the identity automorphism id_A of A. That is, there exists a unitary u in A such that $\alpha = \mathrm{Ad}\, u \circ \gamma \circ \beta \circ \gamma^{-1}$ and there exists a homotopy consisting of automorphisms of A between γ and id_A.*

In order to prove the theorem we shall show a so-called stability of aperiodic automorphisms (Lemma 8) from the Rohlin property. If we establish Lemma 8 then, combining the separability of A, we can apply the Evans–Kishimoto intertwining argument ([**12**, Theorem 4.1] or [**31**, Theorem 5.1]) and conclude the theorem. To show the stability we first discuss a certain property of C^*-algebras which says that an almost central unitary path can be replaced by an almost central and rectifiable one of length smaller than a universal constant without changing its end points. This kind of property is also argued in [**4, 11, 31**]. Furthermore, in our case the replacement can be done continuously with respect to a homotopy of such paths. For a rectifiable path u in a C^*-algebra, we denote by $L(u)$ the length of u. The next lemma is nothing but a slight modification of [**16**, Lemma 5.1].

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LEMMA 6. *Let A be a unital C^*-algebra and let B be a unital C^*-subalgebra of A, which is isomorphic to the Cuntz algebra O_2. Then for any unitaries $u_0,\ u_1 \in C[0,1]\otimes(B'\cap A)$, there exists a unitary $v \in C[0,1]^2 \otimes A$ such that*

$$v(s,0) = u_0(s), \quad v(s,1) = u_1(s), \quad L(v(s,\cdot)) \le \frac{8\pi}{3},$$
$$\|[v(s,t),x]\| \le 4\|[u_1(s),x]\| + 5\|[u_0(s),x]\|$$

for any $x \in B'\cap A$ and $s,t \in [0,1]$.

Proof. First fix $s \in [0,1]$. Following the proof of [**16**, Lemma 5.1], we find a unitary $v_s \in C[0,1]\otimes A$ such that

$$v_s(0) = 1, \quad v_s(1) = u_0(s)^* u_1(s), \quad L(v_s) \le \frac{8\pi}{3},$$
$$\|[v_s(t),x]\| \le 4\|[u_0(s)^* u_1(s),x]\| \quad (x \in B'\cap A, t \in [0,1]).$$

Moreover, examining how to make v_s in that proof, we can easily check that $v_s(t)$ is jointly continuous with respect to s and t. Set $v(s,t) = u_0(s)v_s(t)$. Then v has a desired property. □

Using Lemma 6 and the technique in [**37**, Theorem 3.4], we prove the above-mentioned property for separable nuclear unital purely infinite simple C^*-algebras.

THEOREM 7. *Let A be a separable nuclear unital purely infinite simple C^*-algebra. For any finite subset F of A and $\varepsilon > 0$, there exist a finite subset G of A and $\delta > 0$ satisfying the following condition: for any unitary $u \in C[0,1]^2 \otimes A$ with*

$$\|[u(s,t),y]\| < \delta$$

for all $s,t \in [0,1]$ and $y \in G$, there is a unitary $v \in C[0,1]^2 \otimes A$ such that

$$v(s,0) = u(s,0), \quad v(s,1) = u(s,1), \quad L(v(s,\cdot)) < 6\pi,$$
$$\|[v(s,t),x]\| < \varepsilon$$

for all $s,t \in [0,1]$ and $x \in F$. Furthermore, if F is empty then it is possible that G is empty.

Proof. Let $\otimes_1^\infty O_\infty$ denote the infinite C^*-tensor product of the Cuntz algebra O_∞, which is also isomorphic to O_∞ [**25**]. For $N \in \mathbb{N}$, $\otimes_1^N O_\infty$ and $\otimes_N^\infty O_\infty$ denote the N-times and the infinite C^*-tensor product subalgebras of $\otimes_1^\infty O_\infty$, respectively. By [**25**] we have an isomorphism

$$\beta : \left(\bigotimes_1^\infty O_\infty\right) \otimes A \longrightarrow A.$$

Let F and ε be given as in the statement of the theorem, then we may assume that there is a positive integer N_1 such that

$$\beta^{-1}(x) \in \left(\bigotimes_1^{N_1} O_\infty\right) \otimes A \quad (x \in F).$$

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We set $G = F$ and $\delta = \varepsilon/27$. If u is a unitary in $C[0, 1]^2 \otimes A$, which satisfies the above condition for this G and δ, then we may further assume that there is a positive integer N_2 such that

$$\beta^{-1}(u(s, t)) \in \left(\bigotimes_1^{N_2} O_\infty\right) \otimes A \quad (s, t \in [0, 1])$$

simply because $[0, 1]^2$ is compact. Set $N = \max\{N_1, N_2\}$ and we take a nonzero projection e in $\otimes_{N+1}^\infty O_\infty$ satisfying $[e] = -[1]$ in $K_0(\otimes_{N+1}^\infty O_\infty)$. Here we note that

$$[e \otimes 1_A, \beta^{-1}(x)] = 0, \quad [e \otimes 1_A, \beta^{-1}(u(s, t))] = 0 \quad (x \in F, s, t \in [0, 1]).$$

By the continuity of u, we have a positive integer n and a partition $(0 =)t_0 < t_1 < \cdots < t_{n-1} < t_n(= 1)$ of [0,1] such that

$$\|u(s, t_i) - u(s, t_{i-1})\| < \frac{\varepsilon}{3} \quad (s \in [0, 1], i = 1, \ldots, n).$$

For this n we have a partial isometry $w \in (\otimes_{N+1}^\infty O_\infty) \otimes M_{2n+1}$ such that

$$w^* w = \operatorname{diag}(1, \overbrace{e, 1, \ldots, e, 1}^{2n}),$$
$$ww^* = \operatorname{diag}(1, \overbrace{0, 0, \ldots, 0, 0}^{2n}),$$

where $\operatorname{diag}(x_1, \ldots, x_k)$ stands for the diagonal matrix with the diagonal entries $x_1, \ldots, x_k$. Define p by $\beta(e \otimes 1_A)$ and W by $(\beta \otimes \mathrm{id}_{M_{2n+1}})(w)$. Then by an easy computation, we know W is in the following form:

$$W = \begin{bmatrix} W_0 & W_1 & \cdots & W_{2n} \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

with

$$W_{2k}^* W_{2k} = 1_A \ (k = 0, \ldots, n), \quad W_{2k-1}^* W_{2k-1} = p \ (k = 1, \ldots, n),$$
$$W_k^* W_l = 0 \ (k \neq l), \quad W_0 W_0^* + W_1 W_1^* + \cdots + W_{2n} W_{2n}^* = 1_A.$$

Here we note again that

$$[W_i, x] = 0, \quad [W_i, u(s, t)] = 0 \quad (x \in F, s, t \in [0, 1]).$$

Since $[\operatorname{diag}(e, 1)] = 0$ in $K_0(\otimes_{N+1}^\infty O_\infty)$, we can find a unital embedding of the Cuntz algebra O_2 into

$$\operatorname{diag}(e, 1) M_2\left(\bigotimes_{N+1}^\infty O_\infty\right) \operatorname{diag}(e, 1).$$

Note that the last algebra is a unital C^*-subalgebra of

$$\operatorname{diag}(e \otimes 1_A, 1) M_2\left(\left(\bigotimes_1^\infty O_\infty\right) \otimes A\right) \operatorname{diag}(e \otimes 1_A, 1)$$

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which is isomorphic (via $\beta \otimes \mathrm{id}_{M_2}$) to

$$\mathrm{diag}(p, 1)M_2(A)\,\mathrm{diag}(p, 1).$$

Henceforth, we have a unital embedding of O_2 into $\mathrm{diag}(p, 1)M_2(A)\,\mathrm{diag}(p, 1)$. Moreover, in this embedding the elements

$$\mathrm{diag}(xp, x)\ (x \in F), \qquad \mathrm{diag}(u(s, t)p, u(s, t))\ (s, t \in [0, 1])$$

commute with the elements in O_2. Hence, by Lemma 6, we find for each $k = 1, \ldots, n$ a unitary

$$v_k \in C[0, 1]^2 \otimes \mathrm{diag}(p, 1)M_2(A)\,\mathrm{diag}(p, 1)$$

such that for each $s \in [0, 1]$,

$$v_k(s, 0) = \mathrm{diag}(u(s, 0)p, u(s, 0)), \qquad v_k(s, 1) = \mathrm{diag}(u(s, t_k)p, u(s, t_k)),$$
$$\|[v_k(s, t), \mathrm{diag}(xp, x)]\| \le 4\|[u(s, t_k), x]\| + 5\|[u(s, 0), x]\|$$
$$\le 9\delta = \tfrac{1}{3}\varepsilon \quad (x \in F)$$

and $L(v_k(s, \cdot)) = 8\pi/3$. Here we define, for each $s, t \in [0, 1]$,

$$V(s, t) = \mathrm{diag}(u(s, 0), v_1(s, t), \ldots, v_n(s, t))$$

which is a unitary of

$$\mathrm{diag}(1, p, 1, \ldots, p, 1)M_{2n+1}(A)\,\mathrm{diag}(1, p, 1, \ldots, p, 1)$$

and define $v(s, t) = WV(s, t)W^*$ which is a unitary of

$$\mathrm{diag}(1, 0, \ldots, 0)M_{2n+1}(A)\,\mathrm{diag}(1, 0, \ldots, 0).$$

In particular, we may regard $v(s, t)$ as an element of A. Then it follows that for each $s \in [0, 1]$,

$$v(s, 0) = WV(s, 0)W^* = \sum_{i=0}^{2n} W_i u(s, 0)W_i^* = u(s, 0)\sum_{i=0}^{2n} W_i W_i^* = u(s, 0),$$
$$v(s, 1) = WV(s, 1)W^* = u(s, 0)W_0 W_0^* + \sum_{k=1}^{n} u(s, t_k)(W_{2k-1}W_{2k-1}^* + W_{2k}W_{2k}^*),$$
$$\|[v(s, t), x]\| = \|[WV(s, t)W^*, W\,\mathrm{diag}(x, xp, x, \ldots, xp, x)W^*]\|$$
$$\le \frac{\varepsilon}{3} \quad (t \in [0, 1], x \in F)$$

and $L(v(s, \cdot)) \le 8\pi/3$.

In a similar fashion we can apply the above method to $u(s, 1)$ instead of $u(s, 0)$. Then we obtain for each $k = 1, \ldots, n$ a unitary

$$v'_k \in C[0, 1]^2 \otimes \mathrm{diag}(1, p)M_2(A)\,\mathrm{diag}(1, p)$$

such that, for each $s \in [0, 1]$,

$$v'_k(s, 0) = \mathrm{diag}(u(s, 1), u(s, 1)p), \qquad v'_k(s, 1) = \mathrm{diag}(u(s, t_k), u(s, t_k)p),$$
$$\|[v'_k(s, t), \mathrm{diag}(x, xp)]\| \le \tfrac{1}{3}\varepsilon \quad (x \in F)$$

and $L(v_k'(s,\cdot)) = 8\pi/3$. Likewise we define, for each $s, t \in [0,1]$,

$$V'(s,t) = \mathrm{diag}(v_1(s,t), \dots, v_n(s,t), u(s,1)), \quad v'(s,t) = W'V(s,t)W'^*,$$

where W' indicates

$$\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ W_0 & W_1 & \cdots & W_{2n} \end{bmatrix}.$$

Then $v'(s,t)$ is a unitary in $\mathrm{diag}(0_{2n}, 1)M_{2n+1}(A)\,\mathrm{diag}(0_{2n}, 1)$, so we may also regard it as an element of A. We can easily verify that, for each $s \in [0,1]$,

$$v'(s,1) = \sum_{k=1}^{n} u(s,t_k)(W_{2k-2}W_{2k-2}^* + W_{2k-1}W_{2k-1}^*) + u(s,1)(W_{2n}W_{2n}^*),$$

$$v'(s,0) = u(s,1), \quad \|[v'(s,t), x]\| \le \frac{\varepsilon}{3} \quad (t \in [0,1], x \in F)$$

and $L(v'(s,\cdot)) \le 8\pi/3$. We compare $v(s,1)$ with $v'(s,1)$;

$$\begin{aligned}\|v(s,1) - v'(s,1)\| &= \left\| u(s,0)W_0W_0^* + \sum_{k=1}^{n} u(s,t_k)(W_{2k}W_{2k}^* - W_{2k-2}W_{2k-2}^*) - u(s,1)W_{2n}W_{2n}^* \right\| \\ &= \left\| \sum_{k=1}^{n-1} (u(s,t_{k-1}) - u(s,t_k))W_{2k-2}W_{2k-2}^* \right\| \le \frac{\varepsilon}{3}.\end{aligned}$$

Accordingly $v(s,1)$ is very close to $v'(s,1)$, hence we can connect $v(s,1)$ to $v'(s,1)$ by a unitary $v'' \in C[0,1]^2 \otimes A$ such that

$$\|[v''(s,t), x]\| \le 2\|v(s,1) - v'(s,1)\| + \|[v(s,1), x]\| \le \varepsilon \quad (x \in F)$$

and $L(v''(s,\cdot)) \le 2\arcsin(\varepsilon/6)$. Joining v, v'' and v' canonically, we have a desired unitary of $C[0,1]^2 \otimes A$ and complete the proof. □

Once the above theorem is obtained, the following lemma (which is called the stability of automorphisms) can be proved in a way that has become standard [**19**, Theorem 1, **31**, Proposition 3.4].

LEMMA 8. *Let A be a separable nuclear unital purely infinite simple C^*-algebra and let α be an automorphism of A. If α is aperiodic then α has the following property.*

For any finite subset F of A and $\varepsilon > 0$, there exist a finite subset G of A and $\delta > 0$ which satisfy the following condition: if u is a unitary in $C[0,1]^2 \otimes A$ such that

$$u(s,0) = u(0,t) = 1, \quad \|[u(s,t), y]\| < \delta$$

for all $s, t \in [0,1]$ and $y \in G$, then there exists a unitary $v \in C[0,1] \otimes A$ such that

$$\|u(s,1) - v(s)\alpha(v(s))^*\| < \varepsilon,$$
$$v(0) = 1, \quad \|[v(s), x]\| < \varepsilon$$

for all $s \in C[0,1]$ and $x \in F$. Furthermore, if F is empty then it is possible that G is empty.

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Proof. Let F and ε be given as above. We take a sufficiently large $N \in \mathbb{N}$ such that $6\pi/(N-1) < \varepsilon/2$. Temporarily fix $\varepsilon_1 > 0$ (precisely how small ε_1 must be, will become clear in the course of the proof). Applying Theorem 7 to $\bigcup_{k=0}^{N} \alpha^k(F)$ and ε_1, we choose a finite subset G_1 of A and $\delta_1 > 0$ satisfying the conditions stated there. For a unitary $u \in C[0,1]^2 \otimes A$ we set

$$u^{(k)}(s,t) = \begin{cases} u(s,t)\alpha(u(s,t)) \dots \alpha^{k-1}(u(s,t)) & (k \geq 1) \\ 1 & (k = 0). \end{cases}$$

Then we can choose a finite subset G of A and $\delta > 0$ such that, for any unitary $u \in C[0,1]^2 \otimes A$ satisfying $\|[u(s,t), y]\| < \delta$ $(s, t \in [0,1], y \in G)$, we have

$$\|[u^{(k)}(s,t), x]\| < \min\{\delta_1, \varepsilon_1\} \quad (s, t \in [0,1], x \in G_1 \cup F, k = 1, \dots, N+1).$$

We claim that these G and δ satisfy the requirement. In order to verify this, let u be a unitary in $C[0,1]^2 \otimes A$ such that

$$u(s,0) = u(0,t) = 1, \quad \|[u(s,t), y]\| < \delta \quad (s, t \in [0,1], y \in G).$$

By the preceding consideration we can apply Theorem 7 to $u^{(N)}$ and $u^{(N+1)}$, respectively, and obtain unitaries $v, w \in C[0,1]^2 \otimes A$ such that

$$v(s,0) = u^{(N)}(s,0), \quad v(s,1) = u^{(N)}(s,1), \quad \|[v(s,t), x]\| < \varepsilon_1,$$
$$\|v(s,t) - v(s,t')\| < 6\pi|t - t'| \quad (s, t, t' \in [0,1], x \in \cup_{k=0}^{N} \alpha^k(F))$$

and that the similar conditions hold for w and $u^{(N+1)}$. Since α has the Rohlin property, we can choose projections $e_0, \dots, e_{N-1}, f_0, \dots, f_N$ as in Theorem 1 which almost commute with

$$\left\{\alpha^{i-(N-1)}\left(v\left(s, \frac{i}{N-1}\right)\right) \,\middle|\, i = 0, \dots, N-1, s \in [0,1]\right\},$$
$$\left\{\alpha^{j-N}\left(w\left(s, \frac{j}{N}\right)\right) \,\middle|\, j = 0, \dots, N, s \in [0,1]\right\},$$
$$\{u^{(k)}(s,1) \mid k = 1, \dots, N+1, \ s \in [0,1]\}$$

and F up to within ε_1. Here we define, for each $s \in [0,1]$,

$$U(s) = \sum_{i=0}^{N-1} u^{(i+1)}(s,1)\alpha^{i-(N-1)}\left(v\left(s, \frac{i}{N-1}\right)\right)^* e_i$$
$$+ \sum_{j=0}^{N} u^{(j+1)}(s,1)\alpha^{j-N}\left(w\left(s, \frac{j}{N}\right)\right)^* f_j.$$

Then we can check that $U(s)$ is close to a unitary when $N\varepsilon_1$ is sufficiently small, and furthermore check that $U(s)\alpha(U(s))^*$ is close to $u(s,1)$ up to within $6\pi/(N-1) + 2N\varepsilon_1$. Therefore, taking the polar decomposition of $U(s)$ (which can be done continuously with respect to $s \in [0,1]$), we have a desired unitary path. □

Now we show Theorem 5 by applying the Evans–Kishimoto intertwining argument to our setting as in the proof of [**31**, Theorem 5.1].

Proof of Theorem 5. We fix a countable subset $\{x_k \mid k \geq 1\}$ of $U(A)$, whose linear span is dense in A, and set $B_n = \{x_k \mid k = 1, \dots, n\}$. Replacing β by $\mathrm{Ad}\, u(0) \circ \beta$ and $u(t)$ by $u(t)u(0)^*$, we may assume that $u(0) = 1$. Let $\varepsilon > 0$ be given.

First, we shall construct $t_n > 0$, unitaries $u_n, w_n \in C[0,1] \otimes A$, a finite subset G_n of A and $\delta_n > 0$ for each $n \geq 1$, which satisfy the following conditions.

$(1, n)$ $\|\sigma_{n-1}(x) - \mathrm{Ad}\, u^{(n-1)}(t)^{\varepsilon(n-1)} \circ \sigma_{n-2}(x)\| < 2^{-n}\varepsilon$ $(x \in B_n, t \geq t_n)$.

$(2, n)$ $\|\sigma_{n-1}(y) - \mathrm{Ad}\, u^{(n-1)}(t)^{\varepsilon(n-1)} \circ \sigma_{n-2}(y)\| < 2^{-1}\delta_{n-1}$ $(y \in G_{n-1}, t \geq t_n)$.

$(3, n)$ $\sigma_n = \lim_{t\to\infty} \mathrm{Ad}\, u^{(n)}(t)^{\varepsilon(n)} \circ \sigma_{n-1}$ (pointwise).

$(4, n)$ $u^{(n-1)}(t_n s)^{\varepsilon(n-1)} = w_n(s) u_n(s) \sigma_{n-2}(u_n(s))^*$ $(s \in [0,1])$.

$(5, n)$ $u_n(0) = 1$, $\|w_n(s) - 1\| < 2^{-n}\varepsilon$ $(s \in [0,1])$.

$(6, n)$ $\|[u_n(s), x]\| < 2^{-n}\varepsilon$ $(s \in [0,1], x \in F_{n-2})$.

$(7, n)$ For any $u \in U(C[0,1]^2 \otimes A)$ with

$$u(s,0) = u(0,t) = 1, \quad \|[u(s,t), y]\| < \delta_n \quad (s, t \in [0,1], y \in \sigma_n(G_n)),$$

there exists a $v \in U(C[0,1] \otimes A)$ such that

$$\|u(s,1)v(s)\sigma_n(v(s))^* - 1\| < 2^{-(n+2)}\varepsilon, \quad v(0) = 1,$$
$$\|[v(s), x]\| < 2^{-(n+2)}\varepsilon \quad (s \in [0,1], x \in F_n).$$

In the above conditions we denote $(-1)^n$ by $\varepsilon(n)$ and define unitaries $u^{(n)} \in C[0,1] \otimes A$ and automorphisms σ_n of A by

$$u^{(0)} = u,$$
$$u^{(n)}(t)^{\varepsilon(n-1)} = u^{(n-1)}(t + t_n)^{\varepsilon(n-1)} \cdot u^{(n-1)}(t_n)^{\varepsilon(n)} \quad (n \geq 1),$$
$$\sigma_{-1} = \beta, \quad \sigma_0 = \alpha,$$
$$\sigma_n = \mathrm{Ad}\, u^{(n-1)}(t_n)^{\varepsilon(n-1)} \circ \sigma_{n-2} \quad (n \geq 1),$$

and set

$$F_{-1} = \emptyset, \quad F_0 = \emptyset,$$
$$F_n = B_n \cup \{u_k(s) \mid k = n, n-2, \dots, s \in [0,1]\} \quad (n \geq 1),$$
$$G_0 = \emptyset, \quad \delta_0 = 1.$$

(In particular, $(2, n)$ is a trivial statement when $n = 1$, as is $(6, n)$ when $n = 1, 2$.) We also regard all the conditions (except $(3, n)$) as trivial when $n = 0$. Note that $(3, 0)$ denotes the hypothesis of this theorem.

Let $n \geq 1$. Suppose that we have constructed $t_k, u_k, w_k, G_k, \delta_k$ for each $k \leq n-1$. Then we proceed as follows. Since $\sigma_{n-1} = \lim_{t\to\infty} \mathrm{Ad}\, u^{(n-1)}(t)^{\varepsilon(n-1)} \circ \sigma_{n-2}$, there is a $t_n > 0$ such that $(1, n)$ and $(2, n)$ hold. Then by definition,

$$\mathrm{Ad}\, u^{(n)}(t)^{\varepsilon(n-1)} \circ \sigma_n = \mathrm{Ad}\, u^{(n-1)}(t + t_n)^{\varepsilon(n-1)} \circ \sigma_{n-2}.$$

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Note that the right-hand term converges to σ_{n-1} (as $t \to \infty$) from $(3, n-1)$. This implies $(3, n)$. By using $(2, n-1)$ we have that

$$\|\sigma_{n-2}(y) - \mathrm{Ad}\, u^{(n-2)}(t + t_{n-1})^{\varepsilon(n-2)} \circ \sigma_{n-3}(y)\| < 2^{-1}\delta_{n-2} \quad (y \in G_{n-2}, t \geq 0).$$

Thus again by definition,

$$\|\sigma_{n-2}(y) - \mathrm{Ad}\, u^{(n-1)}(t)^{\varepsilon(n-2)} \circ \sigma_{n-1}(y)\| < 2^{-1}\delta_{n-2} \quad (y \in G_{n-2}, t \geq 0).$$

Since $u^{(n-1)}(0) = 1$, we have

$$\|\sigma_{n-2}(y) - \mathrm{Ad}\, u^{(n-1)}(t)^{\varepsilon(n-2)} \circ \sigma_{n-2}(y)\| < \delta_{n-2} \quad (y \in G_{n-2}, t \geq 0).$$

Hence we can apply $(7, n-2)$ (when $n = 1, 2$, which means Lemma 8 in the case when $F, G = \emptyset$) for $u^{(n-1)}(t_n(\cdot)(\cdot))^{\varepsilon(n-1)} \in U(C[0,1]^2 \otimes A)$ and find $u_n, w_n \in U(C[0,1] \otimes A)$ which satisfy $(4, n)$,$(5, n)$ and $(6, n)$. Since F_n is compact, we can also apply Lemma 8 to this F_n and $2^{-(n+2)}\varepsilon$, and obtain a finite subset G_n of A, $\delta_n > 0$ satisfying $(7, n)$. We thus complete the induction.

Secondly, we show outer conjugacy between α and β using the previous paragraph. From $(6, n)$ $(n \geq 3)$, we can define automorphisms $\gamma_0(s)$ and $\gamma_1(s)$ of A by pointwise limits

$$\gamma_0(s) = \lim_{k\to\infty} \mathrm{Ad}\, u_{2k}(s)u_{2k-2}(s)\cdots u_2(s),$$
$$\gamma_1(s) = \lim_{k\to\infty} \mathrm{Ad}\, u_{2k+1}(s)u_{2k-1}(s)\cdots u_1(s)$$

for each $s \in [0, 1]$. Since $u_k(0) = 1$ $(k \geq 1)$, both $\gamma_0(1)$ and $\gamma_1(1)$ are isotopic to id_A. On the other hand, by a direct computation it is verified that

$$\sigma_{2k} = \mathrm{Ad}\, w'_{2k} \circ \mathrm{Ad}(u_{2k}(1)u_{2k-2}(1)\cdots u_2(1)) \circ \alpha \circ \mathrm{Ad}(u_{2k}(1)u_{2k-2}(1)\cdots u_2(1))^*$$

where

$$w'_{2k} \equiv w_{2k}(1) \cdot \mathrm{Ad}\, u_{2k}(1)(w_{2k-2}(1)) \cdot \mathrm{Ad}(u_{2k}(1)u_{2k-2}(1))(w_{2k-4}(1)) \cdot \cdots$$
$$\cdots \cdot \mathrm{Ad}(u_{2k}(1)u_{2k-2}(1)\cdots u_4(1))(w_2(1)).$$

From $(5, n)$, (w'_{2k}) converges uniformly to a unitary $W_0 \in A$ with $\|W_0 - 1\| < \varepsilon$. Hence (σ_{2k}) converges pointwise to an automorphism $\mathrm{Ad}\, W_0 \circ \gamma_0(1) \circ \alpha \circ \gamma_0(1)^{-1}$. Similarly (σ_{2k+1}) converges pointwise to an automorphism $\mathrm{Ad}\, W_1 \circ \gamma_1(1) \circ \beta \circ \gamma_1(1)^{-1}$ for some unitary $W_1 \in A$ with $\|W_1 - 1\| < \varepsilon$. From $(1, 2k+1)$ and $(1, 2k)$ we have

$$\|\sigma_{2k}(x) - \sigma_{2k+1}(x)\| < 2^{-(2k+1)}\varepsilon \quad (x \in B_{2k+1}),$$
$$\|\sigma_{2k-1}(x) - \sigma_{2k}(x)\| < 2^{-2k}\varepsilon \quad (x \in B_{2k}).$$

Therefore, we conclude that

$$\mathrm{Ad}\, W_0 \circ \gamma_0(1) \circ \alpha \circ \gamma_0(1)^{-1} = \mathrm{Ad}\, W_1 \circ \gamma_1(1) \circ \beta \circ \gamma_1(1)^{-1}.$$

We have thus proved the theorem. □

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Combining Theorem 5 and the classification theory due to Phillips, we reach our classification theorem of aperiodic automorphisms of a nuclear purely infinite simple C^*-algebra.

THEOREM 9. *Let A be a separable nuclear unital purely infinite simple C^*-algebra and let α and β be aperiodic automorphisms of A. Then the following three conditions are equivalent:*

(1) $[\alpha] = [\beta]$ *in* $KK(A, A)$*;*

(2) α *is asymptotically unitarily equivalent to* β*;*

(3) α *is outer conjugate to* β *with respect to an automorphism* γ *which is homotopic to the identity automorphism of* A.

Moreover, under the same assumption the following three conditions are also equivalent to each other:

(1′) *there exists an invertible element* $\eta \in KK(A, A)$ *such that* $[\alpha] = \eta \cdot [\beta] \cdot \eta^{-1}$ *and* $[1_A] \cdot \eta = [1_A]$*;*

(2′) *there exist a continuous mapping* u *from* $[0, \infty)$ *into* $U(A)$ *and an automorphism* γ *of* A *such that* $\alpha = \lim_{t\to\infty} \mathrm{Ad}\, u(t) \circ \gamma \circ \beta \circ \gamma^{-1}$ *pointwise;*

(3′) α *is outer conjugate to* β.

Proof. The equivalence between (1) and (2) follows from the results of Phillips [**38**, Lemma 1.3.3, Theorems 4.1.3 and 4.1.4]. It is also shown that (3) implies (1). Theorem 5 says that (2) implies (3). Similarly one verifies the equivalence among (1′), (2′) and (3′) by using [**38**, Corollary 4.2.2]. □

Remark 10. (i) It is known that any pair (G_0, G_1) of countable abelian groups can be described by $(K_0(A), K_1(A))$ of a separable nuclear unital purely infinite simple C^*-algebra A [**38**, Theorem 4.2.5] and that any (unit-preserving) invertible element of $KK(A, A)$ can be lifted to an automorphism of A [**38**, Corollary 4.2.2]. Hence there exist two aperiodic automorphisms of some separable nuclear unital purely infinite simple C^*-algebra A, which induce two different elements of $KK(A, A)$ but the same element of $KL(A, A)$ [**39**] by using the universal coefficient theorem [**41**]. This means that outer conjugacy of aperiodic automorphisms is not in general characterized by their KL elements.

(ii) Theorem 9 is still valid for any separable non-unital nuclear purely infinite simple C^*-algebra A, if one removes the condition $[1_A] \cdot \eta = [1_A]$ from (1′) and regards the unitaries considered there as the elements of the unitization of A.

Finally, we present a corollary which is a direct consequence of [**25**, Theorem 3.14] and Theorem 9 above. The author thanks Masaki Izumi for pointing out this fact.

COROLLARY 11. *Let A be a separable nuclear unital purely infinite simple C^*-algebra and let α be an automorphism of A. If α is aperiodic then α absorbs any automorphism β of O_∞, that is, there exist a unitary u in $O_\infty \otimes A$ and an isomorphism γ from A onto $O_\infty \otimes A$ such that*

$$\beta \otimes \alpha = \mathrm{Ad}\, u \circ \gamma \circ \alpha \circ \gamma^{-1}.$$

Proof. Let φ be a unital homomorphism from A into $O_\infty \otimes A$, which is given by $\varphi(a) = 1 \otimes a$ for $a \in A$. Then by [**25**, Theorem 3.14] there exists a unital homomorphism ψ from $O_\infty \otimes A$ into A such that

$$[\varphi \circ \psi] = [\mathrm{id}_{O_\infty \otimes A}] \quad \text{in } KK(O_\infty \otimes A, O_\infty \otimes A),$$
$$[\psi \circ \varphi] = [\mathrm{id}_A] \quad \text{in } KK(A, A).$$

By virtue of Phillips's classification theorem [**38**, Corollary 4.2.2], we have an isomorphism γ from A onto $O_\infty \otimes A$ such that

$$[\gamma] = [\varphi] \quad \text{in } KK(O_\infty \otimes A, A), \quad [\gamma^{-1}] = [\psi] \quad \text{in } KK(A, O_\infty \otimes A).$$

Accordingly,

$$[\gamma] \cdot [\beta \otimes \alpha] \cdot [\gamma^{-1}] = [\varphi] \cdot [\beta \otimes \alpha] \cdot [\psi] = [\psi \circ (\beta \otimes \alpha) \circ \varphi]$$
$$= [\psi \circ \varphi \circ \alpha] = [\alpha].$$

Therefore, we obtain the result by Theorem 9 and complete the proof. □

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