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A categorical analogue of the monoid semiring construction

PETER HINES[†]

Department of Computer Science, University of York, York, United Kingdom
Email: peter.hines@cs.york.ac.uk

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This paper introduces and studies a categorical analogue of the familiar monoid semiring construction. By introducing an axiomatisation of summation that unifies notions of summation from algebraic program semantics with various notions of summation from the theory of analysis, we demonstrate that the monoid semiring construction generalises to cases where both the monoid and the semiring are categories. This construction has many interesting and natural categorical properties, and natural computational interpretations.

1. Introduction

The monoid semiring construction – in particular, the special case of the group ring construction – is one of the most familiar and useful algebraic constructions. This paper places this construction within a significantly more general categorical setting, where the monoid is generalised to a category (with a suitable smallness condition) and the semiring is replaced by a category equipped with a suitable notion of partial summation on hom-sets.

1.1. *Cauchy products and monoid semirings*

In the formal theory of power series, an infinite power series over some complex variable z, given as $P = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots$, may be treated as simply a function $P : \mathbb{N} \to \mathbb{C}$. Given another formal power series $Q : \mathbb{N} \to \mathbb{C}$ over the same variable, their *convolution*, or *Cauchy product*, is the formal power series $(Q * P) : \mathbb{N} \to \mathbb{C}$ given by

$$(Q * P)(n) = \sum_{n=y+x} q(y)p(x).$$

A formal power series $P : \mathbb{N} \to \mathbb{C}$ converges absolutely within the unit disk $\{\|z\| \leqslant 1\}$ when the sum $\sum_{n\in\mathbb{N}} P(n)$ converges absolutely, and it is a straightforward result of analysis (Titchmarsh 1983) that the Cauchy product of two formal power series that converge

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absolutely within the unit disk itself converges absolutely within the unit disk (and much more general conditions may also lead to convergence – see Titchmarsh (1983)).

When restricting formal power series to the case where only a finite number of coefficients are non-zero (that is, polynomials in some complex variable z), convergence is guaranteed not only within the unit disk, but for all $z \in \mathbb{C}$. Algebraically, this naturally generalises to the familiar theory of monoid semirings (Golan 1999).

Definition 1.1 (monoid semirings). Let $(M, \cdot)$ be a monoid and $(R, \times, +, 1_R, 0_R)$ be a unital semiring. The **monoid semiring** $R[M]$ is the unital semiring whose elements are functions $\eta : M \to R$ such that $\|\{m : \eta(m) \neq 0\}_{m\in M}\| < \infty$. The multiplication and addition in this semiring are given by

$$(\eta \times \mu)(m) = \sum_{m=qp} \eta(q)\mu(p)$$

$$(\eta + \mu)(m) = \eta(m) + \mu(m).$$

The additive identity is the function $0(m) = 0_R \ \forall m \in M$, and the multiplicative identity is the function

$$1(m) = \begin{cases} 1_R & m = 1_M \\ 0_R & \text{otherwise.} \end{cases}$$

When $(M, \cdot)$ is a group, $R[M]$ is called the **group semiring**; similarly, when R is a ring, $R[M]$ is called the **monoid (or group) ring**.

This paper generalises the above construction of monoid semirings in two ways:

- The finite sums of a semiring are replaced by a more general axiomatic summation of (possibly infinite) indexed families.
- The monoids $(M, \cdot)$ and $(R, \times)$ are replaced by categories. Thus, the unital ring R is replaced by a category with some appropriate notion of summation on homsets.

2. An axiomatic notion of summation

For the programme outlined above, we replace semirings with categories equipped with a partial summation on hom-sets. The overall intention of this paper is to provide a categorical analogue of the monoid semiring construction that generalises the usual theory, but is also applicable to categories used in algebraic program semantics. As discussed in Appendix A, the axiomatisations of summation commonly used within algebraic program semantics have properties that rule out various analytic notions of summation, such as absolute convergence of real or complex sums.

We therefore introduce a very general axiomatisation of summation that includes, as special cases, various notions of summation from both theoretical computer science (in particular, algebraic program semantics) and analysis. By comparison with the other notions of summation discussed in Appendix A, this is a very weak axiomatisation – in particular, the expressive power we require comes from both the axioms we now present, and the axioms for the interaction of summation and composition given in Section 3.

Definition 2.1 (partial commutative monoids (PCMs)). Given sets M and I, an **I-indexed family of elements** of M is defined to be a function $x : I \to M$. For simplicity, we denote this by $\{x_i\}_{i \in I}$. Throughout this paper, we restrict ourselves to countable (that is, either finite or denumerably infinite) indexing sets, and hence **countably indexed families**.

A **partial commutative monoid** or **PCM** is a non-empty set M together with a partial function Σ from indexed families of M to elements of M. An indexed family of elements of M is said to be **summable** when it is in the domain of Σ, and summation is required to satisfy the following two axioms:

— **Unary sum axiom:** Any family $\{x_i\}_{i \in I}$, where $I = \{i'\}$ is a singleton set, is summable, and $\sum_{i \in I} x_i = x_{i'}$.

— **Weak partition-associativity axiom:** Let $\{x_i\}_{i \in I}$ be a *summable* family and $\{I_j\}_{j \in J}$ be a countable partition† of I. Then $\{x_i\}_{i \in I_j}$ is summable for every $j \in J$, as is $\{\sum_{i \in I_j} x_i\}_{j \in J}$, and

$$\sum_{i \in I} x_i = \sum_{j \in J} \left(\sum_{i \in I_j} x_i \right).$$

Given a summable family $x = \{x_i\}_{i \in I}$, we may write $\Sigma(x)$ (unambiguously) as $\sum_{i \in I} x_i$. In particular, if $I = \{1, \ldots, n\}$, we write $\Sigma(x) = x_1 + x_2 + x_3 + \cdots + x_n$, and if $I = \mathbb{N}$, $\Sigma(x) = x_1 + x_2 + x_3 + \cdots$. Notice that, by weak partition-associativity, we may equate different partitions of a summable family x, for example:

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots &= x_1 + (x_2 + x_3 + \cdots + x_n + \cdots) \\ &= (x_1 + x_2) + (x_3 + x_4) + \cdots + (x_n + x_{n+1}) + \cdots \\ &= (x_1 + x_3 + x_5 + \cdots) + (x_2 + x_4 + x_6 + \cdots). \end{aligned}$$

Remark 2.2 (WPA axiom). The above weak partition-associativity axiom is clearly a weakening of the usual partition-associativity axiom from algebraic program semantics (Manes and Arbib 1986; Haghverdi 2000), where the two-sided implication is weakened to a one-sided version (see Appendix A for more details). However, in this weakened form it is also familiar from traditional analysis. For example, Hille (1982, page 108) states and proves the following property of absolute convergence of real numbers:

> 'An absolutely convergent series may be split into mutually exclusive subseries, finite or infinite in number. The sum of these subseries is equal to the sum of the original series.'

(Note that the prior assumption of an absolutely convergent series in this quotation means that this statement is not equivalent to the usual partition-associativity axiom described in Definition A.1). In Appendix A, we give various examples of PCMs from both analysis and algebraic program semantics, and compare this formalism to other axiomatisations of summation used in various fields.

We will first demonstrate that the indexed summation of a PCM is invariant under isomorphism of indexing sets.

† Following Manes and Arbib (1986), we also allow countably many I_j to be empty.

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Proposition 2.3. Let $x : I \to M$ and $y : J \to M$ be countably indexed families and let $\varphi : I \to J$ be a bijection satisfying $y \circ \varphi = x$. Then $\Sigma(y)$ is defined exactly when $\Sigma(x)$ is defined, in which case they are equal.

Proof. For arbitrary $i \in I$, the set $J_i = \{\varphi(i)\}$ is a singleton, and hence, by the unary sum axiom, is summable. As indexing families are countable, $\{J_i \mid i \in I\}$ is a countable partition of J. Let us now assume that $\{y_j\}_{j\in J}$ is summable. We deduce that:

$$\begin{aligned} \sum_{j\in J} y_j &= \sum_{i\in I}\sum_{j\in J_i} y_j && \text{(weak partition-associativity)} \\ &= \sum_{i\in I} y_{\varphi(i)} && \text{(definition and unary sum axiom)} \\ &= \sum_{i\in I} x_i. && \text{(definition)} \end{aligned}$$

Alternatively, if we assume that $\{y_j\}_{j\in J}$ is not summable, the assumption that $\{x_i\}_{i\in I}$ is summable will (by interchanging the roles of x and y in the above argument) imply the summability of $\{y_j\}_{j\in J}$, which gives a contradiction. Therefore, $\{y_j\}_{j\in J}$ is summable exactly when $\{x_i\}_{i\in I}$ is summable, in which case their sums are equal. □

Note the similarity of this notion with either the permutation-independence of *absolute convergence* of real sums (Definition A.3) or the notion of *unconditional convergence* of sums in Banach space (Definition A.6).

The following basic properties of PCMs will be used throughout.

Proposition 2.4. Let (M, Σ) be a PCM. Then

(1) **Summable subfamilies:** Let $\{x_i\}_{i\in I}$ be a summable family of M. Then any subfamily $\{x_i\}_{i\in K}$, where $K \subseteq I$, is also summable.
(2) **Existence of zero:** The empty set is summable, and $x + \{\} = x = \{\} + x$ for all $x \in M$. Hence it is a zero for M, and we write $0 = \sum\{\}$.
(3) **Sums of zeros:** For any index set I, let $0_I : I \to M$ denote the constantly zero family (so $0_I(i) = 0$, for all $i \in I$). Then 0_I is summable, and $\Sigma_I 0_I = 0$. More generally, for any element $x \in M$, $x + 0 + 0 + 0 + \cdots = x$ (where $0 + 0 + \cdots$ denotes (the sum of) either a finite or infinite sequence of 0's).

Proof. The proofs of (1) and (2) are based on very similar proofs (for the special case of partially additive monoids – see Appendix A) presented in Manes and Arbib (1986).

(1) (Summable subfamilies) Any subset $K \subseteq I$ defines a partition of I, namely $\{K, I \setminus K\}$. By weak partition-associativity, $\sum_{i\in K} x_i$ exists.
(2) (Existence of zero) As M is by definition non-empty, the unary sum axiom implies that the set of summable families is also non-empty. The empty family is a subfamily of any summable family. Hence, letting $K = \varnothing$ in the partition above, we see that the empty family $\{\}$ is summable. It is then immediate that $\sum\{\} = 0$ is a zero for the summation operation, and $0 + x = x = x + 0$ exists for arbitrary $x \in M$.

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(3) (Sums of zeros) We pick any partition of I whose first cell is I itself, and the remaining cells are empty (the number of empty cells is either finite or infinite, depending on whether we want a finite or infinite sum of 0's). For example, we write $I = I_1 \uplus (\uplus_{n>1} I_n)$, where $I_1 = I$, $I_i = \emptyset$, if $i > 1$. If $x = \{x_i\}_{i \in I}$ is an I-indexed *summable* family, then, by weak partition-associativity, we have

$$\sum_{i \in I} x_i = \sum_{i \in I_1} x_i + \sum_{n>1} (\sum_{i \in I_n} x_i)$$
$$= \sum_{i \in I_1} x_i + 0 + 0 + \cdots .$$

We now pick a singleton family $\{x\}$, so $\Sigma(x) = x$, and the result follows. □

We now define homomorphisms of PCMs, and show that the class of all PCMs, together with this notion of homomorphism, forms a category.

Definition 2.5 (PCM homomorphisms and the category of PCMs). A **homomorphism** of PCMs is a function $f : (M, \Sigma) \to (N, \Sigma')$ satisfying the following natural property:

Given a summable family $\{m_i\}_{i \in I}$ of (M, Σ), we have $\{f(m_i)\}_{i \in I}$ is a summable family of (N, Σ'), and $f(\Sigma_{i \in I} m_i) = \Sigma'_{i \in I} f(m_i)$.

Proposition 2.6. The class of all PCMs, together with the above notion of homomorphism, forms a category, which we denote by **PCM**.

Proof. First note that for a PCM (M, Σ^M), the identity function $1_M : M \to M$ is a PCM homomorphism. We next consider PCM homomorphisms

$$f : (A, \Sigma^A) \to (B, \Sigma^B)$$
$$g : (B, \Sigma^B) \to (C, \Sigma^C),$$

together with a summable family $\{a_i\}_{i \in I}$ of A. Then the function $gf : A \to C$ satisfies

$$g\left(f\left(\Sigma^A_{i \in I} a_i\right)\right) = g\left(\Sigma^B_{i \in I} f(a_i)\right) = \Sigma^C_{i \in I} gf(a_i).$$

(Note that these sums are required to exist, by the definition of PCM homomorphism.) Thus gf is a PCM homomorphism from (A, Σ^A) to (C, Σ^C). Finally, associativity of composition follows from the associativity of composition for functions. □

Examples of PCMs are given in Appendix A. For the program outlined in Section 1, we now require categories whose hom-sets are PCMs, together with a specified interaction between summation and composition.

3. Categories with a notion of summation on hom-sets

We now introduce a certain class of categories whose hom-sets are PCMs, together with axioms for the interaction of summation and composition.

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Definition 3.1 (PCM-categories). We define a **PCM-category** to be a locally small† category $\mathscr{C}$, together with, for all $X, Y \in Ob(\mathscr{C})$, a partial function $\Sigma^{(X,Y)}$ from countably indexed families over $\mathscr{C}(X,Y)$ to $\mathscr{C}(X,Y)$ (we will often omit the superscript when it is clear from the context).

This class of partial functions is required to satisfy the following axioms:

(1) **PCM-structure on hom-sets:** $\left(\mathscr{C}(X,Y), \Sigma^{(X,Y)}\right)$ is a PCM, for all $X, Y \in Ob(\mathscr{C})$.

(2) **Strong distributivity:** Given summable families

$$\{f_i \in \mathscr{C}(X,Y)\}_{i\in I}$$
$$\{g_j \in \mathscr{C}(Y,Z)\}_{j\in J},$$

we have

$$\{g_j f_i \in \mathscr{C}(X,Z)\}_{(j,i)\in J\times I}$$

is a summable family satisfying

$$\sum_{(j,i)\in J\times I}^{(X,Z)} g_j f_i = \left(\sum_{j\in J}^{(Y,Z)} g_j\right)\left(\sum_{i\in I}^{(X,Y)} f_i\right)$$

We consider examples of PCM-categories in Appendix A, and properties of PCM-categories in Section 3.1.

Remark 3.2 (PCM-categories and categorical enrichment). A very natural question at this point is to ask whether a 'PCM-category' is a category enriched over some suitable (monoidal, or closed) category of PCMs – see Section 10 for details.

3.1. *Properties of PCM-categories*

As may be expected, the strong distributivity property, together with the unary sum axiom, implies the usual left- and right-distributivity laws.

Proposition 3.3. Let $\left(\mathscr{C}, \Sigma^{(\_,\_)}\right)$ be a PCM category and $\{g_i \in \mathscr{C}(Y,Z)\}_{i\in I}$ be a summable family. Then, for all arrows $f \in \mathscr{C}(X,Y)$ and $h \in \mathscr{C}(Z,T)$, we have

$$\{hg_i \in \mathscr{C}(Y,T)\}_{i\in I}$$
$$\{g_i f \in \mathscr{C}(X,Z)\}_{i\in I}$$

are summable families, and

$$h\left(\sum_{i\in I} g_i\right) = \sum_{i\in I}(hg_i)$$
$$\left(\sum_{i\in I} g_i\right) f = \sum_{i\in I}(g_i f).$$

† That is, we allow for a proper class of objects, but require that all homsets are indeed sets.

Proof. Consider the index set $A = \{a'\}$, and the indexed family $\{h_a\}_{a\in A}$ given by $h_{a'} = h$. By the unitary sum axiom $h = \sum_{a\in A} h_a$, so

$$h\sum_{i\in I} g_i = \left(\sum_{a\in A} h_a\right)\left(\sum_{j\in J} g_j\right).$$

By strong distributivity,

$$\left(\sum_{a\in A} h_a\right)\left(\sum_{j\in J} g_j\right) = \sum_{(a,j)\in A\times J} h_a g_j.$$

As A is a single element set, $A \times J \cong J$, and $h_a = h$. Therefore, by Proposition 2.3,

$$h\sum_{j\in J} g_j \;=\; \sum_{j\in J} hg_j.$$

The proof for the opposite distributive law is almost identical. □

Corollary 3.4. Every PCM-category has zero arrows.

Proof. Let $(\mathscr{C}, \Sigma^{(\_,\_)})$ be a PCM-category. For all $X, Y \in Ob(\mathscr{C})$, we define 0_{XY} to be the sum of the empty set $\{\}_{XY} \subseteq \mathscr{C}(X, Y)$. Then, by the above distributive laws, for all $f \in \mathscr{C}(Y, Z)$, we have $f0_{XY} = f(\sum\{\}_{XY})$, so $f0_{XY} = (\sum\{\}_{XZ}) = 0_{XZ}$. Similarly, $0_{XY}g = 0_{WY}$ for all $g \in \mathscr{C}(W, X)$. □

Remark 3.5 (the usual treatment of distributivity). The usual approach in algebraic program semantics is to take the above left- and right-distributivity laws as axiomatic, and use the (much stronger) notion of summation to prove an analogue of strong distributivity. This is described in Appendix A. We do not take this approach because the axiomatisation of summation it requires is too strong – it imposes the *positivity property* that $x + y = 0 \Rightarrow x = 0 = y$. Were we to have taken this approach, it would have meant ruling out many of the motivating examples for the Cauchy product construction, including group rings and convergent polynomials over real and complex variables.

Instead, as we demonstrate by example in Appendix A, neither the PCM axiomatisation nor strong distributivity implies the positivity property.

We now consider some implications of strong distributivity.

Proposition 3.6. Let $\left(\mathscr{C}, \Sigma^{(\_,\_)}\right)$ be a PCM-cat, and

$$\{g_j \in \mathscr{C}(Y, Z)\}_{j\in J}$$
$$\{f_i \in \mathscr{C}(X, Y)\}_{i\in I}$$

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be summable families. Then

$$\sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right)$$

$$\sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right)$$

are both defined, and

$$\sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right) = \sum_{(i,j)\in I\times J} g_j f_i = \sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right).$$

Proof. By the strong distributivity property, the family $\{g_j f_i \in \mathscr{C}(X,Z)\}_{(j,i)\in J\times I}$ is summable. Now consider the partition of $J\times I$ given by $\{\{(j,i)\}_{j\in J}\}_{i\in I}$. By the weak partition-associativity axiom, for arbitrary fixed $i \in I$, the family $\{g_j f_i\}_{j\in J}$ is summable, as is $\{\sum_{j\in J} g_j f_i\}_{i\in I}$, and

$$\sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right) = \sum_{(i,j)\in I\times J} g_j f_i.$$

The dual identity

$$\sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right) = \sum_{(i,j)\in I\times J} g_j f_i$$

follows by partitioning $J\times I$ as $\{\{(j,i)\}_{i\in I}\}_{j\in J}$, and therefore

$$\sum_{i\in I}\left(\sum_{j\in J} g_j f_i\right) = \sum_{(i,j)\in I\times J} g_j f_i = \sum_{j\in J}\left(\sum_{i\in I} g_j f_i\right).$$

□

Proposition 3.7. Let $\left(\mathscr{C}, \Sigma^{(\_,\_)}\right)$ be a PCM-category and $\{s_i \in \mathscr{C}(X,X)\}_{i\in I}$ be a summable family. Then for all $n > 0$, the family

$$\{s_{i_n} s_{i_{n-1}} \dots s_{i_2} s_{i_1} \in \mathscr{C}(X,X)\}_{(i_n,\dots i_1)\in I^n}$$

is summable, as are all its subfamilies.

Proof. The proof is by induction. The result is trivially true for $n = 1$. Now assume it holds for some $k > 0$. Then by strong distributivity,

$$\{s_i s_{i_k} \dots s_{i_1} \in \mathscr{C}(X,X)\}_{(i,(i_k,\dots i_1))\in I\times I^k}$$

is also summable, and our result follows by induction. Finally, we complete the proof by recalling the summable subfamilies property (Proposition 2.4). □

Corollary 3.8. Let $\left(\mathscr{C}, \Sigma^{(\_,\_)}\right)$ be a PCM-category and $F = \{f_i \in \mathscr{C}(X,X)\}_{i\in I}$ be a summable family containing the identity. Then:

(1) Arbitrary finite subsets of the submonoid of $\mathscr{C}(X,X)$ generated by F are summable.

(2) Let F' denote the indexed subfamily given by removing all occurrences of 1_X from F. When there exists some word w in the subsemigroup generated by F' satisfying $w = 1_X$, we have:

(a) The sum $\sum_{i=1}^{M} 1_X$ exists, for all $M \in \mathbb{N}$.

(b) For all $f \in \mathscr{C}(X,Y)$ and $g \in \mathscr{C}(W,X)$, the sums

$$\sum_{i=1}^{M} f \in \mathscr{C}(X,Y)$$

$$\sum_{i=1}^{M} g \in \mathscr{C}(W,X)$$

exist, for all $M \in \mathbb{N}$.

Proof.

(1) Consider a finite subset $T \subseteq F^* \subseteq \mathscr{C}(X,X)$, where F^* denotes the free monoid on F. As T is finite, there exists some $K \in \mathbb{N}$ such that each $t \in T$ may be written as a distinct word of no more than K members of F. However, since F contains the identity, each word of T may be written as a distinct word of exactly K members of $\{f_i\}_{i \in I}$. Thus, the result follows by Proposition 3.7 above and the summable subfamilies property (Proposition 2.4).

(2) We now assume the additional condition on F given above:

(a) We write $w = 1_X$ as a word of K elements of F'. Then, by Proposition 3.7, the family

$$\{1_X^{K(M-N)} w 1_X^{KN}\}_{N=1..M}$$

is summable. However,

$$1_X^{K(M-N)} w 1_X^{KN} = 1_X$$

for all $N = 1..M$. Therefore

$$\sum_{N=1}^{M} 1_X$$

exists, as does

$$\sum_{N=1}^{M'} 1_X$$

for all $0 < M' < M$ by the summable subfamilies property (Proposition 2.4).

(b) By distributivity (Proposition 3.3), $\{f 1_X \in \mathscr{C}(X,Y)\}_{i=1..M}$ exists, so $\sum_{i=1}^{M} f$ exists. The proof for arbitrary $g \in \mathscr{C}(W,X)$ is similar. □

3.2. *The category of PCM-categories*

The class of all PCM-categories is itself a category.

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Definition 3.9 (PCM-functors and the category $\mathbf{Cat}_\Sigma$). Given PCM-categories $\mathscr{C}, \mathscr{D}$, we say that a functor $\Gamma : \mathscr{C} \rightarrow \mathscr{D}$ is a **PCM-functor** when:

— Given a summable family $\{f_i \in \mathscr{C}(X,Y)\}_{i \in I}$, we have $\{\Gamma(f_i) \in \mathscr{D}(\Gamma(X), \Gamma(Y))\}$ is a summable family, and

$$\Gamma\left(\sum_{i \in I} f_i\right) = \sum_{i \in I} \Gamma(f_i).$$

We denote the category of all PCM-categories and PCM-functors by $\mathbf{Cat}_\Sigma$.

Proposition 3.10. $\mathbf{Cat}_\Sigma$ is well defined.

Proof. First note that identity functors on PCM-categories are trivially PCM-functors. To prove compositionality, consider two PCM-functors

$$\Gamma \in \mathbf{Cat}_\Sigma(\mathscr{C}, \mathscr{D})$$
$$\Delta \in \mathbf{Cat}_\Sigma(\mathscr{D}, \mathscr{E}).$$

By definition, for any summable family $\{f_i \in \mathscr{C}(X,Y)\}_{i \in I}$, the family

$$\{\Gamma(f_i) \in \mathscr{D}(\Gamma(X), \Gamma(Y))\}_{i \in I}$$

is summable, as is

$$\{\Delta\Gamma(f_i) \in \mathscr{E}(\Delta\Gamma(X), \Delta\Gamma(Y))\}_{i \in I}.$$

Then, also by the definition of PCM-functors,

$$\Delta\left(\Gamma\left(\sum_{i \in I} f_i\right)\right) = \Delta\left(\sum_{i \in I} \Gamma(f_i)\right) = \sum_{i \in I} \Delta\Gamma(f_i),$$

so $\Delta\Gamma$ is also a PCM-functor. Finally, associativity follows from the usual associative property for functors, so $\mathbf{Cat}_\Sigma$ is well defined. □

We also have finite products of PCM-categories.

Proposition 3.11. The category $\mathbf{Cat}_\Sigma$ has finite products.

Proof. Consider $\mathscr{C}, \mathscr{D} \in Ob(\mathbf{Cat}_\Sigma)$. We define their product $\mathscr{C} \times \mathscr{D}$ in a similar way to the usual product of categories: objects are pairs (A, X), where $A \in Ob(\mathscr{C})$ and $X \in Ob(\mathscr{D})$. The homset $(\mathscr{C} \times \mathscr{D})((A,X),(B,Y))$ is exactly the Cartesian product $\mathscr{C}(A,B) \times \mathscr{D}(X,Y)$, with the usual component-wise composition.

We still need to consider summation on homsets. The projections $\pi_1 : \mathscr{C} \times \mathscr{D} \rightarrow \mathscr{C}$ and $\pi_2 : \mathscr{C} \times \mathscr{D} \rightarrow \mathscr{D}$ are defined exactly as in the usual product of categories. For non-empty I, a family $\{f_i \in (\mathscr{C} \times \mathscr{D})((A,X),(B,Y))\}_{i \in I}$ is summable exactly when

$$\{\pi_1(f_i) \in \mathscr{C}(A,B)\}_{i \in I}$$
$$\{\pi_2(f_i) \in \mathscr{D}(X,Y)\}_{i \in I}$$

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are summable, in which case,

$$\sum_{i\in I} f_i \;=\; \left(\sum_{i\in I}\pi_1(f_i)\;,\;\sum_{i\in I}\pi_2(f_i)\right) \in (\mathscr{C}\times\mathscr{D})((A,X),(B,Y)).$$

When I is empty, we simply take $\sum_{i\in I} f_i = (0_{AB}, 0_{XY})$.

We now demonstrate that this definition satisfies the required universal property for a categorical product. Given PCM-functors $\Gamma_1 \in \mathbf{Cat}_\Sigma(\mathscr{X},\mathscr{C})$ and $\Gamma_2 \in \mathbf{Cat}_\Sigma(\mathscr{X},\mathscr{D})$, we define $\langle\Gamma_1,\Gamma_2\rangle : \mathscr{X} \to \mathscr{C}\times\mathscr{D}$ by:

— *On objects*:

$$\langle\Gamma_1,\Gamma_2\rangle(R) = (\Gamma_1(R),\Gamma_2(R))$$

for all $R \in Ob(\mathscr{X})$.

— *On arrows*:

$$\langle\Gamma_1,\Gamma_2\rangle(f) = (\Gamma_1(f),\Gamma_2(f)) \in (\mathscr{C}\times\mathscr{D})((\Gamma_1(R),\Gamma_2(R)),(\Gamma_1(S),\Gamma_2(S))$$

for all $f \in \mathscr{X}(R,S)$.

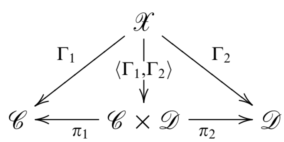
The functoriality of $\langle\Gamma_1,\Gamma_2\rangle$ is immediate. To demonstrate that it is also a PCM-functor, consider a summable family $\{f_i \in \mathscr{X}(R,S)\}_{i\in I}$. Then

$$\{\langle\Gamma_1,\Gamma_2\rangle(f_i)\}_{i\in I} \;=\; \{(\Gamma_1(f_i),\Gamma_2(f_i))\}_{i\in I},$$

which is summable by the definition of summability in $\mathscr{C}\times\mathscr{D}$. By the definition of summation in $\mathscr{C}\times\mathscr{D}$,

$$\sum_{i\in I}\langle\Gamma_1,\Gamma_2\rangle(f_i) \;=\; \left(\sum_{i\in I}\Gamma_1(f_i)\;,\;\sum_{i\in I}\Gamma_2(f_i)\right),$$

so $\langle\Gamma_1,\Gamma_2\rangle$ is also a PCM-functor. Finally, by the usual theory of product categories, the following diagram commutes:



□

4. The categorical Cauchy product

We are now in a position to introduce a categorical analogue of the monoid semiring construction of Definition 1.1. In honour of the original axiomatisation of such products in the theory of formal power series, we refer to this as the *(categorical) Cauchy product*†. However, we first require the following preliminary definition.

† Some new terminology is certainly needed. Starting from the theory of monoid semirings, we will replace both monoids and semirings with categories. However, we wish to avoid replacing the term ‘monoid-semiring’ by ‘category-category’.

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Definition 4.1 (locally countable categories). We say that a category $\mathscr{D}$ is **locally countable** when for all $U, V \in Ob(\mathscr{D})$, the homset $\mathscr{D}(U, V)$ is a countable set. We denote the full subcategory of **Cat**, whose objects are locally countable categories, by **cCat**.

Definition 4.2. Given a PCM-category $\mathscr{C} \in Ob(\mathbf{Cat}_\Sigma)$ and a locally countable category $\mathscr{D} \in Ob(\mathbf{cCat})$, we define their **Cauchy product** $\mathscr{C}[\mathscr{D}] \in Ob(\mathbf{Cat}_\Sigma)$ as follows:

— **Objects:** $Ob(\mathscr{C}[\mathscr{D}]) = Ob(\mathscr{C}) \times Ob(\mathscr{D})$.

— **Arrows:** The homset $\mathscr{C}[\mathscr{D}]((X, U), (Y, V))$ consists of all functions

$$f : \mathscr{D}(U, V) \to \mathscr{C}(X, Y)$$

such that $\{f(a) \in \mathscr{C}(X, Y)\}_{a \in \mathscr{D}(U,V)}$ is a summable family.

— **Composition:** Given

$$g \in \mathscr{C}[\mathscr{D}]((Y, V), (Z, W))$$
$$f \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$$

as functions

$$f : \mathscr{D}(U, V) \to \mathscr{C}(X, Y)$$
$$g : \mathscr{D}(V, W) \to \mathscr{C}(Y, Z),$$

we have $gf \in \mathscr{C}[\mathscr{D}]((X, U), (Z, W))$ is the function from $\mathscr{D}(U, W)$ to $\mathscr{C}(X, Z)$ given by

$$gf(c) = \sum_{\{(b,a):c=ba\} \subseteq \mathscr{D}(V,W) \times \mathscr{D}(U,V)} g(b)f(a).$$

For clarity, we will often use the shorthand notation

$$gf(c) = \sum_{c=ba} g(b)f(a).$$

— **Summation:** An indexed family $\{f_i \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))\}_{i \in I}$ is summable exactly when

$$\{f_i(h) \in \mathscr{C}(X, Y)\}_{(i,h) \in I \times \mathscr{D}(U,V)} \text{ is summable in } \mathscr{C},$$

in which case

$$\left(\sum_{i \in I} f_i\right)(h) \overset{def.}{=} \sum_{i \in I} f_i(h) \in \mathscr{C}(X, Y).$$

Terminology 4.3 In the above definition of the Cauchy product $\mathscr{C}[\mathscr{D}]$, we refer to the PCM-category $\mathscr{C} \in Ob(\mathbf{Cat}_\Sigma)$ as the **base category** and the locally countable category $\mathscr{D} \in Ob(\mathbf{cCat})$ as the **index category**.

We will now prove that the above construction is well defined.

Theorem 4.4. The Cauchy product $\mathscr{C}[\mathscr{D}]$ defined above is a **PCM**-category.

Proof. We first show that $\mathscr{C}[\mathscr{D}]$ is a category, and then consider the indexed summation on homsets.

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We demonstrate that the composition of $\mathscr{C}[\mathscr{D}]$ is well defined, associative, and has identities:

(1) **Composition is well defined:** We assume arrows

$$g \in \mathscr{C}[\mathscr{D}]((Y,V),(Z,W))$$
$$f \in \mathscr{C}[\mathscr{D}]((X,U),(Y,V))$$

that is, functions

$$f : \mathscr{D}(U,V) \to \mathscr{C}(X,Y)$$
$$g : \mathscr{D}(V,W) \to \mathscr{C}(Y,Z)$$

where

$$\{f(a) \in \mathscr{C}(X,Y)\}_{a\in\mathscr{D}(U,V)}$$
$$\{g(b) \in \mathscr{C}(Y,Z)\}_{b\in\mathscr{D}(V,W)}$$

are summable. So we need to show that $(gf)(c) \in \mathscr{C}(X,Z)$ exists for all $c \in \mathscr{D}(U,W)$, and that

$$\left\{ gf(c) = \sum_{c=ba} g(b)f(a) \right\}_{c\in\mathscr{C}(X,Z)}$$

is also a summable family. By definition,

$$\sum_{a\in\mathscr{D}(U,V)} f(a) \in \mathscr{C}(X,Y)$$

exists, as does

$$\sum_{b\in\mathscr{D}(V,W)} g(b) \in \mathscr{C}(Y,Z).$$

The strong distributivity property thus implies the summability of the indexed family

$$P = \{g(b)f(a)\}_{(b,a)\in\mathscr{D}(V,W)\times\mathscr{D}(U,V)}$$

together with the identity

$$\left(\sum_{b\in\mathscr{D}(V,W)} g(b)\right)\left(\sum_{a\in\mathscr{D}(U,V)} f(a)\right) = \sum(P).$$

Given some arbitrary $c \in \mathscr{D}(U,W)$, consider the (possibly empty) subfamily of P_c of P given by $\{g(b)f(a)\}_{ba=c}$. This is a subfamily of P, and thus, by the subfamilies property of Proposition 2.4, it is itself a summable family. Therefore, $(gf)(c) \in \mathscr{C}(X,Z)$ is well defined for all $c \in \mathscr{D}(U,W)$.
Finally, consider the family $\{P_c\}_{c\in\mathscr{D}(U,W)}$. Observe that for distinct $x \neq y \in \mathscr{D}(U,W)$, the intersection of P_x and P_y is empty. Thus, $\{P_c\}_{c\in\mathscr{D}(U,W)}$ is a partition of the summable family P and, by the weak partition-associativity axiom, is itself a summable family satisfying

$$\sum_{c\in\mathscr{D}(U,W)} P_c = \sum_{(b,a)\in\mathscr{D}(V,W)\times\mathscr{D}(U,V)} g(b)f(a).$$

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(2) **Associativity of composition:** Consider arrows

$$h \in \mathscr{C}[\mathscr{D}]((Z,W),(T,P))$$
$$g \in \mathscr{C}[\mathscr{D}]((Y,V),(Z,W))$$
$$f \in \mathscr{C}[\mathscr{D}]((X,U),(Y,V)).$$

By definition,

$$(hg)(r) = \sum_{\{(q,p):r=qp\}\subseteq \mathscr{D}(W,P)\times\mathscr{D}(V,W)} h(q)g(p)$$

and, similarly,

$$(gf)(c) = \sum_{\{(b,a):c=ba\}\subseteq \mathscr{D}(V,W)\times\mathscr{D}(U,V)} g(b)f(a).$$

Therefore, for all $\gamma \in \mathscr{D}(U,P)$,

$$(h(gf))\,(\gamma) = \sum_{\{(\beta,\alpha):\gamma=\beta\alpha\}\subseteq \mathscr{D}(W,P)\times\mathscr{D}(U,W)} h(\beta)\,(gf)\,(\alpha),$$

which, by definition of the composite $gf \in \mathscr{C}[\mathscr{D}]((X,U),(Z,W))$, is given by

$$(h(gf))\,(\gamma) \quad = \sum_{\{(\beta,\alpha):\gamma=\beta\alpha\}\subseteq \mathscr{D}(W,P)\times\mathscr{D}(U,W)} h(\beta)\left(\sum_{\{(b,a):\alpha=ba\}\subseteq \mathscr{D}(V,W)\times\mathscr{D}(U,V)} g(b)f(a)\right).$$

By distributivity (Proposition 3.3), we may write this as

$$(h(gf))\,(\gamma) \quad = \sum_{\{(\beta,\alpha):\gamma=\beta\alpha\}\subseteq \mathscr{D}(W,P)\times\mathscr{D}(U,W)} \left(\sum_{\{(b,a):\alpha=ba\}\subseteq \mathscr{D}(V,W)\times\mathscr{D}(U,V)} h(\beta)g(b)f(a)\right).$$

Conversely, $((hg)f) \in \mathscr{C}[\mathscr{D}]((X,U),(T,P))$ is given by

$$((hg)f)(\nu) = \sum_{\{(\mu,\lambda):\nu=\mu\lambda\}\subseteq \mathscr{D}(V,P)\times\mathscr{D}(U,V)} (hg)(\mu)f(\lambda)$$

for all $\nu \in \mathscr{D}(U,P)$, which, by definition of the composite $hg \in \mathscr{C}[\mathscr{D}]((Y,V),(T,P))$, is given by

$$((hg)f)(\nu) \quad = \sum_{\{(\mu,\lambda):\nu=\mu\lambda\}\subseteq \mathscr{D}(V,P)\times\mathscr{D}(U,V)} \left(\sum_{\{(c,b):\mu=cb\}\subseteq \mathscr{D}(W,P)\times\mathscr{D}(V,W)} h(c)g(b)\right) f(\lambda).$$

Again by distributivity (Proposition 3.3), this may be written as

$$((hg)f)(\nu) \quad = \sum_{\{(\mu,\lambda):\nu=\mu\lambda\}\subseteq \mathscr{D}(V,P)\times\mathscr{D}(U,V)} \left(\sum_{\{(c,b):\mu=cb\}\subseteq \mathscr{D}(W,P)\times\mathscr{D}(V,W)} h(c)g(b)f(\lambda)\right).$$

Now observe that, by the definition of the arrows of $\mathscr{C}[\mathscr{D}]$, the families

$$\{h(c) \in \mathscr{C}(Z,T)\}_{c\in\mathscr{D}(W,P)}$$
$$\{g(b) \in \mathscr{C}(Y,Z)\}_{b\in\mathscr{D}(V,W)}$$
$$\{f(a) \in \mathscr{C}((X,Y)\}_{a\in\mathscr{D}(U,V)}$$

are all summable. Therefore, by the strong distributivity property, the family

$$\{h(c)g(b)f(a)\}_{(c,b,a)\in\mathscr{D}(W,P)\times\mathscr{D}(V,W)\times\mathscr{D}(U,V)}$$

is summable. Given arbitrary $d \in \mathscr{D}(U,P)$, let Q_d be the subfamily of the above indexing set given by

$$Q_d = \{(c,b,a) : d = cba\} \subseteq \mathscr{D}(W,P) \times \mathscr{D}(V,W) \times \mathscr{D}(U,V),$$

which, by the summable subfamilies property, is summable. By the weak partition-associativity axiom, we may partition $\sum_{Q_d} h(c)g(b)f(a)$ in two distinct ways – by relabelling indices, these may be seen to correspond to $((hg)f)(d)$ and $(h(gf))(d)$. Hence $((hg)f)(d) = (h(gf))(d)$ for all $d \in \mathscr{D}(U,P)$, so

$$(hg)f = h(gf) \in \mathscr{C}[\mathscr{D}]((X,U),(T,P)),$$

as required.

(3) **Identity arrows:** We begin by recalling the existence of zero elements in a PCM from Proposition 2.4, and the proof that PCM-categories have zero arrows in Corollary 3.4. At an object $(X,U) \in Ob(\mathscr{C}[\mathscr{D}])$, the identity arrow is given by $1_{(X,U)}$ as follows:

$$1_{(X,U)}(r) = \begin{cases} 1_X \in \mathscr{C}(X,X) & r = 1_U \in \mathscr{D}(U,U) \\ 0_X & \text{otherwise.} \end{cases}$$

From the definition of composition and Proposition 2.4, for all

$$g \in \mathscr{C}[\mathscr{D}]((X,U),(Y,V))$$
$$f \in \mathscr{C}[\mathscr{D}]((W,T),(X,U)),$$

we have

$$\left(g1_{(X,U)}\right)(s) = g(s) \quad \forall s \in \mathscr{D}(U,V)$$
$$\left(1_{(X,U)}f\right)(r) = f(r) \quad \forall r \in \mathscr{D}(T,U).$$

Thus $1_{(X,U)} \in \mathscr{C}[\mathscr{D}]((X,U),(X,U))$ is the identity, as required.

We now just need to show that $\mathscr{C}[\mathscr{D}]$ is not only a category, but a PCM-category:

(1) **Hom-sets are PCMs:** Given objects $(X,U),(Y,V) \in Ob(\mathscr{C}[\mathscr{D}])$, we will show that the summation given in Definition 4.2 above gives a PCM structure to $\mathscr{C}[D]((X,U),(Y,V))$.

— **Unary sum axiom:** Consider an indexed family

$$\{f_i \in \mathscr{C}[\mathscr{D}]((X,U),(Y,V))\}_{i\in\{i'\}} \quad \text{where} \quad f_{i'} = f.$$

We first demonstrate that $\{f_i(a) \in \mathscr{C}(X,Y)\}_{(i,a)\in\{i'\}\times\mathscr{D}(U,V)}$ is summable in $\mathscr{C}$. As $\{i'\}$ is a single-element set, $\{i'\} \times \mathscr{D}(U,V) \cong \mathscr{D}$, and (trivially) $f_i = f$, for all $i \in \{i'\}$.

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Therefore, by Proposition 2.3, the summability of $\{f_i(a) \in \mathscr{C}(X, Y)\}_{(i,a)\in\{i'\}\times\mathscr{D}(U,V)}$ is equivalent to the summability of $\{f(a) \in \mathscr{C}(X, Y)\}_{a\in\mathscr{D}(U,V)}$, and this is summable by the definition of arrows in $\mathscr{C}[\mathscr{D}]$. Thus, singleton families are summable in $\mathscr{C}[\mathscr{D}]((X, U), (Y, V))$. Finally, by the definition of the summation of $\mathscr{C}[\mathscr{D}]$ and the unary sum axiom for the PCM $\left(\mathscr{C}(X, Y), \Sigma^{X,Y}\right)$,

$$\left(\sum_{i\in\{i'\}} f_i\right)(a) = \sum_{i\in\{i'\}} f_i(a) = f(a),$$

and, therefore, $\sum_{i\in\{i'\}} f_i = f \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$.

— **Weak partition-associativity:** Consider a summable family

$$\{f_i \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))\}_{i\in I},$$

and let $\{I_j\}_{j\in J}$ be a partition of I. By the definition of summability in $\mathscr{C}[\mathscr{D}]$, the family $\{f_i(a) \in \mathscr{C}(X, Y)\}_{(i,a)\in I\times\mathscr{D}(U,V)}$ is summable in $\mathscr{C}$. Now consider the family $\{f_{i'}(a) \in \mathscr{C}(X, Y)\}_{(i',a)\in I_j\times\mathscr{D}(U,V)}$. This is a subfamily of a summable family of $\mathscr{C}(X, Y)$ and thus, by the summable subfamilies property (Proposition 2.4), it is itself a summable family. Therefore, by the definition of summability in $\mathscr{C}[\mathscr{D}]$, the family $\{f_{i'} \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))\}_{i\in I'}$ is summable.
Similarly, to show that $\sum_{j\in J}(\sum_{i'\in I_j} f_{i'})$ is summable in $\mathscr{C}[\mathscr{D}]((X, U), (Y, V))$, we note that

$$\left\{\sum_{i'\in I_j} f_{i'}(a) \in \mathscr{C}(X, Y)\right\}_{(j,a)\in J\times\mathscr{D}(U,V)}$$

is summable by the weak partition-associativity axiom applied to the PCM $\left(\mathscr{C}(X, Y), \Sigma^{X,Y}\right)$, and (again, by WPA),

$$\left(\sum_{j\in J}\left(\sum_{i'\in I_J} f_{i'}\right)\right)(a) = \left(\sum_{i\in I} f_i\right)(a) \ \in \mathscr{C}(X, Y)$$

for all $a \in \mathscr{D}(U, V)$, and thus

$$\left(\sum_{j\in J}\left(\sum_{i'\in I_J} f_{i'}\right)\right) = \left(\sum_{i\in I} f_i\right) \ \in \ \mathscr{C}[\mathscr{D}]((X, U), (Y, V)).$$

Hence, the summation on $\mathscr{C}[\mathscr{D}]((X, U), (Y, V))$ satisfies weak partition-associativity.

(2) **Strong Distributive Law:** Consider summable families of $\mathscr{C}[\mathscr{D}]$

$$\{f_i \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))\}_{i\in I}$$
$$\{g_j \in \mathscr{C}[\mathscr{D}]((Y, V), (Z, W))\}_{j\in J}.$$

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Summability of these families is equivalent to the summability of the following families in $\mathscr{C}$:

$$\{f_i(a) \in \mathscr{C}(X,Y)\}_{(i,a)\in I\times\mathscr{D}(U,V)}$$
$$\{g_j(b) \in \mathscr{C}(Y,Z)\}_{(j,b)\in J\times\mathscr{D}(V,W)}.$$

By the strong distributivity law for $\mathscr{C}$, the following family is therefore summable:

$$\{g_j(b)f_i(a) \in \mathscr{C}(X,Z)\}_{(j,b,i,a)\in J\times\mathscr{D}(V,W)\times I\times\mathscr{D}(U,V)}.$$

For all $c \in \mathscr{D}(U,W)$, consider the (possibly empty) subset

$$P_c = \{(j,b,i,a) : c = ba\} \subseteq J \times \mathscr{D}(V,W) \times I \times \mathscr{D}(U,V).$$

Note that $P_c \cap P_{c'} = \emptyset$, for all $c \neq c'$, and

$$\bigcup_{c\in\mathscr{D}(U,W)} P_c = J \times \mathscr{D}(V,W) \times I \times \mathscr{D}(U,V),$$

giving a $\mathscr{D}(U,W)$-indexed partition of the summable family

$$\{g_j(b)f_i(a) \in \mathscr{C}(X,Z)\}_{(j,b,i,a)\in J\times\mathscr{D}(V,W)\times I\times\mathscr{D}(U,V)}.$$

Thus, by the weak partition-associativity property of $(\mathscr{C}(X,Z),\Sigma^{X,Y})$, the family

$$\{g_j(b)f_i(a) \in \mathscr{C}(X,Z) : ba = c\}_{(j,c,i)\in J\times\mathscr{D}(U,W)\times I}$$

is summable, demonstrating that $\{g_jf_i \in \mathscr{C}[\mathscr{D}]((X,U),(Y,V))\}_{(j,i)\in J\times I}$ is summable in $\mathscr{C}[\mathscr{D}]$, as required.

For all $c \in \mathscr{D}(U,W)$, the identity

$$\left(\sum_{j\in J} g_j\right)\left(\sum_{i\in I} f_i\right)(c) = \left(\sum_{(j,i)\in J\times I} g_jf_i\right)(c) \;\; \in \mathscr{C}(X,Z)$$

is then immediate from the existence of both sides of this equation, and the strong distributivity law for $\mathscr{C}$, so

$$\left(\sum_{j\in J} g_j\right)\left(\sum_{i\in I} f_i\right) = \left(\sum_{(j,i)\in J\times I} g_jf_i\right) \;\; \in \mathscr{C}[\mathscr{D}]((X,U),(Z,W)),$$

as required. □

4.1. *The Cauchy product as a bifunctor*

Theorem 4.5. The Cauchy product of Definition 4.2 defines a bifunctor

$$(\_)[\_] : \mathbf{Cat}_\Sigma \times \mathbf{cCat} \to \mathbf{Cat}_\Sigma.$$

That is:

(1) Given $\mathscr{D} \in Ob(\mathbf{cCat})$, we have $(\_)[\mathscr{D}] : \mathbf{Cat}_\Sigma \to \mathbf{Cat}_\Sigma$ is a functor.
(2) Given $\mathscr{C} \in Ob(\mathbf{Cat}_\Sigma)$, we have $\mathscr{C}[\_] : \mathbf{cCat} \to \mathbf{Cat}_\Sigma$ is a functor.

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Proof.

(1) We first demonstrate that for arbitrary $\mathscr{D} \in \mathbf{cCat}$, the map $(\_)[\mathscr{D}] : \mathbf{Cat}_\Sigma \to \mathbf{Cat}_\Sigma$ defines a functor:

— **On objects:** Given a PCM-category $\mathscr{C} \in Ob(\mathbf{Cat}_\Sigma)$, we have $\mathscr{C}[\mathscr{D}] \in Ob(\mathbf{Cat}_\Sigma)$ is as defined in Definition 4.2.

— **On arrows:** Given $\Gamma \in \mathbf{Cat}_\Sigma(\mathscr{C}, \mathscr{E})$, we define the functor

$$(\Gamma[\mathscr{D}]) \in \mathbf{Cat}_\Sigma(\mathscr{C}[\mathscr{D}], \mathscr{E}[\mathscr{D}])$$

as follows:

– *On objects:* For all $(X, U) \in Ob(\mathscr{C}[\mathscr{D}])$, we define $(\Gamma[\mathscr{D}])(X, U) = (\Gamma(X), U)$.

– *On arrows:* Given an arbitrary arrow $f \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$, we define

$$(\Gamma[\mathscr{D}])(f) \in \mathscr{E}[\mathscr{D}]((\Gamma(X), U), (\Gamma(Y), V))$$

by

$$(\Gamma[\mathscr{D}])(f)(r) = \Gamma(f(r)) \in \mathscr{E}(\Gamma(X), \Gamma(Y))$$

for all $r \in \mathscr{D}(U, V)$. It is immediate that this is well defined as an arrow in

$$\mathscr{E}[\mathscr{D}]((\Gamma(X), U), (\Gamma(Y), V))$$

since, as Γ is a **PCM**-functor (that is, an arrow of $\mathbf{Cat}_\Sigma(\mathscr{C}, \mathscr{E})$), we have

$$\sum_{r \in \mathscr{D}(U,V)} f(r) \ \text{ exists} \quad \Rightarrow \quad \sum_{r \in \mathscr{D}(U,V)} \Gamma(f(r)) \ \text{ exists.}$$

To prove compositionality, consider

$$f \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$$
$$g \in \mathscr{C}[\mathscr{D}]((Y, V), (Z, W)).$$

By the definition of composition,

$$gf(c) = \sum_{c=ba} g(b)f(a)$$

for all $c \in \mathscr{D}(U, W)$. However, by the definition of the functor $(\Gamma[\mathscr{D}])$,

$$((\Gamma[\mathscr{D}])(g)(\Gamma[\mathscr{D}])(f))(c) = \sum_{c=ba} \Gamma(g(b))\Gamma(f(a)).$$

By the functoriality of Γ,

$$((\Gamma[\mathscr{D}])(g)(\Gamma[\mathscr{D}])(f))(c) = \sum_{c=ba} \Gamma(g(b)f(a)),$$

and, as Γ is a PCM-functor,

$$((\Gamma[\mathscr{D}])(g)(\Gamma[\mathscr{D}])(f))(c) = \Gamma\left(\sum_{c=ba} g(b)f(a)\right) = (\Gamma[\mathscr{D}])(gf)(c).$$

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Finally, given another functor $\Delta \in \mathbf{Cat}_\Sigma(\mathscr{E}, \mathscr{F})$, we have

– *On objects:*

$$(\Delta[\mathscr{D}])\,(\Gamma[\mathscr{D}])\,(X, U) = (\Delta\Gamma(X), U) = ((\Delta\Gamma)[\mathscr{D}])\,(X, U).$$

– *On arrows:* Given $f \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$, we have

$$(\Delta[\mathscr{D}])\,(\Gamma[\mathscr{D}])\,(f)(r) = \Delta(\Gamma(f))(r) = (\Delta\Gamma(f))(r) = ((\Delta\Gamma)[\mathscr{D}])\,(f)(r).$$

(2) We will now demonstrate that for arbitrary $\mathscr{C} \in \mathbf{Cat}_\Sigma$, the map $\mathscr{C}[\_] : \mathbf{cCat} \to \mathbf{Cat}_\Sigma$ is also functorial:

— **On objects:** Given arbitrary $\mathscr{D} \in Ob(\mathbf{cCat})$, we have $\mathscr{C}[\mathscr{D}]$ is given by Definition 4.2.

— **On arrows:** Given a functor $\Lambda \in \mathbf{cCat}(\mathscr{D}, \mathscr{H})$, we define

$$\mathscr{C}[\Lambda] \in \mathbf{Cat}_\Sigma(\mathscr{C}[\mathscr{D}], \mathscr{C}[\mathscr{H}])$$

by:

– *On objects:* $\mathscr{C}[\Lambda](X, U) = (X, \Lambda(U))$.

– *On arrows:* Given $f \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$, we define

$$(\mathscr{C}[\Lambda])\,(f) \in \mathscr{C}[\mathscr{H}]((X, \Lambda(U)), (Y, \Lambda(V)))$$

by

$$(\mathscr{C}[\Lambda])\,(f)(x) = \sum_{x=\Lambda(a)} f(a) \;\; \in \mathscr{C}(X, Y)$$

for all $x \in \mathscr{H}(\Lambda(U), \Lambda(V))$. This sum is well defined, since $\{f(a)\}_{a\in\mathscr{D}(U,V)}$ is a summable family. Also, note that $\{x = \Gamma(a)\}_{x\in\mathscr{H}(\Gamma(U),\Gamma(V))}$ is a partition of $\mathscr{D}(U, V)$, and thus, by the weak partition-associativity axiom,

$$\{(\mathscr{C}[\Lambda])\,(f)(x)\}_{x\in\mathscr{H}(\Lambda(U),\Lambda(V))}$$

is summable, so

$$(\mathscr{C}[\Lambda])\,(f) \;\in\; \mathscr{C}[\mathscr{H}]((X, \Lambda(U)), (Y, \Lambda(V)))$$

is well defined.

To prove compositionality, consider

$$f \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$$
$$g \in \mathscr{C}[\mathscr{D}]((Y, V), (Z, W)).$$

By the definition of composition, $gf(c) = \sum_{c=ba} g(b)f(a)$ for all $c \in \mathscr{D}(U, W)$, so

$$(\mathscr{C}(\Lambda)(gf))\,(z) = \sum_{z=\Lambda(c)} (gf)(c)$$

for all $z \in \mathscr{H}(\Lambda(U), \Lambda(W))$.

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Now note that, for all

$$y \in \mathscr{H}(\Lambda(V), \Lambda(W))$$
$$x \in \mathscr{H}(\Lambda(U), \Lambda(V)),$$

we have

$$(\mathscr{C}(\Lambda)(g))\,(y) = \sum_{y=\Lambda(b)} g(b)$$
$$(\mathscr{C}(\Lambda)(f))\,(x) = \sum_{x=\Lambda(a)} f(c),$$

so, by the strong distributive law for PCM-categories, and the functoriality of Λ,

$$(\mathscr{C}(\Lambda)(g))\,(\mathscr{C}(\Lambda)(f))\,(z) = \sum_{z=\Lambda(c)} (gf)(c) = (\mathscr{C}(\Lambda)(gf))\,(z) \;\in \mathscr{C}(X, Z).$$

Finally, given another functor $\Omega \in \mathbf{cCat}(\mathscr{H}, \mathscr{K})$, we have:

— *On objects:*

$$(\mathscr{C}[\Omega])\,(\mathscr{C}[\Lambda])\,(X, U) = (X, \Omega\Lambda(U)) = (\mathscr{C}[\Omega\Lambda])\,(X, U).$$

— *On arrows:* Given $f \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$, we have

$$(\mathscr{C}[\Omega])\,(\mathscr{C}[\Lambda])\,(f)(p) = \sum_{p=\Omega(x)\;,\; x=\Lambda(a)} f(a) = \sum_{p=\Omega\Lambda(a)} f(a) = (\mathscr{C}[\Omega\Lambda])\,(f)(p)$$

for all $p \in \mathscr{K}(\Omega\Lambda(U), \Omega\Lambda(V))$. □

Remark 4.6 (is the Cauchy product a monoidal tensor?). Since the Cauchy product is a bifunctor $\mathbf{Cat}_\Sigma \times \mathbf{cCat} \to \mathbf{Cat}_\Sigma$, it is natural to wonder whether, when restricted to locally countable PCM-categories, it is in fact a monoidal tensor. It is also easy to show that this is not the case: consider three locally countable PCM-categories, $\mathscr{C}, \mathscr{D}, \mathscr{E} \in Ob(\mathbf{Cat}_\Sigma)$, and denote their (object-indexed families of) summations by $\Sigma^{\mathscr{C}(\_,\_)}$, $\Sigma^{\mathscr{D}(\_,\_)}$ and $\Sigma^{\mathscr{E}(\_,\_)}$, respectively. Then it is immediate that the structure of $\mathscr{C}[\mathscr{D}[\mathscr{E}]]$ depends on the family of summations $\Sigma^{\mathscr{D}(\_,\_)}$ on the homsets of $\mathscr{D}$, whereas the structure of $(\mathscr{C}[\mathscr{D}])[\mathscr{E}]$ is independent of $\Sigma^{\mathscr{D}(\_,\_)}$. Therefore, in general, $(\mathscr{C}[\mathscr{D}])[\mathscr{E}]$ cannot be equal to $\mathscr{C}[\mathscr{D}[\mathscr{E}]]$, even up to a canonical isomorphism.

Rather, as we will now demonstrate, there exist embeddings of $\mathscr{C}$ into $\mathscr{C}[\mathscr{D}]$ indexed by objects of $\mathscr{D}$, together with embeddings of $\mathscr{D}$ into $\mathscr{C}[\mathscr{D}]$ indexed by objects of $\mathscr{C}$, and an embedding of the product $\mathscr{C} \times \mathscr{D}$ into $\mathscr{C}[\mathscr{D}]$. The embeddings of $\mathscr{C}$ into $\mathscr{C}[\mathscr{D}]$ also have a common left-inverse, giving an indexed family of retractions.

5. Embedding the base category into a Cauchy product

We now give an embedding of the base category $\mathscr{C}$ into the Cauchy product $\mathscr{C}[\mathscr{D}]$, and show that $\mathscr{C}$ is a retract of $\mathscr{C}[\mathscr{D}]$.

We first exhibit a forgetful functor from $\mathscr{C}[\mathscr{D}]$ to $\mathscr{C}$.

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Definition 5.1. Given $\mathscr{D} \in Ob(\mathbf{cCat})$ and $\mathscr{C} \in Ob(\mathbf{Cat}_\Sigma)$, we define $\sigma_{\mathscr{C},\mathscr{D}} : \mathscr{C}[\mathscr{D}] \to \mathscr{C}$ by:

— **On objects:** $\sigma_{\mathscr{C},\mathscr{D}}(X, U) = X$, for all $(X, U) \in Ob(\mathscr{C}[\mathscr{D}])$.
— **On arrows:** Given $h \in \mathscr{C}[D]((X, U), (Y, V))$, we have

$$\sigma_{\mathscr{C},\mathscr{D}}(h) = \sum_{a \in \mathscr{D}(U,V)} h(a) \ \in \ \mathscr{C}(X, Y).$$

Proposition 5.2. $\sigma_{\mathscr{C},\mathscr{D}} : \mathscr{C}[\mathscr{D}] \to \mathscr{C}$ as given above is a PCM-functor.

Proof. First note that, by the definition of arrows in $\mathscr{C}[\mathscr{D}]$, the family $\{h(a)\}_{a \in \mathscr{D}(U,V)}$ is summable for all $h \in \mathscr{C}[D]((X, U), (Y, V))$, so

$$\sigma_{\mathscr{C},\mathscr{D}}(h) = \sum_{a \in \mathscr{D}(U,V)} h(a) \ \in \ \mathscr{C}(X, Y)$$

is well defined.

To prove functoriality, consider $k \in \mathscr{C}[\mathscr{D}]((Y, V), (Z, W))$. We have

$$\sigma(k)\sigma(h) = \left(\sum_{b \in \mathscr{D}(V,W)} k(b) \right) \left(\sum_{a \in \mathscr{D}(U,V)} h(a) \right).$$

By the strong distributivity property,

$$\sigma(k)\sigma(h) = \sum_{(b,a) \in \mathscr{D}(V,W) \times \mathscr{D}(U,V)} k(b)h(a).$$

Conversely, $kh \in \mathscr{C}[\mathscr{D}]((X, U)(Z, W))$ is defined by

$$(kh)(c) = \sum_{c=ba} k(b)h(a)$$

for all $c \in \mathscr{D}(U, W)$.

Now note that $\{(kh)(c)\}_{c \in \mathscr{D}(U,W)}$ is a summable family, and by weak partition-associativity,

$$\sigma(kh) = \sum_{c=ba} k(b)h(a) = \sum_{(b,a) \in \mathscr{D}(V,W) \times \mathscr{D}(U,V)} k(b)h(a) = \sigma(k)\sigma(h).$$

So $\sigma : \mathscr{C}[\mathscr{D}] \to \mathscr{C}$ preserves composition. The proof that it also preserves identities follows from the formula for identities in PCM-categories given in Theorem 4.4:

$$1_{(X,U)}(r) = \begin{cases} 1_X \in \mathscr{C}(X, X) & r = 1_U \in \mathscr{D}(U, U) \\ 0_X & \text{otherwise.} \end{cases}$$

It is immediate that $\sigma(1_{(X,U)}) = 1_X \in \mathscr{C}(X, X)$.

Now consider a summable family $\{f_i \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))\}_{i \in I}$. By the definition of summation in $\mathscr{C}[\mathscr{D}]$, the family $\{f_i(a) \in \mathscr{C}(X, Y)\}_{(i,a) \in I \times \mathscr{D}(U,V)}$ is summable in $\mathscr{C}$, and, again by definition,

$$\left(\sum_{i \in I} f_i \right)(a) \ = \ \sum_{i \in I} f_i(a) \ \in \ \mathscr{C}(X, Y).$$

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Thus, by weak partition-associativity and Proposition 2.3,

$$\sum_{i\in I}\sigma(f_i)=\sum_{i\in I}\left(\sum_{a\in\mathscr{D}(U,V)}f(a)\right)=\sum_{a\in\mathscr{D}(U,V)}\left(\sum_{i\in I}f_i(a)\right)=\sigma\left(\sum_{i\in I}f_i\right).$$

Therefore, $\sigma:\mathscr{C}[\mathscr{D}]\to\mathscr{C}$ is a PCM-functor. □

We will now exhibit a family of embeddings of $\mathscr{C}$ into $\mathscr{C}[\mathscr{D}]$, indexed by objects of $\mathscr{D}$.

Definition 5.3. Let $\mathscr{D}$ be an arbitrary category and $\mathscr{C}$ be a PCM-category. For all $U\in Ob(\mathscr{D})$, we define $\eta_{\mathscr{C},U}:\mathscr{C}\to\mathscr{C}[\mathscr{D}]$ by:

— **On objects:** $\eta_{\mathscr{C},U}(X)=(X,U)$, for all $X\in Ob(\mathscr{C})$.
— **On arrows:** Given $h\in\mathscr{C}(X,Y)$, we have $\eta_{\mathscr{C},U}(h)\in\mathscr{C}[\mathscr{D}]((X,U),(Y,U))$ is the function $\eta_{\mathscr{C},U}(h):\mathscr{D}(U,U)\to\mathscr{C}(X,Y)$ given by

$$\left(\eta_{\mathscr{C},U}(h)\right)(a)=\begin{cases}h & a=1_U\\ 0_{XY} & a\neq 1_U.\end{cases}$$

We will now prove that these maps are injective PCM-functors.

Proposition 5.4. For all $U\in Ob(\mathscr{D})$, the map $\eta_{\mathscr{C},U}:\mathscr{C}\to\mathscr{C}[\mathscr{D}]$ defined above is an injective PCM-functor.

Proof. Given $h\in\mathscr{C}(X,Y)$, we know $\eta_U(h)$ is trivially well defined as an arrow of $\mathscr{C}[\mathscr{D}]((X,U),(Y,U))$ since $\sum_{a\in\mathscr{D}(U,U)}h(a)=h$ by Proposition 2.4. Now consider $k\in\mathscr{C}(Y,Z)$. By the definition of composition in $\mathscr{C}[\mathscr{D}]$,

$$\left(\eta_{\mathscr{C},U}(k)\right)\left(\eta_{\mathscr{C},U}(h)\right)(c)=\sum_{c=ba}\left(\eta_{\mathscr{C},U}(k)\right)(b)\left(\eta_{\mathscr{C},U}(h)\right)(a).$$

However,

$$\left(\eta_{\mathscr{C},U}(k)\right)(b)\left(\eta_{\mathscr{C},U}(h)\right)(a)=\begin{cases}kh & b=a=1_U\\ 0_{XY} & \text{otherwise,}\end{cases}$$

so

$$\left(\eta_{\mathscr{C},U}(k)\right)\left(\eta_{\mathscr{C},U}(h)\right)(c)=\begin{cases}kh & c=1_U\\ 0_{XY} & c\neq 1_U,\end{cases}$$

giving $\eta_{\mathscr{C},U}(k)\eta_{\mathscr{C},U}(h)=\eta_{\mathscr{C},U}(kh)$ as required. It is immediate from the definition that

$$\eta_{\mathscr{C},U}(1_X)=1_{(X,U)}\in\mathscr{C}[\mathscr{D}]((X,U),(X,U))$$

for all $X\in Ob(\mathscr{C})$.

For the summation, consider a summable family $\{f_i\in\mathscr{C}(X,Y)\}_{i\in I}$. By the definition of summability in $\mathscr{C}[\mathscr{D}]$, the family $\{\eta_{\mathscr{C},U}(f_i)\in\mathscr{C}[\mathscr{D}]\}_{i\in I}$ is also summable, and

$$\sum_{i\in I}\eta_{\mathscr{C},U}(f_i)=\eta_{\mathscr{C},U}\left(\sum_{i\in I}f_i\right).$$

The injectivity of $\eta_{\mathscr{C},U}:\mathscr{C}\to\mathscr{C}[U]$ on objects is immediate. To demonstrate the injectivity on arrows, consider $f,f'\in\mathscr{C}(X,Y)$ satisfying $\eta_{\mathscr{C},U}(f)=\eta_{\mathscr{C},U}(f')$. Then, for all

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$a \in \mathscr{D}(U, U)$,

$$\eta_{\mathscr{C},U}(f)(a) = \eta_{\mathscr{C},U}(f')(a).$$

Taking $a = 1_U$, gives $f = f'$, as required. □

We therefore have a family of injective PCM-functors from $\mathscr{C}$ to $\mathscr{C}[\mathscr{D}]$ indexed by the objects of $\mathscr{D}$.

Proposition 5.5. Let $\mathscr{D}$ be an arbitrary category and $\mathscr{C}$ be a PCM-category. Then there exists a family of retractions from $\mathscr{C}$ to $\mathscr{C}[\mathscr{D}]$, indexed by objects of $\mathscr{D}$.

Proof. For arbitrary $U \in Ob(\mathscr{D})$, we will demonstrate that $\sigma_{\mathscr{C},\mathscr{D}}\eta_{\mathscr{C},U} = Id_{\mathscr{C}}$:

— **On objects:**

$$\sigma_{\mathscr{C},\mathscr{D}}\eta_{\mathscr{C},U}(X) = \sigma_{\mathscr{C},\mathscr{D}}(X, U) = X$$

— **On arrows:** Given $f \in \mathscr{C}(X, Y)$, we have

$$\sigma_{\mathscr{C},\mathscr{D}}(\eta_{\mathscr{C},U}(f)) = \sum_{a \in \mathscr{D}(U,U)} (\eta_{\mathscr{C},U}(f))$$

where

$$\eta_{\mathscr{C},U}(f)(a) = \begin{cases} f & a = 1_U \\ 0_{XY} & \text{otherwise.} \end{cases}$$

So, by Proposition 2.4, $\sigma_{\mathscr{C},\mathscr{D}}(\eta_{\mathscr{C},U}(f)) = f \in \mathscr{C}(X, Y)$.

Thus $\sigma_{\mathscr{C},\mathscr{D}} : \mathscr{C}[\mathscr{D}] \to \mathscr{C}$ is left-inverse to <u>all</u> $\eta_{\mathscr{C},U} : \mathscr{C} \to \mathscr{C}[\mathscr{D}]$, so $\mathscr{C}$ is a retract of $\mathscr{C}[\mathscr{D}]$, with retractions indexed by $U \in Ob(\mathscr{D})$. □

6. Embedding the index category into a Cauchy product

In a similar way to the way in which there exists an (object-indexed) family of embeddings of the base category into a Cauchy product, we will now exhibit a family of embeddings of the index category into a Cauchy product, which is indexed by objects of the base category.

Definition 6.1. Let $\mathscr{D}$ be an arbitrary category and $\mathscr{C}$ be a PCM-category. For all $X \in Ob(\mathscr{C})$, we define the functor $\gamma_{X,\mathscr{D}} : \mathscr{D} \to \mathscr{C}[\mathscr{D}]$ by:

— **On objects:** $\gamma_{X,\mathscr{D}}(U) = (X, U)$, for all $U \in Ob(\mathscr{D})$,
— **On arrows:** Given $h \in \mathscr{D}(U, V)$, we have $\gamma_{X,\mathscr{D}}(h) \in \mathscr{C}[\mathscr{D}]((X, U), (X, V))$ is defined by

$$\gamma_{X,\mathscr{D}}(h)(a) = \begin{cases} 1_X & a = h \\ 0_X & \text{otherwise.} \end{cases}$$

Proposition 6.2. $\gamma_{X,\mathscr{D}} : \mathscr{D} \to \mathscr{C}[\mathscr{D}]$ as defined above is an injective functor for all $X \in Ob(\mathscr{C})$.

Proof. By Proposition 2.4, it is immediate that for all $h \in \mathscr{D}(U, V)$, the family $\{\gamma_{X,\mathscr{D}}(h)(a) \in \mathscr{C}(X, X)\}_{a \in \mathscr{D}(U,V)}$ is summable, and hence $\gamma_{X,\mathscr{D}}(h)$ is an arrow of

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$\mathscr{C}[\mathscr{D}]((X,U),(X,V))$. To demonstrate functoriality, consider $k \in \mathscr{D}(U,V)$. Then

$$\left(\gamma_{X,\mathscr{D}}(k)\gamma_{X,\mathscr{D}}(h)\right)(c) = \sum_{c=ba} \gamma_{X,\mathscr{D}}(k)(b)\gamma_{X,\mathscr{D}}(h)(a).$$

However,

$$\gamma_X(k)(b) \begin{cases} 1_X & b = k \\ 0_{XX} & \text{otherwise} \end{cases}$$

$$\gamma_X(h)(a) \begin{cases} 1_X & a = h \\ 0_{XX} & \text{otherwise.} \end{cases}$$

Therefore,

$$(\gamma_X(k)\gamma_X(h))(c) = \begin{cases} 1_X & c = kh \\ 0_{XX} & \text{otherwise,} \end{cases}$$

so $\gamma_{X,\mathscr{D}}(k)\gamma(X,\mathscr{D})(h) = \gamma_{X,\mathscr{D}}(kh)$. The proof that $\gamma_{X,\mathscr{D}}$ also preserves identities is trivial.

To demonstrate injectivity, consider $h, h' \in \mathscr{D}(U,V)$. Then

$$\gamma_X(h)(a) \begin{cases} 1_X & a = h \\ 0_{XX} & \text{otherwise} \end{cases}$$

$$\gamma_X(h')(a) \begin{cases} 1_X & a = h' \\ 0_{XX} & \text{otherwise,} \end{cases}$$

which are identical exactly when $h = h'$. □

We therefore have a family of injective functors from $\mathscr{D}$ to $\mathscr{C}[\mathscr{D}]$ indexed by objects of $\mathscr{C}$.

7. Embedding the product of base and index into the Cauchy product

As well as the above embeddings of the base and index categories into the Cauchy product, there is a straightforward embedding of the product of the base and index categories into the Cauchy product.

Definition 7.1. Given a PCM-Category $\mathscr{C}$ and a locally small category $\mathscr{D}$, we define the functor†

$$(\_ \star \_) : \mathscr{C} \times \mathscr{D} \to \mathscr{C}[\mathscr{D}]$$

as follows:

— **Objects:** Given $X \in Ob(\mathscr{C})$ and $U \in Ob(\mathscr{D})$, we have $X \star U = (X,U) \in Ob(\mathscr{C}[\mathscr{D}])$.

— **Arrows:** Given $f \in \mathscr{C}(X,Y)$ and $g \in \mathscr{D}(U,V)$, we have $f \star g \in \mathscr{C}[\mathscr{D}]((X,U),(Y,V))$ is the function

$$(f \star g)(h) \;=\; \begin{cases} f & g = h \\ 0_{X,Y} & \text{otherwise.} \end{cases}$$

† Note that this is simply a functor, rather than a PCM-functor, since the product category $\mathscr{C} \times \mathscr{D}$ is not a PCM-category. However, the construction relies on the assumption that $\mathscr{C}$ is indeed a PCM-category.

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It is immediate that the family $\{(f \star g)(h)\}_{h \in \mathscr{D}(U,V)}$ is summable, so this is indeed an arrow of $\mathscr{C}[\mathscr{D}]$.

Lemma 1. The operation $(\_ \star \_) : \mathscr{C} \times \mathscr{D} \to \mathscr{C}[\mathscr{D}]$ is indeed a functor.

Proof.

— **Compositionality:** Consider the arrows

$$\begin{aligned} f &\in \mathscr{C}(X, Y) \\ f' &\in \mathscr{C}(Y, Z) \\ g &\in \mathscr{D}(U, V) \\ g' &\in \mathscr{D}(V, W). \end{aligned}$$

By definition, $f'f \star g'g \in \mathscr{C}[\mathscr{D}]((X, U), (Z, W))$ is given by the function

$$(f'f \star g'g)(k) = \begin{cases} f'f & k = g'g \\ 0_{X,Z} & \text{otherwise.} \end{cases}$$

Similarly, the arrows

$$\begin{aligned} f \star g &\in \mathscr{C}[\mathscr{D}]((X, U), (Y, V)) \\ f' \star g' &\in \mathscr{C}[\mathscr{D}]((Y, V), (Z, W)) \end{aligned}$$

are, by definition, the functions

$$(f \star g)(h) = \begin{cases} f & h = g \\ 0_{X,Y} & \text{otherwise} \end{cases}$$

$$(f' \star g')(j) = \begin{cases} f' & j = g' \\ 0_{Y,Z} & \text{otherwise.} \end{cases}$$

Then, by the definition of composition in $\mathscr{C}[\mathscr{D}]$,

$$\left((f' \star g')(f \star g)\right)(k) = \sum_{k=jh} (f' \star g')(j)(f \star g)(h).$$

From the definitions of $f' \star g'$ and $f \star g$, we see that

$$\sum_{k=jh} (f' \star g')(j)(f \star g)(h) = \begin{cases} f'f & k = g'g \\ 0_{X,Z} & \text{otherwise.} \end{cases}$$

Hence, $((f' \star g')(f \star g))(k) = (f'f \star g'g)(k)$, as required.

— **Identities:** By definition, $1_X \star 1_U \in \mathscr{C}[\mathscr{D}]((X, U), (X, U))$ is the function

$$(1_X \star 1_U)(h) = \begin{cases} 1_X & h = 1_U \\ 0_{XX} & \text{otherwise,} \end{cases}$$

which, from Theorem 4.4, is precisely $1_{(X,U)} \in \mathscr{C}[\mathscr{D}]((X, U), (X, U))$.

Corollary 7.2. The functor $(\_ \star \_) : \mathscr{C} \times \mathscr{D} \to \mathscr{C}[\mathscr{D}]$ is an embedding of $\mathscr{C} \times \mathscr{D}$ into $\mathscr{C}[\mathscr{D}]$.

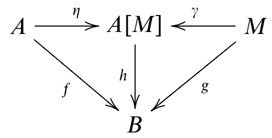

Fig. 1. The universal property for monoid semirings

Proof. By definition, $X \star U \stackrel{def}{=} (X, U)$, so $(\_ \star \_)$ is bijective on objects. To see injectivity on arrows, recall that

$$(f \star g), (h \star k) \in \mathscr{C}[\mathscr{D}]((X, U), (Y, V))$$

are functions from $\mathscr{D}(U, V)$ to $\mathscr{C}(X, Y)$ given in Definition 7.1. From this definition, these are identical exactly when $f = h$ and $g = k$, and hence are equal in $\mathscr{C} \times \mathscr{D}$ – that is,

$$f \star g \;=\; h \star k \quad \Leftrightarrow \quad f \times g \;=\; h \times k,$$

which completes the proof □

8. Universal properties and the Cauchy product

The functors $\eta_{\mathscr{C},U} : \mathscr{C} \to \mathscr{C}[\mathscr{D}]$ and $\gamma_{X,\mathscr{D}} : \mathscr{D} \to \mathscr{C}[\mathscr{D}]$ are clearly (object-indexed) categorical analogues of the usual *augmentation* and *inclusion* maps used in the demonstration of the universal property of the monoid semiring construction (see Steinberger (1993) for a good exposition, albeit in the special case of monoid rings). It is natural to wonder whether an analogous property holds for the categorical Cauchy product. In this section we give an exposition of the usual universal property of monoid semirings, and demonstrate that a straightforward generalisation to the Cauchy product is not possible, except in the trivial one-object case. The computational significance of this is discussed, and we consider the additional structure that would be required in order to have a suitable universal property in the full multi-object case.

8.1. *The universal property of monoid semirings*

The universal property of monoid semirings is a canonical example of a universal property (see, for example, Steinberger (1993)).

Theorem 8.1. Let $(M, \cdot)$ be a monoid, and $(A, \times, +, 1_A, 0_A)$ and $(B, \times_B, 1_B, 0_B)$ be unital semirings. Also, let $f : A \to B$ be a unital semiring homomorphism and $g : (M, \cdot) \to (B, \times_B)$ be a monoid homomorphism. Finally, let $\eta : A \to A[M]$ and $\gamma : M \to A[M]$ be the usual augmentation and inclusion maps. Then there exists a unique unital semiring homomorphism $h : A[M] \to B$ such that the diagram of Figure 1 commutes.

Proof. Given $\alpha : M \to A$, an element of $A[M]$, we define the semiring homomorphism $h : A[M] \to B$ by

$$h(\alpha) \;=\; \sum_{m \in M} f(\alpha(m))g(m).$$

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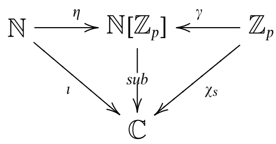

Fig. 2. A canonical example of the universal arrow

The proof that this is a unique unital semiring homomorphism that makes the above diagram commute is then straightforward, and may be found in many algebra texts (for example, Steinberger (1993)). □

Remark 8.2 (interpretation). The usual interpretation of the above universal property is that the unique unital semiring homomorphism making the diagram of Figure 1 commute describes the computational process of *instantiating a free variable*. In order to provide motivation for this interpretation, we will describe a simple example.

Consider the monoid $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ for some prime p, together with the semiring of natural numbers $\mathbb{N}$. The members of the monoid semiring $\mathbb{N}[\mathbb{Z}_p]$ are often written as 'polynomials' in some formal variable z, so the function $f : \mathbb{Z}_p \to \mathbb{N}$ would be written as

$$f(0)z^0 + f(1)z^1 + f(2)z^2 + \cdots + f(p-1)z^{p-1}$$

with the understanding that multiplication of this formal variable is defined by $z^a z^b = z^{a+b\ (mod\ p)}$. Using this polynomial formalism, the augmentation $\eta : \mathbb{N} \to \mathbb{N}[\mathbb{Z}_p]$ and inclusion $\gamma : \mathbb{Z}_p \to \mathbb{N}[\mathbb{Z}_p]$ are given by

$$\eta(n) = nz^0$$
$$\gamma(r) = z^r.$$

We will now consider the usual semiring homomorphism $\iota : \mathbb{N} \to \mathbb{C}$ given by the canonical inclusion, together with the monoid homomorphism $\chi_s : \mathbb{Z}_p \to \mathbb{C}$ defined by $\chi_s(a) = e^{2\pi i \frac{as}{p}}$ for some fixed $s \neq 0 \in \mathbb{N}$.

The universal property of the monoid semiring construction tells us that there is a unique induced universal map $sub : \mathbb{N}[\mathbb{Z}_p] \to \mathbb{C}$ that makes the diagram of Figure 2 commute. From the prescription given in the proof of Theorem 8.1, it is immediate that the action of the universal arrow is given by

$$\begin{array}{c} f(0)z^0 \ + \ f(1)z^1 \ + \ f(2)z^2 \ + \ \cdots \ + \ f(p-1)z^{p-1} \\ \Big\downarrow{\scriptstyle sub} \\ f(0) \ + \ f(1)e^{2\pi i \frac{s}{p}} \ + \ f(2)e^{2\pi i \frac{2s}{p}} \ + \ \cdots \ + \ f(p-1)e^{2\pi i \frac{(p-1)s}{p}} \end{array}$$

(note that we elide the inclusion homomorphism $\iota : \mathbb{N} \to \mathbb{C}$, for clarity). Thus, the induced universal map is simply interpreted as substituting a concrete value for the formal variable z.

An immediate question is whether such a property also exists for the categorical Cauchy product? That is, given a PCM-functor $\Gamma \in \mathbf{Cat}_\Sigma(\mathscr{C}, \mathscr{E})$ together with $\mathscr{D} \in Ob(\mathbf{cCat})$

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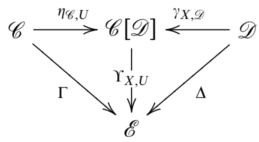


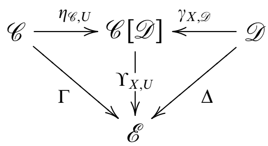
Fig. 3. Can such a universal family of functors exist?

and a functor $\Delta \in \mathbf{Cat}(\mathscr{D}, \mathscr{E})$, does there exist an object-indexed family of functors $\Upsilon_{X,U} \in \mathbf{Cat}_{\Sigma}(\mathscr{C}[\mathscr{D}], \mathscr{E})$ making the diagram of Figure 3 commute for arbitrary choice of $X \in Ob(\mathscr{C})$ and $U \in Ob(\mathscr{D})$?

However, it is straightforward to see simply from the fact that the categorical version is in the multiple-object setting that such a universal property can only ever hold in a very restricted setting, as the following result demonstrates.

Proposition 8.3. We assume the existence of an object-indexed family of functors $\Upsilon_{X,U} : \mathscr{C}[\mathscr{D}] \to \mathscr{E}$ making the diagram of Figure 3 commute for all $X \in Ob(\mathscr{C})$ and $U \in Ob(\mathscr{D})$. Then $\Gamma(X) = \Delta(U)$ for all $X \in Ob(\mathscr{C})$ and $U \in Ob(\mathscr{D})$.

Proof. We fix some arbitrary $X \in Ob(\mathscr{C})$ and $Y \in Ob(\mathscr{D})$ such that the following diagram commutes:



Then for all $P \in Ob(\mathscr{C})$,

$$\eta_{\mathscr{C},U}(P) = (P, U) \in Ob(\mathscr{C}[\mathscr{D}])$$

by the definition of $\eta_{\mathscr{C},U}$, and by the commutativity of the above diagram, $\Upsilon_{X,U}\eta_{\mathscr{C},U} = \Gamma$. Therefore, $\Upsilon_{X,U}(P, U) = \Gamma(P)$. Similarly, $\gamma_{X,\mathscr{D}}(Q) = (X, Q)$ for all $Q \in Ob(\mathscr{D})$, so $\Upsilon_{X,U}(X, Q) = \Delta(Q)$, since $\Upsilon_{X,U}\gamma_{X,\mathscr{D}} = \Delta$. Combining these, we see that

$$\Upsilon_{X,U}(X, U) = \Gamma(X) = \Delta(U).$$

Finally, X and U were chosen arbitrarily, so $\Gamma(X) = \Delta(U)$ for all $X \in Ob(\mathscr{C})$ and $U \in Ob(\mathscr{D})$. □

Remark 8.4 (the multi-object case, and universal arrows). A simple corollary of the above proof is that when such a universal family of functors exists, $\Gamma : \mathscr{C} \to \mathscr{E}$ and $\Delta : \mathscr{D} \to \mathscr{E}$ map all objects of $\mathscr{C}$ and $\mathscr{D}$ to the same object of $\mathscr{E}$ (it is then straightforward, though deeply uninteresting, to demonstrate that in the restricted case where the target in the diagram of Figure 3 is a one-object PCM category, we do indeed have a universal property that is a direct generalisation of that of Theorem 8.1). Note also that the above proof is based entirely on how such a universal family of functors might act on the *objects* of these categories. Therefore, it does not depend on any subtlety about the precise notion of summation used, or even on the definition of composition; rather, it fails on the simple fact that $\mathscr{E}$ has more than one object.

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From an algebraic point of view, this appears to be a serious drawback – indeed, it is not uncommon to *define* the monoid semiring construction simply in terms of the existence of a suitable universal arrow. However, from a more computational point of view, it appears reasonable: from the interpretation given in Remark 8.2, we should think of the construction of a universal arrow as substituting values for variables. When we consider the multi-object setting, the multiplicity of objects gives distinct *types* for both variables and values. The interpretation of Proposition 8.3 is then that, when we try to substitute values for variables, we must do so in a context where all types agree. From a computational, rather than algebraic, point of view this is to be expected!

However, this does not mean that no suitable universal property may exist. The key question is simply which object of $\mathscr{E}$ is the appropriate target of $\Upsilon_{X,U}$? Let us now assume that the PCM-category $\mathscr{E}$ also has a monoidal tensor $(\_\otimes\_)$ that is required to satisfy some universal property for bilinearity, which is analogous to either that of the usual tensor product of Hilbert spaces or the constructions of Bahamonde (1985) for Partially Additive Monoids. In this case, the natural candidate must be the object $\Gamma(X) \otimes \Delta(U) \in Ob(\mathscr{E})$. It therefore seems that questions of universal properties must await a theory of *monoidal* PCM-categories. As we demonstrate in Section 10, such a theory would be key to many reasonable generalisations and applications, both algebraic and computational.

9. Conclusions

We have demonstrated that the monoid-semiring construction can be placed within a significantly more general categorical and multi-object setting. So that this more general theory may be equally applicable in both 'analytic' and 'algebraic' settings, this was done using an axiomatisation of summation that unifies notions from analysis with notions from algebraic program semantics.

10. Future directions

As well as the program outlined above, work continues in several related directions:

— **Categorical enrichment:** A natural question arising from this paper is whether the notion of a 'PCM-category' is in fact an example of categorical enrichment (as in Kelly (1982)) over the category **PCM**? Enrichment requires either a monoidal, or a closed (or monoidal closed) structure. It has recently been demonstrated by the author and P. Scott (Ottawa) that **PCM** is a closed category in the sense of Laplaza (1977), and a monoidal tensor adjoint to the closed structure has been given explicitly by T. Porter (Wales). This monoidal tensor appears to exhibit a universal property for a suitable notion of bilinear maps of PCMs (these are similar to the constructions of Bahamonde (1985) for Partially Additive Monoids). It is expected that a category enriched over this monoidal closed category is exactly a 'PCM-category', as defined in Definition 3.1. This is the subject of ongoing work.

— **The Cauchy product and monoidal structures:** Although we have demonstrated that the Cauchy product is a bifunctor with interesting embedding properties, we have not yet considered the case where either the base category or the index category (or both) have a monoidal tensor. This requires studying the monoidal structure of **PCM** (as above) in order to describe what it means for a PCM-category to have a monoidal tensor. This is undoubtedly an interesting route to explore, and is also important for the applications to semantics described below.

— **PCM-categories, algebraic program semantics and axiomatisations of summation:** From the start, the notion of a PCM-category was intended as a unification of the forms of summation used in algebraic program semantics with more analytic notions of summation used in Hilbert and Banach spaces. A very natural question is therefore how much of the traditional theory may be carried through to this more general setting, and whether such constructions as the Elgot dagger or (presumably partial) particle-style categorical traces may be defined†. Again, the absolute starting point for this is the definition of suitable monoidal tensors, and their interaction with notions of summation.

Note that all the above future directions depend on a detailed study of monoidal tensors and closed structures in both **PCM** and members of $\mathbf{Cat}_\Sigma$. Thus, it appears that pursuing such questions as pure theory may be the most profitable route!

Appendix A. PCMs and PCM-categories

We consider various examples of both PCMs and PCM-categories, as defined in Definitions 2.1 and 3.1, respectively. We also draw some comparisons with other axiomatisations of summation from the field of algebraic program semantics.

A.1. *Examples of PCMs from algebraic program semantics*

Both Σ-*monoids*, and *partially additive monoids*, as introduced in Manes and Benson (1985) and Manes and Arbib (1986) and used in Haghverdi (2000), Abramsky *et al.* (2002) and Haghverdi and Scott (2006), may be given as special cases of PCMs.

Definition A.1 (Σ-monoids, Partially additive monoids). A PCM (M,Σ) is called a **Σ-monoid** when it satisfies the following additional axiom:

— **The (full) partition-associativity axiom:** Let $\{x_i\}_{i\in I}$ be a countably indexed family and $\{I_j\}_{j\in J}$ be a countable partition of I. Then $\{x_i\}_{i\in I}$ is summable if and only if $\{x_i\}_{i\in I_j}$ is summable for every $j\in J$, and $\{\sum_{i\in I_j} x_i\}_{j\in J}$ is summable, in which case

$$\sum_{i\in I} x_i = \sum_{j\in J}\left(\sum_{i\in I_j} x_i\right).$$

† As supporting evidence that this is possible, see Hines and Scott (2012) and Hines (2010), where partial categorical traces based on notions of summation that do not satisfy positivity (but do, however, satisfy the PCM axioms) are used both to model quantum-optics thought experiments and to construct concrete quantum circuits.

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Note that this is a special case of the *weak partition-associativity axiom*, but with a two-way rather than a one-way implication.

A Σ-monoid is called a **Partially Additive Monoid** (PAM) when it satisfies the following additional axiom:

— **The limit axiom:** If $\{x_i\}_{i\in I}$ is a countably indexed family where$\{x_i\}_{i\in F}$ is summable for every finite $F \subseteq I$, then $\{x_i\}_{i\in I}$ is summable.

The following are Partially Additive Monoids, and are therefore examples of PCMs:

— *Partial functions, with the usual summation:* An indexed family of partial functions $\{f_i : X \to Y\}_{i\in I}$ is **summable** exactly when $dom(f_i) \cap dom(f_j) = \emptyset$ for all $i \neq j$. The **sum** is given by

$$\left(\sum_{i\in I} f_i\right)(x) = \begin{cases} f_i(x) & x \in dom(f_i) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

— *Relations, with set-theoretic union:* Any indexed family of relations $\{R_i : X \to Y\}_{i\in I}$ is **summable**, and the **sum** is simply set-theoretic union.

— *Partial injective functions:* The following distinct summations both give a PAM structure to hom-sets of partial injective functions:

  – *The disjointness summation:* An indexed family of partial functions $\{f_i : X \to Y\}_{i\in I}$ is **disjointness-summable** exactly when $dom(f_i)\cap dom(f_j) = \emptyset$ for all $i \neq j$. The **sum** is given by

$$\left(\sum_{i\in I} f_i\right)(x) = \begin{cases} f_i(x) & x \in dom(f_i) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

  – *The overlap summation:* An indexed family of partial functions $\{f_i : X \to Y\}_{i\in I}$ is **overlap-summable** exactly when $x \in dom(f_i) \cap dom(f_j) \Rightarrow f_i(x) = f_j(x)$, for all $i, j \in I$. The **sum** is as given above.

The following example is not a partially additive monoid, but is a Σ-monoid, and thus also an example of a PCM:

— *Absolute convergence on positive cones:* See Selinger (2004) for categories of positive cones, and summation based on the usual summation of positive elements in finite-dimensional vector space.

Given our stated aim of unifying notions of summation from both analysis and algebraic program semantics, both Σ-monoids and Partially Additive Monoids have undesirable properties for our purposes. The *limit axiom* is clearly undesirable for any example based on real or complex numbers: all finite families of complex numbers are summable, but the same is certainly not true (as the limit axiom would imply) for arbitrary countably infinite families.

The full partition-associativity axiom is also undesirable for slightly more subtle reasons, as the following proposition (taken from Manes and Arbib (1986)) demonstrates.

Proposition A.2. Let (M, Σ) be a Σ-monoid and $X = \{x_i\}_{i\in I}$ be a summable family of M satisfying $\sum_{i\in I} x_i = 0$. Then $x_i = 0$ for all $i \in I$.

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Proof. For some $i \in I$, we define $Y = \{x_j\}_{j \neq i \in I}$, so $x_i + \sum Y = 0 = \sum Y + x_i$ by weak partition-associativity. Then, by the full partition-associativity axiom,

$$
\begin{aligned}
x_i &= x_i + 0 + 0 + 0 + \cdots && \text{(by Proposition 2.4)}\\
&= x_i + \left(\sum Y + x_i\right) + \left(\sum Y + x_i\right) + \left(\sum Y + x_i\right) + \cdots \\
& && \text{(by full partition-associativity)}\\
&= \left(x_i + \sum Y\right) + \left(x_i + \sum Y\right) + \left(x_i + \sum Y\right) + \cdots \\
&= 0 + 0 + \cdots \;=\; 0. && \text{(by Proposition 2.4)}
\end{aligned}
$$

Hence $x_i = 0$, but since i was chosen arbitrarily, we have $x_k = 0$ for all $k \in I$. □

A.2. *Positivity, computation and the PCM axioms*

From a certain point of view, positivity seems to be a natural property of the notions of summation used in theoretical computer science. Taking the 'sum' of a family of arrows in a category is often interpreted in a very domain-theoretic manner as looking at the total information provided by all these arrows. The challenge to this intuition comes from the field of quantum computation, where summing amplitudes leads to both constructive and destructive interference effects. For example, Deutsch *et al.* (1999) says:

> 'Amplitudes are complex numbers and may cancel each other, which is referred to as destructive interference, or enhance each other, referred to as constructive interference. The basic idea of quantum computation is to use quantum interference to amplify the correct outcomes and to suppress the incorrect outcomes of computations.'

From this point of view, at least, enforcing positivity in models of quantum computation would seem to rule out the phenomena that distinguish quantum computation. A good example is provided by the quantum Fourier transform[†], which is based on group homomorphisms $\chi : \mathbb{Z}_n \to \mathbf{Hilb}(H,H)$ satisfying $\sum_{j=0}^{n-1} \chi(j) = 0_H$. Clearly, assuming positivity will only allow for the trivial homomorphism.

Finally, see Hines (2010) for an application of category theory to quantum circuits that relies on both summing linear maps and composition based on convolved (that is, Cauchy) products.

A.3. *Non-positive examples of PCMs*

The proof of positivity for Sigma monoids given in Proposition A.2 does *not* apply to general PCMs, as it depends on the two-way implication in the (full) partition-associativity axiom. We will give various examples of PCMs that need not be either Partial Additive Monoids or Sigma-monoids. Many of these are based on the theory of Cauchy sequences (Hobson 1957; Titchmarsh 1983), and various analytic notions of summability, such as absolute convergence of real or complex sums.

[†] Quantum Fourier transforms are required in, for example, Shor's algorithm (Shor 1999) and quantum period-finding generally (Nielsen and Chuang 2000).

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Definition A.3. Let $\sum_{j=0}^{\infty} a_j$ be a formal (that is, not necessarily convergent) series of real numbers. The nth partial sum is defined by $A_n = \sum_{j=0}^{n} a_j$. When $lim_{n\to\infty}(A_n)$ exists, the infinite series is said to **converge**. Note that convergence is *permutation-dependent*. Let $\sum_{j=0}^{\infty} a_j$ and $\sum_{k=0}^{\infty} b_k$, be series satisfying $b_k = a_{\sigma(j)}$ for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$. Then the convergence of $\sum_{j=0}^{\infty} a_j$ is not enough to guarantee the convergence of $\sum_{k=0}^{\infty} b_k$.

A convergent series $\sum_{j=0}^{\infty} a_j$ is said to **converge absolutely** when it satisfies the additional property that the non-negative series $\sum_{j=0}^{\infty} |a_j|$ is convergent. Alternatively, a convergent series $\sum_{j=0}^{\infty} a_j$ is said to be **permutation-independent** when $\sum_{i=0}^{\infty} a_{\sigma(i)}$ converges for arbitrary permutations $\sigma : \mathbb{N} \to \mathbb{N}$.

The following result is straightforward.

Lemma 2. The real line, with summation defined by convergence, is *not* a PCM.

Proof. From Proposition 2.3, summation in a PCM must satisfy permutation independence. □

A standard result of analysis is that for real and complex numbers, absolute convergence is equivalent to permutation-independence†. Real numbers with summation defined by absolute convergence thus provides our first example of a PCM not satisfying the positivity property.

Proposition A.4. The real number line $\mathbb{R}$, together with summation defined by absolute convergence, satisfies the axioms for a PCM.

Proof. It is trivial that the unary sum axiom is satisfied. To see that the weak partition-associativity axiom is also satisfied, see Hille (1982, page 108) – this reference is also cited in Remark 2.2 of this paper. See also Kadets and Kadets (1991, Theorem 8, page 84). □

The following corollary is then immediate.

Corollary A.5. The complex plane $\mathbb{C}$, together with the summation defined by absolute convergence, satisfies the PCM axioms.

The above results may be extended to finite-dimensional Hilbert and Banach spaces, with no substantial obstacles. However, it is more satisfactory to consider summation in the general setting, and restrict to these examples as special cases. As a preliminary, we need some additional analytic notions of summation. The following definition is taken from Day (1973).

Definition A.6. Let X be an arbitrary Banach space. A series $\sum_{i=0}^{\infty} x_i$ is said to **converge absolutely** when $\sum_{i=0}^{\infty} \|x_i\| < \infty$. Alternatively, is is said to **converge unconditionally** when the series $\sum_{j=0}^{\infty} x_{\sigma(j)}$ converges for arbitrary permutations $\sigma : \mathbb{N} \to \mathbb{N}$. (Unconditional summability can simply be thought of as permutation-independent convergence, but in

† We emphasise that this is specific to the real and complex planes, and finite-dimensional spaces, but in more general settings, absolute convergence and permutation-independence are distinct concepts. In particular, infinite-dimensional Banach spaces or abstract topological groups provide counterexamples, as discussed after Definition A.6.

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a more general setting.) It is standard that for any unconditionally convergent series, all rearrangements have the same sum; also, every subseries of an unconditionally convergent series is itself unconditionally convergent (Thorpe 1968).

Theorem A.7. In an arbitrary Banach space, absolute convergence implies unconditional convergence, but the converse is not generally true. However, in finite-dimensional Banach spaces, absolute convergence and unconditional convergence are equivalent.

Proof. This is a standard result of analysis – see, for example, Kadets and Kadets (1991, Theorem 1.3.3). □

Definition A.8. Let X be an arbitrary Banach space. A series $\sum_{i=0}^{\infty} a_i$ is **subseries convergent** when the partial sums $A_n = \sum_{j=0}^{n} \alpha(j)a_j$ form a Cauchy sequence for arbitrary choice of $\alpha : \mathbb{N} \to \{0,1\}$. Subseries convergence is often defined informally as 'each subseries of $\sum_{i=0}^{\infty} a_i$ converges'.

In Banach spaces, subseries convergence provides a nice characterisation of unconditional convergence, as the following classic theorem demonstrates.

Proposition A.9. Let X be an arbitrary Banach space. A series $\sum_{i=0}^{\infty} x_i$ is unconditionally convergent if and only if it is subseries convergent.

Proof. This is a corollary of the classic result of Orlicz (1933). See Day (1973) for a textbook proof (in English), and Lahiri and Das (2002) for a nice elementary proof. □

Note that the above result depends on the sequential completeness of Banach spaces, and thus in more general settings, subseries-convergence and unconditional convergence are not equivalent concepts. In particular, conditions equivalent to subseries convergence were introduced in Gelfand (1938) as 'strong unconditional convergence'[†].

Corollary A.10. Absolutely convergent series in finite-dimensional Banach spaces are subseries-convergent.

Proof. This follows from Theorem A.7. □

As may be expected, subseries-convergent series satisfy the weak partition-associativity axiom. To prove this, we first need some unsurprising technical results.

Definition A.11. Let $\sum_{j=0}^{\infty} y_j$ be an infinite series and $\sum_{i=0}^{\alpha} x_i$ be a finite or infinite series in some Banach space X. We say that $\sum_{j=0}^{\infty} y_j$ is a 0-padding of $\sum_{i=0}^{\alpha} x_i$ when there exists some injection $\eta : \{0,\dots\alpha\} \to \mathbb{N}$ such that for all $j \in \mathbb{N}$,

$$y_j = \begin{cases} x(i) & j = \eta(i) \\ 0 & \text{otherwise.} \end{cases}$$

[†] See McArthur (1961) for many conditions equivalent to subseries-convergence, including the definitions of Gelfand (1938).

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Lemma 3. Let $\sum_{i=0}^{\alpha} x_i$ be a finite or countably infinite subseries-summable series in some Banach space X, and $\sum_{j=0}^{\infty} y_j$ be a zero-padding of $\sum_{i=0}^{\infty} x_i$. Then $\sum_{j=0}^{\infty} y_j$ is subseries convergent and $\sum_{j=0}^{\infty} y_j = \sum_{i=0}^{\alpha} x_i$.

Proof. We assume that $\sum_{i=0}^{\infty} x_i$ is an infinite series, since otherwise the result is trivial. Consider arbitrary $\beta : \mathbb{N} \to \{0,1\}$, along with the sum $\sum_{j=0}^{\infty} \beta(j)y_j$, and let $\eta : \mathbb{N} \to \mathbb{N}$ be the embedding satisfying

$$y_j = \begin{cases} x(i) & j = \eta(i) \\ 0 & \text{otherwise.} \end{cases}$$

Then, since the partial sums $\{\sum_{i=0}^{n} x_i\}_{n\in\mathbb{N}}$ form a Cauchy sequence, so do the partial sums $\{\sum_{j=0}^{m} y_j\}_{m\in\mathbb{N}}$. Thus, as $\beta : \mathbb{N} \to \{0,1\}$ was chosen arbitrarily, we deduce that $\sum_{j=0}^{\infty} y_j$ is subseries convergent, as required. The equivalence of the two sums is then immediate. □

This technical lemma then allows us to appeal to some standard results to demonstrate that subseries-convergent series in Banach spaces satisfy the weak partition-associativity axiom.

Theorem A.12. Let $\sum_{i=0}^{\infty} x_i$ be a subseries-convergent series in an arbitrary Banach space X, and let $\{I_j\}_{j\in J}$ be a countable partition of I. Then $\{x_i\}_{i\in I_j}$ is subseries-convergent, as is $\{\sum_{i\in I_j} x_i\}_{j\in J}$, and $\sum_{i\in I} x_i = \sum_{j\in J}(\sum_{i\in I_j} x_i)$.

Proof. This result is proved for partitions into countably infinite sets in Thorpe (1968, Lemma 2). The case where certain of these partitions are finite appears to be implicitly assumed in Thorpe (1968) – for a formal justification, we may consider zero-padding the original sequence to replace finite subsums by infinite sums with a finite number of non-zero summands, and appealing to Lemma 3. □

We may now list a number of PCMs that do not, or are not required to, satisfy the positivity condition.

(1) *Absolute convergence of real or complex numbers:* Absolute convergence of countable sums in the real or complex plane is a motivating example for the theory of PCMs and PCM-categories. It arises as a special case of absolute convergence in finite-dimensional Hilbert spaces, as below.

(2) *Absolute convergence in finite-dimensional Hilbert spaces:* All finitely indexed families are summable, with the usual summation. A countably indexed family $\{\psi_i\}_{i\in\mathbb{N}}$ is **summable** exactly when the sequence $\{\sum_{i=1}^{n} \|\psi_i\|\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

(3) *Subseries-summable summation in arbitrary Banach spaces:* All finitely indexed families are summable, with the usual summation. A countably indexed family $\{b_i\}_{i\in\mathbb{N}}$ is **summable** exactly when the series $\sum_{i=0}^{\infty} b_i$ is subseries-summable in the sense of Definition A.8, in which case the sum is the limit of the Cauchy sequence $\{\sum_{i=1}^{n} b_i\}_{n\in\mathbb{N}}$.

(4) *The unit ball summation in finite-dimensional Banach spaces:* Let $\mathscr{B}$ be a finite-dimensional Banach space, and denote the unit ball by $Ball(\mathscr{B}) = \{b \in \mathscr{B} : \|b\| \leqslant 1\}$. An indexed family $\{b_i\}_{i\in I}$ is summable exactly when $\sum_{i\in I} \|b_i\| \leqslant 1$, in which case its sum is the usual Banach space summation.

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(5) *Any abelian monoid:* Given an abelian monoid $(M, +, 0_M)$, the following are distinct PCM-structures:

— *The finite families summation:* An indexed set $\{m_i\}_{i\in I}$ is summable exactly when the subfamily of non-zero elements $\{m_j\}_{j\in J\subseteq I}$ is a finite family. The sum is defined by

$$\sum_{i\in I} m_i = \begin{cases} \sum_{j\in J} m_j & J \neq \{\} \\ 0_M & \text{otherwise,} \end{cases}$$

where the finitary sum on the right-hand side is the usual sum of the abelian group.

— *The K-bounded summation:* This is as above, but where the summable families are those with at most K non-zero elements.

It is almost immediate from the commutativity and associativity of composition in M that these both satisfy the PCM axioms.

(6) *Any abelian group:* These can be considered with either of the above, the finite families summation or K-bounded summation, as a special case of the abelian monoid examples.

A.4. *Examples of PCM-categories*

A PCM-category is defined in Definition 3.1 to be a category $\mathscr{C}$ where each hom-set has a specified PCM structure, together with the *strong distributivity* axiom that connects composition and summation. This states that, given summable families $\{g_j \in \mathscr{C}(Y, Z)\}_{j\in I}$ and $\{f_i \in \mathscr{C}(X, Y)\}_{i\in I}$, we have $\{g_j f_i \in \mathscr{C}(X, Z)\}_{(j,i)\in\mathscr{C}(X,Z)}$ is summable and

$$\left(\sum_{j\in J} g_j\right)\left(\sum_{i\in I} f_i\right) = \sum_{(j,i)\in J\times I} g_j f_i.$$

— *The real or complex numbers, with multiplication and absolute convergence:* Given two absolutely convergent sums of real or complex numbers, $\sum_{i\in I} r_i$ and $\sum_{j\in J} s_j$, we have, by the definition of absolute convergence, that $\sum_{(i,j)\in I\times J} r_i s_j$ exists and

$$\left(\sum_{i\in I} r_i\right)\left(\sum_{j\in J} s_j\right) = \sum_{(i,j)\in I\times J} r_i s_j.$$

Therefore, $(\mathbb{R}, \times)$ or $(\mathbb{C}, \times)$, with this indexed summation, is a one-object PCM-category.

— *Linear maps on finite-dimensional Hilbert space, with composition and uniform convergence:* This follows similarly to the above examples. Note that the hom-set of maps between two finite-dimensional Hilbert spaces is itself a finite-dimensional Hilbert space, and the distinct notions of convergence with respect to various operator norms all coincide in the finite-dimensional case. The strong distributivity property is a classic result of analysis (see, for example Swartz (1992) for a general setting).

— *Any ring, with the finite families summation:* Let $(R, \times, +)$ be a ring. Then $(R, \times)$ is a monoid, and thus a one-object category. Its unique homset (that is, the elements of

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R) is an abelian monoid, $(R, +)$. Hence we may take the the finite families summation $\Sigma_{<\infty}$ of Section A.3 to get the PCM $(R, \Sigma_{<\infty})$. Given two summable families $\{s_j\}_{j\in J}$ and $\{r_i\}_{i\in I}$, the family $\{s_j r_i\}_{(j,i)\in J\times I}$ has a finite number of non-zero elements, and hence is summable. Given the summability of the required families, the identity

$$\left(\sum_{j\in I} s_j\right)\left(\sum_{i\in I} r_i\right) = \sum_{(j,i)\in J\times I} s_j r_i$$

is then straightforward from the definition of summation in terms of the addition in the ring $(R, \times, +)$.

Note that just because the homsets of a category are PCMs does not necessarily mean we have a PCM-category – we also need the strong distributivity condition of Definition 3.1. For example, consider a unital ring $(R, \times, +, 1, 0)$. The multiplicative monoid $(R, \times, 1)$ is trivially a one-object category, and as demonstrated in Section A.3, we may give the additive abelian monoid $(R, +, 0)$ a PCM structure using the K-bounded summations $\Sigma_{\leqslant K}$ where a family is summable exactly when it has no more than K non-zero elements. However, for $K > 1$, the K-bounded summation does *not* in general make $(R, \times, \Sigma_{\leqslant K})$ a one-object PCM category. Let us assume that R has no zero-divisors, and consider two summable families containing K non-zero elements $\{s_j\}_{j\in J}$ and $\{r_i\}_{i\in I}$. Then the strong distributivity law does not hold, since $\{s_j r_i\}_{(j,i)\in J\times I}$ is not a summable family as it contains $K^2 > K$ non-zero elements.

We now demonstrate that there is a whole class of examples to be found within the field of algebraic program semantics. The proofs that these are PCM-categories arises from the following straightforward result.

Proposition A.13. Let $\mathscr{C}$ be a category, together with for all $X, Y \in Ob(\mathscr{C})$, a function $\Sigma^{(X,Y)}$ from indexed families over $\mathscr{C}(X, Y)$ to $\mathscr{C}(X, Y)$ such that:

(1) $\left(\mathscr{C}(X, Y), \Sigma^{(X,Y)}\right)$ is a PCM satisfying the additional *full partition-associativity axiom* of Definition A.1.

(2) The usual left and right distributivity conditions are satisfied: that is, if we are given a summable family $\{g_i \in \mathscr{C}(B, C)\}_{i\in I}$ and arbitrary $f \in \mathscr{C}(A, B)$ and $h \in \mathscr{C}(C, D)$, then the families $\{hg_i \in \mathscr{C}(B, D)\}_{i\in I}$ and $\{g_i f \in \mathscr{C}(A, C)\}_{i\in I}$ are summable and

$$h\left(\sum_{i\in I} g_i\right) = \sum_{i\in I}(hg_i)$$

$$\left(\sum_{i\in I} g_i\right) f = \sum_{i\in I}(g_i f).$$

Then $\left(\mathscr{C}, \Sigma^{(\_,\_)}\right)$ satisfies the strong distributivity condition of Definition 3.1, and thus is a PCM-category.

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Proof. We write

$$f = \sum_{i \in I} f_i$$
$$g = \sum_{j \in J} g_j.$$

Then, by left-distributivity and the existence of $\sum_{i \in I} f_i$, we deduce

$$gf = \sum_{i \in I} g f_i.$$

However, $g = \sum_{j \in J} g_j$, so, by full partition-associativity,

$$gf = \sum_{i \in I} \left(\sum_{j \in J} g_j f_i \right).$$

Using the right distributivity law and full partition-associativity,

$$gf = \sum_{j \in J} \left(\sum_{i \in I} g_j f_i \right).$$

Again by full partition-associativity and Proposition 2.3, we may replace the doubly indexed sum by a single indexed sum, giving

$$gf = \sum_{(j,i) \in J \times I} g_j f_i,$$

and thus

$$gf = \left(\sum_{j \in J} g_j \right) \left(\sum_{i \in I} f_i \right) = \sum_{(j,i) \in J \times I} g_j f_i.$$

Therefore $\mathscr{C}$ satisfies strong distributivity, as required. □

Note that the converse is not true: weak partition-associativity together with the strong distributivity law does not, in general, imply the full partition-associativity axiom. This is clear from the failure of positivity in many of the examples given.

Corollary A.14. The Partially Additive Categories (PACs) of Manes and Arbib (1986) are PCM-categories, as are the Unique Decomposition Categories (UDCs) of Haghverdi (2000), Abramsky *et al.* (2002) and Haghverdi and Scott (2006).

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References

Abramsky, S., Haghverdi, E. and Scott, P. (2002) Geometry of interaction and linear combinatory algebras. *Mathematical Structures in Computer Science* **12** (5) 625–665.

Abramsky, S. (2005) Abstract Scalars, Loops, and Free Traced and Strongly Compact Closed Categories. *Springer-Verlag Lecture Notes in Computer Science* **3629** 1–29.

Bahamonde, A. (1985) Tensor Product of Partially-Additive Monoids. *Semigroup Forum* **32** 31–53.

Day, M. (1973) *Normed Linear Spaces* (Third edition), Springer-Verlag.

Deutsch, D., Ekert, A. and Lupacchini, R. (1999) Machines, Logic and Quantum Physics. Available at arXiv:math.HO/9911150 v1.

Haghverdi, E. (2000) *A categorical approach to linear logic, geometry of proofs and full completeness*, Ph.D. Thesis, University of Ottawa.

Haghverdi, E. and Scott, P. (2006) A categorical model for the Geometry of Interaction. *Theoretical Computer Science* **350** (2) 252–274.

Gelfand, I. (1938) Abstrakte Funktionen und lineare Operatoren. *Recueil Math'ematique [Mat. Sbornik] N.S.* **4** (46) volume 2 235–286.

Golan, J. (1999) *Power Algebras Over Semirings: With Applications in Mathematics and Computer Science*, Mathematics and Its Applications **488**, Springer-Verlag.

Hille, E. (1982) *Analytic Function Theory, Volume 1* (Second edition), AMS Chelsea publishing.

Hines, P. (2008a) Machine Semantics. *Theoretical Computer Science* **409** (1) 1–23.

Hines, P. (2008b) Machine Semantics: from Causality to Computational Models. *International Journal of Unconventional Computation* **4** (3) 249–272.

Hines, P. (2010) Quantum Circuit Oracles for Abstract Machine Computations. *Theoretical Computer Science* **411** 1501–1520.

Hines, P. and Scott, P. (2012) Categorical Traces from Single-Photon Linear Optics. In: Abramsky, S. and Mislove, M. (eds.) Mathematical Foundations of Information Flow. *AMS Proceedings of Symposia in Applied Mathematics* **71** 89–124.

Hobson, E. W. (1957) *The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Volume 1* (Second edition), Dover Publications.

Kadets, V. and Kadets, M. (1991) *Rearrangements of Series in Banach Spaces*, American Mathematical Society.

Kelly, G. M. (1982) *Basic Concepts of Enriched Category Theory*, LMS Lecture notes **64**, Cambridge University Press. (Reprinted in Kelly (2005)).

Kelly, G. M. (2005) *Basic Concepts of Enriched Category Theory*, Reprints in Theory and Applications of Categories **10**.

McArthur, C. (1961) A note on Subseries Convergence. *Proceedings of the American Mathematical Society* **12** (4) 540–545.

Manes, E. and Arbib, M. (1986) *Algebraic Approaches to Program Semantics*, Springer-Verlag.

Manes, E. and Benson, D. (1985) The inverse Semigroup of a Sum-Ordered Semiring. *Semigroup Forum* **31** 129–152.

Lahiri, B. K. and Das, P. (2002) Subseries in Banach spaces. *Mathematica Slovaca* **52** (3) 361–368.

Laplaza, M. L. (1977) Coherence in Nonmonoidal Closed Categories. *Transactions of the American Mathematical Society* **230** 293–311.

Nielsen, M. and Chuang, I. (2000) *Quantum Computation and Quantum Information*, Cambridge University Press.

Orlicz, W. (1933) Über unbedingte Konvergenz in Functionenraumen I. *Studia Mathematica* **4** 33–37.

Swartz, C. (1992) Iterated series and the Hellinger-Toeplitz theorem. *Publicacions Matemàtiques* **36** 167–173.

Selinger, P. (2004) Towards a quantum programming language. *Mathematical Structures in Computer Science* **14** (4) 527–586.

Shor, P. (1999) Polynomial time algorithms for prime factorisation and discrete logarithms on a quantum computer. *SIAM review* **41** 303–332.

Steinberger, M. (1993) *Algebra*, Prindle, Weber and Schmidt. (Updated version (2006) available online at `http://math.albany.edu/~mark/algebra.pdf`.)

Titchmarsh, E. C. (1983) *The Theory of Functions* (Second edition), Oxford University Press.

Thorpe, B. (1968) On the equivalence of certain notions of bounded variation. *Journal of the London Mathematical Society* **43** 247–252.