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SHIFTED COISOTROPIC CORRESPONDENCES

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Abstract We define (iterated) coisotropic correspondences between derived Poisson stacks, and construct symmetric monoidal higher categories of derived Poisson stacks, where the i-morphisms are given by i-fold coisotropic correspondences. Assuming an expected equivalence of different models of higher Morita categories, we prove that all derived Poisson stacks are fully dualizable and so determine framed extended TQFTs by the Cobordism Hypothesis. Along the way, we also prove that the higher Morita category of E_n-algebras with respect to coproducts is equivalent to the higher category of iterated cospans.

Keywords: derived algebraic geometry; higher categories; coisotropic structures

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1. Introduction

1.1. Canonical relations

In symplectic geometry, Weinstein [37, 38] has proposed that the 'correct' notion of morphisms between two symplectic manifolds (X, ω_X) and (Y, ω_Y) should be *Lagrangian correspondences* (also known as *canonical relations*), i.e. Lagrangian submanifolds of $(X \times Y, \omega_X - \omega_Y)$. As one piece of evidence for this claim, it is a well-known fact that a smooth map $X \to Y$ is a symplectomorphism if and only if its graph is a Lagrangian correspondence. Under certain transversality hypotheses, it is possible to compose Lagrangian correspondences by taking an intersection, and Weinstein suggested that a 'category' of symplectic manifolds and Lagrangian correspondences should in some sense be a natural domain for geometric quantization. However, in general, it is not possible to compose Lagrangian correspondences (though see [35] for a way to partially circumvent this problem in the context of Floer theory).

Poisson geometry can be viewed as a generalization of symplectic geometry, where we weaken the non-degeneracy condition. In this context, the analogue of Lagrangian correspondences between Poisson manifolds (X, π_X) and (Y, π_Y) is the *coisotropic correspondences*, i.e. coisotropic submanifolds of $(X \times Y, \pi_X - \pi_Y)$. A map $X \to Y$ can be shown to be a Poisson morphism if and only if its graph is a coisotropic correspondence, and Weinstein [36] proved that under suitable transversality hypotheses, these too can be composed by taking intersections.

Symplectic and Poisson structures are also important in algebraic geometry, and here similar problems arise. Indeed, to define symplectic structures on an algebraic scheme X, one requires the cotangent sheaf Ω^1_X to be a vector bundle, which means that the scheme has to be smooth. However, intersections of smooth schemes are not smooth in general. More generally, we may consider a version of symplectic structures where we replace the cotangent sheaf Ω^1_X by the cotangent complex $\mathbb{L}_X$, in which case we require the cotangent complex to be perfect. However, we again run into the problem that the cotangent complexes of intersections of schemes with perfect cotangent complexes are in

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general not themselves perfect. Extra structures on derived Lagrangian intersections of symplectic schemes have been studied in [5] and on derived coisotropic intersections of Poisson schemes in [1].

A way to deal with the problem of non-transverse intersections is to work in the setting of *derived* algebraic geometry, where derived schemes with perfect cotangent complexes *are* stable under intersections. (Foundational references on derived algebraic geometry include [8, 9, 21, 34].) In this setting, analogues of symplectic and Lagrangian structures on derived schemes and, more generally, derived Artin stacks, have been introduced by Pantev, Toën, Vaquié, and Vezzosi [25], while analogues of Poisson and coisotropic structures were introduced by the same authors together with Calaque [7] (see also [24]). More precisely, in the derived setting differential forms naturally form a bicomplex, which allows us to consider *shifted* versions of all of these structures (so that, for example, an s-shifted symplectic form gives an equivalence $\mathbb{T}_X \simeq \mathbb{L}_X[s]$ with a shift by some integer s, instead of an equivalence $\mathbb{T}_X \simeq \mathbb{L}_X$ between the tangent and cotangent complexes of a derived stack X).

The goal of the present paper is to introduce a notion of (iterated) shifted coisotropic correspondences between shifted Poisson stacks and construct higher categories whose objects are shifted Poisson stacks and whose (higher) morphisms are (iterated) shifted coisotropic correspondences.

1.2. The 1-category of derived Poisson stacks

Before we describe the contents of this paper in more detail, it is helpful to first discuss the simplest case of our construction, namely the 1-category $h\mathrm{CoisCorr}_1^s$ of s-shifted coisotropic correspondences.

For this, we must first give a brief sketch of the definition of s-shifted Poisson structures on derived stacks, due to Calaque–Pantev–Toën–Vaquié–Vezzosi [7]. These authors associate to every derived stack X a certain symmetric monoidal stable ∞-category $\mathcal{M}_X$ (thought of as the ∞-category of quasi-coherent complexes on the de Rham stack of X), contravariantly functorial in X, together with a commutative algebra $\mathcal{P}_X^\infty \in \mathcal{M}_X$ (which is only a lax functor of X). (We will review this formalism in more detail in §3.1.) An s-shifted Poisson structure on X is then defined to be a lift of the commutative algebra structure on $\mathcal{P}_X^\infty$ to a $\mathbb{P}_{s+1}$-algebra, where $\mathbb{P}_{s+1}$ is the operad of dg Poisson algebras with a bracket of degree $-s$.

Remark 1.2.1. In other words, we can define an ∞-groupoid of s-shifted Poisson structures on X as the pullback

$$\begin{tikzcd}
\mathrm{Pois}(X,s) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{M}_X) \arrow[d] \\
\{\mathcal{P}_X^\infty\} \arrow[r] & \mathrm{CAlg}(\mathcal{M}_X).
\end{tikzcd}$$

Remark 1.2.2. The additivity theorem for Poisson algebras proved by the third author [32, Theorem 2.22] and, independently, Rozenblyum, says that $\mathbb{E}_n$-algebras in

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$\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C})$ are the same thing as $\mathbb{P}_{n+s}$-algebras in $\mathcal{C}$. Since $\mathbb{E}_n$-algebras in commutative algebras are just commutative algebras, we could have equivalently used $\mathbb{E}_n$-algebras in $\mathbb{P}_{s-n+1}$-algebras in $\mathcal{M}_X$ in our definition of Poisson structures above.

Derived stacks endowed with an s-shifted Poisson structure are the objects of the category $h\mathrm{CoisCorr}_1^s$. Next, to describe its morphisms, we outline the definition of a *coisotropic correspondence* between two s-shifted Poisson stacks X and X'. This is first of all given by a span $X \xleftarrow{f} Y \xrightarrow{g} X'$ of derived stacks. This induces a cospan of commutative algebras in $\mathcal{M}_Y$,

$$f^*\mathcal{P}_X^\infty \to \mathcal{P}_Y^\infty \leftarrow g^*\mathcal{P}_{X'}^\infty.$$

Since the symmetric monoidal structure on commutative algebras is cocartesian, we can equivalently view this cospan as giving $\mathcal{P}_Y^\infty$ the structure of an $(f^*\mathcal{P}_X^\infty, g^*\mathcal{P}_{X'}^\infty)$ bimodule in $\mathrm{CAlg}(\mathcal{M}_Y)$. A coisotropic correspondence is then a lift of this structure to a bimodule in the symmetric monoidal ∞-category $\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_Y)$, where we view the $\mathbb{P}_{s+1}$-algebra structures on $\mathcal{P}_X^\infty$ and $\mathcal{P}_{X'}^\infty$ as associative algebras in $\mathbb{P}_s$-algebras.

Remark 1.2.3. If we have chosen s-shifted Poisson structures for X and X', this means that the ∞-groupoid of compatible coisotropic structures on the span is given by the pullback

$$\begin{tikzcd}
\mathrm{Cois}_{X,X'}(f,g;s) \arrow[r] \arrow[d] & \mathrm{Mod}_{f^*\mathcal{P}_X^\infty, g^*\mathcal{P}_{X'}^\infty}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_Y)) \arrow[d] \\
\{\mathcal{P}_Y^\infty\} \arrow[r] & \mathrm{Mod}_{f^*\mathcal{P}_X^\infty, g^*\mathcal{P}_{X'}^\infty}(\mathrm{CAlg}(\mathcal{M}_Y)).
\end{tikzcd}$$

The coisotropic correspondences are the morphisms in the category $h\mathrm{CoisCorr}_1^s$ of s-shifted coisotropic correspondences. To compose two coisotropic correspondences given by spans

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] & & Y' \arrow[dl] \arrow[dr] & \\
X & & X' & & X''
\end{tikzcd}$$

we first compose the spans in the usual way, by forming a pullback

$$\begin{tikzcd}
 & & Z \arrow[dl] \arrow[dr] & & \\
 & Y \arrow[dl] \arrow[dr] & & Y' \arrow[dl] \arrow[dr] & \\
X & & X' & & X''.
\end{tikzcd}$$

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This pullback induces a pushout square in $\mathrm{CAlg}(\mathcal{M}_Z)$ (see Proposition 3.1.5),

$$\begin{tikzcd} \mathcal{P}^{\infty}_{X'} \arrow[r] \arrow[d] & \mathcal{P}^{\infty}_{Y} \arrow[d] \\ \mathcal{P}^{\infty}_{Y'} \arrow[r] & \mathcal{P}^{\infty}_{Z}, \end{tikzcd}$$

i.e. $\mathcal{P}^{\infty}_Z \simeq \mathcal{P}^{\infty}_Y \otimes_{\mathcal{P}^{\infty}_{X'}} \mathcal{P}^{\infty}_{Y'}$ (where we omit notation for the pullbacks to Z). To compose two coisotropic correspondences, we take the corresponding relative tensor product of the bimodules $\mathcal{P}^{\infty}_Y$ and $\mathcal{P}^{\infty}_{Y'}$ in $\mathbb{P}_s$-algebras. This can be interpreted as forming a composite in the Morita category of algebras and bimodules in $\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_Z)$; this has associative algebras as objects, with morphisms from A to B given by (A, B)-bimodules and composition given by taking relative tensor products.

To construct the ∞-categorical extension of this category, we need a more structured way of defining it. For this, we consider a general notion of 'spans with coefficients'. If $\mathcal{C}$ is an ∞-category with pullbacks, then given a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, we can define an ∞-category $\mathrm{Span}_1(\mathcal{C}; F)$ of spans with coefficients in F such that

- an object of $\mathrm{Span}_1(\mathcal{C}; F)$ is a pair $(c \in \mathcal{C}, x \in F(c))$,
- a morphism from (c, x) to (c', x') is a span $c \xleftarrow{f} d \xrightarrow{g} c'$ in $\mathcal{C}$ together with a morphism $\phi\colon F(f)(x) \to F(g)(x')$ in $F(d)$,
- given another morphism from (c', x') to (c'', x'') corresponding to a span $c' \xleftarrow{f'} d' \xrightarrow{g'} c''$ and a morphism $\psi\colon F(f')(x') \to F(g')(x'')$, their composite is given by composing the spans by taking a pullback

$$\begin{tikzcd} & & e \arrow[dl, "h"'] \arrow[dr, "k"] & & \\ & d \arrow[dl, "f"'] \arrow[dr, "g"] & & d' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ c & & c' & & c'', \end{tikzcd}$$

  and then composing $F(h)(\phi)\colon F(fh)(x) \to F(gh)(x') \simeq F(f'k)(x')$ with $F(k)(\psi)\colon F(f'k)(x') \to F(g'k)(x'')$ in $F(e)$.

We can apply this to the functors $\mathfrak{P}^s_1, \mathfrak{C}_1\colon \mathrm{dSt}^{\mathrm{op}} \to \mathrm{Cat}$ given by $\mathfrak{P}^s_1(X) = \mathfrak{alg}_1(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_X))$ and $\mathfrak{C}_1(X) = \mathfrak{alg}_1(\mathrm{CAlg}(\mathcal{M}_X))$, where $\mathfrak{alg}_1(\mathcal{C})$ denotes the Morita ∞-category of a monoidal ∞-category $\mathcal{C}$ [15]. The forgetful functor from Poisson algebras to commutative algebras induces a functor

$$\mathrm{Span}_1(\mathrm{dSt}; \mathfrak{P}^s_1) \to \mathrm{Span}_1(\mathrm{dSt}; \mathfrak{C}_1).$$

Moreover, using the section $\mathcal{P}^{\infty}_X \in \mathrm{CAlg}(\mathcal{M}_X)$, we can define a functor $\mathrm{Span}_1(\mathrm{dSt}) \to \mathrm{Span}_1(\mathrm{dSt}; \mathfrak{C}_1)$, which takes $X \in \mathrm{dSt}$ to $(X, \mathcal{P}^{\infty}_X)$, and a span $X \xleftarrow{f} Z \xrightarrow{g} Y$ to itself plus $\mathcal{P}^{\infty}_Z$ viewed as an $f^*\mathcal{P}^{\infty}_X$–$g^*\mathcal{P}^{\infty}_Y$–bimodule. This allows us to define the ∞-category $\mathrm{CoisCorr}^s_1$

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as the pullback

$$\begin{tikzcd} \mathrm{CoisCorr}_1^s \arrow[r] \arrow[d] & \mathrm{Span}_1(\mathrm{dSt}; \mathfrak{P}_1^s) \arrow[d] \\ \mathrm{Span}_1(\mathrm{dSt}) \arrow[r] & \mathrm{Span}_1(\mathrm{dSt}; \mathfrak{C}_1). \end{tikzcd}$$

1.3. Overview of results

In §2.3, we use the higher categories of 'spans with local systems' defined in [14] to construct the ∞-categories $\mathrm{Span}_1(\mathcal{C}; F)$ as well as their higher-dimensional cousins $\mathrm{Span}_n(\mathcal{C}; F)$, where F is a functor from $\mathcal{C}$ to the ∞-category of (∞, n)-categories. We then want to define the (∞, n)-category $\mathrm{CoisCorr}_n^s$ as a pullback

$$\begin{tikzcd} \mathrm{CoisCorr}_n^s \arrow[r] \arrow[d] & \mathrm{Span}_n(\mathrm{dSt}; \mathfrak{P}_n^s) \arrow[d] \\ \mathrm{Span}_n(\mathrm{dSt}) \arrow[r] & \mathrm{Span}_n(\mathrm{dSt}; \mathfrak{C}_n), \end{tikzcd}$$

where the functors $\mathfrak{P}_n^s, \mathfrak{C}_n \colon \mathrm{dSt}^{\mathrm{op}} \to \mathrm{Cat}_{(\infty,n)}$ are given by $\mathfrak{P}_n^s(X) = \mathfrak{alg}_n(\mathrm{Alg}_{\mathbb{P}_{s+1-n}}(\mathcal{M}_X))$ and $\mathfrak{C}_n(X) = \mathfrak{alg}_n(\mathrm{CAlg}(\mathcal{M}_X))$, with $\mathfrak{alg}_n(\mathcal{C})$ denoting the Morita (∞, n)-category of $\mathcal{C}$ [15]. However, we need to do some work to construct the functor $\mathrm{Span}_n(\mathrm{dSt}) \to \mathrm{Span}_n(\mathrm{dSt}; \mathfrak{C}_n)$; for this, we prove two results that may be of independent interest.

Theorem 1.3.1 (See Corollary 2.4.11). *Let $\mathrm{Cat}_\infty^{\mathrm{po}}$ be the subcategory of Cat_∞ whose objects are ∞-categories with pushouts and whose morphisms are functors that preserve these. Given a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{po}}$, we can form the functor $\mathrm{Cospan}_n(F) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_{(\infty,n)}$. There is an equivalence of (∞, n)-categories*

$$\mathrm{Span}_n(\mathcal{C}; \mathrm{Cospan}_n(F)) \simeq \mathrm{Cospan}_n(\mathcal{F}),$$

where $\mathcal{F} \to \mathcal{C}^{\mathrm{op}}$ is the cocartesian fibration for F.

Theorem 1.3.2 (See Corollary 2.6.10). *Suppose $\mathcal{C}$ is an ∞-category with finite colimits. Then there is an equivalence of (∞, n)-categories*

$$\mathrm{Cospan}_n(\mathcal{C}) \simeq \mathfrak{alg}_n(\mathcal{C}^{\amalg}).$$

Together, these two results lead to a simplified description of $\mathrm{Span}_n(\mathrm{dSt}; \mathfrak{C}_n)$, which allows us to prove our main result.

Theorem 1.3.3 (See Theorem 3.3.4). *There is a symmetric monoidal (∞, n)-category $\mathrm{CoisCorr}_n^s$ whose objects are derived stacks with s-shifted Poisson structures and whose i-morphisms are i-fold coisotropic correspondences. Assuming all $\mathbb{E}_n$-algebras are fully dualizable, all objects of this (∞, n)-category are fully dualizable.*

It was recently proved by Gwilliam and Scheimbauer [13] that the Morita (∞, n)-category has duals; however, they use a geometric model of this (∞, n)-category, which is not yet known to be equivalent to the algebraic model we use. Assuming this

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comparison (more precisely, see Conjecture 2.5.19), as well as the Cobordism Hypothesis, we have the following corollary.

Corollary 1.3.4. *Every s-shifted derived Poisson stack X determines a framed n-dimensional extended topological quantum field theory*

$$\mathrm{Bord}^{\mathrm{fr}}_{0,n} \to \mathrm{CoisCorr}^s_n.$$

Note that the (∞, n)-category of s-shifted Lagrangian correspondences Lag^s_n has recently been defined in [6].

It is known [7, 27] that s-shifted Poisson structures satisfying a non-degeneracy condition are equivalent to s-shifted symplectic structures in the sense of [25], and similarly that non-degenerate coisotropic structures are equivalent to Lagrangian structures [24, 28]. In §3.4, we explain how we expect these equivalences to generalize to relate $\mathrm{CoisCorr}^s_n$ to a symmetric monoidal (∞, n)-category Lag^s_n of s-shifted symplectic stacks and iterated Lagrangian correspondences, which is constructed in forthcoming work of the first author with Calaque and Scheimbauer [6].

2. Categorical preliminaries

In this section, we carry out the preliminary categorical constructions we require. We begin by briefly reviewing the definitions of (and fixing our notation for) iterated Segal spaces in §2.1 and then recalling the construction of higher categories of spans from [14] in §2.2. In §2.3, we use this to introduce higher categories of spans with coefficients in an (∞, n)-category. For the case of spans with coefficients in cospans, we then provide a simpler description of this construction in §2.4. In §2.5, we recall the definition of the higher Morita category of E_n-algebras from [15], which we use in §2.6 to prove that the higher category of cospans is a higher Morita category.

2.1. Review of iterated Segal spaces

The goal of this subsection is to provide a brief review of the theory of iterated Segal spaces, which was introduced by Barwick in [2]; iterated Segal spaces will be our model for (∞, n)-categories. Our discussion here is mainly intended to fix the notation we use in the rest of the paper; we refer the reader to [14, §§3, 4, 7, 11] for further details and motivation.

Definition 2.1.1. We write $\mathbb{\Delta}$ for the usual simplex category, with objects the ordered sets $[n] := \{0, 1, \dots, n\}$ and order-preserving functions as morphisms. A morphism $\phi\colon [n] \to [m]$ in $\mathbb{\Delta}$ is called *inert* if it is the inclusion of a sub-interval, i.e. if $\phi(i) = \phi(0) + i$ for all i, and *active* if it preserves the end points, i.e. if $\phi(0) = 0$ and $\phi(n) = m$. We write $\mathbb{\Delta}_{\mathrm{int}}$ for the subcategory of $\mathbb{\Delta}$ containing only the inert maps.

Notation 2.1.2. For all n, we have maps in $\mathbb{\Delta}$,

$$\sigma_i : [0] \to [n], \quad \rho_i : [1] \to [n],$$

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where σ_i $(0 \leqslant i \leqslant n)$ sends 0 to i and ρ_i $(0 < i \leqslant n)$ sends 0 and 1 to $i-1$ and i, respectively.

Definition 2.1.3. Let $\mathcal{C}$ be an ∞-category with pullbacks. A *category object* in $\mathcal{C}$ is a functor $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that the natural morphisms induced by the maps σ_i and ρ_i

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are equivalences in $\mathcal{C}$, for all n. We let $\mathrm{Cat}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ spanned by the category objects.

The above definition can be iterated, leading us to the following notion.

Definition 2.1.4. Let $\mathcal{C}$ be an ∞-category with pullbacks. A *n-uple category object* in $\mathcal{C}$ is defined inductively as a category object in the ∞-category of $(n-1)$-category objects. We let $\mathrm{Cat}^n(\mathcal{C})$ denote the ∞-category of n-uple category objects in $\mathcal{C}$, viewed as a full subcategory of $\mathrm{Fun}(\mathbb{\Delta}^{n,\mathrm{op}}, \mathcal{C})$. If $\mathcal{C}$ is the ∞-category $\mathcal{S}$ of spaces, we refer to n-uple category objects as *n-uple Segal spaces*.

Among the n-uple Segal spaces, we can single out those that describe (∞, n)-categories by imposing constancy conditions.

Definition 2.1.5. Let $\mathcal{C}$ be an ∞-category with pullbacks. A *1-fold Segal object* in $\mathcal{C}$ is simply a category object in $\mathcal{C}$. We now say inductively that an *n-fold Segal object* in $\mathcal{C}$ is an n-uple category object X in $\mathcal{C}$ such that

- the restriction $X_{0,\bullet,\ldots,\bullet} \in \mathrm{Cat}^{n-1}(\mathcal{C})$ is constant,
- the restrictions $X_{k,\bullet,\ldots,\bullet} \in \mathrm{Cat}^{n-1}(\mathcal{C})$ are $(n-1)$-fold Segal objects for all k.

We denote by $\mathrm{Seg}_n(\mathcal{C})$ the full subcategory of $\mathrm{Cat}^n(\mathcal{C})$ spanned by n-fold Segal objects. If $\mathcal{C}$ is the ∞-category $\mathcal{S}$ of spaces, we refer to n-fold Segal objects as *n-fold Segal spaces*.

By definition, the category $\mathrm{Seg}_n(\mathcal{C})$ comes equipped with an inclusion functor to $\mathrm{Cat}^n(\mathcal{C})$.

Proposition 2.1.6. *Let $\mathcal{C}$ be an ∞-category with pullbacks. The inclusion $\mathrm{Seg}_n(\mathcal{C}) \to \mathrm{Cat}^n(\mathcal{C})$ admits a right adjoint, which will be denoted by U^n_{Seg}.*

We refer to [14, Proposition 4.12] for a proof. If X is an n-fold Segal object in $\mathcal{C}$, we refer to $U^n_{\mathrm{Seg}} X$ as the *underlying n-uple category object* of X.

To obtain the correct ∞-category of (∞, n)-categories, we must invert the fully faithful and essentially surjective morphisms. By the results of Rezk [29] in the case $n = 1$ and Barwick [2] in general, this localization is given by the full subcategory of *complete* objects, defined as follows.

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Definition 2.1.7. Let X be an n-fold Segal space. We inductively say that X is *complete* if

- the Segal space $X_{\bullet,0,\dots,0}$ is complete in the sense of [29],
- the $(n-1)$-fold Segal space $X_{1,\bullet,\dots,\bullet}$ is complete.

We denote by $\mathrm{CSS}_n(\mathcal{S})$ the full subcategory of $\mathrm{Seg}_n(\mathcal{S})$ spanned by complete n-fold Segal spaces.

We also denote $\mathrm{CSS}_n(\mathcal{S})$ by $\mathrm{Cat}_{(\infty,n)}$; this ∞-category is equivalent to those of other descriptions of (∞, n)-categories by [4].

Notation 2.1.8. Let $\mathcal{D}$ be an n-fold Segal space, and let x and y be objects of $\mathcal{D}$ (i.e. points of $\mathcal{D}_{0,\dots,0}$). Then the $(n-1)$-fold Segal space $\mathcal{D}(x, y)$ of morphisms from x to y is defined by the pullback square

$$\begin{tikzcd}
\mathcal{D}(x, y) \arrow[r] \arrow[d] & \mathcal{D}_1 \arrow[d] \\
\{(x, y)\} \arrow[r] & \mathcal{D}_0 \times \mathcal{D}_0
\end{tikzcd}$$

of $(n-1)$-fold Segal spaces.

Since (complete) n-fold Segal spaces are models for (∞, n)-categories, it is natural to consider a notion of monoidal structures on these objects.

Definition 2.1.9. Let $\mathcal{C}$ be an ∞-category with finite products. An *associative monoid* in $\mathcal{C}$ is a functor $A : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that the natural maps

$$A_n \to A_1 \times \cdots \times A_1$$

are equivalences for all n. We denote by $\mathrm{Mon}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ spanned by associative monoids. Monoids in the categories $\mathrm{Seg}_n(\mathcal{S})$ or $\mathrm{CSS}^n(\mathcal{S})$ will be called *monoidal* n-fold (complete) Segal spaces.

We can once again iterate the above definition.

Definition 2.1.10. Inductively, a *k-uple monoid* in $\mathcal{C}$ is simply defined to be a $(k-1)$-uple monoid in $\mathrm{Mon}(\mathcal{C})$, and we denote by $\mathrm{Mon}_k(\mathcal{C})$ the category of k-uple monoids in $\mathcal{C}$. The k-uple monoids in $\mathrm{Seg}_n(\mathcal{S})$ or in $\mathrm{CSS}_n(\mathcal{S})$ are called *k-uply monoidal* n-fold (complete) Segal spaces.

Remark 2.1.11. Note that iterating Definition 2.1.9 does not produce any new operation, but instead adds commutativity constraints on the already existing one. This is analogous to the Eckmann–Hilton argument and to the Dunn–Lurie additivity (see also Remark 2.1.13).

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Note that there are natural functors $\mathrm{Mon}_k(\mathcal{C}) \to \mathrm{Mon}_{k-1}(\mathcal{C})$ for all k, which are defined by sending $X \in \mathrm{Mon}_k(\mathcal{C})$ to $X_{1,\bullet,\dots,\bullet}\colon \mathbb{\Delta}^{n-1,\mathrm{op}} \to \mathcal{C}$.

Definition 2.1.12. The ∞-category $\mathrm{Mon}_\infty(\mathcal{C})$ of *∞-uple monoids* in $\mathcal{C}$ is defined to be the limit of the diagram

$$\cdots \to \mathrm{Mon}_k(\mathcal{C}) \to \mathrm{Mon}_{k-1}(\mathcal{C}) \to \cdots \to \mathrm{Mon}(\mathcal{C}) \to \mathcal{C}.$$

If $\mathcal{C}$ is $\mathrm{Seg}_n(\mathcal{S})$ or $\mathrm{CSS}_n(\mathcal{S})$, elements in $\mathrm{Mon}_\infty(\mathcal{C})$ will be called *∞-uply monoidal n*-fold (complete) Segal spaces.

Remark 2.1.13. k-uply and ∞-uply monoidal (complete) n-fold Segal spaces can be equivalently described as $\mathbb{E}_n$-algebras and $\mathbb{E}_\infty$-algebras, in the sense of [20] (we refer the reader to [14, Proposition 10.12] for a precise statement). Therefore, k-uply and ∞-uply monoidal (complete) n-fold Segal spaces will be alternatively called *$\mathbb{E}_k$-monoidal* and *symmetric monoidal* (complete) n-fold Segal spaces.

Remark 2.1.14. If $\mathcal{D}$ is an n-fold Segal space and x is an object of $\mathcal{D}$, then $\mathcal{D}(x,x)$ is canonically a monoidal $(n-1)$-fold Segal space. This construction can be iterated so that if we have a sequence of $(n+i)$-fold Segal spaces $\mathcal{D}_i$ $(0 \leqslant i \leqslant m)$ and objects x_i in $\mathcal{D}_i$ such that $\mathcal{D}_i(x_i,x_i) \simeq \mathcal{D}_{i-1}$, then $\mathcal{D}_0$ is an $\mathbb{E}_m$-monoidal n-fold Segal space (where we may have $m = \infty$).

We now briefly recall what it means for an (∞,n)-category to have adjoints and duals (see [18] or [14, §11] for more details).

Definition 2.1.15. Let $\mathcal{D}$ be a 2-fold Segal space, and let $h_2\mathcal{D}$ denote its homotopy 2-category. A 1-morphism in $\mathcal{D}$ is an (left or right) *adjoint* if its image in $h_2\mathcal{D}$ is one. We say that $\mathcal{D}$ *has adjoints for 1-morphisms* if every 1-morphism in $\mathcal{D}$ is both a left and a right adjoint. If $\mathcal{D}$ is an n-fold Segal space, we similarly say that $\mathcal{D}$ has adjoints for 1-morphisms if its underlying 2-fold Segal space has adjoints for 1-morphisms; by induction, we then say that $\mathcal{D}$ has adjoints for i-morphisms for $i > 1$ if $\mathcal{D}(x,y)$ has adjoints for $(i-1)$-morphisms for all objects x, y. If $\mathcal{D}$ has adjoints for i-morphisms for all $1 \leqslant i < n$, we simply say that $\mathcal{D}$ *has adjoints*, while a (k-uply) monoidal n-fold Segal space *has duals* if it has adjoints when viewed as an $(n+1)$-fold Segal space.

We need to know that these properties are preserved under pullbacks, which is a consequence of the following observation.

Proposition 2.1.16. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D}_1 \arrow[d] \\ \mathcal{D}_2 \arrow[r] & \mathcal{E} \end{tikzcd}$$

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be a pullback in the ∞-category $\mathrm{CSS}_2(\mathcal{S})$ of $(\infty,2)$-categories. Then a morphism in $\mathcal{C}$ has a left (right) adjoint if and only if its images in $\mathcal{D}_1$ and $\mathcal{D}_2$ have left (right) adjoints.

Proof. Let Adj denote the free adjunction 2-category. This is described explicitly in [30], where it is proved that an adjunction in an $(\infty,2)$-category $\mathcal{K}$ is equivalent to a functor $\mathrm{Adj} \to \mathcal{K}$, from which it is clear that any functor of $(\infty,2)$-categories must preserve adjunctions. It thus suffices to show the 'if' direction, which we do for the case of left adjoints.

Let $l\colon \Delta^1 \to \mathrm{Adj}$ denote the inclusion of the 1-morphism that is a left adjoint. By [30, Theorem 4.4.18], for any $(\infty,2)$-category $\mathcal{K}$, the fibres of

$$l^*\colon \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{Adj},\mathcal{K}) \to \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1,\mathcal{K})$$

are either empty or contractible, and a 1-morphism in $\mathcal{K}$ is a left adjoint precisely when the fibre is non-empty. Moreover, our pullback square gives a commutative cube

$$\begin{tikzcd}
\mathrm{Map}(\mathrm{Adj},\mathcal{C}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Map}(\mathrm{Adj},\mathcal{D}_1) \arrow[dr] \arrow[dd] & \\
& \mathrm{Map}(\mathrm{Adj},\mathcal{D}_2) \arrow[rr] \arrow[dd] & & \mathrm{Map}(\mathrm{Adj},\mathcal{E}) \arrow[dd] \\
\mathrm{Map}(\Delta^1,\mathcal{C}) \arrow[rr] \arrow[dr] & & \mathrm{Map}(\Delta^1,\mathcal{D}_1) \arrow[dr] & \\
& \mathrm{Map}(\Delta^1,\mathcal{D}_2) \arrow[rr] & & \mathrm{Map}(\Delta^1,\mathcal{E}),
\end{tikzcd}$$

where the top and bottom faces are pullbacks. Given a 1-morphism f in $\mathcal{C}$, we get a pullback square of fibres, which shows that if the images of f in $\mathcal{D}_1$ and $\mathcal{D}_2$ are left adjoints, then f is a left adjoint. □

Since the notions of 'having duals' and 'having adjoints' are defined in terms of adjunctions in $(\infty,2)$-categories, we get the following as an immediate consequence.

Corollary 2.1.17. *Let*

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D}_1 \arrow[d] \\
\mathcal{D}_2 \arrow[r] & \mathcal{E}
\end{tikzcd}$$

be a pullback of symmetric monoidal (∞,n)-categories.

(i) *If $\mathcal{D}_1$ and $\mathcal{D}_2$ have adjoints, then $\mathcal{C}$ has adjoints.*

(ii) *If $\mathcal{D}_1$ and $\mathcal{D}_2$ have duals, then $\mathcal{C}$ has duals.*

Remark 2.1.18. In the case of duals for objects, this is also a consequence of [20, Proposition 4.6.1.11].

2.2. Review of higher categories of spans

In this subsection, we will review the definition of higher categories of iterated spans from [14].

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Definition 2.2.1. We write Σ^n for the partially ordered set of pairs (i, j) with $0 \leqslant i \leqslant j \leqslant n$, with $(i, j) \leqslant (i', j')$ if $i \leqslant i'$ and $j' \leqslant j$. A map $\phi\colon [n] \to [m]$ in $\mathbb{\Delta}$ determines a functor $\Sigma^n \to \Sigma^m$ taking (i, j) to $(\phi(i), \phi(j))$, yielding a functor $\Sigma^\bullet\colon \mathbb{\Delta} \to \mathrm{Cat}$. We write $\Sigma^{i_1,\ldots,i_n} := \Sigma^{i_1} \times \cdots \times \Sigma^{i_n}$, which gives functors $\Sigma^{\bullet,\ldots,\bullet}\colon \mathbb{\Delta}^n \to \mathrm{Cat}$. We let $\widehat{\Sigma}^n \to \mathbb{\Delta}^{n,\mathrm{op}}$ denote the cartesian fibration for this functor.

Definition 2.2.2. If $\mathcal{C}$ is an ∞-category, we let $\overline{\mathrm{SPAN}}^+(\mathcal{C}) \to \mathbb{\Delta}^{n,\mathrm{op}}$ be the cocartesian fibration for the functor $\mathrm{Fun}(\Sigma^{\bullet,\ldots,\bullet}, \mathcal{C})\colon \mathbb{\Delta}^{n,\mathrm{op}} \to \mathrm{Cat}_\infty$. We also write $\overline{\mathrm{SPAN}}(\mathcal{C}) \to \mathbb{\Delta}^{n,\mathrm{op}}$ for its underlying left fibration, corresponding to the functor $\mathrm{Map}(\Sigma^{\bullet,\ldots,\bullet}, \mathcal{C})\colon \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{S}$.

Remark 2.2.3. The ∞-category $\overline{\mathrm{SPAN}}^+(\mathcal{C})$ has a universal property by [11, Proposition 7.3]. For any ∞-category $\mathcal{K}$ over $\mathbb{\Delta}^{n,\mathrm{op}}$, we have a natural equivalence

$$\mathrm{Map}_{/\mathbb{\Delta}^{n,\mathrm{op}}}(\mathcal{K}, \overline{\mathrm{SPAN}}^+(\mathcal{C})) \simeq \mathrm{Map}(\mathcal{K} \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\Sigma}^n, \mathcal{C}).$$

Definition 2.2.4. Let Λ^i denote the full subcategory of Σ^i spanned by the pairs (i, j) with $j - i \leqslant 1$. These subcategories are preserved by inert maps in $\mathbb{\Delta}$, giving a functor $\Lambda^\bullet\colon \mathbb{\Delta}_{\mathrm{int}} \to \mathrm{Cat}$. Similarly, we define $\Lambda^{i_1,\ldots,i_n} := \Lambda^{i_1} \times \cdots \times \Lambda^{i_k}$, which gives a functor $\Lambda^{\bullet,\ldots,\bullet}\colon \mathbb{\Delta}^n_{\mathrm{int}} \to \mathrm{Cat}$ with a natural transformation $\Lambda^{\bullet,\ldots,\bullet} \to \Sigma^{\bullet,\ldots,\bullet}|_{\mathbb{\Delta}^n_{\mathrm{int}}}$.

Definition 2.2.5. Let $\mathcal{C}$ be an ∞-category with pullbacks. We say a functor $f\colon \Sigma^{i_1,\ldots,i_k} \to \mathcal{C}$ is *cartesian* if it is a right Kan extension of its restriction to $\Lambda^{i_1,\ldots,i_k}$. We write $\mathrm{SPAN}_n^+(\mathcal{C})$ and $\mathrm{SPAN}_n(\mathcal{C})$ for the full subcategories of $\overline{\mathrm{SPAN}}_n^+(\mathcal{C})$ and $\overline{\mathrm{SPAN}}_n(\mathcal{C})$, respectively, spanned by the cartesian functors.

We then have the following:

- The restricted projection $\mathrm{SPAN}_n^+(\mathcal{C}) \to \mathbb{\Delta}^{n,\mathrm{op}}$ is a cocartesian fibration, by [14, Corollary 5.12].
- The corresponding functor $\mathbb{\Delta}^{n,\mathrm{op}} \to \mathrm{Cat}_\infty$ is an n-uple category object, by [14, Proposition 5.14].

Similarly, $\mathrm{SPAN}_n(\mathcal{C})$ is an n-uple Segal space.

Definition 2.2.6. We let $\mathrm{Span}_n(\mathcal{C}) := U^n_{\mathrm{Seg}}(\mathrm{SPAN}_n(\mathcal{C}))$ be the underlying n-fold Segal space of $\mathrm{SPAN}_n(\mathcal{C})$.

Notation 2.2.7. If $\mathcal{C}$ is an ∞-category with pushouts, we also write $\mathrm{COSPAN}_n(\mathcal{C}) := \mathrm{SPAN}_n(\mathcal{C}^{\mathrm{op}})$ and $\mathrm{Cospan}_n(\mathcal{C}) := \mathrm{Span}_n(\mathcal{C}^{\mathrm{op}})$.

We have the following results from [14]:

- The n-fold Segal space $\mathrm{Span}_n(\mathcal{C})$ is complete, by [14, Corollary 8.5].
- For objects, $x, y \in \mathcal{C}$, the $(n-1)$-fold Segal space of maps $\mathrm{Span}_n(\mathcal{C})(x, y)$ is naturally equivalent to $\mathrm{Span}_{n-1}(\mathcal{C}_{/x,y})$, by [14, Proposition 8.3]. Here $\mathcal{C}_{/x,y} := \mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}_{/y}$ is the ∞-category of spans $x \leftarrow c \to y$ with x and y fixed.

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- As a consequence, if $\mathcal{C}$ has a terminal object (i.e. $\mathcal{C}$ has all finite limits), then the (∞, n)-category $\mathrm{Span}_n(\mathcal{C})$ has a natural symmetric monoidal structure, as in [14, Proposition 12.1].
- The symmetric monoidal (∞, n)-category $\mathrm{Span}_n(\mathcal{C})$ has duals, by [14, Corollary 12.5].

Following [14, Section 6], we also consider a variant of the definition of $\mathrm{Span}_n(\mathcal{C})$, giving a higher category of 'iterated spans with local systems' in a category object in $\mathcal{C}$.

Notation 2.2.8. Let $\Pi\colon \widehat{\Sigma} \to \mathbb{\Delta}^{\mathrm{op}}$ be the functor taking $([n], (i, j))$ to $[j-i]$, and a morphism $([n], (i, j)) \to ([m], (i', j'))$ given by a morphism $\phi\colon [m] \to [n]$ in $\mathbb{\Delta}$ such that $(i, j) \leqslant (\phi(i'), \phi(j'))$ to the morphism $[j'-i'] \to [j-i]$ given by $s \mapsto \phi(i'+s)-i$. We write Π^n for the product of n copies of Π, and $\Pi_I\colon \Sigma^I \to \mathbb{\Delta}^{n,\mathrm{op}}$ for its restriction to Σ^I.

Definition 2.2.9. Let $\mathcal{C}$ be an ∞-category with pullbacks. Given a functor $F\colon \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{C}$, we write $\overline{\mathrm{SPAN}}_n^+(\mathcal{C}; F) \to \mathbb{\Delta}^{n,\mathrm{op}}$ for the cocartesian fibration corresponding to the functor $I \mapsto \mathrm{Fun}(\Sigma^I, \mathcal{C})_{/F\circ\Pi_I}$.

Remark 2.2.10. $\overline{\mathrm{SPAN}}_n^+(\mathcal{C}; F)$ can also be described as the ∞-category of commutative diagrams

$$\begin{tikzcd} \Sigma^I \arrow[r] \arrow[d, "\Pi_I"] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ \mathbb{\Delta}^{n,\mathrm{op}} \arrow[r, "F"] & \mathcal{C}. \end{tikzcd}$$

If we write $\mathcal{C}_{//F}$ for the pullback

$$\begin{tikzcd} \mathcal{C}_{//F} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ \mathbb{\Delta}^{n,\mathrm{op}} \arrow[r, "F"] & \mathcal{C}, \end{tikzcd}$$

this means we can describe $\overline{\mathrm{SPAN}}_n^+(\mathcal{C}; F)$ as the pullback

$$\begin{tikzcd} \overline{\mathrm{SPAN}}_n^+(\mathcal{C}; F) \arrow[r] \arrow[d] & \overline{\mathrm{SPAN}}_n^+(\mathcal{C}_{//F}) \arrow[d] \\ \mathbb{\Delta}^{n,\mathrm{op}} \arrow[r] & \overline{\mathrm{SPAN}}_n^+(\mathbb{\Delta}^{n,\mathrm{op}}), \end{tikzcd}$$

where the bottom horizontal map is the section of $\overline{\mathrm{SPAN}}_n^+(\mathbb{\Delta}^{n,\mathrm{op}}) \to \mathbb{\Delta}^{n,\mathrm{op}}$ corresponding to Π^n under the equivalence

$$\mathrm{Map}_{/\mathbb{\Delta}^{n,\mathrm{op}}}(\mathbb{\Delta}^{n,\mathrm{op}}, \overline{\mathrm{SPAN}}^+(\mathbb{\Delta}^{n,\mathrm{op}})) \simeq \mathrm{Map}(\widehat{\Sigma}^n, \mathbb{\Delta}^{n,\mathrm{op}})$$

of Remark 2.2.3.

Definition 2.2.11. Suppose $\mathcal{C}$ is an ∞-category with pullbacks and $F\colon \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{C}$ is an n-uple category object. (Then $\Pi_I F\colon \Sigma^I \to \mathcal{C}$ is cartesian for all I by [14, Lemma 6.4].)

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We define $\mathrm{SPAN}_n^+(\mathcal{C}; F)$ as the pullback

$$\begin{tikzcd}
\mathrm{SPAN}_n^+(\mathcal{C}; F) \arrow[r] \arrow[d] & \overline{\mathrm{SPAN}}_n^+(\mathcal{C}; F) \arrow[d] \\
\mathrm{SPAN}_n^+(\mathcal{C}) \arrow[r] & \overline{\mathrm{SPAN}}_n^+(\mathcal{C}).
\end{tikzcd}$$

Then $\mathrm{SPAN}_n^+(\mathcal{C}; F) \to \mathbb{\Delta}^{n,\mathrm{op}}$ is a cocartesian fibration, being a fibre product of cocartesian fibrations over $\mathbb{\Delta}^{n,\mathrm{op}}$ along functors that preserve cocartesian morphisms. Moreover, it corresponds to an n-uple category object in Cat_∞ by [14, Proposition 6.7]. We write $\mathrm{SPAN}_n(\mathcal{C}; F)$ for the underlying left fibration, which corresponds to an n-uple Segal space, and $\mathrm{Span}_n(\mathcal{C}; F)$ for its underlying n-fold Segal space.

Remark 2.2.12. Using the description of the right adjoint U^n_{Seg} in terms of iterated pullbacks in the proof of [14, Proposition 4.12], it is easy to see that for an n-uple category object $F: \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{C}$, we have

$$\mathrm{Span}_n(\mathcal{C}; F) \simeq U^n_{\mathrm{Seg}}\mathrm{SPAN}_n(\mathcal{C}; F) \simeq \mathrm{Span}_n(\mathcal{C}; U^n_{\mathrm{Seg}}F).$$

If $\mathcal{C}$ is an ∞-category with finite limits, and ξ, η are objects of $\mathrm{Span}_n(\mathcal{C}; F)$, corresponding to morphisms $\xi: x \to F_{0,\ldots,0}$, $\eta: y \to F_{0,\ldots,0}$ in $\mathcal{C}$, then by [14, Proposition 9.3], we can identify the $(n-1)$-fold Segal space of maps $\mathrm{Span}_n(\mathcal{C}; F)(\xi, \eta)$ with $\mathrm{Span}_{n-1}(\mathcal{C}; F_{\xi,\eta})$, where $F_{\xi,\eta}$ is the functor $\mathbb{\Delta}^{n-1,\mathrm{op}} \to \mathcal{C}$ defined as the pullback

$$\begin{tikzcd}
F_{\xi,\eta} \arrow[r] \arrow[d] & F_1 \arrow[d] \\
x \times y \arrow[r, "\xi\times\eta"] & F_0 \times F_0.
\end{tikzcd}$$

Here it will be convenient to slightly reformulate this, using the following observation.

Lemma 2.2.13. *Suppose $\mathcal{C}$ is an ∞-category with pullbacks. Given a functor $F: \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{C}_{/x}$, there is a natural equivalence*

$$\mathrm{Span}_n(\mathcal{C}_{/x}; F) \simeq \mathrm{Span}_n(\mathcal{C}; F).$$

Proof. The commutative square

$$\begin{tikzcd}
(\mathcal{C}_{/x})^{\Delta^1} \arrow[r] \arrow[d, "\mathrm{ev}_1"] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\
\mathcal{C}_{/x} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

is cartesian; pulling back along $F: \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{C}_{/x}$, we get a natural equivalence $\mathcal{C}_{//F} \simeq (\mathcal{C}_{/x})_{//F}$, and hence a natural equivalence $\overline{\mathrm{SPAN}}_n(\mathcal{C}; F) \simeq \overline{\mathrm{SPAN}}_n(\mathcal{C}_{/x}; F)$, which restricts to an equivalence $\mathrm{Span}_n(\mathcal{C}; F) \simeq \mathrm{Span}_n(\mathcal{C}_{/x}; F)$. $\square$

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As a consequence, we may identify $\mathrm{Span}_n(\mathcal{C}; F)(\xi, \eta)$ with $\mathrm{Span}_{n-1}(\mathcal{C}_{/x\times y}; F_{\xi,\eta})$; it is easy to see that this identification is compatible with the identification $\mathrm{Span}_n(\mathcal{C})(x, y) \simeq \mathrm{Span}_{n-1}(\mathcal{C}_{/x\times y})$.

It follows that if F is a functor to symmetric monoidal n-fold Segal objects in $\mathcal{C}$, then $\mathrm{Span}_n(\mathcal{C}; F)$ is a symmetric monoidal n-fold Segal space (see [14, Proposition 13.1]).

If $\mathcal{X}$ is an ∞-topos, we can make sense of *complete* n-fold Segal objects in $\mathcal{X}$, and of (symmetric monoidal) n-fold Segal objects *having adjoints* (and *having duals*). We then have the following.

- If $F : \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{X}$ is complete, then $\mathrm{Span}_n(\mathcal{X}; F)$ is a complete n-fold Segal space by [14, Corollary 9.7].
- If F has adjoints, then so does $\mathrm{Span}_n(\mathcal{X}; F)$, by [14, Theorem 3.3].
- If F is a symmetric monoidal complete n-fold Segal object in $\mathcal{X}$ that has duals, then $\mathrm{Span}_n(\mathcal{X}; F)$ has duals.

Here we only consider $\mathcal{X}$ of the form $\mathcal{P}(\mathcal{C})$ for some ∞-category $\mathcal{C}$, in which case all these notions are given objectwise in $\mathcal{C}$ by the usual notions for n-fold Segal spaces.

2.3. Spans with coefficients

We now introduce higher categories of spans with coefficients as a variant of the constructions above.

Definition 2.3.1. Suppose $\mathcal{C}$ is a small ∞-category. Given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Seg}_n(\mathcal{S})$, we define the n-fold Segal space of *spans in* $\mathcal{C}$ *with coefficients in* F as the pullback

$$\begin{tikzcd}
\mathrm{Span}_n(\mathcal{C}; F) \arrow[r] \arrow[d] & \mathrm{Span}_n(\mathcal{P}(\mathcal{C}); F') \arrow[d] \\
\mathrm{Span}_n(\mathcal{C}) \arrow[r] & \mathrm{Span}_n(\mathcal{P}(\mathcal{C})),
\end{tikzcd}$$

where F' is F regarded as a functor $\mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{P}(\mathcal{C})$, $\mathrm{Span}_n(\mathcal{P}(\mathcal{C}); F')$ is the ∞-category of spans in the ∞-topos $\mathcal{P}(\mathcal{C})$ with coefficients in F', and the bottom horizontal functor is induced by the Yoneda embedding. We define the variants $\mathrm{SPAN}_n(\mathcal{C}; F)$, etc. similarly.

Remark 2.3.2. From the definition of $\overline{\mathrm{SPAN}}_n(\mathcal{P}(\mathcal{C}); F')$, we see that $\overline{\mathrm{SPAN}}_n(\mathcal{C}; F)$ has the following description: its fibre at I is the space of commutative diagrams

$$\begin{tikzcd}
\Sigma^I \arrow[rr] \arrow[d] \arrow[dd, bend right=60, "\Pi_I"'] & & \mathcal{P}(\mathcal{C})^{\Delta^1} \arrow[d] \\
\mathcal{C} \times \mathbb{\Delta}^{n,\mathrm{op}} \arrow[rr, "Y \times F'"] \arrow[d] & & \mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{C}) \\
\mathbb{\Delta}^{n,\mathrm{op}}, & &
\end{tikzcd}$$

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where Y denotes the Yoneda embedding. If we define $\mathcal{F} \to \mathcal{C} \times \mathbb{\Delta}^{n,\mathrm{op}}$ by the pullback square

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{P}(\mathcal{C})^{\Delta^1} \arrow[d] \\ \mathcal{C} \times \mathbb{\Delta}^{n,\mathrm{op}} \arrow[r, "Y \times F'"] & \mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{C}), \end{tikzcd}$$

then we can equivalently describe $\overline{\mathrm{SPAN}}_n(\mathcal{C}; F)$ as the space of commutative diagrams

$$\begin{tikzcd} \Sigma^I \arrow[r] \arrow[dr, "\Pi_I"] & \mathcal{F} \arrow[d] \\ & \mathbb{\Delta}^{n,\mathrm{op}}. \end{tikzcd}$$

(Here $\mathcal{F} \to \mathcal{C} \times \mathbb{\Delta}^{n,\mathrm{op}}$ is the bifibration (see §A.1) corresponding to F viewed as a functor $\mathcal{C}^{\mathrm{op}} \times \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{S}$.) From this, we obtain an alternative definition of $\mathrm{SPAN}_n(\mathcal{C}; F)_I$ as the pullback

$$\begin{tikzcd} \mathrm{SPAN}_n(\mathcal{C}; F) \arrow[r] \arrow[d] & \overline{\mathrm{SPAN}}(\mathcal{F}) \arrow[d] \\ \mathrm{SPAN}_n(\mathcal{C}) \arrow[r] & \overline{\mathrm{SPAN}}_n(\mathcal{C}) \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \overline{\mathrm{SPAN}}_n(\mathbb{\Delta}^{n,\mathrm{op}}), \end{tikzcd}$$

where the bottom horizontal map is the fibre product of the inclusion $\mathrm{SPAN}_n(\mathcal{C}) \to \overline{\mathrm{SPAN}}_n(\mathcal{C})$ with the functor $\mathbb{\Delta}^{n,\mathrm{op}} \to \overline{\mathrm{SPAN}}_n(\mathbb{\Delta}^{n,\mathrm{op}})$ corresponding to $\Pi\colon \widehat{\Sigma} \to \mathbb{\Delta}^{n,\mathrm{op}}$.

Remark 2.3.3. Given a functor $F\colon \mathcal{C} \to \mathrm{Cat}^n(\mathcal{S})$, it follows from Remark 2.2.12 that we have

$$\mathrm{Span}_n(\mathcal{C}; F) \simeq U^n_{\mathrm{Seg}} \mathrm{SPAN}_n(\mathcal{C}; F) \simeq \mathrm{Span}_n(\mathcal{C}; U^n_{\mathrm{Seg}} F).$$

Lemma 2.3.4.

(i) *If $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Seg}_{n-1}(\mathcal{S})$ lands in the full subcategory $\mathrm{Cat}_{(\infty,n)}$ of complete n-fold Segal spaces, then $\mathrm{Span}_n(\mathcal{C}; F)$ is a complete Segal space.*

(ii) *If F is a functor from $\mathcal{C}^{\mathrm{op}}$ to symmetric monoidal n-fold Segal spaces, then $\mathrm{Span}_n(\mathcal{C}; F)$ is symmetric monoidal.*

(iii) *If F is a functor from $\mathcal{C}^{\mathrm{op}}$ to (∞, n)-categories with adjoints, then $\mathrm{Span}_n(\mathcal{C}; F)$ has adjoints.*

(iv) *If F is a functor from $\mathcal{C}^{\mathrm{op}}$ to symmetric monoidal (∞, n)-categories with duals, then $\mathrm{Span}_n(\mathcal{C}; F)$ has duals.*

Proof. In case (i), $F'\colon \mathbb{\Delta}^{n,\mathrm{op}} \to \mathcal{P}(\mathcal{C})$ is a complete n-fold Segal object of $\mathcal{P}(\mathcal{C})$, so $\mathrm{Span}_n(\mathcal{P}(\mathcal{C}); F')$ is a complete n-fold Segal space by [14, Proposition 9.2]. The n-fold Segal space $\mathrm{Span}_n(\mathcal{C}; F)$ is therefore complete as the limit of a diagram of complete objects, computed in n-fold Segal spaces, is complete. Similarly, in case (ii), F' is a symmetric monoidal n-fold Segal object in $\mathcal{P}(\mathcal{C})$, and so $\mathrm{Span}_n(\mathcal{P}(\mathcal{C}); F')$ is symmetric monoidal by [14, Proposition 13.1]. Moreover, the functors in the pullback square defining $\mathrm{Span}_n(\mathcal{C}; F)$

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are naturally symmetric monoidal, and the forgetful functor from symmetric monoidal n-fold Segal spaces to n-fold Segal spaces preserves limits. Parts (iii) and (iv) follow similarly using Corollary 2.1.17 together with [14, Theorem 13.3 and Corollary 13.4]. □

Proposition 2.3.5. *Suppose ξ, η are objects of $\mathrm{Span}_n(\mathcal{C}; F)$ corresponding to pairs $(x \in \mathcal{C}, \xi \in F(x))$, $(y \in \mathcal{C}, \eta \in F(y))$. If we define $F_{\xi,\eta}\colon \mathcal{C}_{/x,y} \to \mathrm{Seg}_{n-1}(\mathcal{S})$ to be the functor that takes $x \leftarrow z \rightarrow y$ to the pullback*

$$\begin{tikzcd}
F_{\xi,\eta}(z) \arrow[rr] \arrow[d] & & F(z)_1 \arrow[d] \\
\{(\xi,\eta)\} \arrow[r] & F(x)_0 \times F(y)_0 \arrow[r] & F(z)_0 \times F(z)_0,
\end{tikzcd}$$

then there is a natural equivalence of $(n-1)$-fold Segal spaces

$$\mathrm{Span}_n(\mathcal{C}; F)(\xi, \eta) \simeq \mathrm{Span}_{n-1}(\mathcal{C}_{/x,y}; F_{\xi,\eta}).$$

Proof. From the definition of $\mathrm{Span}_n(\mathcal{C}; F)$ as a pullback, it follows that we have a pullback square

$$\begin{tikzcd}
\mathrm{Span}_n(\mathcal{C}; F)(\xi, \eta) \arrow[r] \arrow[d] & \mathrm{Span}_n(\mathcal{P}(\mathcal{C}); F')(\xi', \eta') \arrow[d] \\
\mathrm{Span}_n(\mathcal{C})(x, y) \arrow[r] & \mathrm{Span}_n(\mathcal{P}(\mathcal{C}))(Y(x), Y(y)),
\end{tikzcd}$$

where ξ' is the morphism $Y(x) \to F_{0,\ldots,0}$ corresponding to $\xi \in F_{0,\ldots,0}(x)$, and similarly for η'. By [14, Proposition 9.3], we can identify $\mathrm{Span}_n(\mathcal{P}(\mathcal{C}); F')(\xi', \eta')$ with $\mathrm{Span}_{n-1}(\mathcal{P}(\mathcal{C}); F'_{\xi',\eta'})$, where $F'_{\xi',\eta'}$ is the $(n-1)$-fold Segal object in $\mathcal{P}(\mathcal{C})$ defined as the pullback

$$\begin{tikzcd}
F'_{\xi',\eta'} \arrow[r] \arrow[d] & F_1 \arrow[d] \\
Y(x) \times Y(y) \arrow[r, "\xi' \times \eta'"] & F_0 \times F_0.
\end{tikzcd}$$

Here $F'_{\xi',\eta'}$ is naturally a functor $\mathbb{\Delta}^{n-1,\mathrm{op}} \to \mathcal{P}(\mathcal{C})_{/Y(x)\times Y(y)}$, and so by Lemma 2.2.13, we can equivalently identify this with $\mathrm{Span}_{n-1}(\mathcal{P}(\mathcal{C})_{/Y(x)\times Y(y)}; F'_{\xi',\eta'})$, compatibly with the identification of $\mathrm{Span}_n(\mathcal{P}(\mathcal{C}))(Y(x), Y(y))$ with $\mathrm{Span}_{n-1}(\mathcal{P}(\mathcal{C})_{/Y(x)\times Y(y)})$ from [14, Proposition 8.3]. We thus have a pullback square

$$\begin{tikzcd}
\mathrm{Span}_n(\mathcal{C}; F)(\xi, \eta) \arrow[r] \arrow[d] & \mathrm{Span}_{n-1}(\mathcal{P}(\mathcal{C})_{/Y(x)\times Y(y)}; F'_{\xi',\eta'}) \arrow[d] \\
\mathrm{Span}_{n-1}(\mathcal{C}_{/x,y}) \arrow[r] & \mathrm{Span}_{n-1}(\mathcal{P}(\mathcal{C})_{/Y(x)\times Y(y)}).
\end{tikzcd}$$

The canonical functor $\mathcal{P}(\mathcal{C}_{/x,y}) \to \mathcal{P}(\mathcal{C})_{/Y(x)\times Y(y)}$ is an equivalence (since $\mathcal{C}_{/x,y} \to \mathcal{C}$ is the right fibration for $Y(x) \times Y(y)$), and under this equivalence, the functor $F'_{\xi',\eta'}$ corresponds to $(F_{\xi,\eta})'$. Our pullback square is therefore equivalent to that defining $\mathrm{Span}_{n-1}(\mathcal{C}_{/x,y}; F_{\xi,\eta})$, as required. □

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Remark 2.3.6. In the special case where $\mathcal{C}$ has a terminal object $*$ and $x \simeq y \simeq *$ so that ξ and η are objects of $F(*)$, we can identify $F_{\xi,\eta}$ with the functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Seg}_{n-1}(\mathcal{S})$ taking $c \in \mathcal{C}$ to the mapping $(n-1)$-fold Segal space $F(c)(f^*\xi, f^*\eta)$, where f denotes the unique map $c \to *$.

2.4. Spans with coefficients in cospans

Suppose $\mathcal{C}$ is an ∞-category with pullbacks, and consider a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{po}}$ to the ∞-category of small ∞-categories with pushouts. Then we have a functor $\mathrm{COSPAN}_n(F)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^n(\mathcal{S})$, and we can consider the n-uple Segal space $\mathrm{SPAN}_n(\mathcal{C}; \mathrm{COSPAN}_n(F))$. Our goal in this subsection is to give a simpler description of this n-uple Segal space.

Proposition 2.4.1. *Let $\mathcal{F} \to \mathcal{C}^{\mathrm{op}}$ be the cocartesian fibration corresponding to a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{po}}$. Then there is a natural equivalence*

$$\mathrm{SPAN}_n(\mathcal{C}; \mathrm{COSPAN}_n(F)) \simeq \mathrm{COSPAN}_n(\mathcal{F}).$$

Our starting point is the following description of spans with coefficients in cospans.

Proposition 2.4.2. *Given $\phi\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{po}}$ with corresponding cocartesian fibration $\mathcal{F} \to \mathcal{C}^{\mathrm{op}}$, then $\overline{\mathrm{SPAN}}_n(\mathcal{C}; \overline{\mathrm{COSPAN}}_n(\phi))_I$ is equivalent to the space of diagrams of the form*

$$\begin{tikzcd}
\mathrm{Tw}^\ell(\Sigma^I) \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\Sigma}_{n,\mathrm{op}} \arrow[r, "\alpha"] \arrow[d] & \mathcal{F} \arrow[d] \\
\Sigma^{I,\mathrm{op}} \arrow[r, "\gamma^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}}
\end{tikzcd}$$

such that α takes every morphism of $\mathrm{Tw}^\ell(\Sigma^I) \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\Sigma}_{n,\mathrm{op}}$ that lies over a cartesian morphism in $\widehat{\Sigma}_{n,\mathrm{op}}$ to a cocartesian morphism in $\mathcal{F}$, where $\widehat{\Sigma}_{n,\mathrm{op}} \to \mathbb{\Delta}^{n,\mathrm{op}}$ is the cartesian fibration for $I \mapsto \Sigma^{I,\mathrm{op}}$. Here $\mathrm{Tw}^\ell(\Sigma^I)$ denotes the left fibration version of the twisted arrow category of Σ^I; see Definition A.2.2.

Proof. Let $\mathcal{X} \to \mathcal{C} \times \mathbb{\Delta}^{n,\mathrm{op}}$ be the bifibration corresponding to

$$(c, I) \mapsto \overline{\mathrm{COSPAN}}_n^+(\phi(c))_I.$$

Then by Remark 2.3.2, we can identify $\overline{\mathrm{SPAN}}_n(\mathcal{C}; \overline{\mathrm{COSPAN}}_n^+(\phi))_I$ with the space of commutative diagrams

$$\left\{ \begin{tikzcd}
\Sigma^I \arrow[r] \arrow[dr, "\Pi_I"] & \mathcal{X} \arrow[d] \\
 & \mathbb{\Delta}^{n,\mathrm{op}}
\end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd}
\Sigma^I \arrow[r] \arrow[dr] \arrow[ddr, bend right, "\Pi_I"'] & \mathcal{X} \arrow[d] \\
 & \mathcal{C} \times \mathbb{\Delta}^{n,\mathrm{op}} \arrow[d] \\
 & \mathbb{\Delta}^{n,\mathrm{op}}
\end{tikzcd} \right\}.$$

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Now Corollary A.2.6 identifies this with the space of commutative diagrams

$$\left\{\begin{tikzcd} \mathrm{Tw}^{\ell}(\mathbb{\Sigma}^I) \arrow[r] \arrow[d] & \mathcal{X}^{\ell} \arrow[d] \\ \mathbb{\Sigma}^{I,\mathrm{op}} \times \mathbb{\Sigma}^I \arrow[r] \arrow[d] & \mathcal{C}^{\mathrm{op}} \times \mathbb{\Delta}^{n,\mathrm{op}} \arrow[d] \\ \mathbb{\Sigma}^I \arrow[r, "\Pi_I"] & \mathbb{\Delta}^{n,\mathrm{op}} \end{tikzcd}\right\}.$$

Here $\mathcal{X}^{\ell} \to \mathcal{C}^{\mathrm{op}} \times \mathbb{\Delta}^{n,\mathrm{op}}$ is the underlying left fibration of the cocartesian fibration for the functor $(c, I) \mapsto \mathrm{Fun}(\mathbb{\Sigma}^I, \phi(c))$, and so Corollary A.3.10 identifies this space with that of commutative squares

$$\left\{\begin{tikzcd} \mathrm{Tw}^{\ell}(\mathbb{\Sigma}^I) \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\mathbb{\Sigma}}_{n,\mathrm{op}} \arrow[r, "\alpha"] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathbb{\Sigma}^{I,\mathrm{op}} \arrow[r] & \mathcal{C}^{\mathrm{op}} \end{tikzcd}\right\},$$

such that α takes every morphism of $\mathrm{Tw}^{\ell}(\mathbb{\Sigma}^I) \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\mathbb{\Sigma}}_{n,\mathrm{op}}$ that lies over a cartesian morphism in $\widehat{\mathbb{\Sigma}}_{n,\mathrm{op}}$ to a cocartesian morphism in $\mathcal{F}$. □

Notation 2.4.3. We use the abbreviation

$$\mathbb{X}^I := \mathrm{Tw}^{\ell}(\mathbb{\Sigma}^I) \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\mathbb{\Sigma}}_{n,\mathrm{op}}.$$

Corollary 2.4.4. $\mathrm{SPAN}_n(\mathcal{C}; \mathrm{COSPAN}_n(\phi))_I$ *is the space of commutative diagrams*

$$\begin{tikzcd} \mathrm{Tw}^{\ell}(\mathbb{\Sigma}^I) \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\mathbb{\Sigma}}_{n,\mathrm{op}} \arrow[r, "\alpha"] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathbb{\Sigma}^{I,\mathrm{op}} \arrow[r, "\gamma^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}}, \end{tikzcd}$$

where

(1) $\gamma \colon \mathbb{\Sigma}^I \to \mathcal{C}$ *is cartesian, i.e. is a right Kan extension of its restriction to* $\mathbb{\Lambda}^I$*,*

(2) α *takes every morphism of* $\mathrm{Tw}^{\ell}(\mathbb{\Sigma}^I) \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\mathbb{\Sigma}}_{n,\mathrm{op}}$ *that lies over a cartesian morphism in* $\widehat{\mathbb{\Sigma}}_{n,\mathrm{op}}$ *to a cocartesian morphism in* $\mathcal{F}$*,*

(3) *for every morphism* $i \colon A \to B$ *in* $\mathbb{\Sigma}^I$*, the diagram*

$$\mathbb{\Sigma}^{\pi_I(B),\mathrm{op}} \simeq \{i\} \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\mathbb{\Sigma}}_{n,\mathrm{op}} \to \{\gamma(A)\} \times_{\mathcal{C}^{op}} \mathcal{F} \simeq \phi(\gamma(A))$$

is cocartesian, i.e. is a left Kan extension of its restriction to $\mathbb{\Lambda}^{\pi_I(B),\mathrm{op}}$. □

In order to use this description to prove Proposition 2.4.1, we need to relate diagrams of shape $\mathbb{X}^I$ to diagrams of shape $\mathbb{\Sigma}^{I,\mathrm{op}}$ in $\mathcal{F}$. This we will do in two steps, using the following explicit description of the category $\mathbb{X}^n$.

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Lemma 2.4.5.

(i) *The category $\Sigma^n \times_{\Delta^{\mathrm{op}}} \widehat{\Sigma}_{1,\mathrm{op}}$ is equivalent to the partially ordered set of quadruples of integers (a, b, c, d), $0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant n$, where $(a, b, c, d) \leqslant (a', b', c', d')$ if and only if*

$$a \leqslant a' \leqslant b' \leqslant b \leqslant c \leqslant c' \leqslant d' \leqslant d.$$

This corresponds to a cartesian morphism in $\widehat{\Sigma}_{1,\mathrm{op}}$ if and only if $b' = b$, $c' = c$, and the projection to Σ^n is given by $(a, b, c, d) \mapsto (a, d)$.

(ii) *The category $\mathrm{Tw}^{\ell}(\Sigma^n)$ is equivalent to the partially ordered set of quadruples of integers (a, b, c, d), $0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant n$, where $(a, b, c, d) \leqslant (a', b', c', d')$ if and only if*

$$a' \leqslant a \leqslant b \leqslant b' \leqslant c' \leqslant c \leqslant d \leqslant d',$$

i.e. the opposite of the partially ordered set in (i). The projections $\mathrm{Tw}^{\ell}(\Sigma^n) \to \Sigma^{n,\mathrm{op}}, \Sigma^n$ are given by $(a, b, c, d) \mapsto (a, d), (b, c)$, respectively.

(iii) *The category $\mathbb{X}^n \simeq \mathrm{Tw}^{\ell}(\Sigma^n) \times_{\Delta^{\mathrm{op}}} \widehat{\Sigma}_{1,\mathrm{op}}$ is equivalent to the partially ordered set of sextuples of integers (a, b, c, d, e, f), where $0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant e \leqslant f \leqslant n$, where $(a, b, c, d, e, f) \leqslant (a', b', c', d', e', f')$, if and only if*

$$a' \leqslant a \leqslant b \leqslant b' \leqslant c' \leqslant c \leqslant d \leqslant d' \leqslant e' \leqslant e \leqslant f \leqslant f'.$$

This corresponds to a cartesian morphism in $\widehat{\Sigma}_{1,\mathrm{op}}$ if and only if $c' = c$, $d' = d$. The projections to $\mathrm{Tw}^{\ell}(\Sigma^n)$ and $\Sigma^n \times_{\Delta^{\mathrm{op}}} \widehat{\Sigma}_{1,\mathrm{op}}$ are given by

$$(a, b, c, d, e, f) \mapsto (b, c, d, e), \quad (a, b, d, e),$$

respectively.

Proof. Since $\widehat{\Sigma}_{1,\mathrm{op}} \to \Delta^{n,\mathrm{op}}$ is the cartesian fibration for the functor $[n] \in \Delta \mapsto \Sigma^{n,\mathrm{op}} \in \mathrm{Cat}$, the category $\widehat{\Sigma}_{1,\mathrm{op}}$ has object pairs $([n], (i, j))$ with $0 \leqslant i \leqslant j \leqslant n$, with a morphism $([n], (i, j)) \to ([m], (i', j'))$ given by a morphism $\phi\colon [m] \to [n]$ in Δ such that $(i, j) \leqslant (\phi(i'), \phi(j'))$ in $\Sigma^{n,\mathrm{op}}$, i.e. $(\phi(i'), \phi(j')) \leqslant (i, j)$ in Σ^n, or $i \leqslant \phi(i') \leqslant \phi(j') \leqslant j$.

On the other hand, the functor $\Pi_n\colon \Sigma^n \to \Delta^{\mathrm{op}}$ takes (i, j) to $[j - i]$, so an object of the fibre product $\Sigma^n \times_{\Delta^{\mathrm{op}}} \widehat{\Sigma}_{1,\mathrm{op}}$ is a pair $((i, j), (i', j'))$ with $0 \leqslant i \leqslant j \leqslant n$ and $0 \leqslant i' \leqslant j' \leqslant j - i$. Identifying this with the quadruple $(i, i' + i, j' + i, j)$, we get a bijection between the objects of $\Sigma^n \times_{\Delta^{\mathrm{op}}} \widehat{\Sigma}_{1,\mathrm{op}}$ and the set of quadruples (a, b, c, d) with $0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant n$.

A morphism $((i, j), (i', j')) \to ((k, l), (k', l'))$ is unique if it exists, and corresponds to the inequalities $(i, j) \leqslant (k, l)$ and $(k' + k - i, l' + k - i) \leqslant (i', j')$ (since the corresponding inclusion $[l - k] \hookrightarrow [j - i]$ in Δ is given by $t \mapsto t + k - i$), i.e.

$$i \leqslant k \leqslant l \leqslant j, \qquad k' + k - i \leqslant i' \leqslant j' \leqslant l' + k - i,$$

which we can rewrite as

$$i \leqslant k \leqslant k' + k \leqslant i' + i \leqslant j' + j \leqslant l' + k \leqslant l \leqslant j.$$

Equivalently, there is a unique morphism $(a, b, c, d) \to (a', b', c', d')$ if and only if $a \leqslant a' \leqslant b' \leqslant b \leqslant c \leqslant c' \leqslant d' \leqslant d$, as required to prove (i).

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To prove (ii), observe that an object of $\mathrm{Tw}^\ell(\Sigma^n)$ is a morphism $(i, j) \to (i', j')$ in Σ^n, which we can identify with a quadruple (i, i', j', j) with $0 \leqslant i \leqslant i' \leqslant j' \leqslant j \leqslant n$. Now a morphism from $(i, j) \to (i', j')$ to $(k, l) \to (k', l')$ in $\mathrm{Tw}^\ell(\Sigma^n)$ is a commutative diagram

$$\begin{tikzcd} (i,j) \arrow[d] & (k,l) \arrow[l] \arrow[d] \\ (i',j') \arrow[r] & (k',l'), \end{tikzcd}$$

which corresponds to the inequalities

$$i \leqslant i' \leqslant j' \leqslant j, \qquad i' \leqslant k' \leqslant l' \leqslant j'$$
$$k \leqslant i \leqslant j \leqslant l, \qquad k \leqslant k' \leqslant l' \leqslant l,$$

which we can combine into the single chain of inequalities

$$k \leqslant i \leqslant i' \leqslant k' \leqslant l' \leqslant j' \leqslant j \leqslant l,$$

which proves (ii).

To prove (iii), observe that the fibre product $\mathrm{Tw}^\ell(\Sigma^n) \times_{\mathbb{\Delta}^{\mathrm{op}}} \widehat{\Sigma}_{1,\mathrm{op}}$ is equivalently the fibre product $\mathrm{Tw}^\ell(\Sigma^n) \times_{\Sigma^n} (\Sigma^n \times_{\mathbb{\Delta}^{\mathrm{op}}} \widehat{\Sigma}_{1,\mathrm{op}})$ of the categories considered in (i) and (ii). We can therefore identify an object of this category with a pair $((a, b, c, d), (i, j, k, l))$, where $(b, c) = (i, l)$, or equivalently a sextuple (a, b, j, k, c, d) with $0 \leqslant a \leqslant b \leqslant j \leqslant k \leqslant c \leqslant d \leqslant n$. The inequalities in (i) and (ii) also combine to give the inequalities in (iii) as the criterion for a morphism to exist in this partially ordered set. □

Definition 2.4.6. Let $\mathbb{T}^n$ denote the partially ordered set of quadruples (a, c, d, f) with $0 \leqslant a \leqslant c \leqslant d \leqslant f \leqslant n$, where $(a, c, d, f) \leqslant (a', c', d', f')$ if and only if $a' \leqslant a \leqslant f \leqslant f'$ and $c' \leqslant c \leqslant d \leqslant d'$. We then define functors $\alpha_n : \mathbb{X}^n \to \mathbb{T}^n$ and $\beta_n : \Sigma^{n,\mathrm{op}} \to \mathbb{T}^n$ by

$$\alpha_n(a, b, c, d, e, f) = (a, c, d, f),$$
$$\beta_n(i, j) = (i, i, j, j).$$

We also define $\gamma_n : \mathbb{T}^n \to \Sigma^{n,\mathrm{op}}$ by $(a, c, d, f) \mapsto (a, f)$; note that $\gamma_n \circ \beta_n = \mathrm{id}$. For $I = ([i_1], \ldots, [i_n]) \in \mathbb{\Delta}^{n,\mathrm{op}}$, we set $\mathbb{T}^I := \prod_j \mathbb{T}^{i_j}$, and we similarly define α_I, β_I, and γ_I as products.

Proposition 2.4.7. *Let C_n denote the set of morphisms $(a, c, d, f) \to (a', c', d', f')$ in $\mathbb{T}^n$ such that $c = c'$, $d = d'$, and similarly let C_I denote the product of these morphisms viewed as morphisms in $\mathbb{T}^I$. Then composition with the functor β_I induces an equivalence*

$$\mathrm{Map}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{T}^I, \mathcal{F}) \to \mathrm{Map}(\Sigma^{I,\mathrm{op}}, \mathcal{F}),$$

where $\mathrm{Map}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{T}^I, \mathcal{F})$ denotes the space of commutative squares

$$\begin{tikzcd} \mathbb{T}^I \arrow[r, "f"] \arrow[d, "\gamma_I"'] & \mathcal{F} \arrow[d] \\ \Sigma^{I,\mathrm{op}} \arrow[r] & \mathcal{C}^{\mathrm{op}}, \end{tikzcd}$$

where f takes the morphisms in C_I to cocartesian morphisms in $\mathcal{F}$.

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Proof. To prove this, we will show that the morphism of marked simplicial sets

$$(\mathrm{N}\mathbb{\Sigma}^{I,\mathrm{op}})^\flat \to (\mathrm{N}\mathbb{T}^I, C_I)$$

is marked anodyne in the cocartesian sense, i.e. dual to that of [17, Definition 3.1.1.1]. Marked anodyne morphisms are closed under the cartesian product of marked simplicial sets by [17, Proposition 3.1.2.3], so it suffices to prove the case $n = 1$. We will do this using a filtration of $\mathrm{N}\mathbb{T}^I$; to define this, it is convenient to first make up some terminology and notation:

- We say a simplex of $\mathrm{N}\mathbb{T}^I$ is *old* if it is contained in the simplicial subset $\mathrm{N}\mathbb{\Sigma}^{I,\mathrm{op}}$, and *new* otherwise.
- If $\sigma\colon \Delta^n \to \mathrm{N}\mathbb{T}^I$ is a non-degenerate new simplex, corresponding to a sequence of morphisms $A_0 \xrightarrow{f_1} A_1 \to \cdots \to A_n$, we define $\nu(\sigma)$ to be the integer such that $A_i \in \beta_n(\mathrm{N}\mathbb{\Sigma}^{I,\mathrm{op}})$ for $i < \nu(\sigma)$ and $A_{\nu(\sigma)} \notin \beta_n(\mathrm{N}\mathbb{\Sigma}^{I,\mathrm{op}})$.
- If σ is a non-degenerate new n-simplex as above, we say that σ is *long* if $\nu(\sigma) > 0$ and the morphism $A_{\nu(\sigma)-1} \to A_{\nu(\sigma)}$ is in C_n, and *short* otherwise.
- If σ is a long new non-degenerate $(n+1)$-simplex, then we say that σ is *associated* to the short new non-degenerate n-simplex $d_{\nu(\sigma)-1}\sigma$. Observe that for every short new non-degenerate n-simplex, there is a *unique* long new non-degenerate $(n+1)$-simplex associated to it.

We let $\mathfrak{F}_n$ be the smallest simplicial subset of $\mathrm{N}\mathbb{T}^I$ containing $\mathfrak{F}_{n-1}$ (where we start with $\mathfrak{F}_{-1}$ containing only the old simplices) together with the short new non-degenerate n-simplices and the long new non-degenerate $(n+1)$-simplices. We then have a filtration of marked simplicial sets

$$\beta_n(\mathrm{N}\mathbb{\Sigma}^{I,\mathrm{op}}) = \mathfrak{F}_{-1} \subseteq \mathfrak{F}_0 \subseteq \cdots \subseteq \mathrm{N}\mathbb{T}^I,$$

where we implicitly regard all these simplicial sets as marked by those of their edges that lie in C_n. Since $\mathrm{N}\mathbb{T}^I$ is the union of the simplicial subsets $\mathfrak{F}_i$, it suffices to show that the morphisms $\mathfrak{F}_{i-1} \hookrightarrow \mathfrak{F}_i$ are all marked anodyne.

Next, we define a subsidiary filtration

$$\mathfrak{F}_{N-1} = \mathfrak{G}_{N,N+1} \subseteq \mathfrak{G}_{N,N} \subseteq \cdots \subseteq \mathfrak{G}_{N,0} = \mathfrak{F}_N,$$

where $\mathfrak{G}_{N,m}$ contains $\mathfrak{F}_{N-1}$ together with those short new non-degenerate N-simplices σ such that $\nu(\sigma) \geqslant m$, as well as their associated $(N+1)$-simplices. Then it suffices to show that the inclusions $\mathfrak{G}_{N,m} \hookrightarrow \mathfrak{G}_{N,m-1}$ are all marked anodyne.

Consider now a short new non-degenerate N-simplex σ with associated $(N+1)$-simplex σ'. Then we observe that

- $d_{\nu(\sigma)}\sigma' = \sigma$,
- $d_i\sigma'$ is a long N-simplex if $i \neq \nu(\sigma), \nu(\sigma)+1$, and so is in $\mathfrak{F}_{N-1}$,
- $\nu(d_{\nu(\sigma)+1}\sigma') = \nu(\sigma)+1$, so $d_{\nu(\sigma)+1}\sigma'$ lies in $\mathfrak{G}_{N,\nu(\sigma)+1}$.

Thus we have pushouts

$$\begin{tikzcd} \coprod_\sigma \Lambda^{N+1}_m \arrow[r] \arrow[d] & \coprod_\sigma \Delta^{N+1} \arrow[d] \\ \mathfrak{G}_{N,m+1} \arrow[r] & \mathfrak{G}_{N,m}, \end{tikzcd}$$

where the coproducts are over all short new non-degenerate N-simplices σ such that $\nu(\sigma) = m$. If $m > 0$, then the top horizontal morphism is inner anodyne, and if $m = 0$, then for every σ the edge $0 \to 1$ in Λ^{N+1}_0 is sent to an edge of $\mathrm{N}\mathbb{T}^n$ that lies in C_n. Hence the top horizontal morphism is still marked anodyne. □

Let us also write $\mathrm{Map}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{X}^I, \mathcal{F})$ for the space of commutative squares

$$\begin{tikzcd} \mathbb{X}^I \arrow[r, "f"] \arrow[d] & \mathcal{F} \arrow[d] \\ \Sigma^{I,\mathrm{op}} \arrow[r, "g^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}}, \end{tikzcd}$$

where f takes the morphisms that lie over cartesian morphisms in $\widehat{\Sigma}_{n,\mathrm{op}}$ to cocartesian morphisms in $\mathcal{F}$. Then composition with α_I and β_I gives natural maps

$$\mathrm{Map}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{X}^I, \mathcal{F}) \leftarrow \mathrm{Map}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{T}^I, \mathcal{F}) \xrightarrow{\sim} \mathrm{Map}(\Sigma^{I,\mathrm{op}}, \mathcal{F}).$$

Now we define $\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{X}^I, \mathcal{F})$ to be the subspace of such squares where

- $g \colon \Sigma^I \to \mathcal{C}$ is cartesian,
- for every morphism $i \colon A \to B$ in Σ^I, the diagram

$$\Sigma^{\pi_I(B),\mathrm{op}} \simeq \{i\} \times_{\Delta^{n,\mathrm{op}}} \widehat{\Sigma}_{n,\mathrm{op}} \to \{g(A)\} \times_{\mathcal{C}^{op}} \mathcal{F} \simeq \phi(g(A))$$

  is cocartesian,

and we also define $\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{T}^I, \mathcal{F})$ to be the subspace of functors that restrict under β_I to cocartesian functors $\Sigma^{I,\mathrm{op}} \to \mathcal{F}$.

Proposition 2.4.8. *The maps given by composition with α_I and β_I restrict to maps*

$$\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{X}^I, \mathcal{F}) \leftarrow \mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{T}^I, \mathcal{F}) \xrightarrow{\sim} \mathrm{Map}^{\mathrm{cocart}}(\Sigma^{I,\mathrm{op}}, \mathcal{F}).$$

We need the well-known description of colimits in a cocartesian fibration, which we spell out as follows.

Lemma 2.4.9. *Suppose $\pi : \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration and $\mathcal{I}$ is a small ∞-category such that*

(i) *$\mathcal{B}$ has colimits of shape $\mathcal{I}$,*

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(ii) *each fibre* $\mathcal{E}_b$ *has colimits of shape* $\mathcal{I}$,

(iii) *the cocartesian pushforward functor* $f_!\colon \mathcal{E}_b \to \mathcal{E}_{b'}$ *preserves colimits of shape* $\mathcal{I}$ *for all morphisms* $f\colon b \to b'$ *in* $\mathcal{B}$.

Then $\mathcal{E}$ *has colimits of shape* $\mathcal{I}$. *The colimit of a diagram* $p\colon \mathcal{I} \to \mathcal{E}$ *is computed by*

(1) *extending* $\pi p\colon \mathcal{I} \to \mathcal{B}$ *to a colimit diagram* $q\colon \mathcal{I}^{\triangleright} \to \mathcal{B}$,

(2) *taking the cocartesian pushforward* $p'\colon \mathcal{I} \to \mathcal{E}_{q(\infty)}$ *of the diagram* p *along the morphisms* $q(i) \to q(\infty)$,

(3) *computing the colimit of* p' *in the fibre* $\mathcal{E}_{q(\infty)}$.

Proof. By [17, Corollary 4.3.1.11], the assumptions imply that there exists a lift $\bar{p}\colon \mathcal{I}^{\triangleright} \to \mathcal{E}$ over q, which is a π-colimit diagram. Combining [17, Propositions 4.3.1.9 and 4.3.1.10], we see that this π-colimit is equivalent to the colimit of the pushed-forward diagram p' in the fibre $\mathcal{E}_{q(\infty)}$. On the other hand, since q is a colimit diagram in $\mathcal{B}$, [17, Proposition 4.3.1.5(2)] shows that $\bar{p}$ is a π-colimit diagram if and only if it is a colimit diagram in $\mathcal{E}$. □

Proof of Proposition 2.4.8. We must check that composition with α_I takes a commutative square in $\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{T}^I, \mathcal{F})$ to one in $\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{X}^I, \mathcal{F})$. Since $\mathcal{F} \to \mathcal{C}^{\mathrm{op}}$ is the cocartesian fibration corresponding to a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{po}}_{\infty}$, Lemma 2.4.9 implies that a commutative square in $\mathcal{F}$ is a pushout if and only if it projects to a pushout square in $\mathcal{C}^{\mathrm{op}}$ and its cocartesian pushforward to the fibre over the terminal object is a pushout square in that fibre. This implies in particular that composition with $\mathcal{F} \to \mathcal{C}^{\mathrm{op}}$ takes cocartesian diagrams in $\mathcal{F}$ to cocartesian diagrams in $\mathcal{C}^{\mathrm{op}}$. Thus it remains only to show that for every morphism $i\colon A \to B$ in Σ^I, the diagram

$$\Sigma^{\pi_I(B),\mathrm{op}} \simeq \{i\} \times_{\mathbb{\Delta}^{n,\mathrm{op}}} \widehat{\Sigma}_{n,\mathrm{op}} \to \mathbb{T}^I \times_{\Sigma^{I,\mathrm{op}}} \{A\} \to \{g(A)\} \times_{\mathcal{C}^{op}} \mathcal{F} \simeq \phi(g(A))$$

is cocartesian. But this diagram is a cocartesian pushforward to the fibre $g(A)$ of the diagram

$$\Sigma^{\pi_I(B),\mathrm{op}} \to \Sigma^{I,\mathrm{op}} \xrightarrow{\beta_I} \mathbb{T}^I \to \mathcal{F},$$

which is cocartesian by [14, Proposition 5.9], and is therefore cocartesian, using again the description of pushouts in $\mathcal{F}$.

Consequently, we see that the functors α_I and β_I induce a morphism of n-uple Segal spaces

$$\mathrm{COSPAN}_n(\mathcal{F}) \to \mathrm{SPAN}_n(\mathcal{C}; \mathrm{COSPAN}_n(F)).$$

To see that this is an equivalence, we need the following observation.

Lemma 2.4.10. *The functor* $\alpha_1\colon \mathbb{X}^1 \to \mathbb{T}^1$ *exhibits* $\mathbb{T}^1$ *as the localization of* $\mathbb{X}^1$ *at the morphisms* $(0, 0, 0, 0, 1, 1) \to (0, \ldots, 0, 1)$ *and* $(0, 0, 1, 1, 1, 1) \to (0, 1, \ldots, 1)$.

Proof. We can depict the partially ordered set $\mathbb{X}^1$ as

$$\begin{gathered}(0, \ldots, 0) \to (0, \ldots, 0, 1) \leftarrow (0, \ldots, 0, 1, 1) \to (0, 0, 0, 1, 1, 1)\\ \leftarrow (0, 0, 1, \ldots, 1) \to (0, 1, \ldots, 1) \leftarrow (1, \ldots, 1)\end{gathered}$$

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and $\mathbb{T}^1$ as

$$(0,0,0,0) \to (0,0,0,1) \to (0,0,1,1) \leftarrow (0,1,1,1) \leftarrow (1,1,1,1).$$

The result is clear from this description since both decompose as pushouts of free categories. □

Proof of Proposition 2.4.1. We have a morphism of n-uple Segal spaces

$$\mathrm{COSPAN}_n(\mathcal{F}) \to \mathrm{SPAN}_n(\mathcal{C}; \mathrm{COSPAN}_n(F)).$$

To see that this is an equivalence, it suffices to show that it is an equivalence on fibres $\mathrm{COSPAN}_n(\mathcal{F})_I \to \mathrm{SPAN}_n(\mathcal{C}; \mathrm{COSPAN}_n(F))_I$, where $I = ([i_1], \ldots, [i_n])$ with $i_j = 0$ or 1 for all j, which follows from the previous lemma.

Corollary 2.4.11. *Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be a functor such that $F(x)$ has finite colimits for $x \in \mathcal{C}$ and $F(f)\colon F(x) \to F(y)$ preserves finite colimits for every morphism $f\colon x \to y$ in $\mathcal{C}^{\mathrm{op}}$. Then there is a symmetric monoidal equivalence of symmetric monoidal (∞, n)-categories*

$$\mathrm{Cospan}_n(\mathcal{F}) \xrightarrow{\sim} \mathrm{Span}_n(\mathcal{C}; \mathrm{Cospan}_n(F)).$$

Proof. Since the functors α_I and β_I are defined as cartesian products, we have a commutative diagram of equivalences

$$\begin{tikzcd}
\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{k,I,\mathrm{op}}}(\mathbb{X}^{k,I}, \mathcal{F}) \arrow[r, "\sim"] & \mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{k,\mathrm{op}}}(\mathbb{X}^{k}, \mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{X}^{I}, \mathcal{F})) \\
\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{k,I,\mathrm{op}}}(\mathbb{T}^{k,I}, \mathcal{F}) \arrow[r, "\sim"] \arrow[u, "\sim"] \arrow[d, "\sim"'] & \mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{k,\mathrm{op}}}(\mathbb{T}^{k}, \mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{T}^{I}, \mathcal{F})) \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
\mathrm{Map}(\Sigma^{k,I,\mathrm{op}}, \mathcal{F}) \arrow[r, "\sim"] & \mathrm{Map}(\Sigma^{k,\mathrm{op}}, \mathrm{Map}(\Sigma^{I,\mathrm{op}}, \mathcal{F})),
\end{tikzcd}$$

with the notation in the right-hand column interpreted so that it makes sense. Setting $k = 1$ and taking the fibres (via the maps $\Sigma^0 \amalg \Sigma^0 \to \Sigma^1$, etc.) at the constant maps to the initial object (which we denote with the subscript $(\emptyset, \emptyset)$), we get a commutative diagram of equivalences

$$\begin{tikzcd}
\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{1,I,\mathrm{op}}}(\mathbb{X}^{1,I}, \mathcal{F})_{(\emptyset,\emptyset)} \arrow[r, "\sim"] & \mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{X}^{I}, \mathcal{F}) \\
\mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{1,I,\mathrm{op}}}(\mathbb{T}^{1,I}, \mathcal{F})_{(\emptyset,\emptyset)} \arrow[r, "\sim"] \arrow[u, "\sim"] \arrow[d, "\sim"'] & \mathrm{Map}^{\mathrm{cocart}}_{C/\Sigma^{I,\mathrm{op}}}(\mathbb{T}^{I}, \mathcal{F}) \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
\mathrm{Map}(\Sigma^{1,I,\mathrm{op}}, \mathcal{F})_{(\emptyset,\emptyset)} \arrow[r, "\sim"] & \mathrm{Map}(\Sigma^{I,\mathrm{op}}, \mathcal{F}).
\end{tikzcd}$$

From this we see that on the underlying n-fold Segal objects, the equivalences of Proposition 2.4.1 are compatible under delooping, i.e. we have commutative squares of equivalences

$$\begin{tikzcd} \mathrm{Span}_{n+1}(\mathcal{C};\mathrm{Cospan}_{n+1}(F))(\emptyset,\emptyset) \arrow[r, "\sim"] & \mathrm{Span}_n(\mathcal{C};\mathrm{Cospan}_n(F)) \\ \mathrm{Cospan}_{n+1}(\mathcal{F})(\emptyset,\emptyset) \arrow[r, "\sim"] \arrow[u, "\sim"] & \mathrm{Cospan}_n(\mathcal{F}). \arrow[u, "\sim"] \end{tikzcd}$$

It follows that the equivalence $\mathrm{Span}_n(\mathcal{C};\mathrm{Cospan}_n(F)) \xrightarrow{\sim} \mathrm{Cospan}_n(\mathcal{F})$ is symmetric monoidal, as required. $\square$

Remark 2.4.12. Let F be as above, and suppose $\sigma\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{F}$ is a section that takes finite limits in $\mathcal{C}$ to colimits in $\mathcal{F}$. Then it follows from the equivalence of Corollary 2.4.11 that σ induces a symmetric monoidal functor of (∞, n)-categories

$$\mathrm{Span}_n(\mathcal{C}) \to \mathrm{Span}_n(\mathcal{C};\mathrm{Cospan}_n(F)).$$

2.5. Review of higher Morita categories

In this subsection, we will briefly recall the definition of the higher Morita category of $\mathbb{E}_n$-algebras in an $\mathbb{E}_n$-monoidal ∞-category, as constructed in [15].

Definition 2.5.1. A Δ^n*-monoidal* ∞*-category* is a cocartesian fibration $\mathcal{V}^{\otimes} \to \Delta^{n,\mathrm{op}}$ such that the corresponding functor $\Delta^{n,\mathrm{op}} \to \mathrm{Cat}_\infty$ is an n-uple monoid in Cat_∞, in the sense of Definition 2.1.10. We will abuse notation by writing $\mathcal{V}$ for $\mathcal{V}^{\otimes}_{(1,\dots,1)}$ and just saying that '$\mathcal{V}$ is a Δ^n-monoidal ∞-category'.

Remark 2.5.2. As a special case of Remark 2.1.13, the notion of Δ^n-monoidal ∞-category is equivalent to that of $\mathbb{E}_n$-monoidal ∞-category considered in [20].

Notation 2.5.3. We say a morphism in $\Delta^n := \Delta^{\times n}$ is inert or active if each of its components in Δ is inert or active, respectively, in the sense of Definition 2.1.1.

Definition 2.5.4. Suppose $\mathcal{V}$ is a Δ^n-monoidal ∞-category. Then a $\Delta^{n,\mathrm{op}}$*-algebra* in $\mathcal{V}$ is a section

$$\begin{tikzcd} \mathcal{V}^{\otimes} \arrow[d] \\ \Delta^{n,\mathrm{op}} \arrow[u, bend left, "A"] \end{tikzcd}$$

such that A takes inert morphisms in $\Delta^{n,\mathrm{op}}$ to cocartesian morphisms in $\mathcal{V}^{\otimes}$.

Remark 2.5.5. It follows from the Dunn–Lurie additivity theorem that $\Delta^{n,\mathrm{op}}$-algebras in $\mathcal{V}$ are the same thing as $\mathbb{E}_n$-algebras; see [15, Corollary A.27].

Definition 2.5.6. More generally, if $\mathcal{O}$ is an ∞-category over $\Delta^{n,\mathrm{op}}$ with a suitable notion of inert morphisms residing over the inert morphisms in $\Delta^{n,\mathrm{op}}$, we can define $\mathcal{O}$*-algebras*

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in a Δ^n-monoidal ∞-category $\mathcal{V}$ as commutative triangles

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "A"] \arrow[dr] & & \mathcal{V}^{\otimes} \arrow[dl] \\ & \Delta^{n,\mathrm{op}} & \end{tikzcd},$$

where A takes inert morphisms in $\mathcal{O}$ to cocartesian morphisms in $\mathcal{V}^{\otimes}$. In particular, this definition makes sense if $\mathcal{O}$ is a *(generalized)* Δ^n*-*∞*-operad*, in the sense of [15, Definition 5.8]. We write $\mathrm{Alg}^n_{\mathcal{O}}(\mathcal{V})$ for the full subcategory of $\mathrm{Fun}_{/\Delta^{n,\mathrm{op}}}(\mathcal{O}, \mathcal{V}^{\otimes})$ spanned by the $\mathcal{O}$-algebras.

Example 2.5.7. For any object I in $\Delta^{n,\mathrm{op}}$, the slice category $\Delta^{n,\mathrm{op}}_{/I} := ((\Delta^{\times n})_{/I})^{\mathrm{op}}$ is a generalized Δ^n-∞-operad via the forgetful functor. Algebras for $\Delta^{\mathrm{op}}_{/[1]}$ in $\mathcal{V}$ correspond to a pair of associative algebras and a bimodule between them, while $\Delta^{n,\mathrm{op}}_{/I}$-algebras where $I = (1, \dots, 1, 0, \dots, 0)$ with k 1's correspond to k-fold iterated bimodules in $\mathbb{E}_{n-k}$-algebras in $\mathcal{V}$; these are the k-morphisms in the higher Morita category. On the other hand, $\Delta^{\mathrm{op}}_{/[2]}$-algebras correspond to a triple of associative algebras A_0, A_1, A_2, together with A_i–A_j-bimodules M_{ij} for all $0 \leqslant i < j \leqslant 2$, as well as an A_1-bilinear map $M_{01} \otimes M_{12} \to M_{02}$, or equivalently a map $M_{01} \otimes_{A_1} M_{12} \to M_{02}$ of A_0–A_2-bimodules.

Example 2.5.8. Let $\mathbb{A}_{/[n]}$ denote the full subcategory of $\Delta_{/[n]}$ spanned by the morphisms $\phi\colon [m] \to [n]$ such that $\phi(i+1) - \phi(i) \leqslant 1$ for all i. For $I = (i_1, \dots, i_n)$ in Δ^n, we set $\mathbb{A}^{n,\mathrm{op}}_{/I} := \prod_{t=1}^n \mathbb{A}^{\mathrm{op}}_{/[i_t]}$; then $\mathbb{A}^{n,\mathrm{op}}_{/I}$ is a generalized Δ^n-∞-operad via the forgetful functor to $\Delta^{n,\mathrm{op}}$. A $\mathbb{A}^{\mathrm{op}}_{/[2]}$-algebra in $\mathcal{V}$ corresponds to a triple of associative algebras, A_0, A_1, A_2, together with an A_0–A_1-bimodule M_{01} and an A_1–A_2-bimodule M_{12}.

Definition 2.5.9. If S is some class of ∞-categories, we say that a Δ^n-monoidal ∞-category $\mathcal{V}^{\otimes}$ is *compatible with* S*-shaped colimits* if $\mathcal{V}$ has S-shaped colimits and the tensor product functor

$$\mathcal{V}^{\times 2} \simeq \mathcal{V}^{\otimes}_{(2,1,\dots,1)} \to \mathcal{V}$$

coming from the map $(2, 1 \dots, 1) \to (1, \dots, 1)$ preserves S-shaped colimits in each variable. (The n tensor products obtained in this way by permuting $(2, 1, \dots, 1)$ can all be shown to be equivalent, so the definition does not depend on the choice of this map.)

Definition 2.5.10. Let $\tau_I\colon \mathbb{A}^{n,\mathrm{op}}_{/I} \to \Delta^{n,\mathrm{op}}_{/I}$ be the inclusion. Composition with τ_I induces a functor $\tau_I^*\colon \mathrm{Alg}^n_{\Delta^{n,\mathrm{op}}_{/I}}(\mathcal{V}) \to \mathrm{Alg}^n_{\mathbb{A}^{n,\mathrm{op}}_{/I}}(\mathcal{V})$. If $\mathcal{V}$ is compatible with Δ^{op}-colimits, then this functor has a fully faithful left adjoint $\tau_{I,!}$. We say a $\Delta^{n,\mathrm{op}}_{/I}$-algebra is *composite* if it is in the essential image of this functor, or equivalently if the counit map $\tau_{I,!}\tau_I^* A \to A$ is an equivalence.

Example 2.5.11. A $\Delta^{\mathrm{op}}_{/[2]}$-algebra as in Example 2.5.7 is composite if and only if the morphism $M_{01} \otimes_{A_1} M_{12} \to M_{02}$ is an equivalence, i.e. if and only if the $\Delta^{\mathrm{op}}_{/[2]}$-algebra presents M_{02} as the *composite* of M_{01} and M_{12} in the higher Morita category.

Definition 2.5.12. Suppose $\mathcal{V}$ is a $\mathbb{A}^n$-monoidal ∞-category compatible with $\mathbb{A}^{n,\mathrm{op}}$-colimits. There is a functor $\mathbb{A}^{n,\mathrm{op}} \to \mathrm{Cat}_\infty$ taking I to $\mathrm{Alg}^n_{\mathbb{A}^{n,\mathrm{op}}_{/I}}(\mathcal{V})$ and a morphism $\phi\colon I \to J$ to the functor given by composition with the functor $\mathbb{A}^{n,\mathrm{op}}_{/I} \to \mathbb{A}^{n,\mathrm{op}}_{/J}$ defined by composing with ϕ. We let

$$\overline{\mathfrak{ALG}}_n(\mathcal{V}) \to \mathbb{A}^{n,\mathrm{op}}$$

be the corresponding cocartesian fibration, and write $\mathfrak{ALG}_n(\mathcal{V})$ for the full subcategory of $\overline{\mathfrak{ALG}}_n(\mathcal{V})$ spanned by the composite $\mathbb{A}^{n,\mathrm{op}}_{/I}$-algebras for all I.

We can now state the main result of [15].

Theorem 2.5.13 [15, Theorem 5.31]. *For $\mathcal{V}$ as above, the restricted functor $\mathfrak{ALG}_n(\mathcal{V}) \to \mathbb{A}^{n,\mathrm{op}}$ is a cocartesian fibration, and the corresponding functor is an n-uple category object in* Cat_∞. □

Remark 2.5.14. The assumption that $\mathcal{V}$ is compatible with $\mathbb{A}^{\mathrm{op}}$-colimits can be weakened to the assumption that $\mathcal{V}$ 'has good relative tensor products' in the sense of [15, Definition 5.18]. In particular, it is not necessary that $\mathcal{V}$ has all simplicial colimits, only those that occur when forming relative tensor products. For example, if $\mathcal{V}$ is equipped with the cocartesian symmetric monoidal structure, then the relative tensor products are given by pushouts, and it is enough to assume that $\mathcal{V}$ has finite colimits.

Remark 2.5.15. An n-uple category object in Cat_∞ gives, by viewing ∞-categories as complete Segal spaces, an $(n+1)$-uple Segal space. From this, we can obtain an $(n+1)$-fold Segal space via Proposition 2.1.6.

Notation 2.5.16. We write $\mathfrak{Alg}_n(\mathcal{V})$ for the completion of the underlying $(n+1)$-fold Segal space $U^{n+1}_{\mathrm{Seg}}\mathfrak{ALG}_n(\mathcal{V})$ of $\mathfrak{ALG}_n(\mathcal{V})$. Thus $\mathfrak{Alg}_n(\mathcal{V})$ is an $(\infty, n+1)$-category; we write $\mathfrak{alg}_n(\mathcal{V})$ for its underlying (∞, n)-category. Equivalently, $\mathfrak{alg}_n(\mathcal{V})$ is the completion of the underlying n-fold Segal space of the n-uple Segal space corresponding to the left fibration obtained by forgetting the non-cocartesian morphisms in $\mathfrak{ALG}_n(\mathcal{V})$.

We then have the following results from [15], which we state for $\mathfrak{alg}_n(\mathcal{V})$, this being the version of the higher Morita category relevant to this paper.

Theorem 2.5.17 [15, Theorem 5.49]. $\mathfrak{alg}_n(\mathcal{V})(A, B) \simeq \mathfrak{alg}_{n-1}(\mathrm{Mod}_{A,B}(\mathcal{V}))$. □

Corollary 2.5.18. *If $\mathcal{V}$ is an $\mathbb{E}_{n+m}$-monoidal ∞-category, then $\mathfrak{alg}_n(\mathcal{V})$ is $\mathbb{E}_m$-monoidal. In particular, if $\mathcal{V}$ is symmetric monoidal, so is $\mathfrak{alg}_n(\mathcal{V})$.*

We now discuss two conjectures that will be relevant to our understanding of the higher category of derived Poisson stacks.

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Conjecture 2.5.19. *Suppose $\mathcal{V}$ is a symmetric monoidal ∞-category compatible with $\mathbb{\Delta}^{n,\mathrm{op}}$-colimits. Then the symmetric monoidal (∞, n)-category $\mathfrak{alg}_n(\mathcal{V})$ has duals (in the sense of Definition 2.1.15, i.e. its objects are dualizable and all i-morphisms have adjoints for $1 \leqslant i < n$). In particular, all objects of $\mathfrak{alg}_n(\mathcal{V})$ are fully dualizable.*

Remark 2.5.20. This conjecture has been proved by Gwilliam and Scheimbauer in [13] for a closely related model $\mathfrak{alg}_n^{FA}(\mathcal{V})$ of the higher Morita (∞, n)-category, defined using factorization algebras. It is expected that there is an equivalence $\mathfrak{alg}_n^{FA}(\mathcal{V}) \simeq \mathfrak{alg}_n(\mathcal{V})$ when $\mathcal{V}$ is *pointed*, i.e. the unit of the monoidal structure is the initial object, and more generally that there is an equivalence

$$\mathfrak{alg}_n^{FA}(\mathcal{V}) \simeq \mathfrak{alg}_n(\mathcal{V}_{I/}).$$

Since the forgetful functor $\mathcal{V}_{I/} \to \mathcal{V}$ induces a symmetric monoidal functor $\mathfrak{alg}_n(\mathcal{V}_{I/}) \to \mathfrak{alg}_n(\mathcal{V})$, the result of Gwilliam–Scheimbauer together with such a hypothetical equivalence would imply that $\mathfrak{alg}_n(\mathcal{V})$ has duals (since duals and adjoints are preserved by any functor, and i-morphisms in $\mathfrak{alg}_n(\mathcal{V})$ for $i < n$ are naturally pointed, and so lift uniquely to $\mathfrak{alg}_n(\mathcal{V}_{I/})$).

Conjecture 2.5.21. *If $\mathcal{V}$ is a pointed $\mathbb{E}_n$-monoidal ∞-category (i.e. the unit is the initial object), then the $(n+1)$-fold Segal space $U_{\mathrm{Seg}}^{n+1}\mathfrak{ALG}_n(\mathcal{V})$ is complete.*

Remark 2.5.22. Completeness of an n-fold Segal space $\mathcal{X}$ is equivalent to completeness of the underlying Segal space $\mathcal{X}_{\bullet,0,\ldots,0}$ and of the $(n-1)$-fold Segal spaces of maps $\mathcal{X}(x, y)$. In the case of $U_{\mathrm{Seg}}^{n+1}\mathfrak{ALG}_n(\mathcal{V})$, both the underlying Segal space and the $(n-1)$-fold Segal spaces of maps can themselves be described as higher Morita categories (pointed if $\mathcal{V}$ is pointed). By induction, this means that it suffices to prove the conjecture in the case $n = 1$.

Remark 2.5.23. It is shown in [33, §3.2.9] that for a pointed monoidal ∞-category, the degeneracy map from the space of objects of $U_{\mathrm{Seg}}^{2}\mathfrak{ALG}_1(\mathcal{V})$ to the space of equivalences is surjective on π_0, i.e. in the pointed case every Morita equivalence comes from an equivalence of algebras in $\mathcal{V}$. (More precisely, Scheimbauer proves the analogue of this statement for the factorization algebra model, but the proof also works for the algebraic model.) Conjecture 2.5.21 then amounts to the assertion that this essentially surjective map is in fact an equivalence.

2.6. Iterated cospans as a higher Morita category

Suppose $\mathcal{C}$ is an ∞-category with finite colimits. Then we can define an (∞, n)-category $\mathrm{Cospan}_n(\mathcal{C})$ of iterated cospans in $\mathcal{C}$ as in §2.2. We can also view $\mathcal{C}$ as a symmetric monoidal ∞-category via coproducts, and hence define an (∞, n)-category $\mathfrak{alg}_n(\mathcal{C}^{\amalg})$ of $\mathbb{E}_n$-algebras in $\mathcal{C}$. In this subsection, we will show that there is an equivalence

$$\mathrm{Cospan}_n(\mathcal{C}) \simeq \mathfrak{alg}_n(\mathcal{C}^{\amalg}).$$

For I in $\mathbb{\Delta}^n$, the space $\mathrm{Cospan}_n(\mathcal{C})_I$ is defined as a subspace of the underlying space of the ∞-category $\mathrm{Fun}(\Sigma^I, \mathcal{C})$, while $\mathfrak{alg}_n(\mathcal{C}^{\amalg})_I$ is similarly obtained (before completion)

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from $\mathrm{Alg}^n_{\mathbb{\Delta}^{n,\mathrm{op}}_{/I}}(\mathcal{C}^{\amalg})$. We will prove the equivalence of (∞, n)-categories by finding a natural equivalence of ∞-categories

$$\mathrm{Alg}^n_{\mathbb{\Delta}^{n,\mathrm{op}}_{/I}}(\mathcal{C}^{\amalg}) \xrightarrow{\sim} \mathrm{Fun}(\Sigma^I, \mathcal{C})$$

and a compatible equivalence between $\mathbb{\Lambda}^{n,\mathrm{op}}_{/I}$-algebras and functors from $\mathbb{\Lambda}^I$.

We first consider the case $n = 1$, which breaks down into three steps:

(1) We define a $\mathbb{\Delta}$-∞-operad BM_i and a natural map $\mathbb{\Delta}^{\mathrm{op}}_{/[i]} \to \mathrm{BM}_i$ of generalized $\mathbb{\Delta}$-∞-operads such that for any monoidal ∞-category $\mathcal{V}$, the induced functor

$$\mathrm{Alg}^1_{\mathrm{BM}_i}(\mathcal{V}) \to \mathrm{Alg}^1_{\mathbb{\Delta}^{\mathrm{op}}_{/[i]}}(\mathcal{V})$$

is an equivalence.

(2) We define a unital $\mathbb{\Delta}$-∞-operad BM^*_i and a natural map $\mathrm{BM}_i \to \mathrm{BM}^*_i$ such that for any monoidal ∞-category $\mathcal{V}$, the induced functor

$$\mathrm{Alg}^1_{\mathrm{BM}^*_i}(\mathcal{V}) \to \mathrm{Alg}^1_{\mathrm{BM}_i}(\mathcal{V})_{I/}$$

is an equivalence.

(3) We have a natural equivalence $(\mathrm{BM}^*_i)_{[1]} \simeq \Sigma^{i,\mathrm{op}}$, so using the non-symmetric version of [20, Proposition 2.4.3.16] for any ∞-category $\mathcal{C}$ with coproducts, we have an equivalence

$$\mathrm{Alg}^1_{\mathrm{BM}^*_i}(\mathcal{C}^{\amalg}) \simeq \mathrm{Fun}(\Sigma^{i,\mathrm{op}}, \mathcal{C}).$$

The case of $n > 1$ will then be obtained from this by induction.

Definition 2.6.1. Let BM_n be the non-symmetric operad with objects x_{ij} where $0 \leqslant i \leqslant j \leqslant n$ and multimorphisms given by

$$\mathrm{Hom}(x_{i_1 j_1}, \ldots, x_{i_k j_k}; x_{st}) = \begin{cases} *, & s = i_1, j_1 = i_2, \ldots, j_k = t, k > 0, \\ *, & s = t, k = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $\mathrm{BM}^{\otimes}_n$ denotes its (non-symmetric) category of operators, there is a natural map

$$\mathbb{\Delta}^{\mathrm{op}}_{/[n]} \to \mathrm{BM}^{\otimes}_n$$

over $\mathbb{\Delta}^{\mathrm{op}}$, taking $(i_0, \ldots, i_k)$ to $(x_{i_0 i_1}, \ldots x_{i_{k-1} i_k})$.

Lemma 2.6.2. *Composition with the functor $\mathbb{\Delta}^{\mathrm{op}}_{/[n]} \to \mathrm{BM}^{\otimes}_n$ induces an equivalence*

$$\mathrm{Alg}^1_{\mathrm{BM}_n}(\mathcal{V}) \to \mathrm{Alg}^1_{\mathbb{\Delta}^{\mathrm{op}}_{/[n]}}(\mathcal{V})$$

for all monoidal ∞-categories $\mathcal{V}$.

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Proof. Without loss of generality, we may assume that $\mathcal{V}$ is compatible with colimits, as any monoidal ∞-category is a full subcategory of one that is (possibly passing to a larger universe if $\mathcal{V}$ is large and not presentable). Then we have a commutative square

$$\begin{tikzcd} \mathrm{Alg}^1_{\mathrm{BM}_n}(\mathcal{V}) \arrow[r] \arrow[d] & \mathrm{Alg}^1_{\mathbb{\Delta}^{\mathrm{op}}_{/[n]}}(\mathcal{V}) \arrow[d] \\ \mathrm{Fun}((\mathrm{BM}_n)_{[1]}, \mathcal{V}) \arrow[r] & \mathrm{Fun}((\mathbb{\Delta}^{\mathrm{op}}_{/[n]})_{[1]}, \mathcal{V}). \end{tikzcd}$$

Here the vertical arrows are both monadic right adjoints (e.g. by [10, Corollary A.5.6]), and the bottom horizontal arrow is an equivalence, since $(\mathbb{\Delta}^{\mathrm{op}}_{/[n]})_{[1]}$ and $(\mathrm{BM}_n)_{[1]}$ are both the set of pairs (i, j) with $0 \leqslant i \leqslant j \leqslant n$. To show that the top horizontal arrow is an equivalence, it now suffices by [20, Corollary 4.7.3.16] to check that the natural map between free algebras for the two monads is an equivalence. From the formula for (non-symmetric) operadic left Kan extensions (see [10, §A.4]), we see that the corresponding monads are given by

$$T_{\mathbb{\Delta}^{\mathrm{op}}_{/[n]}}\Phi(i, i') \simeq \coprod_{k=0}^{\infty} \coprod_{\substack{(j_0,\ldots,j_k) \\ i=j_0, j_k=i'}} \Phi(j_0, j_1) \otimes \cdots \otimes \Phi(j_{k-1}, j_k) \simeq T_{\mathrm{BM}_n}\Phi,$$

which gives the desired equivalence. □

Definition 2.6.3. Let BM^*_n be the non-symmetric operad with objects x_{ij} with $0 \leqslant i \leqslant j \leqslant n$, and multimorphisms given by

$$\mathrm{Hom}(x_{i_1 j_1}, \ldots, x_{i_k j_k}; x_{st}) = \begin{cases} *, & s \leqslant i_1 \leqslant j_1 \leqslant i_2 \leqslant \cdots \leqslant j_k \leqslant t, k > 0, \\ *, & s \leqslant t, k = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

There is an obvious map $\pi_n \colon \mathrm{BM}_n \to \mathrm{BM}^*_n$. We let $\mathrm{BM}^{*,\otimes}_n \to \mathbb{\Delta}^{\mathrm{op}}$ be the category of operators for BM^*_n and denote the induced map $\mathrm{BM}^{\otimes}_n \to \mathrm{BM}^{*,\otimes}_n$ also by π_n.

Remark 2.6.4. The operad BM^*_n is unital, i.e. every object has a unique nullary operation. By the non-symmetric variant of [20, Proposition 2.3.1.11], this means that for every monoidal ∞-category $\mathcal{V}$, the forgetful functor

$$\mathrm{Alg}^1_{\mathrm{BM}^*_n}(\mathcal{V})_{I/} \simeq \mathrm{Alg}^1_{\mathrm{BM}^*_n}(\mathcal{V}_{I/}) \to \mathrm{Alg}^1_{\mathrm{BM}^*_n}(\mathcal{V})$$

is an equivalence. In particular, the unit I, equipped with its unique BM^*_n-algebra structure, is initial in $\mathrm{Alg}^1_{\mathrm{BM}^*_n}(\mathcal{V})$.

Proposition 2.6.5. *The functor π_n induces an equivalence*

$$\mathrm{Alg}_{\mathrm{BM}^*_n}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{BM}_n}(\mathcal{V})_{I/}$$

for every monoidal ∞-category $\mathcal{V}$.

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Proof. Since $I \in \mathrm{Alg}_{\mathrm{BM}_n}(\mathcal{V})$ is the image of the initial object of $\mathrm{Alg}_{\mathrm{BM}_n^*}(\mathcal{V})$, the functor

$$\pi_n^*\colon \mathrm{Alg}_{\mathrm{BM}_n^*}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{BM}_n}(\mathcal{V})$$

factors uniquely through the forgetful functor from $\mathrm{Alg}_{\mathrm{BM}_n}(\mathcal{V})_{I/}$.

We may again assume, without loss of generality, that $\mathcal{V}$ is compatible with small colimits. Then we have a commutative square

$$\begin{tikzcd}
\mathrm{Alg}^1_{\mathrm{BM}_n}(\mathcal{V})_{I/} \arrow[r] \arrow[d] & \mathrm{Alg}^1_{\mathrm{BM}_n^*}(\mathcal{V}) \arrow[d] \\
\mathrm{Fun}((\mathrm{BM}_n)_{[1]}, \mathcal{V}) \arrow[r] & \mathrm{Fun}((\mathrm{BM}_n^*)^{\simeq}_{[1]}, \mathcal{V}).
\end{tikzcd}$$

Here the vertical arrows are both monadic right adjoints; for the left one, this is because it factors as a composite $\mathrm{Alg}^1_{\mathrm{BM}_n}(\mathcal{V})_{I/} \to \mathrm{Alg}^1_{\mathrm{BM}_n}(\mathcal{V}) \to \mathrm{Fun}((\mathrm{BM}_n)_{[1]}, \mathcal{V})$, where both functors are not only monadic right adjoints but also preserve sifted colimits. Moreover, the bottom horizontal arrow is clearly an isomorphism (note that we use the underlying *groupoid* of $(\mathrm{BM}_n^*)_{[1]}$). Therefore, we may again use [20, Corollary 4.7.3.16] to show that the top horizontal morphism is an equivalence by comparing the free algebras for the two monads. The left adjoint to the left-hand functor takes Φ to $F_{\mathrm{BM}_n}(\Phi) \amalg I$, where the coproduct is taken in BM_n-algebras. The formula for F_{BM_n} identifies I with $F_{\mathrm{BM}_n}(\delta)$, where

$$\delta(i, j) \simeq \begin{cases} I, & j = i + 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since F_{BM_n} preserves colimits, this means we have

$$F_{\mathrm{BM}_n}(\Phi) \amalg I \simeq F_{\mathrm{BM}_n}(\Phi \amalg \delta),$$

and so

$$\begin{aligned}
(F_{\mathrm{BM}_n}(\Phi) \amalg I)(i, i') &\simeq \coprod_{k=0}^{\infty} \coprod_{\substack{(j_0,\dots,j_k) \\ i=j_0,\, j_k=i'}} (\Phi \amalg \delta)(j_0, j_1) \times \cdots \times (\Phi \amalg \delta)(j_{k-1}, j_k) \\
&\simeq \coprod_{k=0}^{\infty} \coprod_{\substack{(j_0,\dots,j_k) \\ i=j_0,\, j_k=i'}} \coprod_{\substack{S \subseteq \{1,\dots,k\} \\ j_s = j_{s-1}+1,\, s\in S}} \bigotimes_{s \notin S} \Phi(j_{s-1}, j_s).
\end{aligned}$$

In this coproduct, we have a term of the form $\Phi(i_1, j_1) \otimes \cdots \otimes \Phi(i_k, j_k)$ whenever

$$i \leqslant i_1 \leqslant j_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant j_k \leqslant i',$$

corresponding to $(i, i+1, \dots i_1 - 1, i_1, i_2, i_2 + 1, \dots, j_k, j_k + 1, \dots, j - 1, i')$ with S identifying the pairs not of the form (i_t, j_t). This gives equivalences

$$(F_{\mathrm{BM}_n}(\Phi) \amalg I)(i, i') \simeq I \amalg \coprod_{k=1}^{\infty} \coprod_{i \leqslant i_1 \leqslant \cdots \leqslant j_k \leqslant i'} \Phi(i_1, j_1) \otimes \cdots \otimes \Phi(i_k, j_k) \simeq F_{\mathrm{BM}_n^*}(i, i'),$$

where the second equivalence again comes from the formula for operadic Kan extensions. □

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Corollary 2.6.6. *If $\mathcal{C}$ is an ∞-category with finite coproducts, then there is a natural equivalence of ∞-categories*

$$\mathrm{Alg}^1_{\Delta^{\mathrm{op}}_{/[n]}}(\mathcal{C}^{\amalg}) \simeq \mathrm{Fun}(\Sigma^{n,\mathrm{op}}, \mathcal{C}).$$

Proof. Since $\mathcal{C}^{\amalg}$ is unital, by Lemma 2.6.2 and Proposition 2.6.5, we have natural equivalences

$$\mathrm{Alg}^1_{\Delta^{\mathrm{op}}_{/[n]}}(\mathcal{C}^{\amalg}) \simeq \mathrm{Alg}^1_{\mathrm{BM}^*_n}(\mathcal{C}^{\amalg}).$$

Using the non-symmetric analogue of [20, Proposition 2.4.3.9], it follows that the restriction functor $\mathrm{Alg}^1_{\mathrm{BM}^*_n}(\mathcal{C}^{\amalg}) \to \mathrm{Fun}((\mathrm{BM}^*_n)_{[1]}, \mathcal{C})$ is an equivalence. (Alternatively, we can use [20, Proposition 2.4.3.9] together with the formula for symmetrizations of ordinary non-symmetric operads from [10, Corollary 3.7.8], which does not change the fibre over [1].) Finally, observe that by definition, $(\mathrm{BM}^*_n)_{[1]}$ is precisely the partially ordered set $\Sigma^{n,\mathrm{op}}$. □

Remark 2.6.7. A variant of the same argument gives a similar equivalence

$$\mathrm{Alg}^1_{\Lambda^{\mathrm{op}}_{/[n]}}(\mathcal{C}^{\amalg}) \simeq \mathrm{Fun}(\Lambda^{n,\mathrm{op}}, \mathcal{C}),$$

compatible with that of Corollary 2.6.6 in the sense that we have a commutative square

$$\begin{tikzcd}
\mathrm{Alg}^1_{\Delta^{\mathrm{op}}_{/[n]}}(\mathcal{C}^{\amalg}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Fun}(\Sigma^{n,\mathrm{op}}, \mathcal{C}) \arrow[d] \\
\mathrm{Alg}^1_{\Lambda^{\mathrm{op}}_{/[n]}}(\mathcal{C}^{\amalg}) \arrow[r, "\sim"] & \mathrm{Fun}(\Lambda^{n,\mathrm{op}}, \mathcal{C}).
\end{tikzcd}$$

Passing to left adjoints, we see that under these equivalences, the composite $\Delta^{\mathrm{op}}_{/[n]}$-algebras in $\mathcal{C}^{\amalg}$ correspond precisely to the functors $\Sigma^{n,\mathrm{op}} \to \mathcal{C}$ that are left Kan extended from $\Lambda^{n,\mathrm{op}}$. Since the equivalences are natural in $[n] \in \Delta$, this implies the following.

Corollary 2.6.8. *If $\mathcal{C}$ is an ∞-category with finite colimits, we have a natural equivalence*

$$\mathfrak{ALG}_1(\mathcal{C}^{\amalg}) \to \mathrm{COSPAN}^+_1(\mathcal{C})$$

of category objects in Cat_∞, *and so an equivalence*

$$\mathfrak{alg}_1(\mathcal{C}^{\amalg}) \to \mathrm{Cospan}_1(\mathcal{C})$$

of ∞-categories.

We can now prove the general case by induction.

Corollary 2.6.9. *If $\mathcal{C}$ is an ∞-category with finite coproducts, then we have a natural equivalence*

$$\mathfrak{ALG}_n(\mathcal{C}^{\amalg}) \simeq \mathrm{COSPAN}^+_n(\mathcal{C})$$

of n-uple category objects in Cat_∞.

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Proof. If $\mathcal{V}$ is a $\mathbb{A}^{n+m}$-monoidal ∞-category, then $\mathrm{Alg}_{\mathbb{A}^{m,\mathrm{op}}_{/J}}(\mathcal{V})$ has a natural $\mathbb{A}^n$-monoidal structure, given objectwise by the tensor product in $\mathcal{V}$, such that there is a natural equivalence

$$\mathrm{Alg}^n_{\mathbb{A}^{n,\mathrm{op}}_{/I}}(\mathrm{Alg}^m_{\mathbb{A}^{m,\mathrm{op}}_{/J}}(\mathcal{V})) \simeq \mathrm{Alg}^{n+m}_{\mathbb{A}^{n+m,\mathrm{op}}_{/(I,J)}}(\mathcal{V}),$$

by [15, Corollary A.77].

Suppose we have a natural equivalence

$$\mathrm{Alg}_{\mathbb{A}^{n-1,\mathrm{op}}_{/J}}(\mathcal{C}^{\amalg}) \simeq \mathrm{Fun}(\Sigma^{J,\mathrm{op}}, \mathcal{C}).$$

The canonical symmetric monoidal structure on the left-hand side corresponds to the cocartesian structure on the right, since this is the unique symmetric monoidal structure given objectwise in $\Sigma^{J,\mathrm{op}}$ by the coproduct in $\mathcal{C}$. For $I = ([i], J)$ in $\mathbb{A}^n$, using Corollary 2.6.6, we then have a natural equivalence

$$\begin{aligned}\mathrm{Alg}^n_{\mathbb{A}^{n,\mathrm{op}}_{/I}}(\mathcal{C}^{\amalg}) &\simeq \mathrm{Alg}^1_{\mathbb{A}^{\mathrm{op}}_{/[i]}}(\mathrm{Alg}^{n-1}_{\mathbb{A}^{n-1,\mathrm{op}}_{/J}}(\mathcal{C}^{\amalg})) \simeq \mathrm{Alg}^1_{\mathbb{A}^{\mathrm{op}}_{/[i]}}(\mathrm{Fun}(\Sigma^{J,\mathrm{op}}, \mathcal{C})^{\amalg}) \\ &\simeq \mathrm{Fun}(\Sigma^{i,\mathrm{op}}, \mathrm{Fun}(\Sigma^{J,\mathrm{op}}, \mathcal{C})) \simeq \mathrm{Fun}(\Sigma^{I,\mathrm{op}}, \mathcal{C}).\end{aligned}$$

As in Remark 2.6.7, we also have a compatible equivalence $\mathrm{Alg}^n_{\mathbb{A}^{n,\mathrm{op}}_{/I}}(\mathcal{C}^{\amalg}) \simeq \mathrm{Fun}(\mathbb{A}^{I,\mathrm{op}}, \mathcal{C})$ and hence a natural equivalence

$$\mathfrak{ALG}_n(\mathcal{C}^{\amalg})_I \simeq \mathrm{COSPAN}^+_n(\mathcal{C})_I,$$

as required. □

Passing to underlying n-fold Segal spaces, since the symmetric monoidal structures are defined by delooping in both cases, we get the following corollary.

Corollary 2.6.10. *If $\mathcal{C}$ is an ∞-category with finite coproducts, then we have an equivalence of symmetric monoidal (∞, n)-categories*

$$\mathfrak{alg}_n(\mathcal{C}^{\amalg}) \simeq \mathrm{Cospan}_n(\mathcal{C}).$$

3. Higher categories of coisotropic correspondences

Our goal in this section is to introduce the notion of (iterated) coisotropic correspondences, and to construct higher categories where these are the (higher) morphisms. In §3.1, we give a brief outline of the theory of formal localization in derived algebraic geometry, as developed in [7]. We then review the notions of Poisson structures on derived stacks and coisotropic structures on morphisms of derived stacks, also from [7], in §3.2. We will avoid going into the technical details of the various constructions, and we refer the reader to [7] and to [26] for a more complete and precise treatment of the subject. In §3.3, we first define coisotropic correspondences between derived Poisson stacks, and then use the results of the previous section to construct (∞, n)-categories of derived Poisson stacks and iterated coisotropic correspondences. We finish by briefly discussing the expected relation of our higher categories to higher categories of symplectic derived stacks in §3.4.

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3.1. Derived stacks and formal localization

We fix a base field k of characteristic 0. Let $\mathsf{cdga}^{\leqslant 0}$ denote the ∞-category of commutative algebras in non-positively graded cochain complexes of k-modules. We write dSt for the ∞-category of *derived stacks*, i.e. étale sheaves of (large) spaces on $\mathsf{cdga}^{\leqslant 0}$. Representable (pre)sheaves give a fully faithful functor $(\mathsf{cdga}^{\leqslant 0})^{\mathrm{op}} \to \mathrm{dSt}$, and we write $\mathrm{dAff} \simeq (\mathsf{cdga}^{\leqslant 0})^{\mathrm{op}}$ for its image; objects of dAff will be called *derived affine schemes*.

We denote by $\mathbf{dArt} \subset \mathrm{dSt}$ the full subcategory of derived Artin stacks locally of finite presentation. This is a convenient ∞-category of derived stacks X, which admit perfect cotangent complexes $\mathbb{L}_X$. The dual will be denoted by $\mathbb{T}_X$.

Consider the inclusion functor

$$i : \mathsf{calg}^{\mathrm{red}} \to \mathsf{cdga}^{\leqslant 0},$$

where $\mathsf{calg}^{\mathrm{red}}$ is the full sub-∞-category of discrete reduced commutative k-algebras. The ∞-category $\mathsf{calg}^{\mathrm{red}}$ can be endowed with the étale topology, and we let $\mathsf{St}_{\mathrm{red}}$ be the ∞-category of stacks on the associated site. By restriction, we immediately get a functor of ∞-categories

$$i^* : \mathrm{dSt} \to \mathsf{St}_{\mathrm{red}},$$

which has both a left adjoint $i_!$ and a right adjoint i_*, as i is both continuous and cocontinuous.

Definition 3.1.1.

- The functor
$$(-)_{\mathrm{dR}} := i_* i^* : \mathrm{dSt} \to \mathrm{dSt}$$
is called the *de Rham stack functor*.
- The functor
$$(-)_{\mathrm{red}} := i_! i^* : \mathrm{dSt} \to \mathrm{dSt}$$
is called the *reduced stack functor*.

Note that by adjunction, for any $X \in \mathrm{dSt}$, we have canonical morphisms $X \to X_{\mathrm{dR}}$ and $X_{\mathrm{red}} \to X$. One can prove that if $X \in \mathrm{dSt}$ is a derived stack, then X_{dR} is simply given by

$$\begin{array}{rccc} X_{\mathrm{dR}} : & (\mathrm{dAff})^{\mathrm{op}} & \longrightarrow & \mathcal{S} \\ & A & \longmapsto & X(A^{\mathrm{red}}), \end{array}$$

where A^{red} is the reduced k-algebra $\mathrm{H}^0(A)/\mathrm{Nilp}(\mathrm{H}^0(A))$. On the other hand, if $\operatorname{Spec} A \in \mathrm{dAff}$ is affine, then $(\operatorname{Spec} A)_{\mathrm{red}} \simeq \operatorname{Spec}(A^{\mathrm{red}})$.

The theory of formal localization mainly deals with the study of the projection $X \to X_{\mathrm{dR}}$. This map is of particular interest, as its fibres are precisely the formal completions of X at its points. More concretely, let $\operatorname{Spec} A \to X_{\mathrm{dR}}$ be an A-point of X_{dR}, and let X_A be the fibre product

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] & X_{\mathrm{dR}}. \end{tikzcd}$$

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It can be shown (see [7, Proposition 2.1.8]) that X_A is equivalent to the formal completion of the map $\mathsf{Spec}\, A^{\mathrm{red}} \to X \times \mathsf{Spec}\, A$. This is easily seen to imply that $(X_A)_{\mathrm{red}} \simeq \mathsf{Spec}\, A^{\mathrm{red}}$. In other words, one can think of X_A as a sort of 'formal thickening' of $\mathsf{Spec}\, A^{\mathrm{red}}$. By the properties of the de Rham stack, the map $\mathsf{Spec}\, A \to X_{\mathrm{dR}}$ corresponds to a map $\mathsf{Spec}\, A^{\mathrm{red}} \to X$, which is induced by the map $\mathsf{Spec}\, A^{\mathrm{red}} \simeq (X_A)_{\mathrm{red}} \to X_A$, so that we get a commutative diagram

$$\begin{tikzcd} & X_A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathsf{Spec}\, A^{\mathrm{red}} \arrow[ur] \arrow[r] & \mathsf{Spec}\, A \arrow[r] & X_{\mathrm{dR}} \end{tikzcd}$$

of derived stacks, where the square on the right is cartesian.

The upshot of the above discussion is that we can think of $X \to X_{\mathrm{dR}}$ as a family of formal derived stacks and, more explicitly, as the family of formal completions of X at its closed points. By the general theorem of [19], these formal completions correspond to dg Lie algebras. However, these dg Lie algebras do not extend to form a sheaf of dg Lie algebras over X_{dR}. Instead, the Chevalley–Eilenberg complexes of these dg Lie algebras extend globally, thus producing a sheaf of graded mixed algebras over X_{dR}.

We are interested in studying prestacks on X_{dR}, that is to say functors out of the ∞-category $(\mathrm{dAff}_{/X_{\mathrm{dR}}})^{op}$. Let $\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}}$ be the ∞-category of graded mixed dg modules (i.e. the ∞-category underlying the model category of these considered in [7]). For notational convenience, we give the following definition.

Definition 3.1.2. Let X be a derived stack.

- We denote by $\mathcal{D}_X$ the ∞-category of prestacks of ind-objects in graded mixed dg modules on X_{dR}, that is to say
$$\mathcal{D}_X := \mathrm{Fun}((\mathrm{dAff}_{/X_{\mathrm{dR}}})^{\mathrm{op}}, \mathsf{Ind}(\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}})).$$
We consider this as a symmetric monoidal ∞-category with respect to the pointwise tensor product coming from $\mathsf{Ind}(\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}})$.
- We denote by $\mathcal{A}_X$ the ∞-category of prestacks of graded mixed cdgas in ind-objects on X_{dR}, that is to say
$$\mathcal{A}_X := \mathrm{Fun}((\mathrm{dAff}_{/X_{\mathrm{dR}}})^{\mathrm{op}}, \mathrm{CAlg}(\mathsf{Ind}(\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}}))).$$
Equivalently, since the tensor product on $\mathcal{D}_X$ is pointwise, we have
$$\mathcal{A}_X \simeq \mathrm{CAlg}(\mathcal{D}_X).$$

Note that both assignments $X \mapsto \mathcal{D}_X$ and $X \mapsto \mathcal{A}_X$ are functorial, in the sense that if we have a map $f : X \to Y$ of derived stacks, we immediately get a functor $f^* : \mathcal{D}_Y \to \mathcal{D}_X$ (and similarly for $\mathcal{A}_X$), simply given by pullback of prestacks. Equivalently, we can encode these functors into cocartesian fibrations $\mathcal{D} \to \mathrm{dSt}^{\mathrm{op}}$ and $\mathcal{A} \to \mathrm{dSt}^{\mathrm{op}}$.

Consider the following ind-object in the ∞-category $\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}}$:

$$k(\infty) := \{k(0) \to k(1) \to \cdots \to k(i) \to k(i+1) \to \cdots\},$$

where $k(i)$ is the graded mixed module simply given by k sitting in degree 0 and weight i, together with the trivial mixed structure. The maps $k(i) \to k(i+1)$ are the canonical morphisms in the ∞-category of graded mixed modules.

The ind-object $k(\infty)$ is a commutative algebra in the category $\mathbf{Ind}(\epsilon\text{-dg}^{\mathrm{gr}})$, and it can be used to define two fundamental prestacks on X_{dR}.

Definition 3.1.3. (1) The *twisted crystalline structure sheaf* of X is defined to be

$$\begin{array}{rccc} \mathbb{D}^{\infty}_{X_{\mathrm{dR}}}: & (\mathrm{dAff}_{/X_{\mathrm{dR}}})^{\mathrm{op}} & \longrightarrow & \mathrm{CAlg}(\mathbf{Ind}(\epsilon\text{-dg}^{\mathrm{gr}})) \\ & (\mathbf{Spec}\, A \to X_{\mathrm{dR}}) & \longmapsto & \mathrm{DR}(A^{\mathrm{red}}/A) \otimes_k k(\infty). \end{array}$$

(2) The *twisted prestack of principal parts* of X is defined as

$$\begin{array}{rccc} \mathcal{P}^{\infty}_{X}: & (\mathrm{dAff}_{/X_{\mathrm{dR}}})^{\mathrm{op}} & \longrightarrow & \mathrm{CAlg}(\mathbf{Ind}(\epsilon\text{-dg}^{\mathrm{gr}})) \\ & (\mathbf{Spec}\, A \to X_{\mathrm{dR}}) & \longmapsto & \mathrm{DR}(\mathbf{Spec}\, A^{\mathrm{red}}/X_A) \otimes_k k(\infty). \end{array}$$

Both prestacks $\mathbb{D}^{\infty}_{X_{\mathrm{dR}}}$ and $\mathcal{P}^{\infty}_{X}$ are functorial in X, in the sense that they can be interpreted as sections of the cocartesian fibration $\mathcal{A} \to \mathrm{dSt}^{\mathrm{op}}$. We will denote the corresponding sections by $\mathbb{D}^{\infty}$ and $\mathcal{P}^{\infty}$, respectively. Note however that given a map of derived stacks $f: X \to Y$, we have $f^*\mathbb{D}^{\infty}_{Y_{\mathrm{dR}}} \simeq \mathbb{D}^{\infty}_{X_{\mathrm{dR}}}$, but in general $f^*\mathcal{P}^{\infty}_{Y}$ is not equivalent to $\mathcal{P}^{\infty}_{X}$. In other words, the section $\mathbb{D}^{\infty}$ is cocartesian, while $\mathcal{P}^{\infty}$ is not. We remark however that there is always an induced map $f^*_{\mathcal{P}}: f^*\mathcal{P}^{\infty}_{Y} \to \mathcal{P}^{\infty}_{X}$.

For every derived stack X, there is a natural map $\mathbb{D}^{\infty}_{X_{\mathrm{dR}}} \to \mathcal{P}^{\infty}_{X}$ in the category $\mathcal{A}_X$, which one can view as endowing $\mathcal{P}^{\infty}_{X}$ with the structure of a $\mathbb{D}^{\infty}_{X_{\mathrm{dR}}}$-algebra.

For notational convenience, we give the following definition.

Definition 3.1.4. The cocartesian fibration associated to the functor

$$X \mapsto \mathrm{Mod}_{\mathbb{D}^{\infty}_{X_{\mathrm{dR}}}}(\mathcal{D}_X)$$

will be denoted as $\mathcal{M} \to \mathrm{dSt}^{\mathrm{op}}$.

Note that by definition, we have an equivalence $\mathcal{A}_X \simeq \mathrm{CAlg}(\mathcal{D}_X)$, which in turn gives an equivalence

$$(\mathcal{A}_X)_{\mathbb{D}^{\infty}_{X_{\mathrm{dR}}}/} \simeq \mathrm{CAlg}(\mathrm{Mod}_{\mathbb{D}^{\infty}_{X_{\mathrm{dR}}}}(\mathcal{D}_X)) \simeq \mathrm{CAlg}(\mathcal{M}_X).$$

Thus, $\mathcal{P}^{\infty}_{X}$ can be viewed as an object of $\mathrm{CAlg}(\mathcal{M}_X)$, and $\mathcal{P}^{\infty}$ as a section of the cocartesian fibration $\mathcal{M}_{\mathrm{CAlg}} \to \mathrm{dSt}^{\mathrm{op}}$ corresponding to $\mathrm{CAlg}(\mathcal{M}_{(-)})$.

By a slight abuse of notation, we denote by $\mathcal{M}_{\mathrm{CAlg}} \to \mathbf{dArt}^{\mathrm{op}}$ the restriction of the cocartesian fibration $\mathcal{M}_{\mathrm{CAlg}} \to \mathrm{dSt}^{\mathrm{op}}$ to the full subcategory of derived Artin stacks locally of finite presentation. The following is the key input we will need to apply the results of the previous section to coisotropic correspondences.

Proposition 3.1.5. *Suppose that the diagram*

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{g} & Z \end{array}$$

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is a pullback of derived Artin stacks locally of finite presentation. Then the induced diagram

$$\begin{tikzcd} f^*q^*\mathcal{P}_Z^\infty \arrow[r] \arrow[d] & f^*\mathcal{P}_X^\infty \arrow[d] \\ p^*\mathcal{P}_Y^\infty \arrow[r] & \mathcal{P}_W^\infty \end{tikzcd}$$

is a pushout in the ∞*-category* $\mathrm{CAlg}(\mathcal{M}_W)$*.*

Proof. Since $\mathrm{CAlg}(\mathcal{M}_W) \simeq \mathrm{CAlg}(\mathcal{D}_W)_{\mathbb{D}^\infty_{W_{\mathrm{dR}}}/}$ and the forgetful functor to $\mathrm{CAlg}(\mathcal{D}_W)$ detects weakly contractible colimits, it suffices to show that the underlying diagram in $\mathrm{CAlg}(\mathcal{D}_W)$ is a pushout. But by definition, $\mathcal{D}_W$ is a functor ∞-category, equipped with the pointwise symmetric monoidal structure, and so we have an equivalence

$$\mathrm{CAlg}(\mathcal{D}_W) \simeq \mathrm{Fun}((\mathrm{dAff}_{/W_{\mathrm{dR}}})^{\mathrm{op}}, \mathrm{CAlg}(\mathbf{Ind}(\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}}))).$$

It is therefore enough to check that the diagram is a pushout when evaluated at each object of $\mathrm{dAff}_{/W_{\mathrm{dR}}}$. In other words, given a point $\mathbf{Spec}\,A \to W_{\mathrm{dR}}$, we need to show that the diagram

$$\begin{tikzcd} \mathcal{P}_Z^\infty(A) \arrow[r] \arrow[d] & \mathcal{P}_X^\infty(A) \arrow[d] \\ \mathcal{P}_Y^\infty(A) \arrow[r] & \mathcal{P}_W^\infty(A) \end{tikzcd}$$

is a pushout in $\mathrm{CAlg}(\mathbf{Ind}(\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}}))$. Unravelling the definition of the twisted prestack of principal parts, we are left with proving that the diagram

$$\begin{tikzcd} \mathrm{DR}(\mathbf{Spec}\,A^{\mathrm{red}}/Z_A) \arrow[r] \arrow[d] & \mathrm{DR}(\mathbf{Spec}\,A^{\mathrm{red}}/X_A) \arrow[d] \\ \mathrm{DR}(\mathbf{Spec}\,A^{\mathrm{red}}/Y_A) \arrow[r] & \mathrm{DR}(\mathbf{Spec}\,A^{\mathrm{red}}/W_A) \end{tikzcd}$$

is a pushout of graded mixed commutative algebras. The forgetful functor

$$\mathrm{CAlg}(\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}}) \longrightarrow \mathrm{CAlg}(\mathrm{dg}^{\mathrm{gr}})$$

creates colimits. Hence it suffices to show that the above square is a pushout in the category of graded commutative algebras. The derived stacks X_A, Y_A, Z_A, W_A are algebraizable in the sense of [7, Definition 2.2.1], so by [7, Proposition 2.2.7], we have an equivalence $\mathrm{DR}(\mathbf{Spec}\,A^{\mathrm{red}}/X_A) \simeq \mathrm{Sym}_{A^{\mathrm{red}}}(\mathbb{L}_{\mathbf{Spec}\,A^{\mathrm{red}}/X_A}[-1])$ of graded commutative algebras and similarly for other stacks.

Therefore we need to prove that the square

$$\begin{tikzcd} \mathrm{Sym}_{A^{\mathrm{red}}}(\mathbb{L}_{\mathbf{Spec}\,A^{\mathrm{red}}/Z_A}[-1]) \arrow[r] \arrow[d] & \mathrm{Sym}_{A^{\mathrm{red}}}(\mathbb{L}_{\mathbf{Spec}\,A^{\mathrm{red}}/X_A}[-1]) \arrow[d] \\ \mathrm{Sym}_{A^{\mathrm{red}}}(\mathbb{L}_{\mathbf{Spec}\,A^{\mathrm{red}}/Y_A}[-1]) \arrow[r] & \mathrm{Sym}_{A^{\mathrm{red}}}(\mathbb{L}_{\mathbf{Spec}\,A^{\mathrm{red}}/W_A}[-1]) \end{tikzcd}$$

is a pushout of graded commutative algebras. Since the functor $\mathrm{Sym}_{A^{\mathrm{red}}}(-)$ commutes with colimits, it is enough to prove that

$$\begin{array}{ccc} \mathbb{L}_{\mathrm{Spec}\, A^{\mathrm{red}}/Z_A} & \longrightarrow & \mathbb{L}_{\mathrm{Spec}\, A^{\mathrm{red}}/X_A} \\ \downarrow & & \downarrow \\ \mathbb{L}_{\mathrm{Spec}\, A^{\mathrm{red}}/Y_A} & \longrightarrow & \mathbb{L}_{\mathrm{Spec}\, A^{\mathrm{red}}/W_A} \end{array}$$

is a pushout square. But this follows directly from [24, Lemma 3.5]. □

Corollary 3.1.6. *The section* $\mathcal{P}^\infty\colon \mathrm{dArt}^{\mathrm{op}} \to \mathcal{M}_{\mathrm{CAlg}}$ *preserves finite colimits.*

Proof. Proposition 3.1.5 implies, via Lemma 2.4.9, that $\mathcal{P}^\infty$ preserves pushouts. It thus only remains to show that it preserves the initial object, i.e. that $\mathcal{P}^\infty_{\mathrm{Spec}\, k}$ is the initial object of $\mathrm{CAlg}(\mathcal{M}_{\mathrm{Spec}\, k})$, or equivalently that the canonical map $\mathbb{D}^\infty_{(\mathrm{Spec}\, k)_{\mathrm{dR}}} \to \mathcal{P}^\infty_{\mathrm{Spec}\, k}$ is an equivalence. The functor $\mathbb{D}^\infty_{(\mathrm{Spec}\, k)_{\mathrm{dR}}}$ sends $\mathrm{Spec}\, A \in \mathrm{dAff}$ to $\mathrm{DR}(A^{\mathrm{red}}/A) \otimes_k k(\infty)$. Similarly, the functor $\mathcal{P}^\infty_{\mathrm{Spec}\, k}$ sends $\mathrm{Spec}\, A \in \mathrm{dAff}$ to $\mathrm{DR}(A^{\mathrm{red}}/(\mathrm{Spec}\, k)_A) \otimes_k k(\infty)$. But by definition, $(\mathrm{Spec}\, k)_A \cong \mathrm{Spec}\, A$, so the map $\mathbb{D}^\infty_{(\mathrm{Spec}\, k)_{\mathrm{dR}}} \to \mathcal{P}^\infty_{\mathrm{Spec}\, k}$ is an equivalence. □

3.2. Poisson and coisotropic structures

In this subsection, we recall the notions of Poisson and coisotropic structures in the context of derived algebraic geometry. Let **dg** be the symmetric monoidal model category of cochain complexes of k-modules. We will often work with an arbitrary symmetric monoidal ∞-category $\mathcal{C}$ satisfying a set of assumptions (see [7, Section 1.1]; in particular, we refer there for a proof that the ∞-categories we consider here satisfy the assumptions).

Assumption 3.2.1. Let **C** be a symmetric monoidal model category, which is combinatorial as a model category. Assume the following:

(1) **C** is tensored over **dg** compatibly with the model and symmetric monoidal structures.

(2) For any cofibration $j\colon X \to Y$, any object $A \in \mathbf{C}$ and any morphism $u\colon A \otimes X \to C$, the pushout square

$$\begin{array}{ccc} C & \longrightarrow & D \\ {\scriptstyle u}\uparrow & & \uparrow \\ A \otimes X & \xrightarrow{\mathrm{id}\otimes j} & A \otimes Y \end{array}$$

is a homotopy pushout.

(3) For a cofibrant object $X \in \mathbf{C}$, the functor $X \otimes (-)\colon \mathbf{C} \to \mathbf{C}$ preserves weak equivalences.

(4) **C** is a tractable model category.

(5) Weak equivalences in **C** are stable under filtered colimits and finite products.

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We denote by $\mathcal{C}$ the localization of $\mathbf{C}$ with respect to weak equivalences, which is a k-linear presentably symmetric monoidal ∞-category. We will abuse notation and just say '$\mathcal{C}$ satisfies Assumption 3.2.1', without explicitly mentioning the model category $\mathbf{C}$.

Recall that $\mathbb{P}_{s+1}$ is the dg operad controlling s-shifted Poisson algebras (i.e. commutative algebras together with a compatible Lie bracket of degree $-s$); the notation is chosen so that $\mathbb{P}_n$ is the cohomology of the little discs operad $\mathbb{E}_n$ for $n \geqslant 2$. The operad $\mathbb{P}_{s+1}$ can be used to define Poisson structures on commutative algebras (see [22, Theorem 3.2], [7, Theorem 1.4.9] and [23, Theorem 4.5]).

Definition 3.2.2. Let $\mathcal{C}$ be a k-linear symmetric monoidal ∞-category satisfying Assumption 3.2.1. We define $\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C})$ to be the localization of the category of $\mathbb{P}_{s+1}$-algebras in $\mathbf{C}$ along weak equivalences.

By construction, we have a forgetful functor

$$\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C}) \longrightarrow \mathrm{CAlg}(\mathcal{C}).$$

Definition 3.2.3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category as above. Let $A \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra. The space $\mathrm{Pois}(A, s)$ of s-shifted Poisson structures on A is the fibre of

$$\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$$

taken at the point corresponding to the given commutative structure on A.

Note that the operad $\mathbb{P}_{s+1}$ has an involution given by changing the sign of the bracket, which preserves the map from the commutative operad. Therefore, it induces an involution on $\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C})$, which we consider as passing to the opposite $\mathbb{P}_{s+1}$-algebra and, similarly, an involution on $\mathrm{Pois}(A, s)$ that we denote by $\pi_A \mapsto -\pi_A$.

Let X be a derived Artin stack locally of finite presentation. Recall from the previous section that one can associate to X an ∞-category $\mathcal{M}_X$, which in the language of [7] corresponds to $\mathbb{D}^{\infty}_{X_{\mathrm{dR}}}$-modules. Moreover, one has a canonical object in $\mathrm{CAlg}(\mathcal{M}_X)$, given by $\mathcal{P}^{\infty}_X$. We can define Poisson structures on X in the following way (see [7, Theorem 3.1.2]).

Definition 3.2.4. With notations as above, the space $\mathrm{Pois}(X, s)$ of s-shifted Poisson structures on X is defined to be the space $\mathrm{Pois}(\mathcal{P}^{\infty}_X, s)$, where $\mathcal{P}^{\infty}_X$ is considered as a commutative algebra in the ∞-category $\mathcal{M}_X = \mathrm{Mod}_{\mathbb{D}^{\infty}_{X_{\mathrm{dR}}}}(\mathcal{D}_X)$ of $\mathbb{D}^{\infty}_{X_{\mathrm{dR}}}$-modules.

The notion of shifted Poisson structure on a derived stack admits a relative version. To state this, we will use the following result (Poisson additivity) proved in [32, Theorem 2.22].

Theorem 3.2.5. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category satisfying Assumption 3.2.1. Then there is an equivalence*

$$\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C}) \simeq \mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C}))$$

of symmetric monoidal ∞-categories satisfying the following compatibilities:

(1) *It is equivariant with respect to the involution on $\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C})$ given by passing to the opposite $\mathbb{P}_{s+1}$-algebra and the involution on $\mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C}))$ given by passing to the opposite associative algebra.*

(2) *It is compatible with the forgetful functors to $\mathrm{CAlg}(\mathcal{C})$, i.e. the diagram*

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C})) \arrow[d] \\ \mathrm{CAlg}(\mathcal{C}) \arrow[r, "\sim"] & \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C})) \end{tikzcd}$$

commutes.

Corollary 3.2.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category satisfying Assumption 3.2.1. Then there is an equivalence*

$$\mathrm{Alg}_{\mathbb{P}_{s+n}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C}))$$

of symmetric monoidal ∞-categories.

For a symmetric monoidal ∞-category $\mathcal{C}$, we denote by $\mathbf{LMod}(-)$ the ∞-category of pairs (A, M) of an algebra $A \in \mathrm{Alg}(\mathcal{C})$ and a left A-module $M \in \mathcal{C}$. Note that there is an equivalence

$$\mathbf{LMod}(\mathrm{CAlg}(\mathcal{C})) \simeq \mathrm{Mor}(\mathrm{CAlg}(\mathcal{C}))$$

of ∞-categories by [20, Proposition 2.4.3.16] since the tensor product in $\mathrm{CAlg}(\mathcal{C})$ is the coproduct. As a consequence, we get a forgetful functor

$$\mathbf{LMod}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C})) \to \mathrm{Mor}(\mathrm{CAlg}(\mathcal{C}))$$

defined for every integer s.

Definition 3.2.7. Let $\mathcal{C}$ be a symmetric monoidal ∞-category satisfying Assumption 3.2.1. Let $\phi\colon A \to B$ be a morphism of commutative algebras in $\mathcal{C}$. The space $\mathrm{Cois}(\phi, s)$ of s-shifted coisotropic structures on ϕ is the fibre of

$$\mathbf{LMod}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C})) \to \mathrm{Mor}(\mathrm{CAlg}(\mathcal{C}))$$

taken at the point corresponding to ϕ.

We have a forgetful functor

$$\mathbf{LMod}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C})) \longrightarrow \mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C}),$$

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where the last equivalence is given by Theorem 3.2.5, and this is compatible with the forgetful functor to $\mathrm{CAlg}(\mathcal{C})$. Therefore, we obtain a forgetful map

$$\mathrm{Cois}(\phi, s) \longrightarrow \mathrm{Pois}(A, s),$$

i.e. an s-shifted coisotropic structure on $A \to B$ encodes an s-shifted Poisson structure on A together with some extra data.

Remark 3.2.8. It is also possible to give another definition of a shifted coisotropic structure. Namely, in [31] and [23], the authors describe a 2-coloured operad $\mathbb{P}_{[s+1,s]}$. An important feature of this operad is that any $\mathbb{P}_{[s+1,s]}$-algebra (A, B) has an underlying morphism $A \to B$ of commutative algebras.

More specifically, there is a natural morphism of 2-coloured operads $\mathrm{Comm}^{\Delta^1} \to \mathbb{P}_{[s+1,s]}$, where Comm^{Δ^1} is the operad of morphisms of commutative algebras. In turn, this induces a forgetful functor $\mathrm{Alg}_{\mathbb{P}_{[s+1,s]}} \to \mathrm{Mor}(\mathrm{CAlg})$, and one can define s-shifted coisotropic structures in terms of the fibre of this functor.

This alternative definition has the advantage of being somewhat more explicit, and it is proved to be equivalent to Definition 3.2.7 in [32, Section 3].

Similarly to what we did in Definition 3.2.4, we can now extend the notion of shifted coisotropic structure to general morphisms of derived stacks. Let $f\colon Z \to X$ be a morphism between derived Artin stacks locally of finite presentation. We have an induced symmetric monoidal functor $f^*\colon \mathcal{M}_X \to \mathcal{M}_Z$ and a natural map $f^*_{\mathcal{P}}\colon f^*\mathcal{P}^\infty_X \to \mathcal{P}^\infty_Z$ in $\mathrm{CAlg}(\mathcal{M}_Z)$. We can now give the following definition, which is [24, Definition 2.1].

Definition 3.2.9. Let $f\colon Z \to X$ be a morphism of derived Artin stacks locally of finite presentation. The space $\mathrm{Cois}(f, s)$ of s-shifted coisotropic structures on f is the pullback

$$\begin{tikzcd} \mathrm{Cois}(f, s) \arrow[r] \arrow[d] & \mathrm{Pois}(X, s) \arrow[d] \\ \mathrm{Cois}(f^*_{\mathcal{P}}, s) \arrow[r] & \mathrm{Pois}(f^*\mathcal{P}^\infty_X, s). \end{tikzcd}$$

In other words, an s-shifted coisotropic structure on a map $f\colon Z \to X$ of derived stacks is given by an s-shifted Poisson structure on X, together with a compatible $\mathbb{P}_{[s+1,s]}$-structure on the morphism $f^*_{\mathcal{P}}\colon f^*\mathcal{P}^\infty_X \to \mathcal{P}^\infty_Z$.

3.3. Coisotropic correspondences

We begin by giving the definition of what we mean by a shifted coisotropic correspondence.

Definition 3.3.1. Let

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

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be a correspondence of derived Artin stacks locally of finite presentation. The space $\mathrm{Cois}(f, g; s)$ of s-shifted coisotropic structures on the correspondence (f, g) is the pullback

$$\begin{tikzcd} \mathrm{Cois}(f, g; s) \arrow[r] \arrow[d] & \mathrm{Pois}(X, s) \times \mathrm{Pois}(Y, s) \arrow[d] \\ \mathrm{Cois}((f, g); s) \arrow[r] & \mathrm{Pois}(X \times Y, s), \end{tikzcd}$$

where (f, g) is the induced map $Z \to X \times Y$, and the vertical morphism on the right sends a pair of Poisson structures (π_X, π_Y) on X and Y to the Poisson structure $\pi_X - \pi_Y$ on $X \times Y$.

The notion of an s-shifted coisotropic correspondence can be reinterpreted in a nice algebraic manner. Namely, consider an s-shifted coisotropic correspondence

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

between derived Artin stacks locally of finite presentation. The s-shifted Poisson structures on X and Y correspond to $\mathbb{P}_{s+1}$-structures on $\mathcal{P}_X^\infty$ and $\mathcal{P}_Y^\infty$. By Poisson additivity, we can think of $\mathcal{P}_X^\infty$ and $\mathcal{P}_Y^\infty$ as associative algebras in the ∞-category of $\mathbb{P}_s$-algebras. In other words, they are objects of the ∞-categories $\mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_X))$ and $\mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_Y))$, respectively. Moreover, these s-shifted Poisson structures allow us to enhance $f^*\mathcal{P}_X^\infty \otimes g^*\mathcal{P}_Y^\infty$ to an algebra object in $\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_Z)$.

Next, the s-shifted coisotropic structure on $Z \to X \times Y$ endows $\mathcal{P}_Z^\infty$ with a left module structure over $f^*\mathcal{P}_X^\infty \otimes g^*\mathcal{P}_Y^\infty$ in $\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_Z)$. In other words, $\mathcal{P}_Z^\infty$ becomes an $(f^*\mathcal{P}_X^\infty, g^*\mathcal{P}_Y^\infty)$-bimodule.

In this sense, coisotropic correspondences give a geometric incarnation of bimodules. This fact is the main motivation for our Morita approach to the construction of the ∞-category of coisotropic correspondences.

Remark 3.3.2. Note that a coisotropic morphism from X to Y corresponds to X viewed as a coisotropic correspondence from $\mathrm{Spec}\, k$ to Y.

Following §2.2, we have a symmetric monoidal (∞, n)-category $\mathrm{Span}_n(\mathrm{dArt})$, which has the following informal description:

- Its objects are derived Artin stacks locally of finite presentation.
- A 1-morphism from X to Y is given by a correspondence $X \leftarrow Z \to Y$.
- Higher morphisms are given by iterated correspondences.

The symmetric monoidal structure on $\mathrm{Span}_n(\mathrm{dArt})$ is given by the product of derived Artin stacks with the unit given by the terminal object $* = \mathrm{Spec}\, k$. Each object $X \in \mathrm{Span}_n(\mathrm{dArt})$ is canonically self-dual with the evaluation and coevaluation maps given by

$$X \times X \xleftarrow{\Delta} X \longrightarrow * , \qquad * \longleftarrow X \xrightarrow{\Delta} X \times X$$

(see [14, Lemma 12.3]).

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Next, using the notation introduced in the same section, we have a functor

$$\mathfrak{C}_n := \mathrm{Cospan}_n(\mathrm{CAlg}(\mathcal{M}))\colon \mathbf{dArt}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_{(\infty,n)}.$$

This sends a derived stack X to the (∞, n)-category $\mathfrak{C}_n(X) := \mathrm{Cospan}_n(\mathrm{CAlg}(\mathcal{M}_X))$, which has the following informal description:

- Its objects are commutative algebra objects in $\mathcal{M}_X$.
- A 1-morphism from A to B is given by a cospan $A \to C \leftarrow B$ of commutative algebras in $\mathcal{M}_X$.
- Higher morphisms are given by iterated cospans.

Following Section 2.3, we can also combine the two (∞, n)-categories we have introduced above into a symmetric monoidal (∞, n)-category $\mathrm{Span}_n(\mathbf{dArt}; \mathfrak{C}_n)$ whose objects are pairs (X, A) of a derived stack $X \in \mathbf{dArt}$ and a commutative algebra $A \in \mathcal{M}_X$. By Corollary 3.1.6, the section $\mathcal{P}^{\infty}\colon \mathbf{dArt}^{\mathrm{op}} \to \mathcal{M}_{\mathrm{CAlg}}$ preserves finite colimits; so by Corollary 2.4.11 and Remark 2.4.12, it induces a symmetric monoidal functor

$$\mathrm{Span}_n(\mathbf{dArt}) \longrightarrow \mathrm{Span}_n(\mathbf{dArt}; \mathfrak{C}_n).$$

The cocartesian monoidal structure on $\mathrm{CAlg}(\mathcal{M}_X)$ corresponds to the usual tensor product of algebras; so by Corollary 2.6.10, we have an equivalence of diagrams of symmetric monoidal (∞, n)-categories

$$\mathfrak{C}_n \simeq \mathfrak{alg}_n(\mathrm{CAlg}(\mathcal{M})),$$

where $\mathfrak{alg}_n(-)$ is the Morita (∞, n)-category of $\mathbb{E}_n$-algebras. Therefore, we have an equivalence of symmetric monoidal (∞, n)-categories

$$\mathrm{Span}_n(\mathbf{dArt}; \mathfrak{alg}_n(\mathrm{CAlg}(\mathcal{M}))) \simeq \mathrm{Span}_n(\mathbf{dArt}; \mathfrak{C}_n).$$

Next, the forgetful functor

$$\mathrm{Alg}_{\mathbb{P}_{s-n+1}}(\mathcal{M}_X) \longrightarrow \mathrm{CAlg}(\mathcal{M}_X)$$

is symmetric monoidal; so we obtain a forgetful functor of diagrams of symmetric monoidal (∞, n)-categories

$$\mathfrak{P}^s_n := \mathfrak{alg}_n(\mathrm{Alg}_{\mathbb{P}_{s-n+1}}(\mathcal{M})) \longrightarrow \mathfrak{alg}_n(\mathrm{CAlg}(\mathcal{M})) \simeq \mathfrak{C}_n,$$

and hence a forgetful functor of symmetric monoidal (∞, n)-categories

$$\mathrm{Span}_n(\mathbf{dArt}; \mathfrak{P}^s_n) \longrightarrow \mathrm{Span}_n(\mathbf{dArt}; \mathfrak{C}_n).$$

Definition 3.3.3. The (∞, n)-category $\mathrm{CoisCorr}^s_n$ of s-shifted coisotropic correspondences is the pullback

$$\begin{tikzcd} \mathrm{CoisCorr}^s_n \arrow[r] \arrow[d] & \mathrm{Span}_n(\mathbf{dArt}; \mathfrak{P}^s_n) \arrow[d] \\ \mathrm{Span}_n(\mathbf{dArt}) \arrow[r] & \mathrm{Span}_n(\mathbf{dArt}; \mathfrak{C}_n) \end{tikzcd}$$

of (∞, n)-categories.

Let $\mathcal{M}_{\mathbb{P}_{s+1}} \to \mathbf{dArt}^{\mathrm{op}}$ be the cocartesian fibration corresponding to the functor

$$\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{M}) \simeq \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_{\mathbb{P}_{s+1-n}}(\mathcal{M})),$$

where the latter equivalence is given by Corollary 3.2.6. For a complete n-fold Segal space $\mathcal{C}$, we denote by $\mathcal{C}^{\simeq} := \mathcal{C}_{0,\dots,0} \in \mathcal{S}$ the space of objects of $\mathcal{C}$. The space $(\mathrm{CoisCorr}_n^s)^{\simeq}$ is given by the pullback of spaces of objects obtained from the defining pullback of (∞, n)-categories. Since the n-fold Segal spaces of iterated spans are already complete, we have $\mathrm{Span}_n(\mathbf{dArt})^{\simeq} \simeq \mathbf{dArt}^{\simeq}$, and by Proposition 2.4.1, we have

$$\mathrm{Span}_n(\mathbf{dArt}; \mathfrak{C}_n)^{\simeq} \simeq \mathcal{M}_{\mathrm{CAlg}}^{\simeq}.$$

Using Lemma 2.3.4, we can similarly identify $\mathrm{Span}_n(\mathbf{dArt}; \mathfrak{alg}_n(\mathrm{Alg}_{\mathbb{P}_{s-n+1}}(\mathcal{M})))^{\simeq}$ in terms of the fibration for $\mathfrak{alg}_n(\mathrm{Alg}_{\mathbb{P}_{s-n+1}}(\mathcal{M}))^{\simeq}$. The symmetric monoidal ∞-category $\mathrm{Alg}_{\mathbb{P}_{s-n+1}}(\mathcal{M})$ is pointed; so if we assume Conjecture 2.5.21, then we can identify this space with $\mathcal{M}_{\mathbb{P}_{s+1}}^{\simeq}$. With this assumption, we thus get a pullback square

$$\begin{tikzcd}
(\mathrm{CoisCorr}_n^s)^{\simeq} \arrow[r] \arrow[d] & \mathcal{M}_{\mathbb{P}_{s+1}}^{\simeq} \arrow[d] \\
\mathbf{dArt}^{\simeq} \arrow[r, "\mathcal{P}^{\infty}"] & \mathcal{M}_{\mathrm{CAlg}}^{\simeq}
\end{tikzcd}$$

of spaces. Therefore, the space of objects of $\mathrm{CoisCorr}_n^s$ coincides with the space of derived Artin stacks X equipped with a lift of $\mathcal{P}_X^{\infty} \in \mathrm{CAlg}(\mathcal{M}_X)$ to a $\mathbb{P}_{s+1}$-algebra in $\mathcal{M}_X$, i.e. an s-shifted Poisson structure $\pi \in \mathrm{Pois}(X, s)$. One may analyse in a similar way the space of 1-morphisms, so let us present an informal summary:

- Objects of $\mathrm{CoisCorr}_n^s$ are derived Artin stacks $X \in \mathbf{dArt}$ together with an s-shifted Poisson structure $\pi_X \in \mathrm{Pois}(X, s)$.
- Its morphisms from (X, π_X) to (Y, π_Y) are given by correspondences $X \xleftarrow{f} Z \xrightarrow{g} Y$ of derived Artin stacks equipped with an s-shifted coisotropic structure $\gamma_Z \in \mathrm{Cois}(f, g; s)$ compatible with the given s-shifted Poisson structures π_X and π_Y.
- Higher morphisms are given by iterated correspondences.

Note that the diagram defining $\mathrm{CoisCorr}_n^s$ is a diagram of symmetric monoidal (∞, n)-categories. Therefore, $\mathrm{CoisCorr}_n^s$ carries a natural symmetric monoidal structure. This symmetric monoidal structure can also be defined by delooping, i.e. we have equivalences

$$\mathrm{CoisCorr}_n^s(*, *) \simeq \mathrm{CoisCorr}_{(\infty,n-1)}^{s-1}.$$

Theorem 3.3.4. *Assuming Conjecture 2.5.19, the symmetric monoidal (∞, n)-category $\mathrm{CoisCorr}_n^s$ has duals (i.e. its objects are dualizable and all i-morphisms for $i < n$ have adjoints).*

Proof. The symmetric monoidal (∞, n)-category $\mathrm{Span}_n(\mathbf{dArt})$ has duals by [14, Theorem 12.4 and Corollary 12.5].

Assuming Conjecture 2.5.19, the symmetric monoidal (∞, n)-categories $\mathfrak{C}_n(X)$ and $\mathfrak{P}_n^s(X)$ have duals for any derived Artin stack $X \in \mathbf{dArt}$. Thus, by Lemma 2.3.4, the symmetric monoidal (∞, n)-categories $\mathrm{Span}_n(\mathbf{dArt}; \mathfrak{P}_n^s)$ and $\mathrm{Span}_n(\mathbf{dArt}; \mathfrak{C}_n)$ have duals.

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The claim therefore follows from Corollary 2.1.17. □

Remark 3.3.5. Unwinding the definitions, if X is an s-shifted derived Poisson stack, viewed as an object of $\mathrm{CoisCorr}_1^s$, then the dual $X^\vee$ has the same underlying derived stack X, but its Poisson structure corresponds to the reversed multiplication on $\mathcal{P}_X^\infty$, viewed as an associative algebra in $\mathrm{Alg}_{\mathbb{P}_s}(\mathcal{M}_X)$. By Theorem 3.2.5, in terms of $\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{M}_X)$, this amounts to taking the negative of the Poisson bracket. Thus a coisotropic correspondence from X to Y is equivalent to a coisotropic correspondence from $\mathbf{Spec}\ k$ to $X^\vee \times Y$, or (using Remark 3.3.2) a coisotropic morphism to $X^\vee \times Y$.

3.4. Relationship with Lagrangian correspondences

In this section, we sketch a conjectural relationship between our (∞, n)-category of s-shifted coisotropic correspondences and the (∞, n)-category of s-shifted Lagrangian correspondences from [14] and [6].

Let $\mathcal{C}$ be a symmetric monoidal ∞-category satisfying Assumption 3.2.1. Then one has the de Rham functor (see [7, Section 1.3])

$$\mathrm{DR}\colon \mathrm{CAlg}(\mathcal{C}) \longrightarrow \mathrm{CAlg}(\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}}),$$

which sends a commutative algebra A to the graded commutative algebra $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathrm{Sym}_A(\mathbb{L}_A[-1]))$ equipped with the de Rham differential. One can therefore define the functors

$$\mathcal{A}^2(s), \mathcal{A}^{2,\mathrm{cl}}(s)\colon \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{S})$$

of s-shifted two-forms and closed s-shifted two-forms by

$$\mathcal{A}^2(A, s) = \mathrm{Hom}_{\mathbf{dg}^{\mathrm{gr}}}(k(2)[-s-2], \mathrm{DR}(A))$$
$$\mathcal{A}^{2,\mathrm{cl}}(A, s) = \mathrm{Hom}_{\epsilon\text{-}\mathrm{dg}^{\mathrm{gr}}}(k(2)[-s-2], \mathrm{DR}(A)),$$

where $k(2)[-s-2]$ is the unit object concentrated in weight 2 and cohomological degree $s+2$ with the trivial mixed structure. Note that by construction, we have a natural forgetful map $\mathcal{A}^{2,\mathrm{cl}}(s) \to \mathcal{A}^2(s)$.

Applying the above construction to $\mathcal{C} = \mathbf{dg}$, the ∞-category of complexes of k-modules, we obtain functors

$$\mathcal{A}^2(s)\colon \mathbf{cdga}^{\leqslant 0} \longrightarrow \mathrm{CAlg}(\mathcal{S}), \quad \mathcal{A}^{2,\mathrm{cl}}(s)\colon \mathbf{cdga}^{\leqslant 0} \longrightarrow \mathrm{CAlg}(\mathcal{S}).$$

Let $\mathcal{A}^2(s), \mathcal{A}^{2,\mathrm{cl}}(s)\colon \mathbf{dArt}^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{S})$ be the corresponding right Kan extensions.

Definition 3.4.1. The (∞, n)-category $\mathrm{IsotCorr}_n^s$ of s-shifted isotropic correspondences is

$$\mathrm{IsotCorr}_n^s := \mathrm{Span}_n(\mathrm{dArt}; \mathcal{A}^{2,\mathrm{cl}}(s)) \simeq \mathrm{Span}_n(\mathrm{dArt}_{/\mathcal{A}^{2,\mathrm{cl}}(s)}).$$

Now suppose D is a finite category with an initial object $\emptyset \in D$ and let $D^{\triangleright}$ be the category obtained by formally adjoining a terminal object $* \in D^{\triangleright}$. Suppose $X\colon D \to \mathbf{dArt}_{/\mathcal{A}^{2,\mathrm{cl}}(s)}$ is a diagram of derived Artin stacks equipped with closed s-shifted two-forms. Then we obtain a diagram $\mathbb{T}_X\colon D \to \mathrm{QCoh}(X_\emptyset)$ whose value on $d \in D$ is given by pulling back $\mathbb{T}_{X_d}$ along the unique map $X_\emptyset \to X_d$. Using the closed two-forms, we can extend this to a diagram $\mathbb{T}_X : D^\triangleright \to \mathrm{QCoh}(X_\emptyset)$ whose value on the final object is $(\mathbb{T}_X)_* := \mathbb{L}_{X_\emptyset}[s]$. We say the diagram $X : D \to \mathrm{dArt}_{/\mathcal{A}^{2,\mathrm{cl}}(s)}$ is non-degenerate if $\mathbb{T}_X : D^\triangleright \to \mathrm{QCoh}(X_\emptyset)$ is a limit diagram.

Definition 3.4.2. The (∞, n)-category Lag^s_n of s-shifted Lagrangian correspondences is the subcategory $\mathrm{Lag}^s_n \subset \mathrm{IsotCorr}^s_n$ consisting of non-degenerate diagrams $\Sigma^{i_1,\dots,i_n} \to \mathrm{dArt}_{/\mathcal{A}^{2,\mathrm{cl}}(s)}$.

Let C be a symmetric monoidal category satisfying Assumption 3.2.1 and $\mathcal{C}$ its localization. We define $\mathrm{Alg}_{\mathbb{P}_{s+1}}(C)^\omega$ to be the category whose objects are $\mathbb{P}_{s+1}$-algebras equipped with a strictly closed two-form ω. We have the following two functors,

$$F_1, F_2 : \mathrm{Alg}_{\mathbb{P}_{s+1}}(C)^\omega \longrightarrow \mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathrm{Mod}_{k[\hbar]/\hbar^2}(C)).$$

- Given a $\mathbb{P}_{s+1}$-algebra $A \in \mathrm{Alg}_{\mathbb{P}_{s+1}}(C)$, we define $F_1(A)$ to be the commutative algebra $A[\hbar]/\hbar^2$ equipped with the bracket $\{a, b\}_\hbar = (1 + \hbar)\{a, b\}$ for $a, b \in A$.
- Given a $\mathbb{P}_{s+1}$-algebra $A \in \mathrm{Alg}_{\mathbb{P}_{s+1}}(C)$ equipped with a closed two-form $\omega = \sum_i f_i \mathrm{d}_{\mathrm{dR}} g_i \wedge \mathrm{d}_{\mathrm{dR}} h_i$, we define $F_2(A)$ to be the commutative algebra $A[\hbar]/\hbar^2$ equipped with the bracket $\{a, b\}_\hbar = \{a, b\} \pm \hbar \sum_i f_i \{g_i, a\}\{h_i, b\}$ with the sign determined by the Koszul sign rule.

Note that both F_1 and F_2 modulo $\hbar$ are given by the forgetful functor $\mathrm{Alg}_{\mathbb{P}_{s+1}}(C)^\omega \to \mathrm{Alg}_{\mathbb{P}_{s+1}}(C)$ and they preserve weak equivalences. Therefore, after localization, they give rise to a diagram of symmetric monoidal ∞-categories

$$\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C})^\omega \underset{F_2}{\overset{F_1}{\rightrightarrows}} \mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathrm{Mod}_{k[\hbar]/\hbar^2}(\mathcal{C})) \longrightarrow \mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C}),$$

where the last functor is given by evaluating at $\hbar = 0$. We denote the limit of the above diagram by $\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C})^{\mathrm{compat}}$. This is the ∞-category of compatible pairs; see [7, Definition 1.4.20] and [27, Definition 1.24].

Definition 3.4.3. The (∞, n)-category $\mathrm{CompCorr}^s_n$ of s-shifted compatible correspondences is the pullback

$$\begin{tikzcd}
\mathrm{CompCorr}^s_n \arrow[r] \arrow[d] & \mathrm{Span}_n(\mathrm{dArt}; \mathfrak{alg}_n(\mathrm{Alg}_{\mathbb{P}_{s-n+1}}(\mathcal{M})^{\mathrm{compat}})) \arrow[d] \\
\mathrm{Span}_n(\mathrm{dArt}) \arrow[r] & \mathrm{Span}_n(\mathrm{dArt}; \mathfrak{C}_n)
\end{tikzcd}$$

of (∞, n)-categories.

Note that by construction, we have a symmetric monoidal forgetful functor

$$\mathrm{CompCorr}^s_n \longrightarrow \mathrm{CoisCorr}^s_n.$$

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We expect that one may define non-degenerate coisotropic correspondences $\mathrm{CoisCorr}_n^{s,\mathrm{nd}} \subset \mathrm{CoisCorr}_n^s$ similarly to $\mathrm{Lag}_n^s \subset \mathrm{IsotCorr}_n^s$. Denote

$$\mathrm{CompCorr}_n^{s,\mathrm{nd}} := \mathrm{CoisCorr}_n^{s,\mathrm{nd}} \times_{\mathrm{CoisCorr}_n^s} \mathrm{CompCorr}_n^s.$$

Conjecture 3.4.4. (1) *The projection* $\mathrm{CompCorr}_n^{s,\mathrm{nd}} \to \mathrm{CoisCorr}_n^{s,\mathrm{nd}}$ *is an equivalence.*

(2) *There is a symmetric monoidal functor* $\mathrm{CompCorr}_n^{s,\mathrm{nd}} \to \mathrm{IsotCorr}_n^s$.

(3) *The previous functor restricts to an equivalence* $\mathrm{CompCorr}_n^{s,\mathrm{nd}} \to \mathrm{Lag}_n^s$.

This conjecture would establish the existence of a symmetric monoidal functor of (∞, n)-categories

$$\mathrm{Lag}_n^s \longrightarrow \mathrm{CoisCorr}_n^s,$$

which is an equivalence onto the subcategory $\mathrm{CoisCorr}_n^{s,\mathrm{nd}}$.

Let us note that there is a forgetful functor from $\mathrm{Alg}_{\mathbb{P}_{s+1}}(\mathcal{C})^{\omega}$ to the ∞-category of commutative algebras equipped with a closed s-shifted two-form. Thus, the second claim is closely related to [12, Conjecture 1.3.1]. Claims (1) and (3) on the level of objects have been proven in [7, Theorem 3.2.4] and [27, Theorem 3.33]. The same claims on the level of 1-morphisms have been proven in [28] and [24, Theorem 4.22].

Remark 3.4.5. In [6], it is also shown that every symplectic derived stack determines an oriented extended TQFT using the *AKSZ construction* (defined in the derived algebro-geometric context in [25]). It is tempting to speculate that there exists an analogue of the AKSZ construction for derived Poisson stacks (cf. [16]), and that this can be used to construct, for every derived Poisson stack, oriented extended TQFTs

$$\mathrm{Bord}_{0,n}^{\mathrm{or}} \to \mathrm{CoisCorr}_n^s.$$

Appendix A. Twisted Arrows and Bifibrations

Our goal in this appendix is to prove two somewhat technical results, Corollary A.2.6 and Proposition A.3.1, which will allow us to describe the higher category of spans with coefficients in cospans in Proposition 2.4.2.

A.1. Bifibrations

We begin with a preliminary discussion of bifibrations, in the following sense.

Definition A.1.1. A *bifibration* $(p, q)\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ consists of a cartesian fibration p and a cocartesian fibration q such that a morphism f in $\mathcal{E}$ is

- p-cartesian if and only if $q(f)$ is an equivalence,
- q-cocartesian if and only if $p(f)$ is an equivalence.

Remark A.1.2. This definition is a model-independent version of [17, Definition 2.4.7.2].

Lemma A.1.3. *Consider a commutative triangle of ∞-categories*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "{(p,q)}"'] & & \mathcal{E}' \arrow[dl, "{(p',q')}"] \\ & \mathcal{A} \times \mathcal{B} & \end{tikzcd},$$

where (p,q) and (p',q') are bifibrations. Then f takes q-cocartesian morphisms to q'-cocartesian morphism, and p-cartesian morphisms to p'-cartesian morphisms.

Proof. This is immediate from the definition as f takes a morphism ϕ in $\mathcal{E}$ such that $p(\phi)$ is an equivalence to the morphism $f(\phi)$ where $p'f(\phi) \simeq p(\phi)$ is an equivalence, and similarly for q. $\square$

Proposition A.1.4. *Suppose $(p,q)\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ is a functor such that p is a cartesian fibration, q is a cocartesian fibration, p takes q-cocartesian morphisms to equivalences, and q takes p-cartesian morphisms to equivalences. Then we have the following:*

(i) *The functor $q_a\colon \mathcal{E}_a \to \mathcal{B}$ on fibres at $a \in \mathcal{A}$ is a cocartesian fibration, and a morphism in $\mathcal{E}_a$ is q_a-cocartesian if and only if its image in $\mathcal{E}$ is q-cocartesian.*

(ii) *The functor $p_b\colon \mathcal{E}_b \to \mathcal{A}$ on fibres at $b \in \mathcal{B}$ is a cartesian fibration, and a morphism in $\mathcal{E}_b$ is p_b-cartesian if and only if its image in $\mathcal{E}$ is p-cartesian.*

Proof. We prove (i); the proof of (ii) is the same. Suppose $x \xrightarrow{\phi} x'$ is a morphism in $\mathcal{E}_a$, i.e. a morphism in $\mathcal{E}$ over $b \to b'$ in $\mathcal{B}$ and id_a in $\mathcal{A}$. Then for $y \in \mathcal{E}$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}}(x', y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, y) \arrow[d] \\ \mathrm{Map}_{\mathcal{A}}(a, py) \times \mathrm{Map}_{\mathcal{B}}(b', qy) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{A}}(a, py) \times \mathrm{Map}_{\mathcal{B}}(b, qy) \arrow[d] \\ \mathrm{Map}_{\mathcal{B}}(b', qy) \arrow[r] & \mathrm{Map}_{\mathcal{B}}(b, qy). \end{tikzcd}$$

Here the bottom square is cartesian (since $p\phi$ is an equivalence in $\mathcal{A}$), and so the top square is cartesian if and only if the outer square is cartesian.

Suppose first that ϕ is q-cocartesian so that the outer square is cartesian for any y. If $py \simeq a$, then we can take fibres in the top square at $\mathrm{id}_a \in \mathrm{Map}_{\mathcal{A}}(a,a) \simeq \mathrm{Map}_{\mathcal{A}}(a, py)$, giving a square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}_a}(x', y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}_a}(x, y) \arrow[d] \\ \mathrm{Map}_{\mathcal{B}}(b', qy) \arrow[r] & \mathrm{Map}_{\mathcal{B}}(b, qy), \end{tikzcd}$$

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which is cartesian since the top square is cartesian. This exhibits ϕ as q_a-cocartesian. Moreover, since q-cocartesian morphisms exist, so do q_a-cocartesian morphisms, i.e. q_a is a cocartesian fibration.

Now suppose that ϕ is q_a-cocartesian. To show that ϕ is also q-cocartesian, we must prove that the top square in the diagram above is cartesian for all $y \in \mathcal{E}$. For a given y, this will follow if we can show that for every map $\psi: a \to py$, the square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}}(x', y)_\psi \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, y)_\psi \arrow[d] \\ \mathrm{Map}_{\mathcal{B}}(b', qy) \arrow[r] & \mathrm{Map}_{\mathcal{B}}(b, qy) \end{tikzcd}$$

of fibres at ϕ is cartesian. Let $\bar{\psi}: \psi^* y \to y$ be a p-cartesian morphism over ψ; then $q\bar{\psi}$ is an equivalence, so this square is equivalent to

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}_a}(x', \psi^* y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}_a}(x, \psi^* y) \arrow[d] \\ \mathrm{Map}_{\mathcal{B}}(b', qy) \arrow[r] & \mathrm{Map}_{\mathcal{B}}(b, qy), \end{tikzcd}$$

and this is cartesian since ϕ is by assumption q_a-cocartesian. □

Corollary A.1.5. *Suppose $(p, q): \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ is as in Proposition A.1.4. Then the following are equivalent:*

(1) *(p, q) is a bifibration.*

(2) *q_a is a left fibration for all $a \in \mathcal{A}$.*

(3) *p_b is a right fibration for all $b \in \mathcal{B}$.*

(4) *The fibre $\mathcal{E}_{a,b}$ is an ∞-groupoid for all $a \in \mathcal{A}, b \in \mathcal{B}$.*

Proof. Part (i) of Proposition A.1.4 implies that (p, q) is a bifibration if and only if every morphism in $\mathcal{E}_a$ is q_a-cocartesian for all a, i.e. q_a is a left fibration. Similarly, part (ii) implies that (1) is equivalent to (3). Finally, since q_a is by assumption a cocartesian fibration, it is a left fibration if and only if its fibres $\mathcal{E}_{a,b}$ are ∞-groupoids for all $b \in \mathcal{B}$; so (2) is equivalent to (4). □

We will now show that we can replace bifibrations by left fibrations, and vice versa, using the following constructions:

Construction A.1.6. (i) Suppose $(p, q): \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ is a bifibration. Then we have a commutative triangle

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "{(p,q)}"] \arrow[dr, "p"'] & & \mathcal{A} \times \mathcal{B} \arrow[dl] \\ & \mathcal{A} & \end{tikzcd},$$

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where the diagonal maps are cartesian fibrations, and the horizontal map takes p-cartesian morphisms to cartesian morphisms for the projection $\mathcal{A}\times\mathcal{B}\to\mathcal{A}$, as these are precisely the morphisms that project to equivalences in $\mathcal{B}$. Let $p^\vee\colon \mathcal{E}^\ell\to\mathcal{A}$ be the cocartesian fibration dual to p; then dualization gives a commutative triangle

$$\begin{tikzcd} \mathcal{E}^\ell \arrow[rr] \arrow[dr, "p^\vee"'] & & \mathcal{A}^{\mathrm{op}}\times\mathcal{B} \arrow[dl] \\ & \mathcal{A}^{\mathrm{op}} & \end{tikzcd},$$

where the diagonal maps are cocartesian fibrations and the horizontal map preserves cocartesian morphisms.

(ii) Suppose $(p,q)\colon \mathcal{F}\to\mathcal{A}^{\mathrm{op}}\times\mathcal{B}$ is a left fibration. Then we have a commutative triangle

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "{(p,q)}"] \arrow[dr, "p"'] & & \mathcal{A}^{\mathrm{op}}\times\mathcal{B} \arrow[dl] \\ & \mathcal{A}^{\mathrm{op}} & \end{tikzcd},$$

where the diagonal maps are cocartesian fibrations. A morphism $\phi\colon x\to x'$ in $\mathcal{F}$ is p-cocartesian if and only if $q(\phi)$ is an equivalence in $\mathcal{B}$: In the commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{F}}(x',y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{F}}(x,y) \arrow[d] \\ \mathrm{Map}_{\mathcal{A}^{\mathrm{op}}}(px',py)\times\mathrm{Map}_{\mathcal{B}}(qx',qy) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{A}^{\mathrm{op}}}(px,py)\times\mathrm{Map}_{\mathcal{B}}(qx,qy) \arrow[d] \\ \mathrm{Map}_{\mathcal{A}^{\mathrm{op}}}(px',py) \arrow[r] & \mathrm{Map}_{\mathcal{A}^{\mathrm{op}}}(px,py), \end{tikzcd}$$

the top square is cocartesian since (p,q) is a left fibration, while the bottom square is cartesian if $q(\phi)$ is an equivalence, and hence such a morphism is p-cocartesian; since such p-cocartesian morphisms always exist, by uniqueness, all p-cocartesian morphisms must map to equivalences in $\mathcal{B}$. Thus (p,q) preserves cocartesian morphisms in the triangle above, and so if $p^\vee\colon \mathcal{F}^b\to\mathcal{A}$ denotes the cartesian fibration dual to p, we get a dual triangle

$$\begin{tikzcd} \mathcal{F}^b \arrow[rr, "{(p^\vee,q')}"] \arrow[dr, "p^\vee"'] & & \mathcal{A}\times\mathcal{B} \arrow[dl] \\ & \mathcal{A} & \end{tikzcd},$$

where the diagonal maps are cartesian fibrations and the horizontal map preserves cartesian morphisms.

Proposition A.1.7. *We keep the notation of Construction A.1.6.*

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(i) *Suppose* $(p,q)\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ *is a bifibration. Then* $(p^\vee, q')\colon \mathcal{E}^\ell \to \mathcal{A}^{\mathrm{op}} \times \mathcal{B}$ *is a left fibration.*

(ii) *Suppose* $(p,q)\colon \mathcal{F} \to \mathcal{A}^{\mathrm{op}} \times \mathcal{B}$ *is a left fibration. Then* $(p^\vee, q')\colon \mathcal{F}^b \to \mathcal{A} \times \mathcal{B}$ *is a bifibration.*

We prove general versions of the criteria we will use to establish this proposition.

Lemma A.1.8. *Suppose given a commutative triangle*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{C} & \end{tikzcd}$$

of functors between ∞*-categories such that we have the following:*

(1) p *and* q *are cartesian fibrations.*

(2) f *takes* p*-cartesian edges to* q*-cartesian edges.*

(3) *For each object* $c \in \mathcal{C}$*, the induced map on fibres* $f_c\colon \mathcal{E}_c \to \mathcal{D}_c$ *is a cartesian fibration.*

(4) *Suppose given a commutative square*

$$\begin{tikzcd} \phi^* e' \arrow[r, "\alpha"] \arrow[d, "\beta"'] & e' \arrow[d, "\gamma"] \\ \phi^* e \arrow[r, "\delta"'] & e \end{tikzcd}$$

in $\mathcal{E}$ *lying over the degenerate square*

$$\begin{tikzcd} c' \arrow[r, "\phi"] \arrow[d, "\mathrm{id}_{c'}"'] & c \arrow[d, "\phi"] \\ c' \arrow[r, "\mathrm{id}_c"'] & c \end{tikzcd}$$

in $\mathcal{C}$*, where* α *and* δ *are* p*-cartesian edges and* γ *is* f_c*-cartesian. Then* β *is* $f_{c'}$*-cartesian. (In other words, the induced functor* $\phi^*\colon \mathcal{E}_c \to \mathcal{E}_{c'}$ *takes* f_c*-cartesian edges to* $f_{c'}$*-cartesian edges.)*

Then f *is also a cartesian fibration.*

Proof. Suppose given $e \in \mathcal{E}$ lying over $d \in \mathcal{D}$ and $c \in \mathcal{C}$ (i.e. $d \simeq f(e)$ and $c \simeq p(e) \simeq q(d)$) and a morphism $\delta\colon d' \to d$ in $\mathcal{D}$ lying over $\gamma\colon c' \to c$ in $\mathcal{C}$. Then we must show that there exists an f-cartesian morphism $e' \to e$ over δ.

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Since p is a cartesian fibration, there exists a p-cartesian morphism $\beta\colon \gamma^*e \to e$ over γ, and as f takes p-cartesian edges to q-cartesian edges, its image in $\mathcal{D}$ is a q-cartesian edge $f(\beta)\colon \gamma^*d \to d$. There is then an essentially unique factorization of δ through $f(\beta)$, as

$$d' \xrightarrow{\alpha} \gamma^*d \xrightarrow{f(\beta)} d.$$

Now α is a morphism in $\mathcal{D}_{c'}$; so since $f_{c'}$ is a cartesian fibration, there exists an $f_{c'}$-cartesian edge $\epsilon\colon \alpha^*\gamma^*e \to \gamma^*e$. We will show that the composite $\beta \circ \epsilon\colon \alpha^*\gamma^*e \to \gamma^*e \to e$ is an f-cartesian morphism over δ.

To see this, we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(x, \alpha^*\gamma^*e) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, \gamma^*e) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, e) \arrow[d] \\
\mathrm{Map}_{\mathcal{D}}(f(x), d') \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(f(x), \gamma^*d) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(f(x), d) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(p(x), c') \arrow[r, "\mathrm{id}"'] & \mathrm{Map}_{\mathcal{C}}(p(x), c') \arrow[r] & \mathrm{Map}_{\mathcal{C}}(p(x), c),
\end{tikzcd}$$

where x is an arbitrary object of $\mathcal{E}$. By [17, Proposition 2.4.4.3], to see that $\beta \circ \epsilon$ is f-cartesian, we must show that the composite of the two upper squares is cartesian. We will prove this by showing that both of the upper squares are cartesian. By construction, β is p-cartesian and $f(\beta)$ is q-cartesian; so the composite of the two right squares and the bottom right square are both cartesian, and hence so is the upper right square.

Since a commutative square of spaces is cartesian if and only if the induced maps on all fibres are equivalences, to see that the upper left square is cartesian, it suffices to show that the square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(x, \alpha^*\gamma^*e)_\mu \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(x, \gamma^*e)_\mu \arrow[d] \\
\mathrm{Map}_{\mathcal{D}}(f(x), d')_\mu \arrow[r] & \mathrm{Map}_{\mathcal{D}}(f(x), \gamma^*d)_\mu
\end{tikzcd}$$

obtained by taking the fibre at $\mu\colon p(x) \to c'$ is cartesian for every map μ. Now taking p- and q-cartesian pullbacks along μ, we can (since f takes p-cartesian morphisms to q-cartesian morphisms) identify this with the square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}_{p(x)}}(x, \mu^*\alpha^*\gamma^*e) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}_{p(x)}}(x, \mu^*\gamma^*e) \arrow[d] \\
\mathrm{Map}_{\mathcal{D}_{p(x)}}(f(x), \mu^*d') \arrow[r] & \mathrm{Map}_{\mathcal{D}_{p(x)}}(f(x), \mu*\gamma^*d).
\end{tikzcd}$$

But this is cartesian since by assumption, the map $\mu^*\alpha^*\gamma^*e \to \mu^*\gamma^*e$ is $f_{p(x)}$-cartesian (because ϵ is $f_{c'}$-cartesian). $\square$

Remark A.1.9. In the situation of Lemma A.1.8, if the maps on fibres f_c are right fibrations for all $c \in \mathcal{C}$, then condition (4) is automatically satisfied since every morphism is f_c-cartesian.

Lemma A.1.10. *Suppose $\pi: \mathcal{E} \to \mathcal{I} \times \mathcal{J}$ is a functor of ∞-categories such that*

(i) *the composite $\pi_{\mathcal{I}}: \mathcal{E} \to \mathcal{I}$ is a cartesian fibration,*

(ii) *for every $i \in \mathcal{I}$, the functor $\pi_i: \mathcal{E}_i \to \mathcal{J}$ on fibres over i is a cocartesian fibration.*

Then the composite $\pi_{\mathcal{J}}: \mathcal{E} \to \mathcal{J}$ is a cocartesian fibration, and π preserves cocartesian morphisms.

Proof. Given $e \in \mathcal{E}$ lying over $j \in \mathcal{J}$ and a morphism $\phi: j \to j'$, we must show that there exists a cocartesian morphism in $\mathcal{E}$ over ϕ with source e. Suppose e lies over $i \in \mathcal{I}$, and let $\bar{\phi}: e \to e'$ be a cocartesian morphism over ϕ in $\mathcal{E}_i$. We will show that $\bar{\phi}$ is also a cocartesian morphism in $\mathcal{E}$. Thus we wish to prove that the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(e', x) \arrow[r, "\bar{\phi}^*"] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(e, x) \arrow[d] \\
\mathrm{Map}_{\mathcal{J}}(j', k) \arrow[r, "\phi^*"'] & \mathrm{Map}_{\mathcal{J}}(j, k)
\end{tikzcd}$$

is cartesian for every $x \in \mathcal{E}$ lying over $k \in \mathcal{J}$. It suffices to prove that the square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(e', x) \arrow[r, "\bar{\phi}^*"] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(e, x) \arrow[d] \\
\mathrm{Map}_{\mathcal{J}}(j', k) \times \mathrm{Map}_{\mathcal{I}}(i, l) \arrow[r, "\phi^* \times \mathrm{id}"'] & \mathrm{Map}_{\mathcal{J}}(j, k) \times \mathrm{Map}_{\mathcal{I}}(i, l)
\end{tikzcd}$$

is cartesian, where x lies over l in $\mathcal{I}$. But to show this, it is enough to show that the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(e', x)_f \arrow[r, "\bar{\phi}^*"] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(e, x)_f \arrow[d] \\
\mathrm{Map}_{\mathcal{J}}(j', k) \arrow[r, "\phi^*"'] & \mathrm{Map}_{\mathcal{J}}(j, k)
\end{tikzcd}$$

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on fibres over $f: i \to l$ is cartesian for all f. Since $\mathcal{E} \to \mathcal{I}$ is a cartesian fibration, we can rewrite this as

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}_i}(e', f^*x) \arrow[r, "\bar{\phi}^*"] \arrow[d] & \mathrm{Map}_{\mathcal{E}_i}(e, f^*x) \arrow[d] \\ \mathrm{Map}_{\mathcal{J}}(j', k) \arrow[r, "\phi^*"'] & \mathrm{Map}_{\mathcal{J}}(j, k), \end{tikzcd}$$

where $f^*x \to x$ is a cartesian morphism over f. But now this square is cartesian since $\bar{\phi}$ is by assumption cocartesian in $\mathcal{E}_i$. The assertion that π preserves cocartesian morphisms amounts to π taking $\pi_{\mathcal{J}}$-cocartesian morphisms to equivalences in $\mathcal{I}$, which is clear from our description of $\pi_{\mathcal{J}}$-cocartesian morphisms. □

Proof of Proposition A.1.7. We first prove case (i). It follows from Corollary A.1.5 and Lemma A.1.8 (using Remark A.1.9) that $(p^\vee, q')$ is a cocartesian fibration. Moreover, the fibre $\mathcal{E}^\ell_{a,b}$ is by construction equivalent to the fibre $\mathcal{E}_{a,b}$, which is an ∞-groupoid. Hence $(p^\vee, q')$ is a left fibration.

In case (ii), Lemma A.1.10 implies that q' is a cocartesian fibration, and that q'-cocartesian morphisms map to equivalences under $p^\vee$. Since we also know that q' takes $p^\vee$-cartesian morphisms to equivalences, Corollary A.1.5 implies that $(p^\vee, q')$ is a bifibration since the fibres $(\mathcal{F}^b)_{a,b} \simeq \mathcal{F}_{a,b}$ are ∞-groupoids.

Remark A.1.11. Dually, we can replace a bifibration $\mathcal{E} \to \mathcal{A} \times \mathcal{B}$ by a right fibration $\mathcal{E}^r \to \mathcal{A} \times \mathcal{B}^{\mathrm{op}}$ and vice versa.

Remark A.1.12. Let $\mathrm{Cat}^{\mathrm{bifib}}_{\infty/\mathcal{A}\times\mathcal{B}}$ denote the full subcategory of $\mathrm{Cat}_{\infty/\mathcal{A}\times\mathcal{B}}$ spanned by the bifibrations, and let similarly $\mathrm{Cat}^{L}_{\infty/\mathcal{C}}$ and $\mathrm{Cat}^{R}_{\infty/\mathcal{C}}$ denote the full subcategories of $\mathrm{Cat}_{\infty/\mathcal{C}}$ spanned by the left and right fibrations, respectively. Since dualizing fibrations is an equivalence of ∞-categories, the constructions in Proposition A.1.7 and their dual versions give equivalences

$$\mathrm{Cat}^{\mathrm{bifib}}_{\infty/\mathcal{A}\times\mathcal{B}} \simeq \mathrm{Cat}^{\mathrm{L}}_{\infty/\mathcal{A}^{\mathrm{op}}\times\mathcal{B}} \simeq \mathrm{Cat}^{\mathrm{R}}_{\infty/\mathcal{A}\times\mathcal{B}^{\mathrm{op}}}.$$

A.2. Sections of bifibrations

In this subsection, we will describe sections of a bifibration in terms of the corresponding left and right fibrations.

Proposition A.2.1. *Let $\mathcal{I}$ be an ∞-category. Then the functor $(\mathrm{ev}_0, \mathrm{ev}_1): \mathcal{I}^{\Delta^1} \to \mathcal{I} \times \mathcal{I}$ is the* free *bifibration on $\mathcal{I}$, in the sense that the map*

$$\mathrm{Map}_{/\mathcal{I}\times\mathcal{I}}(\mathcal{I}^{\Delta^1}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Map}_{/\mathcal{I}\times\mathcal{I}}(\mathcal{I}, \mathcal{E}),$$

induced by composition with the canonical map $\mathrm{const}: \mathcal{I} \to \mathcal{I}^{\Delta^1}$, is an equivalence for every bifibration $\mathcal{E} \to \mathcal{I} \times \mathcal{I}$.

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Proof. By [11, Theorem 4.5], the functor $\mathrm{ev}_0 \colon \mathcal{I}^{\Delta^1} \to \mathcal{I}$ is the free cartesian fibration on $\mathrm{id}_{\mathcal{I}}$. Composition with const therefore induces an equivalence

$$\mathrm{Map}^{\mathrm{cart}}_{/\mathcal{I}}(\mathcal{I}^{\Delta^1}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Map}_{/\mathcal{I}}(\mathcal{I}, \mathcal{C})$$

for any cartesian fibration $\mathcal{C} \to \mathcal{I}$. In our case, we then have a commutative square

$$\begin{tikzcd}
\mathrm{Map}^{\mathrm{cart}}_{/\mathcal{I}}(\mathcal{I}^{\Delta^1}, \mathcal{E}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Map}_{/\mathcal{I}}(\mathcal{I}, \mathcal{E}) \arrow[d] \\
\mathrm{Map}^{\mathrm{cart}}_{/\mathcal{I}}(\mathcal{I}^{\Delta^1}, \mathcal{I} \times \mathcal{I}) \arrow[r, "\sim"'] & \mathrm{Map}_{/\mathcal{I}}(\mathcal{I}, \mathcal{I} \times \mathcal{I}),
\end{tikzcd}$$

where the horizontal maps are equivalences. On the fibre over $(\mathrm{ev}_1, \mathrm{ev}_0) \colon \mathcal{I}^{\Delta^1} \to \mathcal{I} \times \mathcal{I}$ (which corresponds to the diagonal $\Delta \colon \mathcal{I} \to \mathcal{I} \times \mathcal{I}$), we get an equivalence

$$\mathrm{Map}_{/\mathcal{I}\times\mathcal{I}}(\mathcal{I}^{\Delta^1}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Map}_{/\mathcal{I}\times\mathcal{I}}(\mathcal{I}, \mathcal{E})$$

since the morphisms in the source automatically preserve cartesian morphisms by Lemma A.1.3. □

Describing the spaces of sections of a bifibration in terms of the corresponding left and right fibrations turns out to involve the twisted arrow ∞-category.

Definition A.2.2. If $\mathcal{C}$ is an ∞-category, we define $\mathrm{Tw}^r(\mathcal{C})$ as the simplicial space

$$\mathrm{Map}([n] \star [n]^{\mathrm{op}}, \mathcal{C}).$$

Restricting to the factors $[n]$ and $[n]^{\mathrm{op}}$, we get a projection

$$\mathrm{Tw}^r(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

We also define $\mathrm{Tw}^\ell(\mathcal{C}) := \mathrm{Tw}^r(\mathcal{C})^{\mathrm{op}}$, which as a simplicial space is $\mathrm{Map}([n]^{\mathrm{op}} \star [n], \mathcal{C})$.

The following is essentially [20, Proposition 5.2.1.3] or [3, Proposition 1.1]. Since we have defined $\mathrm{Tw}^r(\mathcal{C})$ as a Segal space rather than a quasi-category, we briefly discuss how to adapt the proof to this setting.

Proposition A.2.3.

(i) *If $\mathcal{C}$ is a Segal space, then so is* $\mathrm{Tw}^r\,\mathcal{C}$.

(ii) *If $\mathcal{C}$ is a Segal space, then the morphism* $\mathrm{Tw}^r\,\mathcal{C} \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ *is a right fibration.*

(iii) *If $\mathcal{C}$ is a complete Segal space, then so is* $\mathrm{Tw}^r\,\mathcal{C}$.

Proof. To see that $\mathrm{Tw}^r\,\mathcal{C}$ is a Segal space, it suffices to prove that the morphisms $\epsilon^r(\Lambda^n_i) \to \epsilon^r(\Delta^n)$ for $0 < i < n$ are Segal equivalences (i.e. local equivalences for the localization to Segal spaces), where ϵ^r denotes the colimit-preserving functor $\mathcal{P}(\mathbb{\Delta}) \to \mathcal{P}(\mathbb{\Delta})$ extending $[n] \mapsto [n] \star [n]^{\mathrm{op}}$. This follows from the proof of [20, Proposition 5.2.1.3], where this map is shown to be inner anodyne in simplicial sets.

A morphism $\mathcal{E} \to \mathcal{B}$ of Segal spaces is a right fibration if and only if the commutative square

$$\begin{tikzcd} \mathcal{E}_1 \arrow[r] \arrow[d, "d_0"] & \mathcal{B}_1 \arrow[d, "d_0"] \\ \mathcal{E}_0 \arrow[r] & \mathcal{B}_0 \end{tikzcd}$$

is cartesian. For $\mathrm{Tw}^r\, \mathcal{C} \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, we have $(\mathrm{Tw}^r\, \mathcal{C})_1 \simeq \mathrm{Map}(\epsilon^r(\Delta^1), \mathcal{C})$, where $\epsilon^r(\Delta^1) \simeq \Delta^1 \star \Delta^{1,\mathrm{op}} \simeq \Delta^3$, and the square can be rewritten as

$$\begin{tikzcd} \mathrm{Map}(\Delta^3, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^{\{0,1\}} \amalg \Delta^{\{2,3\}}, \mathcal{C}) \arrow[d] \\ \mathrm{Map}(\Delta^{\{1,2\}}, \mathcal{C}) \arrow[r] & \mathrm{Map}(\Delta^{\{1\}} \amalg \Delta^{\{2\}}, \mathcal{C}). \end{tikzcd}$$

This is cartesian since

$$\Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \Delta^{\{2,3\}} \to \Delta^3$$

is a (generating) Segal equivalence.

It is easy to see that any right fibration $\mathcal{E} \to \mathcal{B}$ is conservative, and so gives a pullback square

$$\begin{tikzcd} \mathcal{E}_1^{\mathrm{eq}} \arrow[r] \arrow[d, "d_0"] & \mathcal{B}_1^{\mathrm{eq}} \arrow[d, "d_0"] \\ \mathcal{E}_0 \arrow[r] & \mathcal{B}_0. \end{tikzcd}$$

Thus if $\mathcal{B}$ is complete, then so is $\mathcal{E}$, which means that (ii) implies (iii). $\square$

Proposition A.2.4. *The projection* $\mathrm{Tw}^r(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ *is the cartesian fibration corresponding to the cocartesian fibration* $\mathrm{ev}_1 \colon \mathcal{C}^{\Delta^1} \to \mathcal{C}$.

Proof. Let $\pi \colon \mathcal{E} \to \mathcal{C}^{\mathrm{op}}$ be this dual cartesian fibration. Observe that we have a commutative triangle

$$\begin{tikzcd} \mathcal{C}^{\Delta^1} \arrow[rr, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \arrow[dr, "\mathrm{ev}_1"'] & & \mathcal{C} \times \mathcal{C} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd},$$

where the downward maps are cocartesian fibrations, and the horizontal map preserves cocartesian morphisms. Dualizing, this corresponds to a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \arrow[dl] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd},$$

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where ϕ preserves cartesian morphisms. We claim that ϕ is in fact a right fibration. To prove this, we first use [17, Proposition 2.4.2.11] to see that ϕ is a locally cartesian fibration since fibrewise over $\mathcal{C}^{\mathrm{op}}$ it is given by $\mathcal{E}_x \simeq (\mathcal{C}^{\Delta^1})_x \simeq \mathcal{C}_{/x} \to \mathcal{C}$, which is a right fibration; since the fibres are moreover spaces, this implies that ϕ is a right fibration.

We can now use [20, Corollary 5.2.1.22] to conclude that $\mathcal{E}$ is equivalent to $\mathrm{Tw}^r(\mathcal{C})$ over $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ if and only if

(i) for $c \in \mathcal{C}$, the fibre $\mathcal{E}_{c,\mathcal{C}}$ has a terminal object,

(ii) for $c \in \mathcal{C}^{\mathrm{op}}$, the fibre $\mathcal{E}_{c,\mathcal{C}^{\mathrm{op}}}$ has a terminal object,

(iii) an object $x \in \mathcal{E}$ over (a, b) is terminal in $\mathcal{E}_{a,\mathcal{C}}$ if and only if it is terminal in $\mathcal{E}_{b,\mathcal{C}^{\mathrm{op}}}$.

In our case, the fibre $\mathcal{E}_{c,\mathcal{C}^{\mathrm{op}}}$ is equivalent to $\mathcal{C}_{/c}$, and the fibre at $c \in \mathcal{C}$ is $(\mathcal{C}_{c/})^{\mathrm{op}}$ (as this is the dualization of the fibre $\mathcal{C}_{c/} \to \mathcal{C}$ of $\mathcal{C}^{\Delta^1} \to \mathcal{C} \times \mathcal{C}$ at c, and dualization preserves pullbacks). Both of these clearly have terminal objects. An element in the fibre over $(a, b) \in \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ can be identified with a morphism $b \to a$, and in both cases, the criterion for this to be a fibrewise terminal object is that this morphism must be an equivalence. □

Corollary A.2.5. *The left and right fibrations corresponding to the bifibration $\mathcal{I}^{\Delta^1} \to \mathcal{I} \times \mathcal{I}$ are the left and right twisted arrow ∞-categories*

$$\mathrm{Tw}^\ell\, \mathcal{I} \to \mathcal{I}^{\mathrm{op}} \times \mathcal{I}, \quad \mathrm{Tw}^r\, \mathcal{I} \to \mathcal{I} \times \mathcal{I}^{\mathrm{op}},$$

respectively.

Proof. By Proposition A.2.4, $\mathrm{Tw}^r\, \mathcal{I} \to \mathcal{I}^{\mathrm{op}}$ is the cartesian fibration corresponding to $\mathrm{ev}_1 : \mathcal{I}^{\Delta^1} \to \mathcal{I}$, and similarly for $\mathrm{Tw}^\ell\, \mathcal{I}$; so this follows from Construction A.1.6. □

From this, we obtain a useful description of the sections of a bifibration.

Corollary A.2.6. *Suppose $\mathcal{E} \to \mathcal{A} \times \mathcal{B}$ is a bifibration. Then for functors $\alpha : \mathcal{C} \to \mathcal{A}$, $\beta : \mathcal{C} \to \mathcal{B}$, the space of sections*

$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ \mathcal{C} \arrow[ur] \arrow[r, "(\alpha,\beta)"'] & \mathcal{A} \times \mathcal{B} \end{tikzcd}$$

is equivalent to the spaces of commutative squares

$$\left\{ \begin{tikzcd} \mathcal{C}^{\Delta^1} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\alpha \times \beta"] & \mathcal{A} \times \mathcal{B} \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} \mathrm{Tw}^r(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{E}^r \arrow[d] \\ \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \arrow[r, "\alpha \times \beta^{\mathrm{op}}"] & \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} \mathrm{Tw}^\ell(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{E}^\ell \arrow[d] \\ \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \arrow[r, "\alpha^{\mathrm{op}} \times \beta"] & \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \end{tikzcd} \right\}.$$

A.3. Fibrations of functor ∞-categories

In this subsection, we will prove the following result.

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Proposition A.3.1. *Let $\mathcal{F} \to \mathcal{I}$ be the cocartesian fibration for a functor $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ and $\mathcal{G} \to \mathcal{I}$ be the cartesian fibration for a functor $G\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. If $\mathcal{H} \to \mathcal{I}$ is the cocartesian fibration for the functor $H := \mathrm{Fun}(F(-), G(-))\colon \mathcal{I} \to \mathrm{Cat}_\infty$, then there is a natural equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathcal{I}}(\mathcal{I}, \mathcal{H}) \simeq \mathrm{Fun}_{\mathcal{I}}(\mathcal{F}, \mathcal{G}).$$

Under this equivalence, the cocartesian sections of $\mathcal{H}$ correspond to the functors $\mathcal{F} \to \mathcal{G}$ that take cartesian morphisms to cocartesian morphisms.

The proof requires understanding a variant of the twisted arrow category.

Definition A.3.2. For an ∞-category $\mathcal{C}$, viewed as a complete Segal space, we define $\mathrm{Tw}_2(\mathcal{C})$ to be the simplicial space

$$\mathrm{Tw}_2^r(\mathcal{C})_n \simeq \mathrm{Map}([n] \star [n]^{\mathrm{op}} \star [n], \mathcal{C}).$$

Since $[n] \star [n]^{\mathrm{op}} \star [n]$ can be identified with the pushout of ∞-categories $([n] \star [n]^{\mathrm{op}}) \amalg_{[n]^{\mathrm{op}}} ([n]^{\mathrm{op}} \star [n])$, the simplicial space $\mathrm{Tw}_2^r(\mathcal{C})$ is given by the pullback

$$\begin{tikzcd}
\mathrm{Tw}_2^r(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Tw}^r(\mathcal{C}) \arrow[d] \\
\mathrm{Tw}^r(\mathcal{C})^{\mathrm{op}} \arrow[r] & \mathcal{C}^{\mathrm{op}}.
\end{tikzcd}$$

This implies in particular that $\mathrm{Tw}_2^r(\mathcal{C})$ is a complete Segal space, i.e. an ∞-category.

Lemma A.3.3. *Let $f\colon \mathcal{E} \to \mathcal{B}$ be any functor of ∞-categories. Then*

$$\mathrm{Tw}^r(\mathcal{B}) \times_{\mathcal{B}} \mathcal{E} \to \mathcal{B}^{\mathrm{op}}$$

is a cartesian fibration, corresponding to the functor $\mathcal{B} \to \mathrm{Cat}_\infty$ given by

$$b \mapsto \mathcal{B}_{/b} \times_{\mathcal{B}} \mathcal{E}.$$

Proof. This functor factors as the composite

$$\mathrm{Tw}^r(\mathcal{B}) \times_{\mathcal{B}} \mathcal{E} \to \mathcal{E} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}},$$

where the first functor is a cartesian fibration, being a pullback of $\mathrm{Tw}^r(\mathcal{B}) \to \mathcal{B} \times \mathcal{B}^{\mathrm{op}}$, and the second is obviously a cartesian fibration. Moreover, we can write $\mathrm{Tw}^r(\mathcal{B}) \times_{\mathcal{B}} \mathcal{E}$ as the fibre product $\mathrm{Tw}^r(\mathcal{B}) \times_{\mathcal{B} \times \mathcal{B}^{\mathrm{op}}} \mathcal{E} \times \mathcal{B}^{\mathrm{op}}$ of cartesian fibrations over $\mathcal{B}^{\mathrm{op}}$. This identifies the corresponding functors as the fibre product of the functors associated to the three factors; as $\mathrm{Tw}^r(\mathcal{B}) \to \mathcal{B}^{\mathrm{op}}$ corresponds to $b \mapsto \mathcal{B}_{/b}$ by Proposition A.2.4 and the two other fibrations correspond to constant functors, this gives the result. □

Lemma A.3.4. *There are natural equivalences of ∞-categories*

$$\mathrm{Tw}^r(\mathcal{C}_{/x}) \simeq (\mathcal{C}_{/x})^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C}),$$

$$\mathrm{Tw}^r(\mathcal{C}_{x/}) \simeq \mathcal{C}_{x/} \times_{\mathcal{C}} \mathrm{Tw}^r(\mathcal{C}).$$

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Proof. We will prove the first equivalence; the proof of the second is similar. By the universal property of $\mathcal{C}_{/x}$ and the definition of the twisted arrow ∞-category, we have a natural pullback square

$$\begin{tikzcd}
\mathrm{Map}([n], \mathrm{Tw}^r(\mathcal{C})_{/x}) \arrow[r] \arrow[d] & \mathrm{Map}([n] \star [n]^{\mathrm{op}} \star [0], \mathcal{C}) \arrow[d] \\
\{x\} \arrow[r] & \mathrm{Map}([0], \mathcal{C}).
\end{tikzcd}$$

On the other hand, we have a pullback square

$$\begin{tikzcd}
\mathrm{Map}([n], (\mathcal{C}_{/x})^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C})) \arrow[r] \arrow[d] & \mathrm{Map}([n]^{\mathrm{op}}, \mathcal{C}_{/x}) \arrow[d] \\
\mathrm{Map}([n] \star [n]^{\mathrm{op}}, \mathcal{C}) \arrow[r] & \mathrm{Map}([n]^{\mathrm{op}}, \mathcal{C}).
\end{tikzcd}$$

We can expand this to a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}([n], (\mathcal{C}_{/x})^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C})) \arrow[r] \arrow[d] & \mathrm{Map}([n]^{\mathrm{op}}, \mathcal{C}_{/x}) \arrow[r] \arrow[d] & \{x\} \arrow[d] \\
\mathrm{Map}([n] \star [n]^{\mathrm{op}} \star [0], \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Map}([n]^{\mathrm{op}} \star [0], \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Map}([0], \mathcal{C}) \\
\mathrm{Map}([n] \star [n]^{\mathrm{op}}, \mathcal{C}) \arrow[r] & \mathrm{Map}([n]^{\mathrm{op}}, \mathcal{C}),
\end{tikzcd}$$

where all the squares are pullbacks. In particular, the composite square in the top row is a pullback, which shows that $\mathrm{Map}([n], (\mathcal{C}_{/x})^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C}))$ is equivalent to $\mathrm{Map}([n], \mathrm{Tw}^r(\mathcal{C})_{/x})$, naturally in $[n]$ and x, as required. □

Lemma A.3.5. *Suppose $\pi: \mathcal{E} \to \mathcal{B}$ is a cartesian fibration whose fibres are all weakly contractible. Then π is both cofinal and coinitial.*

Proof. The functor π is cofinal by [17, Lemma 4.1.3.2]. To see that it is also coinitial, observe that for any functor $F: \mathcal{B} \to \mathcal{C}$ the right Kan extension $\pi_*\pi^*F$ exists, and $\pi_*\pi^*F(b) \simeq \lim_{\mathcal{E}_b} F(b) \simeq F(b)$, where the second equivalence uses that $\mathcal{E}_b$ is weakly contractible; thus $\pi_*\pi^*F \simeq F$. The limit of π^*F over $\mathcal{E}$ is the limit over $\mathcal{B}$ of $\pi_*\pi^*F \simeq F$; hence π is indeed coinitial. □

Lemma A.3.6. *For any ∞-category $\mathcal{C}$, the functors $\mathrm{Tw}^r(\mathcal{C}) \to \mathcal{C}, \mathcal{C}^{\mathrm{op}}$ are both cofinal and coinitial.*

Proof. We know that $\mathrm{Tw}^r(\mathcal{C}) \to \mathcal{C}$ and $\mathrm{Tw}^r(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ are cartesian fibrations, with fibres $(\mathcal{C}_{x/})^{\mathrm{op}}$ and $\mathcal{C}_{/x}$, respectively. These ∞-categories are weakly contractible; hence these functors are both cofinal and coinitial by Lemma A.3.5. □

Lemma A.3.7. *Let $\pi_0, \pi_2\colon \mathrm{Tw}_2^r(\mathcal{C}) \to \mathcal{C}$ be the projections induced by restriction to the first and the second copy of $[n]$ in $[n] \star [n]^{\mathrm{op}} \star [n]$, respectively. Then*

(i) *π_0 is a cartesian fibration, corresponding to the functor $x \mapsto \mathrm{Tw}^r(\mathcal{C}_{x/})^{\mathrm{op}}$,*

(ii) *π_2 is a cocartesian fibration, corresponding to the functor $x \mapsto \mathrm{Tw}^r(\mathcal{C}_{/x})$.*

Proof. From the definition of $\mathrm{Tw}^r_2(\mathcal{C})$, we have a pullback square

$$\begin{tikzcd} \mathrm{Tw}^r_2(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Tw}^r(\mathcal{C}) \arrow[d] \\ \mathrm{Tw}^r(\mathcal{C})^{\mathrm{op}} \arrow[r] & \mathcal{C}^{\mathrm{op}}. \end{tikzcd}$$

Now Lemma A.3.3 applied to $\mathrm{Tw}^r(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ (using the equivalence $\mathrm{Tw}^r(\mathcal{C}) \simeq \mathrm{Tw}^r(\mathcal{C}^{\mathrm{op}})$) gives that π_0 is a cartesian fibration corresponding to the functor

$$x \mapsto (\mathcal{C}_{x/})^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C})^{\mathrm{op}}.$$

Similarly, using the op'ed version of Lemma A.3.3, we see that π_2 is the cocartesian fibration for the functor $x \mapsto (\mathcal{C}_{/x})^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C})$. Now Lemma A.3.4 identifies these functors with $\mathrm{Tw}^r(\mathcal{C}_{-/})^{\mathrm{op}}$ and $\mathrm{Tw}^r(\mathcal{C}_{/-})$, respectively. □

Lemma A.3.8. *Consider a diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\phi"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd},$$

where p and q are cocartesian fibrations and ϕ preserves cocartesian morphisms. If the functor $\phi_b \colon \mathcal{E}_b \to \mathcal{F}_b$ is cofinal for every $b \in \mathcal{B}$, then ϕ is cofinal, as is $\phi \times_{\mathcal{B}} \mathcal{B}'$ for any functor $\mathcal{B}' \to \mathcal{B}$.

Proof. It suffices to check that composition with ϕ^{op} preserves limits for functors $f \colon \mathcal{F}^{\mathrm{op}} \to \mathcal{S}$. But here we have natural equivalences

$$\lim_{\mathcal{F}^{\mathrm{op}}} f \simeq \lim_{b \in \mathcal{B}^{\mathrm{op}}} \lim_{\mathcal{F}_b^{\mathrm{op}}} f|_{\mathcal{F}_b^{\mathrm{op}}} \simeq \lim_{b \in \mathcal{B}^{\mathrm{op}}} \lim_{\mathcal{E}_b^{\mathrm{op}}} (f\phi)|_{\mathcal{F}_b^{\mathrm{op}}} \simeq \lim_{\mathcal{E}^{\mathrm{op}}} f\phi.$$

Since the same condition holds for the pullback of ϕ along any map $\mathcal{B}' \to \mathcal{B}$, any such pullback of ϕ is also cofinal. □

Lemma A.3.9. *There is a natural inclusion of posets $[n] \times [1] \to [n] \star [n]^{\mathrm{op}} \star [n]$, extending the inclusion of two copies of $[n]$, which induces a functor of ∞-categories*

$$\Phi\colon \mathrm{Tw}^r_2(\mathcal{C}) \to \mathcal{C}^{\Delta^1}.$$

This functor is both cofinal and coinitial.

Proof. We have commutative diagrams

$$\begin{tikzcd} \mathrm{Tw}^r_2(\mathcal{C}) \arrow[rr, "\Phi"] \arrow[dr, "\pi_2"'] & & \mathcal{C}^{\Delta^1} \arrow[dl, "\mathrm{ev}_1"] \\ & \mathcal{C} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Tw}^r_2(\mathcal{C}) \arrow[rr, "\Phi"] \arrow[dr, "\pi_0"'] & & \mathcal{C}^{\Delta^1} \arrow[dl, "\mathrm{ev}_0"] \\ & \mathcal{C}. & \end{tikzcd}$$

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In the first diagram, the diagonal morphisms are both cocartesian fibrations while in the second, they are cartesian fibrations. Moreover, the functor Φ clearly preserves cocartesian and cartesian morphisms for these fibrations. To show that the top morphism is cofinal or coinitial, it therefore suffices by Lemma A.3.8 to show that the induced morphisms on fibres are all cofinal in the first diagram and coinitial in the second diagram. At $x \in \mathcal{C}$, we can identify these with the projections $\mathrm{Tw}^r(\mathcal{C}_{/x}) \to \mathcal{C}_{/x}$ and $\mathrm{Tw}^r(\mathcal{C}_{x/})^{\mathrm{op}} \to \mathcal{C}_{x/}$, respectively. These are both cofinal and coinitial by Lemma A.3.6. $\square$

Proof of Proposition A.3.1. By [11, Corollary 7.7], the ∞-category $\mathrm{Fun}_{\mathcal{J}}(\mathcal{I}, \mathcal{H})$ is the limit

$$\lim_{i \to j \in \mathrm{Tw}^r(\mathcal{I})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{I}_{/i}, H(j)) \simeq \lim_{i \to j \in \mathrm{Tw}^r(\mathcal{I})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{I}_{/i} \times F(j), G(j)).$$

Similarly, $\mathrm{Fun}_{\mathcal{J}}(\mathcal{F}, \mathcal{G})$ is the limit

$$\lim_{i \to j \in \mathrm{Tw}^r(\mathcal{I})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{I}_{/i} \times_{\mathcal{J}} \mathcal{F}, G(j)).$$

Here $\mathcal{I}_{/i} \times_{\mathcal{J}} \mathcal{F} \to \mathcal{I}_{/i}$ is a cartesian fibration, equivalent by [11, Corollary 7.6] to the colimit

$$\operatorname*{colim}_{x \to y \in \mathrm{Tw}^r(\mathcal{I}_{/i})} \mathcal{I}_{/x} \times F(y).$$

Thus the ∞-category $\mathrm{Fun}_{\mathcal{J}}(\mathcal{F}, \mathcal{G})$ is the limit

$$\lim_{i \to j \in \mathrm{Tw}^r(\mathcal{I})^{\mathrm{op}}} \lim_{x \to y \in \mathrm{Tw}^r(\mathcal{I}_{/i})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{I}_{/x} \times F(y), G(j)).$$

Let $\mathrm{Tw}^r_3(\mathcal{I})$ denote the pullback $\mathrm{Tw}^r_2(\mathcal{I}) \times_{\mathcal{J}} \mathrm{Tw}^r(\mathcal{I})$, where $\mathrm{Tw}^r_2(\mathcal{I})$ is defined in Definition A.3.2. By Lemma A.3.7, the projection $\mathrm{Tw}^r_3(\mathcal{I}) \to \mathrm{Tw}^r(\mathcal{I})$ is then the cocartesian fibration for the functor taking $i \to j$ in $\mathrm{Tw}^r(\mathcal{I})$ to $\mathrm{Tw}^r(\mathcal{I}_{/i})$. Combining the limits in the expression above, we may therefore identify $\mathrm{Fun}_{\mathcal{J}}(\mathcal{F}, \mathcal{G})$ with the limit

$$\lim_{x \to y \to i \to j \in \mathrm{Tw}^r_3(\mathcal{I})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{I}_{/x} \times F(y), G(j)).$$

We may also identify $\mathrm{Tw}_3(\mathcal{I})$ with the pullback $\mathrm{Tw}^r(\mathcal{I}) \times_{\mathcal{J}^{\mathrm{op}}} \mathrm{Tw}^r_2(\mathcal{I})^{\mathrm{op}}$. The functor whose limit we are taking clearly factors through

$$\mathrm{Tw}^r(\mathcal{I}) \times_{\mathcal{J}^{\mathrm{op}}} \Phi^{\mathrm{op}} \colon \mathrm{Tw}^r(\mathcal{I}) \times_{\mathcal{J}^{\mathrm{op}}} \mathrm{Tw}^r_2(\mathcal{I})^{\mathrm{op}} \to \mathrm{Tw}^r(\mathcal{I}) \times_{\mathcal{J}^{\mathrm{op}}} (\mathcal{J}^{\Delta^1})^{\mathrm{op}},$$

where Φ is the functor of Lemma A.3.9. This functor is cofinal by Lemma A.3.8 since Φ is fibrewise cofinal and coinitial, and so this is the pullback of a fibrewise cofinal morphism of cocartesian fibrations over $\mathcal{J}^{\mathrm{op}}$. This means we may identify $\mathrm{Fun}_{\mathcal{J}}(\mathcal{F}, \mathcal{G})$ with the limit

$$\lim_{x \to y \to j \in \mathrm{Tw}^r(\mathcal{I})^{\mathrm{op}} \times_{\mathcal{J}} \mathcal{J}^{\Delta^1}} \mathrm{Fun}(\mathcal{I}_{/x} \times F(y), G(j)).$$

Now consider the commutative triangle

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "c"] \arrow[dr, equal] & & \mathcal{C}^{\Delta^1} \arrow[dl, "\mathrm{ev}_0"] \\ & \mathcal{C} & \end{tikzcd},$$

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where c is the functor induced by composition with $\Delta^1 \to \Delta^0$, taking an object to its identity morphism. This is a morphism of cartesian fibrations, given on fibres by $\{x\} \to \mathcal{C}_{x/}$, which is clearly coinitial; hence c is itself coinitial, as is its pullback along any morphism to the base $\mathcal{C}$. In particular, the induced functor $\mathrm{Tw}^r(\mathcal{J})^{\mathrm{op}} \to \mathrm{Tw}^r(\mathcal{J})^{\mathrm{op}} \times_{\mathcal{J}} \mathcal{J}^{\Delta^1}$ is coinitial. Thus $\mathrm{Fun}_{\mathcal{J}}(\mathcal{F}, \mathcal{G})$ can finally be identified with

$$\lim_{x \to j \in \mathrm{Tw}^r(\mathcal{J})^{\mathrm{op}}} \mathrm{Fun}(\mathcal{J}_{/x} \times F(j), G(j)),$$

which is the same as our first expression for $\mathrm{Fun}_{\mathcal{J}}(\mathcal{J}, \mathcal{H})$. To identify the cocartesian sections, observe that our work so far shows that the cocartesian fibration $\mathcal{H} \to \mathcal{J}$ has the same universal property as the cocartesian fibration given by (the dual of) [17, Corollary 3.2.2.13], whose cocartesian sections are shown there to be given by functors $\mathcal{F} \to \mathcal{G}$ that take cartesian morphisms to cocartesian ones.

Corollary A.3.10. *Let $\mathcal{E} \to \mathcal{C}$ be a cartesian fibration corresponding to a functor $\epsilon\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, and $\mathcal{F} \to \mathcal{D}$ be a cocartesian fibration corresponding to a functor $\phi\colon \mathcal{D} \to \mathrm{Cat}_\infty$. Then if $\mathcal{G} \to \mathcal{C} \times \mathcal{D}$ is the cocartesian fibration corresponding to $\mathrm{Fun}(\epsilon, \phi)\colon \mathcal{C} \times \mathcal{D} \to \mathrm{Cat}_\infty$, then $\mathcal{G}$ satisfies*

$$\mathrm{Fun}_{/\mathcal{C} \times \mathcal{D}}(\mathcal{J}, \mathcal{G}) \simeq \mathrm{Fun}_{/\mathcal{D}}(\mathcal{J} \times_{\mathcal{C}} \mathcal{E}, \mathcal{F})$$

for any functor $\mathcal{J} \to \mathcal{C} \times \mathcal{D}$. Under this equivalence, a cocartesian morphism in $\mathcal{G}$ corresponds to a functor

$$\Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \mathcal{F}$$

that takes cartesian morphisms for $\Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \Delta^1$ to cocartesian morphisms in $\mathcal{F}$.

Proof. Apply Proposition A.3.1 to the pullback of the fibrations to $\mathcal{J}$. □

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