Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2011.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

95.E (Joshua Lam – student at The Leys School, Cambridge)

Given any positive integer k, construct the following number: $S = 0\cdot[2k]\,[3k]\,[5k]\,[7k]\ldots$ where $[m]$ denotes the decimal representation of m and, each time, the product is taken between the next prime number and k. (For example, when $k = 2$, $S = 0{\cdot}461014\ldots$ by concatenating the sequence 2×2, 3×2, 5×2, $7 \times 2, \ldots$.)

For which k is S rational?

95.F (Ovidiu Furdui)

For positive integers m, n, evaluate the double integral

$$\int_0^\infty \int_0^\infty \frac{\left(e^{-mx} - e^{-my}\right)\left(e^{-nx} - e^{-ny}\right)}{(x - y)^2}\, dx\, dy.$$

95.G (Panagiote Ligouras)

Given a scalene triangle ABC with circumradius R and inradius r, let m_a, m_b, m_c denote the lengths of its medians, h_a, h_b, h_c its altitudes and l_a, l_b, l_c the lengths of its angle bisectors. Prove that

$$\frac{l_a^3\left(m_a^2 - h_a^2\right)}{h_a\left(l_a^2 - h_a^2\right)} + \frac{l_b^3\left(m_b^2 - h_b^2\right)}{h_b\left(l_b^2 - h_b^2\right)} + \frac{l_c^3\left(m_c^2 - h_c^2\right)}{h_c\left(l_c^2 - h_c^2\right)} \geqslant 4R(R + 4r).$$

95.H (Alan Wilson)

Prove that the locus of $x^3 + y^3 + z^3 - 3xyz = 1$ in Euclidean 3-space has the form of a surface of revolution. Show also that the sphere $x^2 + y^2 + z^2 = R^2\ (R > 1)$ intersects the locus $x^3 + y^3 + z^3 - 3xyz = 1$ in precisely two distinct circles of points and determine their radii in terms of R.

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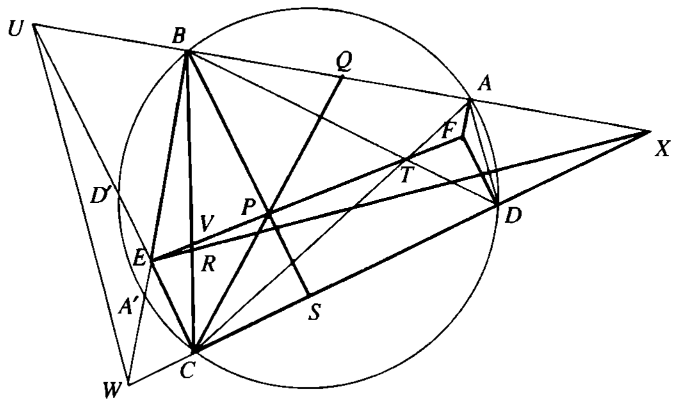
Solutions and comments on **94.I**, **94.J**, **94.K**, **94.L** (November 2010).

94.I (Michael Fox)

The sides AB and CD of a cyclic quadrilateral $ABCD$ meet at X. Points E and F satisfy $\angle EBA = \angle ECD = 90° = \angle FAB = \angle FDC$. P is an arbitrary point on EF; lines AB and CP intersect at Q, BC and EX intersect at R and CD and BP intersect at S. Prove that $\frac{AQ}{QB} \times \frac{BR}{RC} \times \frac{CS}{SD} = 1$.

Although a range of methods was employed by solvers, they tended to follow the same strategy: first prove that $\frac{AQ}{QB} \times \frac{CS}{SD}$ is constant and then establish what the constant is. Michel Bataille and Michael Fox, the proposer whose solution we give below, produced similar synthetic proofs: Michel used inversion (with pole X and sending A to B) for Step 2.



1. The mediators (perpendicular bisectors) of AB and CD meet at the centre O of the circle. But each mediator passes through the midpoint of EF, which therefore is O.
2. Let $T = AC \cap BD$ as shown in the Figure. Rotate ΔAFD through 180° about O; then F maps onto E, and the images of A and D are points A' and D' on the circle. Since $AF \parallel BE$, the line $A'EB$ is straight. Similarly $D'EC$ is straight. The hexagon $AA'BDD'C$ is inscribed in the circle, so by Pascal's theorem the intersections of the pairs of opposite sides, $AA' \cap DD'$, $A'B \cap D'C$ and $BD \cap CA$, are collinear. But these are O, E and T; and since $O \in EF$ it follows that $T \in EF$.
3. Let $U = AB \cap CE$, $V = BC \cap EF$ and $W = CD \cap BE$. With centre C project the range $(UBQA)$ onto EF, i.e. $C(UBQA) = (EVPT)$. Similarly $B(EVPT) = (WCSD)$. Hence $(UBQA)$ and $(WCSD)$ are projective and have the same cross-ratio. Thus $\frac{\frac{AQ}{QB}}{\frac{AU}{UB}} = \frac{\frac{DS}{SC}}{\frac{DW}{WC}}$. So $\frac{AQ}{QB} \times \frac{CS}{SD} = \frac{AU}{UB} \times \frac{CW}{WD}$, a constant.

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4. Since $\angle UBW = 90° = \angle UCW$, $UBCW$ is cyclic, as is $ABCD$. Therefore $\angle BUW = \angle BCD = \angle XAD$, whence $AD // UW$. Consequently $AU / DW = XA / XD$. Also $XA \times XB = XD \times XC$, so $XA/XD = XC/XB$, thus $AU / DW = XC / XB$.
5. The triangles CWE, BUE are similar, hence $CW/BU = CE/BE$. Putting all this together we have $\frac{AU}{UB} \times \frac{CW}{WD} = \frac{AU}{WD} \times \frac{CW}{UB} = \frac{XC}{BX} \times \frac{CE}{EB} = \frac{\Delta XCE}{\Delta XBE}$. But these triangles have a common base EX and their heights are in the ratio RC / BR, so $\frac{AU}{UB} \times \frac{CW}{WD} = \frac{RC}{BR}$, hence $\frac{AQ}{QB} \times \frac{CS}{SD} = \frac{RC}{BR}$, and finally $\frac{AQ}{QB} \times \frac{BR}{RC} \times \frac{CS}{SD} = 1$.

Michel Bataille noted that the proof remains valid, after minor adjustments, if $BC // AD$ or $AC // BD$.

Correct solutions were received from: M. Bataille, S. Dolan, GCHQ Problem Solving Group, M. A. Hennings, G. Howlett, G. B. Trustrum and the proposer M. Fox.

94.J (Stan Dolan)

Some positive integers can be expressed as the sum and product of the same three positive rational numbers. For example,

$$6 = 1 + 2 + 3 = 1 \times 2 \times 3, \qquad 7 = \tfrac{7}{6} + \tfrac{4}{3} + \tfrac{9}{2} = \tfrac{7}{6} \times \tfrac{4}{3} \times \tfrac{9}{2},$$
$$9 = \tfrac{1}{2} + 4 + \tfrac{9}{2} = \tfrac{1}{2} \times 4 \times \tfrac{9}{2}, \qquad 15 = \tfrac{1}{2} + \tfrac{5}{2} + 12 = \tfrac{1}{2} \times \tfrac{5}{2} \times 12.$$

(a) Show that 6 is the smallest positive integer with such a decomposition and find a decomposition for it other than 1 + 2 + 3.

(b) Does 8 have such a decomposition?

(a) Suppose that $n = a + b + c = abc$ with $a, b, c > 0$. Then, by the AM-GM inequality, $n = abc \leqslant \left(\frac{a+b+c}{3}\right)^3 = \frac{n^3}{27}$ so that $n \geqslant \sqrt{27} > 5$.

To generate other decompositions for $n = 6$, Mark Hennings recast the search for rational solutions as follows.

Let $a = \frac{1}{\alpha} > \frac{1}{6}$. Then $b + c = 6 - \frac{1}{\alpha}$, $bc = 6\alpha$ so that $b, c = \frac{1}{2}\left(6 - \frac{1}{\alpha} \pm \frac{1}{\alpha}\sqrt{1 - 12\alpha + 36\alpha^2 - 24\alpha^3}\right) = \frac{1}{2}\left(6 - \frac{1}{\alpha} \pm \frac{\beta}{\alpha}\right)$ where (α, β) is a rational point on the elliptic curve

$$y^2 = 1 - 12x + 36x^2 - 24x^3, \qquad (*)$$

The standard composition rule for rational points on such a curve is $(\alpha_1, \beta_1) * (\alpha_2, \beta_2) = (\alpha_3, \beta_3)$ where

$$\alpha_3 = \begin{cases} \dfrac{3}{2} - \alpha_1 - \alpha_2 - \dfrac{1}{24}\left(\dfrac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2}\right)^2, & \alpha_1 \neq \alpha_2, \\[2ex] \dfrac{3}{2} - 2\alpha_1 - \dfrac{3}{2}\left(\dfrac{6\alpha_1^2 - 6\alpha_1 + 1}{\beta_1}\right)^2, & \alpha_1 = \alpha_2. \end{cases}$$

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The rational points $(1,1)$, $(\frac{1}{2},1)$, $(\frac{1}{3},\frac{1}{3})$ are all on (*) and each yields the $6 = 1 + 2 + 3 = 1 \times 2 \times 3$ decomposition of 6. Then $(1,1) * (\frac{1}{3},\frac{1}{3}) = (\frac{1}{8},\frac{1}{8})$ and $(\frac{1}{2},1) * (\frac{1}{8},\frac{1}{8}) = (\frac{35}{54},\frac{109}{81})$ which yields $6 = \frac{54}{35} + \frac{25}{21} + \frac{49}{15} = \frac{54}{35} \times \frac{25}{21} \times \frac{49}{15}$. Other solutions include

$$6 = \frac{15123}{16159} + \frac{25538}{10153} + \frac{20449}{8023} = \frac{15123}{16159} \times \frac{25538}{10153} \times \frac{20449}{8023},$$

found by Graham Howlett and James Mundie.

(b) 8 does not have such a decomposition.

The GCHQ Problem Solving Group found the following neat, direct argument to show this impossibilty.

Suppose $8 = a + b + c = abc$. We assume that each of these rationals is written in simplest form and with non-negative numerator. Since their product is even, at least one of a, b, c must have even numerator; without loss of generality, we can assume that c has this property. Now

$$8 = bc(8 - b - c) \Rightarrow b^2c + b\left(c^2 - 8c\right) + 8 = 0.$$

For b to be rational, the discriminant of this quadratic must be a (rational) square. Writing c as d / e, we have, for some rational r,

$$r^2 = \left(c^2 - 8c\right)^2 - 32c = c^4 - 16c^3 + 64c^2 - 32c$$

$$= \frac{d^4}{e^4} - 16\frac{d^3}{c^3} + 64\frac{d^2}{e^2} - 32\frac{d}{e} \Rightarrow r^2e^4 = d\left(d^3 - 16d^2e + 64de^2 - 32e^3\right).$$

The latter is clearly an integer square. Let p be a prime dividing $\text{hcf}\left(d, d^3 - 16d^2e + 64de^2 - 32e^3\right)$. Then $p \mid 32e^3$ and since $p \mid d$, $p \nmid e$. Hence $p \mid 32$ and therefore $p = 2$ (we know this case is valid because d is even). Hence d must be one of two forms: $d = 2f^2$ or $d = 4g^2$.

If $d = 2f^2$ we have that, for some integer q_1,

$$8f^6 - 64f^4e + 128f^2e^2 - 32e^3 = q_1^2.$$

Thus q_1 is even, so writing $q_1 = 2q_2$, we have

$$2f^6 - 16f^4e + 32f^2e^2 - 8e^3 = q_2^2.$$

Thus q_2 is even and q_2^2 is divisible by 4. Therefore, f is even. Writing $q_2 = 2q_3$ and $f = 2h$, we have

$$128h^6 - 256h^4e + 128h^2e^2 - 8e^3 = 4q_3^2 \Rightarrow 32h^6 - 64h^4e + 32h^2e^2 - 2e^3 = q_3^2.$$

Thus q_3 is even and q_3^2 is divisible by 4. Thus $2e^3$ is divisible by 4, contradicting the fact that e must be odd.

If $d = 4g^2$ we have that, for some integer q_4,

$$64g^6 - 256g^4e + 256g^2e^2 - 32e^3 = q_4^2.$$

Thus q_4^2 is divisible by 32 and q_4 must be divisible by 8. Thus $32e^3$ is divisible by 64, contradicting the fact that e must be odd.

Thus neither $d = 2f^2$ nor $d = 4g^2$ is possible and so 8 cannot be expressed in the form claimed.

Mark Hennings observed that, from $a + b + c = abc = 8$, we have $(a + b + c)^3 = 64abc$. Since this equation is homogeneous, we can find positive integers x, y, z with no common factor such that $(x + y + z)^3 = 64xyz$. From this equation, it follows that x, y, z are pairwise coprime and hence that $x = p^3$, $y = q^3$, $z = r^3$ with p, q, r pairwise coprime positive integers and $p^3 + q^3 + r^3 = 4pqr$. The non-existence of solutions to this Diophantine equation then follows from results reported in [1].

James Mundie traced the interest in such equations back to the work of Sylvester, [2]. Graham Howlett speculated that an infinite number of positive integers have a decomposition of the type sought.

References

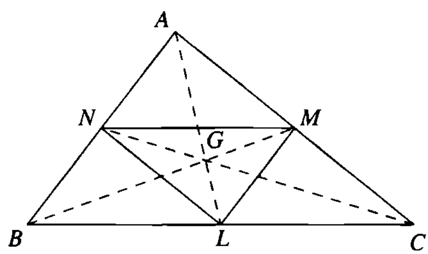
1. E. Dofs, Solutions of $x^3 + y^3 + z^3 = nxyz$, *Acta Arithmetica*, **73.3** (1995).
2. T. Lavrinenko, Solving an indeterminate third degree equation in rational numbers, *Revue d'Histoire des Mathématiques* (2002).

Correct solutions were received from: R. P. C. Forman (part(a)), GCHQ Problem Solving Group, M. A. Hennings, G. Howlett (part (a)), S. N. Maitra (part (a)), J. A. Mundie (part (a)) and the proposer S. Dolan.

94.K (Isaac Sofair)

P is a random point within a plane triangle of sides a, b, c, circumcentre O and circumradius R. Prove that the mean value of OP^2 is $R^2 - \frac{1}{12}(a^2 + b^2 + c^2)$.

Most solvers tackled this popular problem using double integration and encountered a variety of trigonometrical identities *en route*, depending on their choice of axes. Others subsumed the integration into known results on moments of inertia: of these, the solution which follows, based on Neil Curwen's, was particularly neat.



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Consider the triangle ABC as a lamina of unit mass. Then the mean value of OP^2 is the same as I_0, the moment of inertia of the lamina about the axis through O perpendicular to the plane containing the triangle. By the parallel axis theorem $I_0 = I_G + OG^2$ (since triangle ABC has unit mass). In the Figure, L, M, N are the midpoints of the sides of triangle ABC. The four smaller triangles are similar to triangle ABC hence, by parallel axes, adding their moments of inertia about G,

$$I_G = \tfrac{1}{16}I_G + \left[\tfrac{1}{16}I_G + \tfrac{1}{4}\left(\tfrac{1}{3}AL\right)^2\right] + \left[\tfrac{1}{16}I_G + \tfrac{1}{4}\left(\tfrac{1}{3}BM\right)^2\right] + \left[\tfrac{1}{16}I_G + \tfrac{1}{4}\left(\tfrac{1}{3}CN\right)^2\right]$$

so that

$$I_G = \tfrac{1}{27}\left(AL^2 + BM^2 + CN^2\right) = \tfrac{1}{36}\left(a^2 + b^2 + c^2\right)$$

since, from Apollonius' theorem, $AL^2 + BM^2 + CN^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right)$. Also $OG^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$. (A quick proof uses vectors with origin O and $\overrightarrow{OA} = \mathbf{a}$, etc. with $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Then $OG^2 = \frac{1}{9}\left(3R^2 + 2\sum \mathbf{a}.\mathbf{b}\right)$ and $\sum a^2 = \sum(\mathbf{b} - \mathbf{c})^2 = 6R^2 - 2\sum \mathbf{a}.\mathbf{b}$ from which the result follows on eliminating $\sum \mathbf{a}.\mathbf{b}$.) Finally then,

$$I_O = I_G + OG^2 = \tfrac{1}{36}\left(a^2 + b^2 + c^2\right) + R^2 - \tfrac{1}{9}\left(a^2 + b^2 + c^2\right)$$

$$= R^2 - \tfrac{1}{12}\left(a^2 + b^2 + c^2\right), \text{ as required.}$$

Correct solutions were received from: N. Curwen, S. Dolan, GCHQ Problem Solving Group, J. P. Green, M. A. Hennings, G. Howlett, S. N. Maitra, D. A. Quadling, N. Routledge, G. B. Trustrum and the proposer I. Sofair.

94.L (H. A. Shah Ali)

Let π be a permutation of $\{0, 1, \ldots, n-1\}$. For $k = 0, 1, \ldots, n-1$ the *k-circulant* of π is the permutation formed by shifting the entries of π by k positions is a fixed direction.

(i) If each of the k-circulants of π for $k = 0, 1, \ldots, n-1$ has the same number of fixed points, λ, prove that $\lambda = 1$.

(ii) Prove that the case $\lambda = 1$ holds only when n is odd and that the total number of such permutations π is $\frac{1}{2}\phi(n)$, where ϕ is Euler's totient function.

First, an apology. As many readers quickly noticed, the expression $\frac{1}{2}\phi(n)$ in part (ii) is not an exact count, but a lower bound for the number of equivalence classes under 'circulance' of permutations of the required type. As ever, though, respondents were undaunted by this and produced some impressively detailed analyses. I fear that I will not be able to do justice to them in the following short summary, but I am especially grateful to the submissions from the GCHQ Problem Solving Group, Mark Hennings and Norman Routledge: Norman's was notable for the amount he was able to deduce by hand calculations.

(i) Each $0 \leqslant i \leqslant n-1$ is a fixed point for precisely one of the k-circulants of π. Counting all fixed points of all circulants gives $n\lambda = n$ and $\lambda = 1$.

(ii) If $\lambda = 1$, $\pi - id$ must be a permutation (where id denotes the identity and, here and below, working is done mod n). Then

$$0 = \sum_{i=0}^{n-1} \pi(i) - i \equiv \sum_{i=0}^{n-1} i \equiv \frac{1}{2}n(n-1)$$

so that $2 \mid n - 1$ and n is odd.

When $n = 3$, there are 3 such permutations $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$ corresponding to the 3 circulants of $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$. For general odd n, the permutations sought split into $E(n)$ equivalence classes of size n under circulance where a representative of each class may be chosen with $\pi(0) = 0$: for brevity we will call these π 'suitable'. Thus, for $n = 5$, $E(3) = 3$ with the classes represented by the suitable permutations $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 4 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}$. In general, $E(n)$ is odd because the mapping $\pi \to id - \pi$ is an involution on the set of suitable permutations with the single fixed point given by the permutation $\pi(i) = \frac{1}{2}(n+1)i$.

A lower bound for $E(n)$ better than that implied by part (ii) comes from the observation that, if k and $k - 1$ are both coprime to n, then $\pi(i) = ki$ is a suitable permutation. For odd $n \geqslant 5$, $E(n) \geqslant 3$ since $2, \frac{1}{2}(n+1), n-1$ are different possible choices for k. For $n = p^r$ (p an odd prime), there are exactly $p^{r-1}(p-2)$ choices for k so that $E(p^r) \geqslant p^{r-1}(p-2)$. But, for $n \geqslant 7$, this construction does not provide all of the suitable permutations. Mark Hennings observed that the bulk of the permutations sought in the problem consist of $(n-1)$-cycles: counting these would give a route to a better lower bound.

Finally, direct counting by computer gave the following table of values of $E(n)$:

n	3	5	7	9	11	13	15
$E(n)$	1	3	19	225	3441	79259	2424195.

And this is where we hit the buffers! The $E(n)$-sequence is number A003111 in *The On-Line Encyclopedia of Integer Sequences*. Our 'suitable permutations' are there referred to as 'complete mappings of $\mathbb{Z}_n$' and the entry cites a number of recent papers in this area, but no definitive formula for $E(n)$ and only relatively wide asymptotic bounds. There is clearly still plenty of work to be done!

Correct solutions were received from: S. Dolan, GCHQ Problem Solving Group, M. A. Hennings, D. A. Quadling, N. Routledge and the proposer H. A. Shah Ali.

N.J.L.