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Infinitely many arbitrarily small positive solutions for the Dirichlet problem involving the p-Laplacian

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In this paper we present a result of existence of infinitely many arbitrarily small positive solutions to the following Dirichlet problem involving the p-Laplacian,

$$\begin{aligned} -\Delta_p u &= \lambda f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \in \mathbb{R}^N$ is a bounded open set with sufficiently smooth boundary $\partial\Omega$, $p > 1$, $\lambda > 0$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following condition: there exists $\bar{t} > 0$ such that

$$\sup_{t\in[0,\bar{t}\,]} f(\cdot,t) \in L^\infty(\Omega).$$

Precisely, our result ensures the existence of a sequence of a.e. positive weak solutions to the above problem, converging to zero in $L^\infty(\Omega)$.

1. Introduction

In this paper, we consider the problem

$$\left.\begin{aligned} -\Delta_p u &= \lambda f(x,u) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{P_λ}$$

where $\Omega \in \mathbb{R}^N$ is a bounded open set with sufficiently smooth boundary $\partial\Omega$, $p > 1$, Δ_p is the p-Laplacian operator, that is, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $\bar{t} > 0$ such that

$$\sup_{t\in[0,\bar{t}\,]} f(\cdot,t) \in L^\infty(\Omega). \tag{1.1}$$

Precisely, we are interested in the existence of a sequence of a.e. positive weak solutions of (P_λ) converging to zero in $L^\infty(\Omega)$.

A weak solution (P_λ) is any $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}x - \lambda \int_\Omega f(x,u(x))v(x)\,\mathrm{d}x = 0$$

for each $v \in W_0^{1,p}(\Omega)$. Then, for some $\sigma > 0$, our solutions belong to $C^{1+\sigma}(\bar{\Omega})$, as can be proved by standard regularity arguments.

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If u is a weak solution of (P_λ), we say that u is a.e. positive if $m(\{x \in \Omega : u(x) \leqslant 0\}) = 0$, where $m(\cdot)$ is the Lebesgue measure.

The existence of infinitely many solutions for problem (P_λ) has been widely investigated. The most classical results on this topic are essentially based on Ljusternik–Schnirelman theory. In them, the key role is played by the oddness of the nonlinearity. Moreover, in order to check the Palais–Smale condition (or some of its variants), one assumes certain conditions that do not allow an oscillating behaviour of the nonlinearity. We refer, for instance, to [1, 2] as a typical paper of this kind, among the most recent ones.

Multiplicity results for problem (P_λ), when $f(x, \cdot)$ has an oscillating behaviour, are certainly more rare. In this connection, we refer to [6] and [10], where the authors obtain the existence of an unbounded sequence of weak solutions for problem (P_λ) (see also [4, 5]). The existence of infinitely many small solutions to (P_λ) has been studied in [7] by Omari and Zanolin, who proved that if

$$\liminf_{t\to 0^+} \frac{F(t)}{t^p} = 0 \quad \text{and} \quad \limsup_{t\to 0^+} \frac{F(t)}{t^p} = +\infty, \tag{1.2}$$

where $F(t) = \int_0^t f(s)\,\mathrm{d}s$, then, for every $\lambda > 0$, problem (P_λ) has a sequence $\{u_n\}$ of non-zero non-negative weak solutions, satisfying $\max_{\bar{\Omega}} u_n \to 0$ as $n \to +\infty$.

Our main theorem is theorem 2.1, in § 2. Here, we state a particular version of it, when f is independent of x.

THEOREM 1.1. *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions.*

(i′) $\limsup_{t\to 0^+} \dfrac{F(t)}{t^p} = +\infty$, $\liminf_{t\to 0^+} \dfrac{F(t)}{t^p} > -\infty$.

(ii″) *For every $n \in \mathbb{N}$, there exist $\xi_n, \xi_n' \in \mathbb{R}$, with $0 \leqslant \xi_n < \xi_n'$ and $\lim_{n\to+\infty} \xi_n' = 0$, such that*

$$\int_0^{\xi_n} f(s)\,\mathrm{d}s = \sup_{t\in[0,\xi_n']} \int_0^t f(s)\,\mathrm{d}s.$$

Then, for every $\lambda > 0$, problem (P_λ) admits a sequence $\{u_n\}$ of a.e. positive weak solutions strongly convergent to zero and such that

$$\lim_{n\to+\infty} \max_{\bar{\Omega}} u_n = 0.$$

Clearly, theorem 1.1 is a remarkable improvement of the above-mentioned result by Omari and Zanolin. Indeed, not only is the condition $\liminf_{t\to 0^+}(F(t)/t^p) = 0$ replaced by $\liminf_{t\to 0^+}(F(t)/t^p) > -\infty$, but the solutions we find are also a.e. positive rather than simply non-zero and non-negative.

The proof of theorem 2.1 is based on the general approach proposed in [8]. More precisely, we find weak solutions for (P_λ) that are local minima of the underlying energy functional. The technique used to obtain such local minima has been suggested us by the papers of Saint Raymond [10] and Ricceri [9].

This paper is divided into three sections, including this introduction. In the second section we state and prove our result, then the third section is dedicated to the careful study of the conditions of theorem 2.1 with respect to (1.2).

2. Main result

THEOREM 2.1. *Suppose that the function f satisfies the following conditions.*

(i) *For every $n \in \mathbb{N}$, there exist $\xi_n, \xi'_n \in \mathbb{R}$, with $0 \leqslant \xi_n < \xi'_n$ and $\lim_{n\to+\infty} \xi'_n = 0$, such that, for a.e. $x \in \Omega$,*

$$\int_0^{\xi_n} f(x,s)\,\mathrm{d}s = \sup_{t\in[\xi_n,\xi'_n]} \int_0^t f(x,s)\,\mathrm{d}s.$$

(ii) *There exists a non-empty open set $D \subseteq \Omega$, a constant $M \geqslant 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+ \setminus \{0\}$, with $\lim_{n\to+\infty} t_n = 0$, such that*

$$\lim_{n\to+\infty} \frac{\operatorname{ess\,inf}_{x\in D} \int_0^{t_n} f(x,s)\,\mathrm{d}s}{t_n^p} = +\infty$$

and

$$\operatorname*{ess\,inf}_{x\in D}\left(\inf_{t\in[0,t_n]} \int_0^t f(x,s)\,\mathrm{d}s\right) \geqslant -M \operatorname*{ess\,inf}_{x\in D}\left(\int_0^{t_n} f(x,s)\,\mathrm{d}s\right).$$

Then, for every $\lambda > 0$, problem (P_λ) admits a sequence $\{u_n\}$ of a.e. positive weak solutions strongly convergent to zero and such that $\lim_{n\to+\infty} \max_{\bar{\Omega}} u_n = 0$.

Proof. We choose $q \in \,]p-1, ((p-1)N+p)/(N-p)[$ if $p < N$; in other cases it is enough to choose $q > p-1$. From (1.1), it follows that there exist $a > 0$ and $\bar{t} > 0$ such that, for every $0 \leqslant t \leqslant \bar{t}$ and a.e. $x \in \Omega$, one has

$$|f(x,t)| \leqslant a.$$

Moreover, conditions (i) and (ii) imply that, for a.e. $x \in \Omega$,

$$f(x,0) = 0.$$

Without loss of generality, we suppose that, for every $n \in \mathbb{N}$, $\max\{\xi'_n, t_n\} \leqslant \bar{t}$. Let $\lambda > 0$, then we define $g : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x,t) = \begin{cases} f(x,\bar{t}) & \text{if } t > \bar{t}, \\ f(x,t) & \text{if } 0 \leqslant t \leqslant \bar{t}, \\ 0 & \text{if } t < 0. \end{cases}$$

Whence, for a.e. $x \in \Omega$ and $t \in \mathbb{R}$, it turns out that

$$|g(x,t)| \leqslant a. \tag{2.1}$$

Now we consider the following problem:

$$\left.\begin{aligned} -\Delta_p u &= \lambda g(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{$P_{\lambda,g}$}$$

The weak solutions of ($P_{\lambda,g}$) are the critical points of the functional

$$\Phi_\mu(u) = \mu \int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega \left(\int_0^{u(x)} g(x,t)\,\mathrm{d}t\right)\mathrm{d}x, \qquad (u \in W_0^{1,p}(\Omega))$$

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where $\mu = 1/p\lambda$. Owing to (2.1) and the compact embedding of $W_0^{1,p}(\Omega)$ into $L^{q+1}(\Omega)$ (respectively, into $C^0(\bar{\Omega})$, if $p > N$), Φ_μ is well defined, weakly sequentially lower semicontinuous and Gâteaux differentiable in $W_0^{1,p}(\Omega)$.

Fixing $n \in \mathbb{N}$, we set

$$E_n = \{u \in W_0^{1,p}(\Omega) : 0 \leqslant u(x) \leqslant \xi_n' \text{ a.e. in } \Omega\}.$$

Since this set is closed and convex, it is weakly closed. For each $u \in E_n$, one has

$$\Phi_\mu(u) \geqslant -am(\Omega)\xi_n'.$$

Whence Φ_μ is lower bounded in E_n, so we set $\alpha_n = \inf_{E_n} \Phi_\mu$. For every $k \in \mathbb{N}$, there exists $v_k \in E_n$ such that

$$\alpha_n \leqslant \Phi_\mu(v_k) < \alpha_n + \frac{1}{k},$$

then it follows that

$$\begin{aligned} \int_\Omega |\nabla v_k|^p \,\mathrm{d}x &= \frac{1}{\mu}\biggl(\int_\Omega \biggl(\int_0^{v_k(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x + \Phi_\mu(v_k)\biggr) \\ &\leqslant \frac{1}{\mu}\biggl(\int_\Omega \biggl(\int_0^{v_k(x)} a\,\mathrm{d}t\biggr)\,\mathrm{d}x + \alpha_n + \frac{1}{k}\biggr) \\ &\leqslant \frac{1}{\mu}(am(\Omega)\xi_n' + \alpha_n + 1). \end{aligned}$$

Then $\{v_k\}$ is norm bounded in $W_0^{1,p}(\Omega)$. This implies that there exists a subsequence $\{v_{k_m}\}$, weakly convergent to $u_n \in E_n$, being E_n weakly closed. At this point, we exploit the weak sequentially lower semicontinuity of Φ_μ and obtain that $\Phi_\mu(u_n) = \alpha_n$.

We prove that $u_n(x) \in \,]0, \xi_n]$ a.e. in Ω.

Set

$$h(t) = \begin{cases} \xi_n & \text{if } t > \xi_n, \\ t & \text{if } 0 < t \leqslant \xi_n, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Then we define $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ as follows:

$$Tu(x) = h(u(x)) \quad \text{for every } u \in W_0^{1,p}(\Omega) \text{ and } x \in \Omega.$$

The operator T is continuous in $W_0^{1,p}(\Omega)$ (see [3]). Moreover, for every $u \in W_0^{1,p}(\Omega)$, $Tu \in E_n$.

We put $v^* = Tu_n$ and $X = \{x \in \Omega : u_n(x) \notin \,]0, \xi_n]\}$. Then, for a.e. $x \in X$, one has

$$\xi_n < u_n(x) \leqslant \xi_n' \quad \text{or} \quad u_n(x) = 0.$$

However,

$$\int_0^{u_n(x)} g(x,t)\,\mathrm{d}t \leqslant \int_0^{v^*(x)} g(x,t)\,\mathrm{d}t$$

and $|\nabla v^*| = 0$.

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Whence

$$\begin{aligned}\Phi_\mu(v^*) - \Phi_\mu(u_n) &= \mu \int_\Omega (|\nabla v^*|^p - |\nabla u_n|^p)\,\mathrm{d}x - \int_\Omega \biggl(\int_{u_n(x)}^{v^*(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x \\ &= -\mu \int_X |\nabla u_n|^p\,\mathrm{d}x - \int_X \biggl(\int_{u_n(x)}^{v^*(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x \\ &\leqslant -\mu \int_X |\nabla u_n|^p\,\mathrm{d}x.\end{aligned}$$

Since $v^* \in E_n$, it follows that $\Phi_\mu(v^*) - \Phi_\mu(u_n) \geqslant 0$. Then

$$\int_X |\nabla u_n|\,\mathrm{d}x = 0.$$

Whence

$$\|v^* - u_n\|^p = \int_\Omega |\nabla v^* - \nabla u_n|^p\,\mathrm{d}x = \int_X |\nabla u_n|^p\,\mathrm{d}x = 0,$$

which means that $u_n(x) = v^*(x) \in\,]0, \xi_n]$ a.e. in Ω.

Let $u \in W_0^{1,p}(\Omega)$, T be the operator defined above and let

$$X = \{x \in \Omega : u(x) \notin\,]0, \xi_n]\}.$$

We have that if $x \in \Omega \setminus X$, then

$$\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t = 0.$$

Furthermore, if $x \in X$, then one has the following cases.

(a) If $u(x) \leqslant 0$, then

$$\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t = \int_0^{u(x)} g(x,t)\,\mathrm{d}t = 0.$$

(b) If $\xi_n < u(x) \leqslant \xi_n'$, then

$$\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t \leqslant 0.$$

(c) If $u(x) > \xi_n'$, then

$$\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t = \int_{\xi_n}^{u(x)} g(x,t)\,\mathrm{d}t \leqslant \int_{\xi_n}^{u(x)} a\,\mathrm{d}t = a(u(x) - \xi_n).$$

Since the constant

$$C = \sup_{\xi \geqslant \xi_n'} \frac{a(\xi - \xi_n)}{(\xi - \xi_n)^{q+1}}$$

is finite, we have, for a.e. $x \in \Omega$,

$$\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t \leqslant C|u(x) - Tu(x)|^{q+1},$$

then

$$\int_{\Omega}\biggl(\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x \leqslant C\gamma^{q+1}\|u-Tu\|^{q+1},$$

where

$$\gamma = \sup_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{(\int_{\Omega}|u|^{q+1}\,\mathrm{d}x)^{1/(q+1)}}{\|u\|}.$$

Whence, one has

$$\begin{aligned}
\Phi_{\mu}(u)-\Phi_{\mu}(Tu) &= \mu\int_{\Omega}(|\nabla u|^p-|\nabla(Tu)|^p)\,\mathrm{d}x - \int_{\Omega}\biggl(\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x\\
&= \mu\int_{X}|\nabla u|^p\,\mathrm{d}x - \int_{\Omega}\biggl(\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x\\
&= \mu\int_{\Omega}|\nabla u-\nabla(Tu)|^p\,\mathrm{d}x - \int_{\Omega}\biggl(\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x\\
&\geqslant \mu\|u-Tu\|^p - C\gamma^{q+1}\|u-Tu\|^{q+1}.
\end{aligned}$$

Since $Tu\in E_n$, it follows that $\Phi_{\mu}(Tu)\geqslant\Phi_{\mu}(u_n)$. Then we have

$$\Phi_{\mu}(u)\geqslant\Phi_{\mu}(u_n)+\|u-Tu\|^p(\mu-C\gamma^{q+1}\|u-Tu\|^{q+1-p}).$$

Since T is continuous and $q+1-p>0$, there exists $\beta>0$ such that, for every $u\in W_0^{1,p}(\Omega)$ with

$$\|u-u_n\|<\beta, \qquad \|u-Tu\|^{q+1-p}\leqslant\frac{\mu}{2C\gamma^{q+1}}.$$

Then, if $\|u-u_n\|<\beta$, one has

$$\Phi_{\mu}(u)\geqslant\Phi_{\mu}(u_n)+\tfrac{1}{2}\mu\|u-Tu\|^p\geqslant\Phi_{\mu}(u_n),$$

that is, u_n is a local minimum of Φ_{μ}.

For every $n\in\mathbb{N}$ and $u\in E_n$, we have that

$$\Phi_{\mu}(u)\geqslant -am(\Omega)\xi_n'.$$

Then, since $-am(\Omega)\xi_n'\leqslant\alpha_n\leqslant 0$, it follows that

$$\lim_{n\to+\infty}\alpha_n=0.$$

From $u_n\in E_n$ and $\alpha_n=\Phi_{\mu}(u_n)$, it follows that

$$\begin{aligned}
\int_{\Omega}|\nabla u_n|^p\,\mathrm{d}x &= \frac{1}{\mu}\biggl(\int_{\Omega}\biggl(\int_0^{u_n(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x+\Phi_{\mu}(u_n)\biggr)\\
&= \frac{1}{\mu}\biggl(\int_{\Omega}\biggl(\int_0^{u_n(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x+\alpha_n\biggr)\\
&\leqslant \frac{1}{\mu}(am(\Omega)\xi_n'+\alpha_n).
\end{aligned}$$

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Then

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^p \,\mathrm{d}x = 0.$$

We now prove that, for every $n \in \mathbb{N}$, $\alpha_n < 0$. To prove this, fix $n \in \mathbb{N}$, a compact set $K \subset D$ with $m(K) = (M+1)m(D \setminus K)$ and a function $v \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} v(x) &= 1 \quad \text{if } x \in K,\\ 0 \leqslant v(x) &\leqslant 1 \quad \text{if } x \in D \setminus K,\\ v(x) &= 0 \quad \text{if } x \in \Omega \setminus D. \end{aligned}$$

By the former condition of (ii), there exists $\bar{k} \in \mathbb{N}$ such that $t_k \leqslant \xi'_n$ and

$$\operatorname*{ess\,inf}_{x\in D}\int_0^{t_k} g(x,t)\,\mathrm{d}t > \mu\frac{(M+1)\|v\|^p}{m(K)}t_k^p$$

for every $k \in \mathbb{N}$ with $k \geqslant \bar{k}$. Then, taking into account the latter condition of (ii), for every $k \geqslant \bar{k}$, one has

$$\begin{aligned}
&\frac{-\int_\Omega(\int_0^{t_k v(x)} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x}{\int_\Omega |t_k\nabla v|^p\,\mathrm{d}x}\\
&\quad= \frac{-\int_K(\int_0^{t_k} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x - \int_{D\setminus K}(\int_0^{t_k v(x)} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x}{t_k^p\|v\|^p}\\
&\quad\leqslant \frac{\begin{aligned}-\int_K(\operatorname{ess\,inf}_{x\in D}\int_0^{t_k} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x\\ - \int_{D\setminus K}(\operatorname{ess\,inf}_{x\in D}\inf_{t\in[0,t_k]}\int_0^{t} g(x,s)\,\mathrm{d}s)\,\mathrm{d}x\end{aligned}}{t_k^p\|v\|^p}\\
&\quad\leqslant \frac{-\int_K(\operatorname{ess\,inf}_{x\in D}\int_0^{t_k} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x + M\int_{D\setminus K}(\operatorname{ess\,inf}_{x\in D}\int_0^{t_k} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x}{t_k^p\|v\|^p}\\
&\quad= \frac{-(1/(M+1))m(K)\operatorname{ess\,inf}_{x\in D}\int_0^{t_k} g(x,t)\,\mathrm{d}t}{t_k^p\|v\|^p}\\
&\quad< -\mu.
\end{aligned}$$

Whence $t_k v \in E_n$ and $\Phi_\mu(t_k v) < 0$, which implies $\alpha_n < 0$.

So there exists a subsequence of u_n of pairwise distinct elements. Such a subsequence is a sequence of weak solutions for $(P_{\lambda,g})$. On the other hand, we have

$$0 = \operatorname*{ess\,inf}_{x\in\Omega} u_n(x) < \text{ess sup}\, x \in \Omega u_n(x) \leqslant \bar{t}$$

for every $n \in \mathbb{N}$. Then it is a sequence of weak solutions for (P_λ). □

3. Some consequences

Here we give the proof of the theorem 1.1 stated in the introduction.

Proof of theorem 1.1. It is enough to show that (i′) implies (ii). By the former condition of (i′), there exists a sequence of positive numbers $\{t_n\}$, converging to

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zero, such that

$$\lim_{n\to+\infty} \frac{F(t_n)}{t_n^p} \to +\infty.$$

By the latter condition of (i′), there exist $M, \delta > 0$ such that

$$F(t) \geqslant -Mt^p$$

for every $t \in]0, \delta]$. Whence there exists $\nu \in \mathbb{N}$ such that, for every $n > \nu$, one has $t_n \leqslant \delta$ and $F(t_n) > t_n^p$. It turns out that, for $n > \nu$,

$$\inf_{t\in[0,t_n]} F(t) \geqslant -MF(t_n).$$

Then the thesis follows. □

Now we give an example of an application of theorem 2.1, when both conditions (i′) and (1.2) are not satisfied.

EXAMPLE 3.1. Let us consider, for every $\lambda > 0$, the following Dirichlet problem,

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t) = \begin{cases} 9t^{1/2}\sin(1/t^{1/3}) - 2t^{1/6}\cos(1/t^{1/3}) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Thus we have, for every $t \in \mathbb{R}$,

$$F(t) = \int_0^t f(s)\,\mathrm{d}s = \begin{cases} 6t^{3/2}\sin(1/t^{1/3}) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Condition (i) is satisfied. To prove that (ii) holds, let

$$t_n = \frac{8}{\pi^3(1+4n)^3}$$

for every $n \in \mathbb{N}$. Then

$$\lim_{n\to+\infty} \frac{F(t_n)}{t_n^2} = \lim_{n\to+\infty} \frac{6}{\sqrt{t_n}} = +\infty.$$

Moreover, we have

$$\inf_{t\in[0,t_n]} F(t) \geqslant -F(t_n).$$

Then the latter condition of (ii) is also satisfied for $M = 1$. Whence theorem 2.1 ensures the existence of a sequence of a.e. positive and pairwise distinct solutions to the problem that converges to zero. We note that

$$\liminf_{t\to 0^+} \frac{F(t)}{t^2} = -\infty,$$

hence (i′) and (1.2) are not satisfied.

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