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IHARA LEMMA AND LEVEL RAISING IN HIGHER DIMENSION

PASCAL BOYER

Université Sorbonne Paris Nord, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse, France, CoLoss AAPG2019
(boyer@math.univ-paris13.fr)

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Abstract A key ingredient in the Taylor–Wiles proof of Fermat's last theorem is the classical Ihara lemma, which is used to raise the modularity property between some congruent Galois representations. In their work on Sato and Tate, Clozel, Harris and Taylor proposed a generalisation of the Ihara lemma in higher dimension for some similitude groups. The main aim of this paper is to prove some new instances of this generalised Ihara lemma by considering some particular non-pseudo-Eisenstein maximal ideals of unramified Hecke algebras. As a consequence, we prove a level-raising statement.

Key words and phrases: Shimura varieties; torsion in the cohomology; maximal ideal of the Hecke algebra; localised cohomology; Galois representation

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1. Introduction

Let $F = F^+E$ be a CM field with $E/\mathbb{Q}$ quadratic imaginary and F^+ totally real. For $\overline{B}/F$ a central division algebra with dimension d^2 equipped with a involution of second kind $*$ and $\beta \in \overline{B}^{*=-1}$, consider the similitude group $\overline{G}/\mathbb{Q}$ defined for any $\mathbb{Q}$-algebra R by

$$\overline{G}(R) := \left\{(\lambda, g) \in R^\times \times (\overline{B}^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\right\}$$

with $\overline{B}^{op} = \overline{B} \otimes_{F,c} F$, where $c = *_{|F}$ is the complex conjugation and $\sharp_\beta$ the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$. For $p = uu^c$ decomposed in E, we have

$$\overline{G}(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (\overline{B}_v^{op})^\times,$$

where w describes the places of F above u. We suppose the following:

- The associated unitary group $\overline{G}_0(\mathbb{R})$ is compact.
- For any place x of $\mathbb{Q}$ inert or ramified in E, then $G(\mathbb{Q}_x)$ is quasi-split.
- There exists a place v_0 of F above u such that $\overline{B}_{v_0} \simeq D_{v_0,d}$ is the central division algebra over the completion F_{v_0} of F at v_0, with invariant $\frac{1}{d}$.

Fix a prime number $l \neq p$ and consider a finite set S of places of F containing the ramification places Bad of $\overline{B}$. Denote by $\mathbb{T}_S/\overline{\mathbb{Z}}_l$ the unramified Hecke algebra of G outside S. For a cohomological minimal prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_S$, we can associate both a near equivalence class of $\overline{\mathbb{Q}}_l$-automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ and a Galois representation

$$\rho_{\widetilde{\mathfrak{m}}} : G_F := \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

such that the eigenvalues of the Frobenius morphism at an unramified place w are given by the Satake parameter of the local component $\Pi_{\widetilde{\mathfrak{m}},w}$ of $\Pi_{\widetilde{\mathfrak{m}}}$. The semisimple class $\overline{\rho}_{\mathfrak{m}}$ of the reduction modulo l of $\rho_{\widetilde{\mathfrak{m}}}$ depends only on the maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ containing $\widetilde{\mathfrak{m}}$. For all prime x of $\mathbb{Z}$ split in E and a place $w \notin S$ of F above x, we then denote by $P_{\mathfrak{m},w}(X)$ the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_w)$.

1.1 Conjecture (Generalised Ihara lemma by Clozel, Harris and Taylor)**.** *Consider the following:*

- *an open compact subgroup $\overline{U}$ of $\overline{G}(\mathbb{A})$ such that outside S, its local component is the maximal compact subgroup;*
- *a place $w_0 \notin S$ decomposed in E;*
- *a maximal $\mathfrak{m}$ of $\mathbb{T}_S$ such that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.*

Let $\bar{\pi}$ be an irreducible subrepresentation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/\overline{U}^{w_0}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, where $\overline{U} = \overline{U}_{w_0}\overline{U}^{w_0}$; then its local component $\bar{\pi}_{w_0}$ at w_0 is generic.

Remark. In its classical version for GL_2, Ihara's lemma is used to raise the modularity property between some congruent Galois representations; this was also the role of this higher-dimensional version in Clozel, Harris and Taylor's paper on the Sato–Tate conjecture. Shortly afterward, Taylor found an argument to avoid Ihara's lemma. However, this conjecture remains highly interesting (see, e.g., [14, 18]).

The main result of this paper is the following instances of Conjecture 1.1.

1.2 Theorem. *Conjecture 1.1 is true if the maximal ideal $\mathfrak{m}$ verifies the following extra properties:*

(H1) $\mathfrak{m}$ *is KHT-free (see. Remark 1).*

(H2) *The image of $\overline{\rho}_{\mathfrak{m},w_0}$ in its Grothendieck group is multiplicity free*[1] *and does not contain any full Zelevinsky line.*[2]

(H3) $\overline{\rho}_{\mathfrak{m},v_0}$ *is multiplicity free in the following meaning. It corresponds (see. §2.2) by Jacquet–Langlands correspondence to some super-Speh representation* $\mathrm{Speh}_s(\varrho_{v_0})$, *where ϱ_{v_0} is a supercuspidal $\overline{\mathbb{F}}_l$-representation of $GL_g(F_{v_0})$ with $d=sg$ (see. [15, Theorem 3.1.4]. We then ask (see. the notation in §2.2) that*

$$\varrho_{v_0},\varrho_{v_0}\{1\},\cdots,\varrho_{v_0}\{s-1\}$$

be pairwise distinct.

Remark. Concerning (H1), we say that $\mathfrak{m}$ is KHT free if the cohomology groups of the Kottwitz–Harris–Taylor Shimura variety of §2.1, localised at $\mathfrak{m}$, are free. From [8], any of the following properties ensures the KHT-freeness of $\mathfrak{m}$ (see. §3.2):

(1) There exists $w_1\in \mathrm{Spl}(I)$ such that (see. §3.1) the multiset $S_{\mathfrak{m}}(w_1)$ of roots of $P_{\mathfrak{m},w_1}(X)$ does not contain any submultiset of the shape $\{\alpha,q_{w_1}\alpha\}$, where q_{w_1} is the cardinality of the residue field. This hypothesis is called *generic* in [12].

(2) $[F(\exp(2i\pi/l):F]>d$, if we suppose the following property to be true (see. [8, hypothesis 4.17]): if $\theta: G_F\longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ is an irreducible continuous representation such that for all places $w\notin S$ above a prime $x\in\mathbb{Z}$ split in E, then $P_{\mathfrak{m},w}(\theta(\mathrm{Frob}_w))=0$ (resp., $P_{\mathfrak{m}^\vee,w}(\theta(\mathrm{Frob}_w))=0$) and θ is equivalent to $\overline{\rho}_{\mathfrak{m}}$ (resp., $\overline{\rho}_{\mathfrak{m}^\vee}$), where $\mathfrak{m}^\vee$ is the maximal ideal of $\mathbb{T}_S$ associated to the dual multiset of Satake parameters (see. [8, Notation 4.4]). In [17], the authors proved that the previous property is verified in each of the following cases:

- $\overline{\rho}_{\mathfrak{m}}$ is induced from a character of G_K, where K/F is a cyclic Galois extension.
- $l\geq d$ and $SL_d(k)\subset\overline{\rho}_{\mathfrak{m}}(G_F)\subset\overline{\mathbb{F}}_l^\times GL_d(k)$ for some subfield $k\subset\overline{\mathbb{F}}_l$.

(3) $\overline{\rho}_{\mathfrak{m}}$ is irreducible and $[F(\exp(2i\pi/l):F]>d$ [11].

By Chebotarev's theorem, the hypothesis $[F(\exp(2i\pi/l):F]>d$ allows us to pick places v of F such that the order q_v of the residue field of F at v is of order strictly greater than d in $\mathbb{Z}/l\mathbb{Z}$.

[1] In particular, q_{w_1} cannot be congruent to 1 modulo l.

[2] Using the main result of [9], we could take off the condition about not containing a full Zelevinsky line (see. Remark 4.2.2).

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Remark. About (H2), note that the first condition also appears in [13, §4.5] in the statements of level raising. Concerning the second condition of (H2), we can remove it using the main result of [9] (see. Remark 4.2).

To prove theorem 1.2, we first translate such a property to the cohomology group of middle degree of the Kottwitz–Harris–Taylor Shimura variety X_I associated to the similitude group $G/\mathbb{Q}$, such that

- $G(\mathbb{A}^\infty) = \overline{G}(\mathbb{A}^{\infty,p}) \times GL_d(F_{v_0}) \times \prod_{\substack{w|u \\ w \neq v_0}} (\overline{B}_w^{op})^\times$,
- the signatures of $G(\mathbb{R})$ are $(1, d-1) \times (0, d) \times \cdots \times (0, d)$.

In particular, to each prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_S$ is associated a $\overline{\mathbb{Q}}_l$-irreducible automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ of $G(\mathbb{A}_{\mathbb{Q}})$ whose Satake parameters at finite places outside S are prescribed by $\widetilde{\mathfrak{m}}$. We then compute the cohomology groups of the geometric generic fibre of X_I through the spectral sequence of vanishing cycles at the place v_0. Thanks to (H1), the $H^i(X_U, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ are free, and so $H^i(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} = (0)$ for $i \neq d-1$.

Remark. Moreover, (H2) (resp., (H3)) ensures that the graded parts of the filtration of $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, given by the integral version of the weight-monodromy filtration, at the place w_0 (resp., v_0) are also free.

The contribution of the supersingular points of the special fibre at v_0, using (H3), allows us to associate to an irreducible subrepresentation $\overline{\pi}$ of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/\overline{U}^{w_0}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ an irreducible subrepresentation π of $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, such that $\pi^{\infty,v_0} \simeq \bar{\pi}^{\infty,v_0}$. We then try to prove the genericness of π_{w_0} by proving, using (H2), the genericness of irreducible submodules of $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. One ingredient in §4.1 comes from [21, §5], where the hypothesis that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible ensures that the lattices of isotypic components of $H^{d-1}(X_U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}}$, given by the $\overline{\mathbb{Z}}_l$-cohomology, can be written as a tensorial product of stable lattices for $G(\mathbb{A}^\infty)$ and the Galois actions.

Finally, (H2) is needed to control the combinatorics.

Remark. As pointed out to us by M. Harris, the case where the cardinality q_{w_0} of the residue field at w_0 is congruent to 1 modulo l should be of crucial importance for the applications. Meanwhile, our strategy relies on the construction of a filtration of $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ where each graded part verifies the genericness of an irreducible submodule and where these graded parts are parabolically induced. When $q_{w_0} \equiv 1 \bmod l$, parabolically induced $\overline{\mathbb{F}}_l$-representations are often semisimple, and so they cannot verify the genericness of an irreducible submodule. It seems that our approach is not well adapted to treating this fundamental case.

To state our application to level raising, denote by $\mathcal{S}_{w_0}(\mathfrak{m})$ the supercuspidal support of the modulo l reduction of $\Pi_{\widetilde{\mathfrak{m}}, w_0}$ for any prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$: it depends only on $\mathfrak{m}$. By (H2), this support is multiplicity free, and we first break it as $\mathcal{S}_{w_0}(\mathfrak{m}) = \coprod_{\varrho \in \mathcal{Z}} \mathcal{S}_\varrho(\mathfrak{m})$ according to the set of Zelevinsky lines $ZL(\varrho) = \{\varrho\{k\} : \ k \in \mathbb{Z}\}$, where $\mathcal{Z}$ is the set of

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equivalence classes of irreducible supercuspidal $\overline{\mathbb{F}}_l$-representations ϱ of some $GL_{g(\varrho)}(F_{w_0})$ with $1 \leq g(\varrho) \leq d$, under the equivalence relation $\varrho \sim \varrho\{k\}$ for any $k \in \mathbb{Z}$.

For any such ϱ, we then denote $l_1(\varrho) \geq \cdots \geq l_{r(\varrho)}(\varrho) \geq 1$, such that $S_\varrho(\mathfrak{m})$ can be written as the union of $r(\varrho)$ Zelevinsky unlinked segments of length $l_i(\varrho)$,

$$[\varrho\nu^k, \bar{\rho}\nu^{k+l_i(\varrho)-1}] = \{\varrho\nu^k, \varrho\nu^{k+1}, \cdots, \varrho\nu^{k+l_i(\varrho)-1}\}.$$

Then for any minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$, we write its local component $\Pi_{w_0} \simeq \bigtimes_\varrho \Pi_{w_0}(\varrho)$ and $\Pi_{w_0}(\varrho) \simeq \bigtimes_{i=1}^{r(\varrho)} \Pi_{w_0}(\varrho, i)$, where for each $1 \leq i \leq r(\varrho)$ the modulo l reduction of the supercuspidal support of $\Pi_{w_0}(\varrho, i)$ is, with the notations of §3.2, those of the Zelevinsky segment $[\varrho\nu^{\delta_i}, \varrho\nu^{\delta_i+l_i(\varrho)-1}]$.

1.3 Proposition. *Take a maximal ideal $\mathfrak{m}$ verifying hypotheses (H1) and (H2). Let ϱ_0 be such that $\mathcal{S}_{\varrho_0}(\mathfrak{m})$ is nonempty, and consider $1 \leq i \leq r(\varrho_0)$. Then there exist a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and an automorphic representation $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$ such that, with the previous notations, $\Pi_{w_0}(\varrho_0, i)$ is nondegenerate – that is, isomorphic to $\mathrm{St}_{l_i(\varrho)}(\pi_{w_0})$ for some irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation π_{w_0}.*

In particular, when there is only one segment – which is always the case for GL_2 – then the result is optimal.

Remark. In Proposition 1.3, we could also prove that for any such $\widetilde{\mathfrak{m}}$ and any $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$, then $\Pi_{w_0}(\varrho_0, i)$ is nondegenerate, which looks similar to [1, Theorem 2.1], where $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible and decomposed generic, which also imposes that the cohomology groups are free.

2. Shimura variety of Kottwitz–Harris–Taylor type

2.1. Geometry

Recall from the introduction that a prime number l is fixed distinct from all other prime numbers, which will be considered in the following. Let $F = F^+E$ be a CM field with $E/\mathbb{Q}$ imaginary quadratic such that l is unramified, and F^+ totally real with a fixed embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F, we denote by F_v the completion of F at v, with ring of integers $\mathcal{O}_v$, uniformiser ϖ_v and residual field $\kappa(v)$ with cardinality q_v.

Let B be a central division algebra over F of dimension d^2 such that at any place x of F, B_x is either split or a division algebra. We moreover suppose the existence of an involution of second kind $*$ on B such that $*_{|F}$ is the complex conjugation c. For $\beta \in B^{*=-1}$, we denote $\sharp_\beta : x \mapsto \beta x^* \beta^{-1}$ and let $G/\mathbb{Q}$ such that for any $\mathbb{Q}$-algebra R,

$$G(R) = \{(\lambda, g) \in R^\times \times (B^{op} \otimes_\mathbb{Q} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\},$$

with $B^{op} = B \otimes_c F$. If $x = yy^c$ is split in E, then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times Q_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times,$$

where, identifying the places of F^+ above x with those of F above y, we write $x = \prod_i z_i$. Moreover, we can impose the conditions that

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- if x is inert in E then $G(\mathbb{Q}_x)$ is quasi-split,
- the signature of $G(\mathbb{R})$ is $(1,d-1)\times(0,d)\times\cdots\times(0,d)$.

With the notations of the introduction, we have

$$G(\mathbb{A}^\infty)=\overline{G}(\mathbb{A}^{\infty,p})\times\Big(\mathbb{Q}_{p_{v_0}}^\times GL_d(F_{v_0})\times\prod_{\substack{w|u\\ w\neq v_0}}(\overline{B}_w^{op})^\times\Big).$$

2.1.1 Definition. We denote by Bad the set of places w of F such that B_w is nonsplit. Let Spl be the set of finite places w of F not in Bad such that $w_{|\mathbb{Q}}$ is split in E. For such a place w with $p=w_{|\mathbb{Q}}$, we write abusively

$$G(\mathbb{A}^w)=G(\mathbb{A}^p)\times\mathbb{Q}_p^\times\times\prod_{\substack{u|p\\ u\neq w}}(B_u^{op})^\times,$$

and $G(F_w)=GL_d(F_w)$.

Remark. With the notations of the introduction, the role of w in Definition 2.1.1 will be taken by either by v_0, v_1 or w_0.

2.1.2 Notation. For all open compact subgroups U^p of $G(\mathbb{A}^{\infty,p})$ and $m=(m_1,\cdots,m_r)\in\mathbb{Z}_{\geq 0}^r$, we consider

$$U^p(m)=U^p\times\mathbb{Z}_p^\times\times\prod_{i=1}^r\mathrm{Ker}(\mathcal{O}_{B_{v_i}}^\times\longrightarrow(\mathcal{O}_{B_{v_i}}/\mathcal{P}_{v_i}^{m_i})^\times).$$

For w_0 one of the v_i and $n\in\mathbb{N}$, we also introduce $U^{w_0}(n):=U^p(0,\cdots,0,n,0,\cdots,0)$.

We then denote by $\mathcal{I}$ the set of $U^p(m)$ such that there exists a place x for which the projection from U^p to $G(\mathbb{Q}_x)$ does not contain any element with finite order except the identity (see. [19, pp. 90ff.]).

Attached to each $I\in\mathcal{I}$ is a Shimura variety $X_I\to\mathrm{Spec}\,\mathcal{O}_v$ of Kottwitz–Harris–Taylor type, and we denote by $X_{\mathcal{I}}=(X_I)_{I\in\mathcal{I}}$ the projective system: recall that the transition morphisms $r_{J,I}:X_J\to X_I$ are finite flat and even étale when $m_1(J)=m_1(I)$. This projective system is then equipped with a Hecke action of $G(\mathbb{A}^\infty)\times\mathbb{Z}$, where the action of z in the Weil group W_v of F_v is given by $-\deg(z)\in\mathbb{Z}$, $\deg=\mathrm{val}\circ\mathrm{Art}_v^{-1}$, and $\mathrm{Art}_v^{-1}:W_v^{ab}\simeq F_v^\times$ is the Artin isomorphism which sends geometric Frobenius to uniformisers.

2.1.3 Notations (see. [3, §1.3]). Let $I\in\mathcal{I}$. Then we have the following:

- The special fibre of X_I is denoted is $X_{I,s}$, and its geometric special fibre $X_{I,\bar{s}}:=X_{I,s}\times\mathrm{Spec}\,\overline{\mathbb{F}}_p$.
- For $1\leq h\leq d$, $X_{I,\bar{s}}^{\geq h}$ (resp., $X_{I,\bar{s}}^{=h}$) is the closed (resp., open) Newton stratum of height h, defined as the subscheme where the connected component of the universal Barsotti–Tate group is of rank greater than or equal to h (resp., equal to h).

Remark. $X_{I,\bar{s}}^{\geq h}$ is of pure dimension $d-h$. For $1\leq h<d$, the Newton stratum $X_{I,\bar{s}}^{=h}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d-h}(F_v)$, defined as

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the stabiliser of the first h vectors of the canonical basis of F_v^d. Concretely, this means there exists a closed subscheme $X^{=h}_{I,\bar{s},\overline{1_h}}$ stabilised by the Hecke action of $P_{h,d-h}(F_v)$ such that

$$X^{=h}_{I,\bar{s}} \simeq X^{=h}_{I,\bar{s},\overline{1_h}} \times_{P_{h,d-h}(F_v)} GL_d(F_v).$$

2.1.4 Notations. Denote

$$i^h : X^{\geq h}_{I,\bar{s}} \hookrightarrow X^{\geq 1}_{I,\bar{s}}, \quad j^{\geq h} : X^{=h}_{I,\bar{s}} \hookrightarrow X^{\geq h}_{I,\bar{s}}$$

and $j^{=h} = i^h j^{\geq h}$.

2.2. Jacquet–Langlands correspondence and ϱ-type

For a representation π_v of $GL_d(F_v)$ and $n \in \frac{1}{2}\mathbb{Z}$, set $\pi_v\{n\} := \pi_v \otimes q_v^{-n \operatorname{val} \circ \det}$. Recall that the normalised induction of two representations $\pi_{v,1}$ and $\pi_{v,2}$ of, respectively, $GL_{n_1}(F_v)$ and $GL_{n_2}(F_v)$ is

$$\pi_1 \times \pi_2 := \operatorname{ind}_{P_{n_1,n_1+n_2}(F_v)}^{GL_{n_1+n_2}(F_v)} \pi_{v,1}\left\{\frac{n_2}{2}\right\} \otimes \pi_{v,2}\left\{-\frac{n_1}{2}\right\}.$$

A representation π_v of $GL_d(F_v)$ is called *cuspidal* (resp., *supercuspidal*) if it is not a subspace (resp., subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero, these two notions coincide, but this is no more true for $\overline{\mathbb{F}}_l$.

Remark. The modulo l reduction of an irreducible $\overline{\mathbb{Q}}_l$-representation is still irreducible and cuspidal, but not necessarily supercuspidal. In this last case, its supercuspidal support is a Zelevinsky segment associated to some unique inertial equivalent class ϱ, where ϱ is an irreducible supercuspidal $\overline{\mathbb{F}}_l$-representation. Thanks to (H2), we will not be concerned by this subtlety.

2.2.1 Definition. We say that π_v is of type ϱ when the supercuspidal support of its modulo l reduction is contained in the Zelevinsky line of ϱ.

2.2.2 Definition ([4, §1.4] and [24, §9])**.** Let g be a divisor of $d = sg$ and π_v an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$. Then the normalised induced representation

$$\pi_v\left\{\frac{1-s}{2}\right\} \times \pi_v\left\{\frac{3-s}{2}\right\} \times \cdots \times \pi_v\left\{\frac{s-1}{2}\right\}$$

holds a unique irreducible quotient (resp., subspace) denoted by $\mathrm{St}_s(\pi_v)$ (resp., $\mathrm{Speh}_s(\pi_v)$); it is a generalised Steinberg (resp., Speh) representation.

Remark. If χ_v is a character of $F_v^\times$, then $\mathrm{Speh}_s(\chi_v) = \chi_v \circ \det$.

The local Jacquet–Langlands correspondance is a bijection between irreducible essentially square-integrable representations of $GL_d(F_v)$ – that is, representations of the type $\mathrm{St}_s(\pi_v)$ for π_v cuspidal – and irreducible representations of $D_{v,d}^\times$, where $D_{v,d}$ is the central division algebra over F_v with invariant $\frac{1}{d}$.

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2.2.3 Notation. We will denote by $\pi_v[s]_D$ the irreducible representation of $D_{v,d}^\times$ associated to $\mathrm{St}_s(\pi_v^\vee)$ by the local Jacquet–Langlands correspondence.

We denote by $\mathcal{R}_{\overline{\mathbb{F}}_l}(d)$ the set of equivalence classes of irreducible $\overline{\mathbb{F}}_l$-representations of $D_{v,d}^\times$. For $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(d)$, let $\mathcal{C}_{\bar{\tau}}$ be the subcategory of smooth $\mathbb{Z}_l^{nr}$-representations of $D_{v,d}^\times$ with objects whose irreducible subquotients are isomorphic to a subquotient of $\bar{\tau}_{|\mathcal{D}_{v,d}^\times}$. Note that $\mathcal{C}_{\bar{\tau}}$ is a direct factor inside $\mathrm{Rep}^\infty_{\mathbb{Z}_l^{nr}}(D_{v,d}^\times)$, so that every $\mathbb{Z}_l^{nr}$-representation $V_{\mathbb{Z}_l^{nr}}$ of $D_{v,d}^\times$ can be decomposed as a direct sum

$$V_{\mathbb{Z}_l^{nr}} \simeq \bigoplus_{\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(d)} V_{\mathbb{Z}_{l,\bar{\tau}}^{nr}},$$

where $V_{\mathbb{Z}_{l,\bar{\tau}}^{nr}}$ is an object of $\mathcal{C}_{\bar{\tau}}$.

Let π_v be an irreducible cuspidal representation of $GL_g(F_v)$ and fix an integer $s \geq 1$. Then the modulo l reduction of $\mathrm{Speh}_s(\pi)$ is irreducible (see. [15, §2.2.3]).

2.2.4 Notation. When the modulo l reduction of π, denoted by ϱ, is supercuspidal, then we will denote by $\mathrm{Speh}_s(\varrho)$ the modulo l reduction of $\mathrm{Speh}_s(\pi)$: we call it an $\overline{\mathbb{F}}_l$-superspeh representation.

By [15, 3.1.4], we have a bijection

$$\left\{\bar{\mathbb{F}}_l - \text{superspeh irreducible representations of } GL_d(F_v)\right\}$$
$$\simeq$$
$$\left\{\bar{\mathbb{F}}_l - \text{representations irreducible of } D_{v,d}^\times\right\} \quad (2.2.5)$$

compatible with the modulo l reduction in the sense that if π_v is a lifting of ϱ, then the modulo l reduction of $\pi^\vee[s]_D$ matches through the previous bijection with the super-Speh $\mathrm{Speh}_s(\varrho)$.

2.2.6 Definition. An $\overline{\mathbb{F}}_l$-representation of $D_{v,d}^\times$ (resp., an irreducible cuspidal representation of $GL_d(F_v)$) is said to be of type ϱ if all its irreducible subquotients are, through bijection (2.2.5), associated to some super-Speh $\mathrm{Speh}_s(\varrho)$ (resp., its supercuspidal support belongs to the Zelevinsky line of ϱ).

Recall that if $\epsilon(\varrho)$ is the cardinality of the Zelevinsky line associated to ϱ (see. [23, p. 51]), then

$$m(\varrho) = \begin{cases} \epsilon(\varrho), & \text{if } \epsilon(\varrho) > 1; \\ l, & \text{otherwise.} \end{cases}$$

2.2.7 Notation. Let $r(\varrho)$ be the biggest integer i such that l^i divides $\frac{d}{m(\varrho)g}$. We then define

$$g_{-1}(\varrho) = g \quad \text{and} \quad \forall 0 \leq i \leq r(\varrho),\ g_i(\varrho) = m(\varrho) l^i g.$$

We also denote $s_i(\varrho) := \lfloor \frac{d}{g_i(\varrho)} \rfloor$.

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If π_v is an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_k(F_v)$ with type ϱ, then there exists i such that $k = g_i$. We say that π_v is of ϱ-type i, and we denote by $\mathrm{Scusp}_i(\varrho)$ the set of inertial equivalence classes of irreducible cuspidal $\overline{\mathbb{Q}}_l$-representations of ϱ-type i.

2.2.8 Notation. For ϱ an irreducible supercuspidal $\overline{\mathbb{F}}_l$- representation of $GL_g(F_v)$, we denote $\mathcal{R}_\varrho = \coprod_{s\geq 1} \mathcal{R}_\varrho(sg)$, where $\mathcal{R}_\varrho(sg)$ is the set of equivalence classes of irreducible $\overline{\mathbb{F}}_l$-representations of $D^\times_{v,sg}$ of type ϱ.

2.3. Harris–Taylor local systems

Let π_v be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$ and fix $t \geq 1$ such that $tg \leq d$. Thanks to Igusa varieties, Harris and Taylor constructed a local system on $X^{=tg}_{I,\bar{s},\overline{1_h}}$,

$$\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_h}} = \bigoplus_{i=1}^{e_{\pi_v}} \mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_{v,i})_{\overline{1_h}},$$

where $(\pi_v[t]_D)_{|\mathcal{D}^\times_{v,h}} = \bigoplus_{i=1}^{e_{\pi_v}} \rho_{v,i}$ with $\rho_{v,i}$ irreducible. The Hecke action of $P_{tg,d-tg}(F_v)$ is then given through its quotient $GL_{d-tg} \times \mathbb{Z}$. These local systems have stable $\overline{\mathbb{Z}}_l$-lattices, and we will write simply $\mathcal{L}(\pi_v[t]_D)_{\overline{1_h}}$ for any $\overline{\mathbb{Z}}_l$-stable lattice that we do not want to specify.

2.3.1 Notations. For Π_t any representation of GL_{tg} and $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ defined by $\Xi(\frac{1}{2}) = q^{1/2}$, we introduce

$$\widetilde{HT}_1(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]_D)_{\overline{1_h}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

and its induced version

$$\widetilde{HT}(\pi_v, \Pi_t) := \left(\mathcal{L}(\pi_v[t]_D)_{\overline{1_h}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}\right) \times_{P_{tg,d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d-tg}(F_v)$ acts trivially and the action of

$$\left(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v\right) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d-tg}(F_v) \times W_v$$

is given by

- the action of g_v^c on Π_t and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{tg-d}{2}}$ and
- the action of $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ on $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_h}} \otimes \Xi^{\frac{tg-d}{2}}$.

We also introduce

$$HT(\pi_v, \Pi_t)_{\overline{1_h}} := \widetilde{HT}(\pi_v, \Pi_t)_{\overline{1_h}}[d - tg]$$

and the perverse sheaf

$$P(t, \pi_v)_{\overline{1_h}} := j^{=tg}_{\overline{1_h},!*} HT(\pi_v, \mathrm{St}_t(\pi_v))_{\overline{1_h}} \otimes \mathbb{L}(\pi_v),$$

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and their induced versions $HT(\pi_v,\Pi_t)$ and $P(t,\pi_v)$, where

$$j = h = i^h \circ j^{\geq h} : X_{I,\bar{s}}^{=h} \hookrightarrow X_{I,\bar{s}}^{\geq h} \hookrightarrow X_{I,\bar{s}}$$

and $\mathbb{L}^\vee$ is the local Langlands correspondence.

Remark. Recall that π'_v is said to be inertially equivalent to π_v if there exists a character $\zeta : \mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_l^\times$ such that $\pi'_v \simeq \pi_v \otimes (\zeta \circ \mathrm{val} \circ \det)$. Note (see. [3, 2.1.4]) that $P(t,\pi_v)$ depends only on the inertial class of π_v, and

$$P(t,\pi_v) = e_{\pi_v} \mathcal{P}(t,\pi_v),$$

where $\mathcal{P}(t,\pi_v)$ is an irreducible perverse sheaf. When we want to speak of the $\overline{\mathbb{Q}}_l$-versions, we will add it on the notations.

2.3.2 Definition. We say that $HT(\pi_v,\Pi_t)$ or $\mathcal{P}(t,\pi_v)$ is of type ϱ if π_v is.

2.3.3 Lemma. *If $\rho \otimes \sigma$ is a $GL_d(F_v) \times W_v$-equivariant irreducible subquotient of $H^i(X_{I,\bar{s}_v}, \mathcal{P}(\pi_v,t) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$, then*

- *σ is an irreducible subquotient of the modulo l reduction of $\mathbb{L}(\pi_v \otimes \chi_v)$, where χ_v is an uramified character of F_v, and*
- *ρ is an irreducible subquotient of the modulo l reduction of a induced representation of shape $\mathrm{St}_t(\pi_v \otimes \chi_v) \times \psi_v$ where ψ_v is an integral irreducible representation of $GL_{d-tg}(F_v)$.*

Proof. The result follows directly from the description of the actions given previously. □

As usual for σ a representation of W_v and $n \in \frac{1}{2}\mathbb{Z}$, we will denote by $\sigma(n)$ the twisted representation $g \mapsto \sigma(g)|\mathrm{Art}_v^{-1}(g)|^n$, where $|-|$ is the absolute value of F_v.

2.4. Free perverse sheaf

Let $S = \mathrm{Spec}\,\mathbb{F}_q$ and X/S be of finite type; then the usual t-structure on $\mathcal{D}(X,\overline{\mathbb{Z}}_l) := D_c^b(X,\overline{\mathbb{Z}}_l)$ is

$$A \in {}^p\mathcal{D}^{\leq 0}(X,\overline{\mathbb{Z}}_l) \Leftrightarrow \forall x \in X,\ \mathcal{H}^k i_x^* A = 0,\ \forall k > -\dim \overline{\{x\}},$$
$$A \in {}^p\mathcal{D}^{\geq 0}(X,\overline{\mathbb{Z}}_l) \Leftrightarrow \forall x \in X,\ \mathcal{H}^k i_x^! A = 0,\ \forall k < -\dim \overline{\{x\}},$$

where $i_x : \mathrm{Spec}\,\kappa(x) \hookrightarrow X$ and $\mathcal{H}^k(K)$ is the kth sheaf of the cohomology of K.

2.4.1 Notation. Let ${}^p\mathcal{C}(X,\overline{\mathbb{Z}}_l)$ denote the heart of this t-structure with associated cohomology functors ${}^p\mathcal{H}^i$. For a functor T, we denote ${}^pT := {}^p\mathcal{H}^0 \circ T$.

The category ${}^p\mathcal{C}(X,\overline{\mathbb{Z}}_l)$ is abelian equipped with a torsion theory $(\mathcal{T},\mathcal{F})$, where $\mathcal{T}$ (resp., $\mathcal{F}$) is the full subcategory of objects T (resp., F) such that $l^N 1_T$ is trivial for some large enough N (resp., $l.1_F$ is a monomorphism). Applying Grothendieck–Verdier duality, we obtain

$${}^{p+}\mathcal{D}^{\leq 0}(X,\overline{\mathbb{Z}}_l) := \{A \in {}^p\mathcal{D}^{\leq 1}(X,\overline{\mathbb{Z}}_l) :\ {}^p\mathcal{H}^1(A) \in \mathcal{T}\}$$
$${}^{p+}\mathcal{D}^{\geq 0}(X,\overline{\mathbb{Z}}_l) := \{A \in {}^p\mathcal{D}^{\geq 0}(X,\overline{\mathbb{Z}}_l) :\ {}^p\mathcal{H}^0(A) \in \mathcal{F}\},$$

with heart ${}^{p+}\mathcal{C}(X,\overline{\mathbb{Z}}_l)$ equipped with its torsion theory $(\mathcal{F},\mathcal{T}[-1])$.

2.4.2 Definition (see. [6, §1.3]). Let

$$\mathcal{F}(X,\overline{\mathbb{Z}}_l) := {}^p\mathcal{C}(X,\overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{C}(X,\overline{\mathbb{Z}}_l) = {}^p\mathcal{D}^{\leq 0}(X,\overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geq 0}(X,\overline{\mathbb{Z}}_l)$$

be the quasi-abelian category of free perverse sheaves over X.

Remark. For an object L of $\mathcal{F}(X,\overline{\mathbb{Z}}_l)$, we will consider filtrations

$$L_1 \subset L_2 \subset \cdots \subset L_e = L$$

such that for every $1 \leq i \leq e-1$, $L_i \hookrightarrow L_{i+1}$ is a strict monomorphism – that is, L_{i+1}/L_i is an object of $\mathcal{F}(X,\overline{\mathbb{Z}}_l)$.

For a free $L \in \mathcal{F}(X,\Lambda)$, we consider the diagram

$$\begin{array}{ccccccc} & & L & & & & \\ & \nearrow^{\mathrm{can}_{!,L}} & & & \searrow^{\mathrm{can}_{*,L}} & & \\ {}^{p+}j_!j^*L & \overset{+}{\twoheadrightarrow} & {}^pj_{!*}j^*L & \underset{+}{\hookrightarrow\!\!\!\twoheadrightarrow} & {}^{p+}j_{!*}j^*L & \overset{+}{\hookrightarrow} & {}^pj_*j^*L, \end{array}$$

where the bottom is (see. [6, remark following 1.3.12]) the canonical factorisation of ${}^{p+}j_!j^*L \longrightarrow {}^pj_*j^*L$ and where the maps $\mathrm{can}_{!,L}$ and $\mathrm{can}_{*,L}$ are given by the adjunction property. Consider now X equipped with a stratification

$$X = X^{\geq 1} \supset X^{\geq 2} \supset \cdots \supset X^{\geq d},$$

and let $L \in \mathcal{F}(X,\overline{\mathbb{Z}}_l)$. For $1 \leq h < d$, denote $X^{1\leq h} := X^{\geq 1} - X^{\geq h+1}$ and $j^{1\leq h} : X^{1\leq h} \hookrightarrow X^{\geq 1}$. We then define

$$\mathrm{Fil}_!^r(L) := \mathrm{Im}_{\mathcal{F}}\Big({}^{p+}j_!^{1\leq r}j^{1\leq r,*}L \longrightarrow L\Big),$$

which gives a filtration

$$0 = \mathrm{Fil}_!^0(L) \subset \mathrm{Fil}_!^1(L) \subset \mathrm{Fil}_!^1(L) \cdots \subset \mathrm{Fil}_!^{d-1}(L) \subset \mathrm{Fil}_!^d(L) = L.$$

Dually, $\mathrm{CoFil}_{*,r}(L) = \mathrm{Coim}_{\mathcal{F}}\Big(L \longrightarrow {}^pj_*^{1\leq r}j^{1\leq r,*}L\Big)$. Define a cofiltration

$$L = \mathrm{CoFil}_{\mathfrak{G},*,d}(L) \twoheadrightarrow \mathrm{CoFil}_{\mathfrak{G},*,d-1}(L) \twoheadrightarrow \cdots$$
$$\cdots \twoheadrightarrow \mathrm{CoFil}_{\mathfrak{G},*,1}(L) \twoheadrightarrow \mathrm{CoFil}_{\mathfrak{G},*,0}(L) = 0$$

and a filtration

$$0 = \mathrm{Fil}_*^{-d}(L) \subset \mathrm{Fil}_*^{1-d}(L) \subset \cdots \subset \mathrm{Fil}_*^0(L) = L,$$

where $\mathrm{Fil}_*^{-r}(L) := \mathrm{Ker}_{\mathcal{F}}\big(L \twoheadrightarrow \mathrm{CoFil}_{*,r}(L)\big)$.

Remark. These two constructions are exchanged by Grothendieck–Verdier duality, $D(\mathrm{CoFil}_{\mathfrak{G},!,-r}(L)) \simeq \mathrm{Fil}^{-r}_{\mathfrak{G},*}(D(L))$ and $D(\mathrm{CoFil}_{\mathfrak{G},*,r}(L)) \simeq \mathrm{Fil}^r_{\mathfrak{G},!}(D(L))$.

We can also refine the previous filtrations (see. [6, Proposition 2.3.3]) to obtain exhaustive filtrations

$$0 = \mathrm{Fill}_!^{-2^{d-1}}(L) \subset \mathrm{Fill}_!^{-2^{d-1}+1}(L) \subset \cdots$$
$$\cdots \subset \mathrm{Fill}_!^0(L) \subset \cdots \subset \mathrm{Fill}_!^{2^{d-1}-1}(L) = L, \qquad (2.4.3)$$

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such that the graded parts $\mathrm{grr}^k(L)$ are simple over $\overline{\mathbb{Q}}_l$ – that is, they verify ${}^p j_{!*}^{=h} j^{=h,*}\mathrm{grr}^k(L) \hookrightarrow\!\!\!\!\twoheadrightarrow_+ \mathrm{grr}^k(L)$ for some h. Dually, we can construct a cofiltration

$$L = \mathrm{CoFill}_{*,2^{d-1}}(L) \twoheadrightarrow \mathrm{CoFill}_{*,2^{d-1}-1}(L) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{CoFill}_{*,-2^{d-1}}(L) = 0$$

and a filtration $\mathrm{Fill}_*^{-r}(L) := \mathrm{Ker}_{\mathcal{F}}\big(L \twoheadrightarrow \mathrm{CoFill}_{*,r}(L)\big)$.

2.5. Vanishing-cycle perverse sheaf

2.5.1 Notation. For $I \in \mathcal{I}$, let

$$\Psi_{I,\Lambda} := R\Psi_{\eta_v,I}(\Lambda[d-1])\left(\frac{d-1}{2}\right)$$

be the vanishing-cycle autodual perverse sheaf on $X_{I,\bar{s}_v}$. When $\Lambda = \overline{\mathbb{Z}}_l$, we will simply write $\Psi_{\mathcal{I}}$.

Recall the following result of [19] relating $\Psi_{\mathcal{I}}$ with Harris–Taylor local systems:

2.5.2 Proposition (see. [3, §2.4] and [19, Proposition IV.2.2])**.** *There is a $G(\mathbb{A}^{\infty,v}) \times P_{h,d-h}(F_v) \times W_v$-equivariant isomorphism*

$$\mathrm{ind}_{(D_{v,h}^\times)^0 \varpi_v^{\mathbb{Z}}}^{D_{v,h}^\times}\big(\mathcal{H}^{h-d-i}\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}\big)_{|X_{\mathcal{I},\bar{s},\overline{1_h}}^{=h}} \simeq \bigoplus_{\bar{\tau}\in\mathcal{R}_{\overline{\mathbb{F}}_l}(h)} \mathcal{L}_{\overline{\mathbb{Z}}_l,\overline{1_h}}(\mathcal{U}_{\bar{\tau},\mathbb{N}}^{h-1-i}),$$

where

- *$\mathcal{R}_{\overline{\mathbb{F}}_l}(h)$ is the set of equivalence classes of irreducible $\overline{\mathbb{F}}_l$-representations of $D_{v,h}^\times$;*
- *for $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)$ and V a $\overline{\mathbb{Z}}_l$-representation of $D_{v,h}^\times$, $V_{\bar{\tau}}$ denotes (see. [16, §B.2]) the direct factor of V whose irreducible subquotients are isomorphic to a subquotient of $\bar{\tau}_{|\mathcal{D}_{v,h}^\times}$, where $\mathcal{D}_{v,h}$ is the maximal order of $D_{v,h}$;*
- *with the previous notation, $\mathcal{U}_{\bar{\tau},\mathbb{N}}^i := \big(\mathcal{U}_{F_v,\overline{\mathbb{Z}}_l,d}^i\big)_{\bar{\tau}}$; and*
- *the matching between the system indexed by $\mathcal{I}$ and those by $\mathbb{N}$ is given by the map $m_1 : \mathcal{I} \longrightarrow \mathbb{N}$.*

Remark. For $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)$ and a lifting τ which by Jacquet–Langlands correspondence can be written $\tau \simeq \pi[t]_D$ for π irreducible cuspidal, let $\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$ be in the supercuspidal support. Then the inertial class of ϱ depends only on $\bar{\tau}$, and we will use the following notation:

2.5.3 Notation. With the previous notation, we write V_ϱ for $V_{\bar{\tau}}$.

Remark. $\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$ is an object of $\mathcal{F}(X_{\mathcal{I},\bar{s}},\overline{\mathbb{Z}}_l)$. Indeed, by [2, Proposition 4.4.2], $\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$ is an object of ${}^p\mathcal{D}^{\leq 0}(X_{\mathcal{I},\bar{s}},\overline{\mathbb{Z}}_l)$. By [20, Variant 4.4 of Theorem 4.2], we have $D\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l} \simeq \Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$, so that

$$\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l} \in {}^p\mathcal{D}^{\leq 0}(X_{\mathcal{I},\bar{s}},\overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geq 0}(X_{\mathcal{I},\bar{s}},\overline{\mathbb{Z}}_l) = \mathcal{F}(X_{\mathcal{I},\bar{s}},\overline{\mathbb{Z}}_l).$$

2.5.4 Proposition (see. [9, §3.2]). *We have a decomposition*

$$\Psi_I \simeq \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \mathrm{Scusp}_{\mathbb{F}_l}(g)} \Psi_\varrho,$$

where all the Harris–Taylor perverse sheaves of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ *are of type* ϱ.

Remark. In [3], we decomposed $\Psi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ as a direct sum $\bigoplus_{\pi_v} \Psi_{\pi_v}$, where π_v describes the set of equivalent inertial classes of irreducible cuspidal representations. Then $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} \Psi_{\pi_v}$, where $\mathrm{Cusp}(\varrho)$ is the set of equivalent inertial classes of irreducible cuspidal representations of type ϱ in the sense of Definition 2.2.6.

In [10] we give the precise description of $\mathrm{gr}^r_{\mathfrak{S},!}(\Psi_{I,\varrho})$, which is defined over $\overline{\mathbb{Z}}_l$. By construction, they are supported on $X^{\geq r}_{I,\bar{s}_v}$ and trivial if g does not divide r. Otherwise, for $r = tg$ we have

$$\mathrm{ind}_{(D^\times_{v,tg})^0 \varpi_v^{\mathbb{Z}}}^{D^\times_{v,tg}} \Big(j^{=tg,*} \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{I,\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \Big) \simeq \bigoplus_{\substack{i=-1 \\ t_i g_i(\varrho) = tg}}^{r(\varrho)} \bigoplus_{\pi_v \in \mathrm{Scusp}_i(\varrho)} HT(\pi_v, \mathrm{St}_{t_i}(\pi_v)) \otimes \mathbb{L}(\pi_v) \Big(-\frac{t_i - 1}{2} \Big).$$

We can then consider the naive ϱ-filtration

$$\mathrm{Fil}^*_{\varrho, r(\varrho), \overline{\mathbb{Q}}_l}(\Psi, tg) \subset \cdots \subset \mathrm{Fil}^*_{\varrho, -1, \overline{\mathbb{Q}}_l}(\Psi, tg) = j^{=tg,*} \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{I,\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l,$$

where $\mathrm{ind}_{(D^\times_{v,tg})^0 \varpi_v^{\mathbb{Z}}}^{D^\times_{v,tg}} \Big(\mathrm{Fil}^*_{\varrho,k,\overline{\mathbb{Q}}_l}(\Psi, tg) \Big)$ is isomorphic to

$$\bigoplus_{\substack{i=k \\ t_i g_i(\varrho) = tg}}^{r(\varrho)} \bigoplus_{\pi_v \in \mathrm{Scusp}_i(\varrho)} HT(\pi_v, \mathrm{St}_{t_i}(\pi_v)) \otimes \mathbb{L}(\pi_v) \Big(-\frac{t_i - 1}{2} \Big),$$

and the associated integral filtration of $j^{=tg,*} \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{I,\varrho})$, defined by pullback

$$\begin{array}{ccc} \mathrm{Fil}^*_{\varrho,k}(\Psi, tg) & \hookrightarrow & \mathrm{Fil}^*_{\varrho,k,\overline{\mathbb{Q}}_l}(\Psi, tg) \\ \downarrow & & \downarrow \\ j^{=tg,*} \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{I,\varrho}) & \hookrightarrow & j^{=tg,*} \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{I,\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l. \end{array}$$

For $k = -1, \cdots, r(\varrho)$, the graded parts $\mathrm{gr}_{\varrho,k}(\Psi, tg)$ are then of ϱ-type k. We can then refine these filtrations by separating the $\pi_v \in \mathrm{Scusp}_k(\varrho)$ to obtain

$$(0) = \mathrm{Fil}^{*,0}_\varrho(\Psi, tg) \subset \mathrm{Fil}^{*,1}_\varrho(\Psi, tg) \subset \cdots \subset \mathrm{Fil}^{*,r}_\varrho(\Psi, tg) = j^{=tg,*} \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{I,\varrho}).$$

By taking the iterated images of $j_!^{=tg}\operatorname{Fil}^k_\varrho(\Psi, tg) \longrightarrow \operatorname{gr}^{tg}_{\mathfrak{S},!}(\Psi_{I,\varrho})$, we then construct a filtration

$$(0) = \operatorname{Fil}^0_\varrho(\Psi, tg) \subset \operatorname{Fil}^1_\varrho(\Psi, tg) \subset \cdots \subset \operatorname{Fil}^r_\varrho(\Psi, tg) = \operatorname{gr}^{tg}_{\mathfrak{S},!}(\Psi_{I,\varrho}).$$

Finally, we can filter each of these graded parts using an exhaustive filtration of stratification to obtain a filtration of $\Psi_{I,\varrho}$ whose graded parts are $\mathfrak{P}(\pi_v, t)(\frac{1-s_i(\varrho)+2k}{2})$ for $\pi_v \in \operatorname{Scusp}_i(\varrho)$ with $i \geq -1$ and $k = 0, \cdots, s_i(\varrho) - 1$.

3. Cohomology of KHT Shimura varieties

3.1. Localisation at a non-pseudo-Eisenstein ideal

3.1.1 Definition. Let Spl be the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write $\operatorname{Spl}(I)$ for the subset of Spl of places which do not divide the level I.

Let $\operatorname{Unr}(I)$ be the union of

- places $q \neq l$ of $\mathbb{Q}$ inert in E not below a place of Bad and where I_q is maximal and
- places $w \in \operatorname{Spl}(I)$.

3.1.2 Notation. For $I \in \mathcal{I}$ a finite level, write

$$\mathbb{T}_I := \prod_{x \in \operatorname{Unr}(I)} \mathbb{T}_x,$$

where for x a place of $\mathbb{Q}$ (resp., $x \in \operatorname{Spl}(I)$), $\mathbb{T}_x$ is the unramified Hecke algebra of $G(\mathbb{Q}_x)$ (resp., of $GL_d(F_x)$) over $\overline{\mathbb{Z}}_l$.

3.1.3 Example. For $w \in \operatorname{Spl}(I)$, we have

$$\mathbb{T}_w = \overline{\mathbb{Z}}_l[T_{w,i} : \ i = 1, \cdots, d],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(O_w)\operatorname{diag}(\overbrace{\varpi_w, \cdots, \varpi_w}^{i}, \overbrace{1, \cdots, 1}^{d-i})GL_d(O_w) \subset GL_d(F_w).$$

More generally, the Satake isomorphism identifies $\mathbb{T}_x$ with $\overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$, where

- T_x is a split torus,
- W_x is the spherical Weyl group and
- $X^{un}(T_x)$ is the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.

Consider a fixed maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$, and for every $x \in \operatorname{Unr}(I)$ denote by $S_\mathfrak{m}(x)$ the multiset[3] of modulo l Satake parameters at x associated to $\mathfrak{m}$.

[3] A multiset is a set with multiplicities.

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3.1.4 Example. For every $w \in \mathrm{Spl}(I)$, the multiset of Satake parameters at w corresponds to the roots of the Hecke polynomial

$$P_{\mathfrak{m},w}(X) := \sum_{i=0}^{d} (-1)^i q_w^{\frac{i(i-1)}{2}} \overline{T_{w,i}} X^{d-i} \in \overline{\mathbb{F}}_l[X].$$

That is, $S_{\mathfrak{m}}(w) := \{\lambda \in \mathbb{T}_I/\mathfrak{m} \simeq \overline{\mathbb{F}}_l \text{ such that } P_{\mathfrak{m},w}(\lambda) = 0\}$. For a maximal ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_{I^l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, we also have the multiset of Satake parameters

$$S_{\widetilde{\mathfrak{m}}}(w) := \{\lambda \in \mathbb{T}_I \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l/\widetilde{\mathfrak{m}} \simeq \overline{\mathbb{Q}}_l \text{ such that } P_{\widetilde{\mathfrak{m},w}}(\lambda) = 0\}.$$

3.1.5 Notation. Let Π be an irreducible automorphic representation of $G(\mathbb{A})$ of level I, which means here that Π has nontrivial invariants under I and that for every $x \in \mathrm{Unr}(I)$, Π_x is unramified. Then Π defines

- a maximal ideal $\widetilde{\mathfrak{m}}(\Pi)$ of $\mathbb{T}_{I^l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ or
- a minimal prime ideal $\widetilde{\mathfrak{m}}(\Pi)$ of $\mathbb{T}_{I^l}$ contained in a maximal ideal $\mathfrak{m}(\Pi)$ of $\mathbb{T}_{I^l}$, which corresponds to its modulo l Satake parameters.

A minimal prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_{I^l}$ is said to be cohomological if there exists a cohomological automorphic $\overline{\mathbb{Q}}_l$-representation Π of $G(\mathbb{A})$ of level I with $\widetilde{\mathfrak{m}} = \widetilde{\mathfrak{m}}(\Pi)$. Such a Π is not unique, but $\widetilde{\mathfrak{m}}$ defines a unique near equivalence class in the sense of [22]; we denote it by $\Pi_{\widetilde{\mathfrak{m}}}$. Then let

$$\rho_{\widetilde{\mathfrak{m}},\overline{\mathbb{Q}}_l} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

be the Galois representation associated to such a Π thanks to [19, 22], which by the Chebotarev theorem can be defined over some finite extension $K_{\widetilde{\mathfrak{m}}}$ – that is, $\rho_{\widetilde{\mathfrak{m}},\overline{\mathbb{Q}}_l} \simeq \rho_{\widetilde{\mathfrak{m}}} \otimes_{K_{\widetilde{\mathfrak{m}}}} \overline{\mathbb{Q}}_l$.

It is well known that $\rho_{\widetilde{\mathfrak{m}}}$ has stable lattices and the semisimplification of its modulo l reduction is independent of the chosen stable lattice. Moreover, it depends only on the maximal ideal $\mathfrak{m}$; we denote by

$$\overline{\rho}_{\mathfrak{m}} : G_F \longrightarrow GL_d(\overline{\mathbb{F}}_l)$$

its extension to $\overline{\mathbb{F}}_l$. For every $w \in \mathrm{Spl}(I)$, recall that the multiset of eigenvalues of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_w)$ is $S_{\mathfrak{m}}(w)$, obtained from $S_{\widetilde{\mathfrak{m}}}(w)$ by taking the modulo l reduction.

Assume moreover that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Then the $\overline{\mathbb{Q}}_l$-cohomology group $H^{d-1}(X_{U,\bar{\eta}}, \overline{\mathbb{Q}}_l)_{\mathfrak{m}}$ gives a continuous d-dimensional Galois representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\mathbb{T}_{S,\mathfrak{m}}[1/l]),$$

where $\mathrm{Gal}_{F,S}$ is the Galois group of the maximal extension of F which is unramified outside S. As all characteristic Frobenius polynomials take values in $\mathbb{T}_{S,\mathfrak{m}}$, and as $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, using the classical theory of pseudorepresentations we know that $\rho_{\mathfrak{m}}$ takes values in $GL_d(\mathbb{T}_{S,\mathfrak{m}})$.

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3.2. Freeness of the cohomology

From now on we fix a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ verifying one of the following conditions (see. the introduction):

(1) There exists $w_1 \in \mathrm{Spl}(I)$ such that $S_{\mathfrak{m}}(w_1)$ does not contain any submultiset of the shape $\{\alpha, q_{w_1}\alpha\}$, where q_{w_1} is the cardinality of the residue field. This hypothesis is called *generic* in [12].

(2) With $[F(\exp(2i\pi/l) : F] > d$:

- $\overline{\rho}_{\mathfrak{m}}$ is induced from a character of G_K for a cyclic galoisian extension K/F or
- $SL_n(k) \subset \overline{\rho}_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_l^\times GL_n(k)$ for a subfield $k \subset \overline{\mathbb{F}}_l$.

Remark. If the main result of [11] is true, one may only suppose, besides the irreducibility of $\overline{\rho}_{\mathfrak{m}}$, that $[F(\exp(2i\pi/l) : F] > d$.

3.2.1 Theorem (see. [8]). *For $\mathfrak{m}$ as before, the localised cohomology groups $H^i(X_{I,\bar{\eta}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ are free.*

As $X_I \longrightarrow \mathrm{Spec}\, \mathcal{O}_v$ is proper, we have a $G(\mathbb{A}^\infty) \times W_v$-equivariant isomorphism $H^{d-1+i}(X_{I,\bar{\eta}_v}, \overline{\mathbb{Z}}_l) \simeq H^i(X_{I,\bar{s}_v}, \Psi_I)$. Using the previous filtration of Ψ_I, we can compute $H^{p+q}(X_{I,\bar{s}_v}, \Psi_{I,\varrho})_{\mathfrak{m}}$ through a spectral sequence whose entries[4] $E_1^{p,q}$ are the $H^{p+q}(X_{I,\bar{s}_v}, \mathfrak{P}(\pi_v, t)(\frac{1-s_i(\varrho)+2k}{2}))_{\mathfrak{m}}$ for $\pi_v \in \mathrm{Scusp}_i(\varrho)$ with $i \geq -1$ and $k = 0, \cdots, s_i(\varrho) - 1$. Over $\overline{\mathbb{Q}}_l$, it follows from [4], thanks to the hypothesis (1) on $\mathfrak{m}$, that all these cohomology groups are concentrated in degree 0, so that this $\overline{\mathbb{Q}}_l$-spectral sequence degenerates in E_1. In this section, under (H2) we want to prove the same result on $\overline{\mathbb{F}}_l$, which is equivalent to the freeness of $H^j(X_{I,\bar{s}_v}, \mathfrak{P}(\pi_v, t)(\frac{1-s_i(\varrho)+2k}{2}))_{\mathfrak{m}}$.

We need first some notations from [4, §1.2]. For all $t \geq 0$, we denote

$$\Gamma^t := \Big\{(a_1, \cdots, a_r, \epsilon_1, \cdots, \epsilon_r) \in \mathbb{N}^r \times \{\pm\}^r : \ r \geq 1, \ \sum_{i=1}^r a_i = t\Big\}.$$

A element of Γ^t will be denoted by $(\overleftarrow{a_1}, \cdots, \overrightarrow{a_r})$, where the arrow above each integer a_i is $\overleftarrow{a_i}$ (resp., $\overrightarrow{a_i}$) if ϵ_i is negative (resp., positive). We then consider on Γ^t the equivalence relation induced by

$$(\cdots, \overleftarrow{a}, \overleftarrow{b}, \cdots) = (\cdots, \overleftarrow{a+b}, \cdots), \quad (\cdots, \overrightarrow{a}, \overrightarrow{b}, \cdots) = (\cdots, \overrightarrow{a+b}, \cdots)$$

and $(\cdots, \overleftarrow{0}, \cdots) = (\cdots, \overrightarrow{0}, \cdots)$. We denote by $\overrightarrow{\Gamma}^t$ the set of these equivalence classes whose elements are denoted by $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]$.

Remark. In each class $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_k}] \in \overrightarrow{\Gamma}^t$, there exists a unique reduced element $(b_1, \cdots, b_r, \epsilon_1, \cdots, \epsilon_r) \in \Gamma^t$ such that $b_i > 0$ for all $1 \leq i \leq r$ and $\epsilon_i \epsilon_{i+1}$ is negative for $1 \leq i < r$.

[4] We do not need here to give the precise relations between (p, q) and i, t, k in the formula.

3.2.2 Theorem. *Let $(b_1,\cdots,b_r,\epsilon_1,\cdots,\epsilon_r)$ be the reduced element in $[\overleftarrow{a_1},\cdots,\overrightarrow{a_k}]\in\overrightarrow{\Gamma}^{t}$. We then define*

$$\mathcal{S}\Big([\overleftarrow{a_1},\cdots,\overrightarrow{a_k}]\Big)$$

as the subset of permutations σ of $\{0,\cdots,t-1\}$ such that for all $1\leq i\leq r$ with ϵ_i positive (resp., negative) and for all $b_1+\cdots+b_{i-1}\leq k<k'\leq b_1+\cdots+b_i$, then $\sigma^{-1}(k)<\sigma^{-1}(k')$ (resp., $\sigma^{-1}(k)>\sigma^{-1}(k')$).

We also introduce $\mathcal{S}^{op}\Big([\overleftarrow{a_1},\cdots,\overrightarrow{a_k}]\Big)$ by imposing, under the same conditions, $\sigma^{-1}(k)>\sigma^{-1}(k')$ (resp., $\sigma^{-1}(k)<\sigma^{-1}(k')$).

3.2.3 Proposition (see. [24, §2]). *Let g be a divisor of $d=sg$ and π be an irreducible cuspidal representation of $GL_g(F_v)$. There exists a bijection*

$$[\overleftarrow{a_1},\cdots,\overrightarrow{a_r}]\in\overrightarrow{\Gamma}^{s-1}\mapsto[\overleftarrow{a_1},\cdots,\overrightarrow{a_r}]_\pi$$

into the set of irreducible subquotients of the induced representation

$$\pi\left\{\frac{1-s}{2}\right\}\times\pi\left\{\frac{3-s}{2}\right\}\times\cdots\times\pi\left\{\frac{s-1}{2}\right\}$$

characterised by the following property:

$$J_{P_{g,2g,\cdots,sg}}([\overleftarrow{a_1},\cdots,\overrightarrow{a_r}]_\pi)=\sum_{\sigma\in\mathcal{S}\Big([\overleftarrow{a_1},\cdots,\overrightarrow{a_r}]\Big)}\pi\left\{\frac{1-s}{2}+\sigma(0)\right\}\otimes\cdots\otimes\pi\left\{\frac{1-s}{2}+\sigma(s-1)\right\},$$

or equivalently by

$$J_{P^{op}_{g,2g,\cdots,sg}}([\overleftarrow{a_1},\cdots,\overrightarrow{a_r}]_\pi)=\sum_{\sigma\in\mathcal{S}^{op}\Big([\overleftarrow{a_1},\cdots,\overrightarrow{a_r}]\Big)}\pi\left\{\frac{1-s}{2}+\sigma(0)\right\}\otimes\cdots\otimes\pi\left\{\frac{1-s}{2}+\sigma(s-1)\right\}.$$

Remark. With this notation, $\mathrm{St}_s(\pi)$ (resp., $\mathrm{Speh}_s(\pi)$) is $[\overleftarrow{s-1}]_\pi$ (resp., $[\overrightarrow{s-1}]_\pi$).

3.2.4. Lemma. *Let π be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$ such that its modulo l reduction ϱ is supercuspidal. Suppose the cardinality of the Zelevinsky line of ϱ is greater than or equal to s. Then the irreducible subquotients of the modulo l reduction of $[\overleftarrow{a_1},\cdots,\overrightarrow{a_r}]_\pi$ for $[\overleftarrow{a_1},\cdots,\overrightarrow{a_r}]$ describing $\overrightarrow{\Gamma}^{s-1}$ are pairwise distinct.*

Proof. By hypothesis on the cardinality of the Zelevinsky line, all these irreducible $\overline{\mathbb{F}}_l$-subquotients have a nontrivial image under $J_{P_{g,2g,\cdots,sg}}$. The result then follows directly from

- the commutation of Jacquet functors with the modulo l reduction and
- the fact that the $r_l(\pi)\{\frac{1-s}{2}+k\}$ are pairwise distinct for $0\leq k<s$, so that the image under $J_{P_{g,2g,\cdots,sg}}$ of $\pi\{\frac{1-s}{2}\}\times\pi\{\frac{3-s}{2}\}\times\cdots\times\pi\{\frac{s-1}{2}\}$ is multiplicity free.

□

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3.2.5. Notation. We will denote by $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]_\varrho$ any irreducible subquotient of the modulo l reduction of $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]_\pi$.

Remark. If, moreover, the cardinality of the Zelevinsky line of ϱ is strictly greater than s, then (see. [5]) the modulo l reduction of $[\overleftarrow{s-1}]_\pi$ is irreducible and nondegenerate – that is, $[\overleftarrow{s-1}]_\varrho$ is well defined and nondegenerate.

3.2.6. Theorem. *Consider a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_S$ such that for all i, the $\overline{\mathbb{Z}}_l$-module $H^i(X_{U,\bar\eta}, \overline{\mathbb{Z}}_l)_\mathfrak{m}$ is free. We suppose moreover, according to (H2), that the image of $\bar\rho_{\mathfrak{m},w_0}$ in the Grothendieck group is multiplicity free. Then for all $\overline{\mathbb{Z}}_l$-Harris–Taylor local systems $HT(\pi_{w_0}, t)$, the $H^i(X_{U,\bar s_{w_0}}, {}^p j^{=tg}_{!*} HT(\pi_{w_0}, t))_\mathfrak{m}$ are free.*

Remark. In Theorem 3.2.6, we need just the multiplicity-free part of (H2), as it is used in the previous lemma. Note, moreover, that by [7, §4.5], the multiplicity-free hypothesis is necessarily true.

Proof. First denote by $\mathrm{Scusp}_{w_0}(\mathfrak{m})$ the set of inertial equivalence classes of irreducible supercuspidal $\overline{\mathbb{F}}_l$-representations belonging to the supercuspidal support of the modulo l reduction of the local component at w_0 of a representation Π in the near equivalence class $\Pi_\mathfrak{m}$ associated to $\mathfrak{m}$.

We then consider the vanishing-cycle spectral sequence at w_0, localised at $\mathfrak{m}$:

$$H^i(X_{U,\bar\eta_{w_0}}, \overline{\mathbb{Z}}_l)_\mathfrak{m} \simeq \bigoplus_{\varrho \in \mathrm{Scusp}_{w_0}(\mathfrak{m})} H^i(X_{U,\bar s_{w_0}}, \Psi_{I,\varrho})_\mathfrak{m}.$$

Then for every $\varrho \in \mathrm{Scusp}_{w_0}(\mathfrak{m})$, the $H^i(X_{U,\bar s_{w_0}}, \Psi_{I,\varrho})_\mathfrak{m}$ are free. For π_v of type ϱ, the strategy to prove the freeness of $H^i(X_{U,\bar s_{w_0}}, {}^p j^{=tg}_{!*} HT(\pi_{w_0}, t))_\mathfrak{m}$ is to argue by absurdity and produce some torsion cohomology class in one of the $H^i(X_{U,\bar s_{w_0}}, \Psi_{I,\varrho})_\mathfrak{m}$. Then let t be minimal such that there exists $i \neq 0$ with

$$H^i\left(X_{I,\bar s_{w_0}}, \mathcal{P}(\pi_{w_0}, t)\left(\frac{1-t+2k}{2}\right)\right)_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \neq (0)$$

for $0 \leq k < t$, and where $\mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2})$ is a graded part of some filtration of Ψ_ϱ.

Consider, for example, the filtration constructed before using the adjunction $j^{1\leq h}_! j^{1\leq h,*} \longrightarrow \mathrm{Id}$. As remarked before – and considering also $\Psi_{\varrho^\vee}$ and its filtration constructed using $\mathrm{Id} \longrightarrow j^{1\leq h}_* j^{1\leq h,*}$ – we can suppose that such an i is strictly negative, and we denote by i_0 such a minimal i.

By Lemma 2.3.3, an irreducible $GL_d(F_{w_0}) \times W_{w_0}$-equivariant subquotient of $H^i(X_{I,\bar s_{w_0}}, \mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is one of the modulo l reductions of a representation we can write in the following shape:

$$\left([\overleftarrow{a_1}, \cdots, \overrightarrow{a_{i-1}}, \overleftrightarrow{1}, \overleftarrow{t-1}, \overleftrightarrow{1}, \overrightarrow{a_{i+1}}, \cdots \overleftarrow{a_r}]_\pi \times \Upsilon_{w_0}\right) \otimes \mathbb{L}(\pi\{\frac{\delta}{2}\}),$$

where the a_j are some integers, Υ_{w_0} is an irreducible $\overline{\mathbb{Q}}_l$-representation whose modulo l reduction has a supercuspidal support away from those of the previous segment and the following are true:

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- The symbol $\overleftrightarrow{1}$ before (resp., after) the $\overleftarrow{t-1}$ can be $\overleftarrow{1}$ or $\overrightarrow{1}$ if $\sum_{j=1}^{i-1} a_i > 0$ (resp., $\sum_{j=i+1}^{r} a_i > 0$). We will write $a_i = t+1$.
- Let $\left\{\pi\{\frac{\alpha}{2}\}, \pi\{\frac{\alpha}{2}+1\}, \cdots, \pi\{\frac{\alpha}{2}+t-1\}\right\}$ denote the supercuspidal support of $\overleftarrow{t-1}$ inside $[\overleftarrow{a_1}, \cdots, \overleftarrow{t-1}, \cdots \overrightarrow{a_r}]_\pi$. Then $\frac{\delta}{2} = \frac{\alpha}{2} + k$, where k is the integer in $\mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2})$.

Remark. In particular, we can suppose that the previous k is equal to 0.

Consider then such an irreducible subquotient $\tau \times \psi_{w_0} \otimes \sigma$, where

- ψ_{w_0} (resp., σ) is any irreducible subquotient of the modulo l reduction of Υ_{w_0} (resp., $\mathbb{L}(\pi\{\frac{\delta}{2}\})$) and
- τ is an irreducible subquotient of the modulo l reduction of some

$$[\overleftarrow{a_1}, \cdots, \overrightarrow{a_{i-1}}, \overrightarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overrightarrow{a_{i+1}}, \cdots \overrightarrow{a_r}]_\pi.$$

By the previous lemma 3.2.4, we can recover the a_i from τ.

Let us show now that this $\tau \times \psi_{w_0} \otimes \sigma$ is also a subquotient of $H^{i_0}(X_{I,\bar{s}_{w_0}}, \Psi_{I,\varrho})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, which contradicts our hypothesis on $\mathfrak{m}$. Denote $\mathrm{Fil}^{k-1} \subset \mathrm{Fil}^k \subset \Psi_{I,\varrho}$ such that $\mathrm{gr}^k = \mathrm{Fil}^k / \mathrm{Fil}^{k-1} \simeq \mathcal{P}(\pi_{w_0}, t)(\frac{1-t}{2})$. By hypothesis (H2), all the irreducible cuspidal $\overline{\mathbb{Q}}_l$-representations $\pi'_{w_0} \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(\varrho)$ such that one of the $H^i(X_{I,\bar{s}_{w_0}}, \mathcal{P}(\pi'_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \neq (0)$ are necessarily of ϱ-type -1. Then in particular all the Harris–Taylor perverse sheaves $\mathcal{P}(\pi'_{w_0}, t')$ which are subquotients of Fil^{k-1} must verify $t' > t$. The spectral sequence which computes $H^{i_0+1}(X_{I,\bar{s}_{w_0}}, \mathrm{Fil}^{k-1})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, thanks to a filtration of Fil^{k-1}, allows us to describe it as extensions between irreducible subquotients of the modulo l reduction of some

$$\left([\overleftarrow{a_1}, \cdots, \overrightarrow{a_{i-1}}, \overleftrightarrow{1}, \overleftarrow{t'-1}, \overleftrightarrow{1}, \overrightarrow{a_{i+1}}, \cdots \overrightarrow{a_r}]_{\pi'} \times \psi_{w_0}\right) \otimes \mathbb{L}(\pi'\{\frac{\delta'}{2}\})$$

with $t' > t$ and where $\pi'\{\frac{\delta'}{2}\}$ belongs to the supercuspidal support of $\overleftarrow{t'-1}$. But using the inequality $t' > t$, we see that τ cannot be a subquotient of the modulo l reduction of any $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_{i-1}}, \overrightarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overrightarrow{a_{i+1}}, \cdots \overrightarrow{a_r}]_\pi$.

Now using the filtration $\mathrm{Fil}^{k-1} \subset \mathrm{Fil}^k \subset \Psi_\varrho$, to conclude it suffices to look at $H^{i_0-1}(X_{I,\bar{s}_{w_0}}, \Psi_{I,\varrho} / \mathrm{Fil}^k)_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. For the Harris–Taylor perverse sheaves $\mathcal{P}(\pi'_{w_0}, t)(\frac{1-t+2k}{2})$ with $t' > t$ we argue as before, and for the others we invoke the minimality of t and i_0.

3.3. From the Ihara lemma to the cohomology

Recall first that

$$X^{=d}_{I,\bar{s}_{v_0}} = \coprod_{i \in \mathrm{Ker}^1(\mathbb{Q}, G)} X^{=d}_{I,\bar{s}_{v_0},i},$$

and that for a $G(\mathbb{A}^\infty)$-equivariant sheaf $\mathcal{F}_{I,i}$ on $X^{=d}_{I,\bar{s}_{v_0},i}$, its fibre at some compatible system $z_{i,I}$ of supersingular points has an action of $\overline{G}(\mathbb{Q}) \times GL_d(F_{v_0})^0$, where $GL_d(F_{v_0})^0$ is the kernel of the valuation of the determinant, so that (see. [3, Proposition 5.1.1]) as a

$G(\mathbb{A}^\infty) \simeq \overline{G}(\mathbb{A}^{\infty,v_0}) \times GL_d(F_{v_0})$-module, we have

$$H^0(X^{=d}_{I,\bar{s}_{v_0},i}, \mathcal{F}_{I,i}) \simeq \mathrm{ind}_{\overline{G}(\mathbb{Q})}^{\overline{G}(\mathbb{A}^{\infty,v_0})\times\mathbb{Z}} z_i^* \mathcal{F}_{I,i}.$$

Here, $\delta \in \overline{G}(\mathbb{Q}) \mapsto (\delta^{\infty,v_0}, \mathrm{val}\,\mathrm{orn}(\delta_{v_0})) \in \overline{G}(\mathbb{A}^{\infty,v_0}) \times \mathbb{Z}$ and the action of $g_{v_0} \in GL_d(F_{v_0})$ is given by those of $(g_0^{-\mathrm{val}\det g_{v_0}} g_{v_0}, \mathrm{val}\det g_{v_0}) \in GL_d(F_{v_0})^0 \times \mathbb{Z}$, where $g_0 \in GL_d(F_{v_0})$ is any fixed element with $\mathrm{val}\det g_0 = 1$. Moreover (see. [3, Corollaire 5.1.2]), if $z_i^*\mathcal{F}_I$ is provided with an action of the kernel $(D^\times_{v_0,d})^0$ of the valuation of the reduced norm – an action compatible with those of $\overline{G}(\mathbb{Q}) \hookrightarrow D^\times_{v_0,d}$ – then as a $G(\mathbb{A}^\infty)$-module we have

$$H^0(X^{=d}_{I,\bar{s}_{v_0},i}, \mathcal{F}_{I,i}) \simeq \mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}^\infty), \Lambda) \otimes_{D^\times_{v_0,d}} \mathrm{ind}_{(D^\times_{v_0,d})^0}^{D^\times_{v_0,d}} z_i^*\mathcal{F}_{I,i} \tag{3.3.1}$$

3.3.1 Lemma. *Let $\overline{\pi}$ be an irreducible sub-$\overline{\mathbb{F}}_l$-representation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/U^{v_0}, \overline{\mathbb{F}}_l)$. Denote its local component $\bar{\pi}_{v_0}$ at v_0 as $\pi_{v_0}[s]_D$, with π_{v_0} an irreducible cuspidal representation of $GL_g(F_{v_0})$ with $d = sg$. Then $\overline{\pi}^{v_0}$ is a subrepresentation of $H^0(X^{=d}_{U^{v_0},\bar{s}_{v_0}}, HT(\pi^\vee_{v_0}, s)) \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l$.*

Proof. Clearly we have $\overline{\pi}^{v_0} \subset \mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/U^{v_0}, \overline{\mathbb{F}}_l) \otimes \bar{\pi}^\vee_{v_0}$. The result then follows from expression (3.3.1) and the definition of the Harris–Taylor local system $HT(\pi^\vee_{v_0}, s)$ with support on the supersingular stratum. □

3.3.2 Proposition. *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_S$ verifying (H1) and (H3), and let $\bar{\pi}$ be an irreducible sub-$\overline{\mathbb{F}}_l$-representation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/U^v, \overline{\mathbb{F}}_l)_\mathfrak{m}$. Then $\bar{\pi}^{\infty,v}$ is a sub-$\overline{\mathbb{F}}_l$-representation of $H^{d-1}(X_{U,\bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_\mathfrak{m}$.*

Proof. By [15, Theorem 3.1.4], $\bar{\pi}_{v_0}$ is associated, through the modulo l Jacquet–Langlands correspondence, to some super-Speh $\mathrm{Speh}_s(\varrho)$ with ϱ an irreducible supercuspidal representation of $GL_g(F_{v_0})$ with $d = sg$. Recall that $H^i(X_{U,\bar{s}_{v_0}}, \Psi_\varrho)_\mathfrak{m}$ is a direct factor of $H^{d-1}(X_{U,\bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_\mathfrak{m}$, so that it suffices to prove that $\bar{\pi}^{\infty,v}$ is a sub-$\overline{\mathbb{F}}_l$-representation of $H^i(X_{U,\bar{s}_{v_0}}, \Psi_\varrho)_\mathfrak{m}$.

As in the proof of Theorem 3.2.6, but now using (H3), consider the filtration of Ψ_ϱ introduced before so that its graded parts are some Harris–Taylor perverse sheaves of type ϱ and its $\mathfrak{m}$-localised cohomology groups are free concentrated in degree 0. Note in particular that $\mathcal{P}(\pi^\vee_{v_0}, s)(\frac{s-1}{2})$ is its first graded part, so that using the spectral sequence computing $H^i(X_{U,\bar{s}_{v_0}}, \Psi_\rho)_\mathfrak{m}$ with $E_1^{p,q}$ given by the $H^i(X_{U,\bar{s}_{v_0}}, \mathcal{P}(\pi'_{v_0}, t)(\frac{1-t+2k}{2}))_\mathfrak{m}$, we see that

$$H^i(X_{U,\bar{s}_{v_0}}, \mathcal{P}(\pi^\vee_{v_0}, s)(\frac{s-1}{2}))_\mathfrak{m} \hookrightarrow\!\!\!\!\!\rightarrow H^i(X_{U,\bar{s}_{v_0}}, \Psi_\rho)_\mathfrak{m}$$

with free cokernel, so that $H^0(X^{=d}_{U^{v_0},\bar{s}_{v_0}}, \mathcal{P}(\pi^\vee_{v_0}, s)(\frac{s-1}{2}))_\mathfrak{m}$ is a subspace of $H^{d-1}(X_{U,\bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_\mathfrak{m}$. The result then follows from Lemma 3.3.1. □

The strategy to prove the Ihara lemma, under our restrictive hypothesis on $\mathfrak{m}$, is now to prove the same statement on $H^{d-1}(X_{U,\bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_\mathfrak{m}$ – that is, if π^{∞,v_0} is a subspace of it,

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then its local component $\pi_{w_0}^{\infty, v_0}$ at the place w_0 is generic. Finally, our statement of the Ihara lemma will follow from Proposition 4.2.2.

4. Nondegeneracy property for global cohomology

4.1. Global lattices as a tensorial product

From now on we suppose that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

4.1.1 Proposition. *Let $\Pi^{\infty, U} \otimes L_g(\Pi_{v_1}^{\vee})$ be a direct factor of $H^{d-1}(X_{U,\bar{\eta}_{v_1}}, \overline{\mathbb{Q}}_l)_{\mathfrak{m}}$, and consider its lattice given by the $\overline{\mathbb{Z}}_l$-cohomology. Then this lattice is a tensorial product $\Gamma_G \otimes \Gamma_W$ of a stable lattice Γ_G (resp., Γ_W) of $\Pi^{\infty, U}$ (resp., of $L_d(\Pi_{v_1}^{\vee})$).*

Proof. The result is classical, and we resume the arguments of [21, §5]. With [21, Definition 5.2], as $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible, $\Pi^{\infty, U} \otimes L_g(\Pi_{v_1}^{\vee})$ is $\sigma_{\overline{\mathbb{Z}}_l}$-typic, where $\sigma_{\overline{\mathbb{Z}}_l}$ is, up to isomorphism, the only stable $\overline{\mathbb{Z}}_l$-lattice of $L_g(\Pi_{v_1}^{\vee})$. The statement then follows from [21, Proposition 5.4]. □

Reasonably, it should be possible to prove the higher-dimensional version of [21, Theorem 5.6] – that is, to prove that as a $\mathbb{T}_{S,\mathfrak{m}}[\mathrm{Gal}_{F,S}]$-module,

$$H^{d-1}(X_{U,\bar{\eta}}, \mathbb{Z}_l)_{\mathfrak{m}} \simeq \sigma_{\mathfrak{m}} \otimes_{\mathbb{T}_{S,\mathfrak{m}}} \rho_{\mathfrak{m}}$$

for some $\mathbb{T}_{S,\mathfrak{m}}$-module $\sigma_{\mathfrak{m}}$ on which Gal_F acts trivially.

4.2. Proof of the main result

Let $\mathcal{S}(\mathfrak{m})$ be the supercuspidal support of the modulo l reduction of any $\Pi_{\widetilde{m}, w_0}$ in the near equivalence class associated to a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$. Recall that $\mathcal{S}(\mathfrak{m})$ depends only on $\mathfrak{m}$, and by (H2) it is multiplicity free; we decompose it according to the set $\mathcal{Z}$ of Zelevinsky lines defined as the set of equivalence classes of irreducible supercuspidal $\overline{\mathbb{F}}_l$-representations ϱ of some $GL_{g(\varrho)}(F_{w_0})$ with $1 \leq g(\varrho) \leq d$, under the equivalence relation $\varrho \sim \varrho\{k\}$ for any $k \in \mathbb{Z}$:

$$\mathcal{S}(\mathfrak{m}) = \coprod_{\varrho \in \mathcal{Z}} \mathcal{S}_{\varrho}(\mathfrak{m}).$$

Recall that for such ϱ, its associated Zelevinsky line $ZL(\varrho) = \{\varrho\{k\} : k \in \mathbb{Z}\}$ is of cardinality $\epsilon(\varrho)$. We then denote by $l_1(\varrho) \geq \cdots \geq l_{r(\varrho)}(\varrho) > 0$ the integers so that $S_{\varrho}(\mathfrak{m})$ can be written as a disjoint union of $r(\varrho)$ unlinked Zelevinsky segments

$$[\varrho\{\delta_i\}, \varrho\{\delta_i + l_i(\varrho) - 1\}] = \big\{\varrho\{\delta_i\}, \varrho\{\delta_i + 1\}, \cdots, \varrho\{\delta_i + l_i(\varrho) - 1\}\big\}.$$

An irreducible $\overline{\mathbb{F}}_l$-representation τ_{w_0} of $GL_d(F_{w_0})$ whose supercuspidal support is equal to $\mathcal{S}(\mathfrak{m})$ can be written as a full induced $\tau_{w_0} \simeq \bigtimes_{\varrho} \tau_{\varrho}$, where each τ_{ϱ} is also a full induced representation

$$\tau_{\varrho} \simeq \bigtimes_{i=1}^{r(\varrho)} \tau_{\varrho, i}$$

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with $\tau_{\varrho,i}$ of supercuspidal support equal to those of $[\varrho\{\delta_i\},\varrho\{\delta_i+l_i(\varrho)-1\}]$. Using Notation 3.2.5, each of these $\tau_{\varrho,i}$ can be written as

$$\tau_{\varrho,i}\simeq[\overleftarrow{a_1(\varrho)},\overrightarrow{a_2(\varrho)},\cdots,\overrightarrow{a_{t_i(\varrho)}(\varrho)}]_{\varrho\{\delta_i\}},$$

with $\sum_{j=1}^{t_i(\varrho)}a_j=l_i(\varrho)-1$.

4.2.1 Definition. We say (see. Remark 3.2.5) that τ_{w_0} is nondegenerate if for all ϱ and all $1\leq i\leq r(\varrho)$, $\tau_{\varrho,i}\simeq[\overleftarrow{l_i(\varrho)-1}]_{\varrho\{\delta_i\}}$.

4.2.2. Proposition. *Let τ_{w_0} be an irreducible representation of $GL_d(F_{w_0})$ which is a subspace of*

$$H^{d-1}(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\overline{\mathbb{F}}_l)_{\mathfrak{m}}:=\varinjlim_n H^{d-1}(X_{U^{w_0}(n),\bar\eta_{w_0}},\overline{\mathbb{F}}_l)_{\mathfrak{m}}.$$

Then π_{w_0} is nondegenerate.

Proof. Note first that the supercuspidal support of τ_{w_0} must be $\mathcal{S}(\mathfrak{m})$. The exhaustive filtration of Ψ_{ϱ_0} (see. §2.5), whose graded parts are Harris–Taylor perverse sheaves, gives a filtration of $H^{d-1}(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\overline{\mathbb{F}}_l)_{\mathfrak{m}}$, whose graded part are, thanks to Theorem 3.2.6, the

$$H^0\left(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\mathcal{P}(\pi_{w_0},t)\left(\frac{1-t+2k}{2}\right)\right)_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l$$

for $\pi_{w_0}\in\mathrm{Scusp}_{-1}(\varrho)$ with ϱ such that $S_\varrho(\mathfrak{m})$ is nonempty. Then τ_{w_0} must be a subspace of one of these graded parts. We argue by absurdity using the following lemma:

4.2.3. Lemma. *If ρ is a subspace of $[\overleftarrow{t-1}]_{\varrho\{\frac{-\delta}{2}\}}\times\rho'$, then with Notation 3.2.5,*

- *if $\delta=s-t$, then* $\rho=\left[\overleftarrow{t-1},\overrightarrow{1},\overbrace{\cdots}^{s-t-1}\right]_\varrho$;
- *if $\delta=t-s$, then* $\rho=\left[\overbrace{\cdots}^{s-t-1},\overleftarrow{1},\overleftarrow{t-1}\right]_\varrho$;
- *otherwise – that is, if $t-s<\delta<s-t$ – then*

$$\rho=\left[\overbrace{\cdots}^{s-t-\delta-1},\overleftarrow{1},\overleftarrow{t-1},\overrightarrow{1},\overbrace{\cdots}^{s-t+\delta-1}\right]_\varrho.$$

Proof. The result is well known over $\overline{\mathbb{Q}}_l$, and we can easily argue in the same way using

- the fact that all $\varrho\{\frac{1-s}{2}+k\}$ for $0\leq k\leq s-1$ are pairwise distinct and
- the property of commutation between the modulo l reduction and the Jacquet functors.

Consider, for example, the case $t-s<\delta<s-t$. By Frobenius reciprocity we see that the subspace we are looking for is some undetermined irreducible subspace of the modulo l reduction of $\Big[\overbrace{\cdots}^{s-t-\delta-1},\overleftarrow{1},\overleftarrow{t-1},\overrightarrow{1},\overbrace{\cdots}^{s-t+\delta-1}\Big]_{\pi}$. By convention (see. Notation 3.2.5), we denote such a subquotient $\Big[\overbrace{\cdots}^{s-t-\delta-1},\overleftarrow{1},\overleftarrow{t-1},\overrightarrow{1},\overbrace{\cdots}^{s-t+\delta-1}\Big]_{\varrho}$. □

Suppose now, by absurdity, that there exists an irreducible supercuspidal $\overline{\mathbb{F}}_l$-representation ϱ_0 such that τ_{ϱ_0} is degenerate, take i with

$$\tau_{\varrho_0,i}\simeq[\cdots,\overrightarrow{a},\cdots]_{\varrho_0}$$

and let $\beta\in\frac{1}{2}\mathbb{Z}$ be such that $\varrho_0\{\beta\}$ is the supercuspidal corresponding to the end of the arrow $\overrightarrow{a}$. From Proposition 4.1.1 we see that $\tau_{w_0}\otimes\overline{\rho}_{\mathfrak{m}}$ is an $\overline{\mathbb{F}}_l[GL_d(F_{w_0})\times \mathrm{Gal}(\overline{F}/F)]$-submodule of $H^{d-1}(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\overline{\mathbb{F}}_l)_{\mathfrak{m}}$. After restricting the Galois action to the Weil group at w_0, we see that $\tau_{w_0}\otimes\mathbb{L}(\varrho_0\{\beta\})$ has to be an $\overline{\mathbb{F}}_l[GL_d(F_{w_0})\times W_{w_0}]$-submodule of $H^{d-1}(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\overline{\mathbb{F}}_l)_{\mathfrak{m}}$ and, as before, of one of the

$$H^0\left(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\mathcal{P}(\pi_{w_0},t)\left(\frac{1-t+2k}{2}\right)\right)_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l$$

for $\pi_{w_0}\in \mathrm{Scusp}_{-1}(\varrho_0)$. Recall that this last cohomology group is parabolically induced from

$$H^0\left(X^{\geq tg}_{U^{w_0}(\infty),\bar\eta_{w_0},\overline{1_{tg}}},\mathcal{P}_1(\pi_{w_0},t)\left(\frac{1-t+2k}{2}\right)\right)_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l,$$

where by Lemma 2.3.3 every irreducible $\overline{\mathbb{F}}_l[P_{tg,d}(F_{w_0})\times W_{w_0}]$-subquotient of it can be written as $[\overleftarrow{t-1}]_{\varrho_0\{-\frac{\delta}{2}\}}\otimes\tau\otimes\mathbb{L}(\varrho_0\{\alpha\})$, where τ is any irreducible representation of $GL_{d-tg}(F_{w_0})$ and $\alpha\in\frac{1}{2}\mathbb{Z}$ is such that $\varrho_0\{\alpha\}$ belongs to the supercuspidal support of $[\overleftarrow{t-1}]_{\varrho_0\{-\frac{\delta}{2}\}}$.

The contradiction then follows from Lemma 4.2.3.

Finally, our restricted version of the Ihara lemma given in the introduction follows from Propositions 3.3.2 and 4.2.2.

Remark. Note that in the previous proof we used the second part of (H2) to say that the modulo l reduction of $[\overleftarrow{s-1}]_{\pi}$ is irreducible and so any of its subspace is nondegenerate (see. Remark 3.2.5). Using the main result of [9], we have this last property without any hypothesis, so as this is the only place where we use the second part of (H2), we can remove it.

4.3. Level raising

Before dealing with the general case, consider the case $d=2$, and take $l\geq 3$ such that the order of q_{w_0} modulo l is 2. Suppose then, by absurdity, that there exists a maximal ideal $\mathfrak{m}$ such that

(a) for every prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, the local component at w_0 of $\Pi_{\widetilde{\mathfrak{m}}}$ is unramified;

(b) for such a prime ideal, we write $\Pi_{\widetilde{\mathfrak{m}},w_0} \simeq \chi_{w_0,1} \times \chi_{w_0,2}$ and suppose $\chi_{w_0,1}\chi_{w_0,2}^{-1} \equiv \nu$ mod l.

Using (a) and the spectral sequence of vanishing cycles at w_0, we obtain

$$H^1(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} \simeq H^1(X^{=1}_{U,\bar{s}_{w_0}}, \Psi(\overline{\mathbb{F}}_l))_{\mathfrak{m}},$$

where $X^{=1}_{U,\bar{s}_{w_0}}$ is the ordinary locus of the geometric special fibre of X_U at w_0. It is well known that this cohomology group is parabolically induced. Moreover, the only nondegenerate irreducible representation of $GL_d(F_{w_0})$ which is a subquotient of the modulo l reduction of $\chi_{w_0,1} \times \chi_{w_0,1}\nu$ is cuspidal, because of the fact that q_{w_0} is of order 2 modulo l; this nondegenerate representation can not be a subspace of the induced representation $H^1(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. The contradiction is then given by the Ihara lemma.

In higher dimension, recall first the notations of the beginning of the previous section. For a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and an automorphic representation $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$ in the near equivalence class associated to $\widetilde{\mathfrak{m}}$, we write its local component at w_0 as

$$\Pi_{w_0} \simeq \bigtimes_{\varrho} \Pi_{w_0}(\varrho)$$

and $\Pi_{w_0}(\varrho) \simeq \bigtimes_{i=1}^{r(\varrho)} \Pi_{w_0}(\varrho, i)$, where for each $1 \leq i \leq r(\varrho)$, the modulo l reduction of the supercuspidal support of $\Pi_{w_0}(\varrho, i)$ is, with the notations of the previous section, that of the Zelevinsky segment $[\varrho\{\delta_i\}, \varrho\{\delta_i + l_i(\varrho) - 1\}]$.

4.3.1 Proposition. *Take a maximal ideal $\mathfrak{m}$ verifying hypotheses (H1) and (H2). Let ϱ_0 be such that $\mathcal{S}_{\varrho_0}(\mathfrak{m})$ is nonempty, and consider $1 \leq i \leq r(\varrho_0)$. Then there exist a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and an automorphic representation $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$ such that with the previous notation, $\Pi_{w_0}(\varrho_0, i)$ is nondegenerate – that is, it is isomorphic to $\mathrm{St}_{l_i(\varrho)}(\pi_{w_0})$ for some irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation π_{w_0}.*

Remark. In particular, if $\mathcal{S}(\mathfrak{m}) = \mathcal{S}_{\varrho_0}(\mathfrak{m})$ and $r(\varrho_0) = 1$ – that is, the supercuspidal support of the modulo l reduction of the local component at w_0 of any $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$ for any $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ is a Zelevinsky segment – then Π_{w_0} is nondegenerate. This is the case considered in [13, §4.5]. In a forthcoming work, we intend to explain how to raise the level simultaneously for all $1 \leq i \leq r(\varrho_0)$ and all ϱ_0 together.

Proof. For a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$, we write

$$\Pi_{w_0}(\varrho_0, i) \simeq \mathrm{St}_{s_1}(\pi_{w_0,1}) \times \cdots \mathrm{St}_{s_a}(\pi_{w_0,a}),$$

where $s_1 \geq s_2 \geq \cdots \geq s_a \geq 1$ and $\pi_{w_0,1}, \cdots, \pi_{w_0,a}$ are irreducible cuspidal $\overline{\mathbb{Q}}_l$-representations of type ϱ_0 of $GL_{g_i}(F_{w_0})$. We then argue by absurdity: we suppose $a \geq 2$ for all $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and we choose such an $\widetilde{\mathfrak{m}}$ so that s_1 is maximal. The strategy is then, using Lemma 4.2.3, to construct a degenerate $\overline{\mathbb{F}}_l[GL_d(F_{w_0})]$-subspace of $H^{d-1}(X_{U^{w_0}(\infty),\bar{\eta}_{w_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ which contradicts the genericness of irreducible submodules of this cohomology group, which

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was proved before. In [4, §3.6] we prove that for all minimal prime ideals $\widetilde{\mathfrak{m}}' \subset \mathfrak{m}$,

$$H^i(X_{U,\bar{s}_{w_0}}, HT_{\overline{\mathbb{Q}}_l}(\pi_{w_0,1}, t))_{\widetilde{\mathfrak{m}}'} = (0)$$

either if $t > s_1$ or for $t = s_1$, if $i \neq 0$. Consider now the filtration

$$\mathrm{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}) \hookrightarrow\!\!\!\!\to \mathrm{Fil}_*^{1-s_1 g_1}(\Psi_{\varrho_0}) \hookrightarrow\!\!\!\!\to \Psi_{\varrho_0}$$

and recall that, by construction,

- $\mathrm{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0})$ is supported in $X_{I,\bar{s}_{w_0}}^{>s_1 g_1}$ and
- $\mathrm{gr}_*^{1-s_1 g_1}(\Psi_{\varrho_0}) \simeq \bigoplus_{\pi_{w_0} \in \mathrm{Scusp}_{-1}(\varrho_0)} \mathcal{P}(\pi_{w_0}, s_1)(\frac{s_1-1}{2})$.

By Theorem 3.2.6, we know the cohomology groups of Harris–Taylor perverse sheaves to be free, so

- $H^i(X_{U,\bar{s}_{w_0}}, \mathrm{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}))_{\mathfrak{m}} = (0)$ and
- $H^i(X_{U,\bar{s}_{w_0}}, \Psi_{\varrho_0} / \mathrm{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}))_{\mathfrak{m}}$ is free.

Recall, moreover (see. [4, §3.6]), that $\Pi_{w_0} \otimes \mathbb{L}(\pi_{w_0,1})(\frac{s_1-1}{2})$ is a direct factor of

$$H^i\left(X_{U^{w_0}(\infty),\bar{s}_{w_0}}, HT_{\overline{\mathbb{Q}}_l}(\pi_{w_0,1}, s_1)\left(\frac{s_1-1}{2}\right)\right)_{\widetilde{\mathfrak{m}}},.$$

The stable lattice given by the $\overline{\mathbb{Z}}_l$-cohomology looks like $\big(\Gamma(\varrho_0,1) \times \Gamma^{\varrho_0,1}\big) \times \Gamma_W$, where

- $\Gamma(\varrho_0,1)$ is a stable lattice of $\mathrm{St}_{s_1}(\pi_{w_0,1})$,
- $\Gamma^{\varrho_0,1}$ is a stable lattice of $\big(\bigtimes_{\varrho \neq \varrho_0} \Pi_{w_0}(\varrho)\big) \times \big(\bigtimes_{i=2}^{r(\varrho_0)} \Pi_{w_0}(\varrho_0, i)\big)$ and
- Γ_W is a stable lattice of $\mathbb{L}(\pi_{w_0,1})(\frac{s_1-1}{2})$.

The result then follows from Lemma 4.2.3. □

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