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ON PROJECTIVE MANIFOLDS WITH PSEUDO-EFFECTIVE TANGENT BUNDLE

GENKI HOSONO [1], MASATAKA IWAI [2] AND SHIN-ICHI MATSUMURA [3]

[1] *Mathematical Institute, Tohoku University, 6-3, Aramaki Aza-Aoba, Aoba-ku, Sendai 980-8578, Japan* (genki.hosono.a4@tohoku.ac.jp, genki.hosono@gmail.com)

[2] *Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Tokyo, 153-8914, Japan* (masataka@ms.u-tokyo.ac.jp, masataka.math@gmail.com)

[3] *Mathematical Institute, Tohoku University, 6-3, Aramaki Aza-Aoba, Aoba-ku, Sendai 980-8578, Japan* (mshinichi-math@tohoku.ac.jp, mshinichi0@gmail.com)

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Abstract In this paper, we develop the theory of singular Hermitian metrics on vector bundles. As an application, we give a structure theorem of a projective manifold X with pseudo-effective tangent bundle; X admits a smooth fibration $X \to Y$ to a flat projective manifold Y such that its general fibre is rationally connected. Moreover, by applying this structure theorem, we classify all the minimal surfaces with pseudo-effective tangent bundle and study general nonminimal surfaces, which provide examples of (possibly singular) positively curved tangent bundles.

Keywords: singular Hermitian metrics; pseudo-effective vector bundles; tangent bundles; rationally connected varieties; abelian varieties; MRC fibrations; numerically flat vector bundles; splitting of vector bundles; classification of surfaces

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Contents

1. Introduction

The structure theorem for compact Kähler manifolds with semipositive bisectional curvature was established by Howard, Smyth and Wu [26] and Mok [31] after the Frankel conjecture (resp., the Hartshorne conjecture) was solved by Siu and Yau [38] (resp., Mori [32]). Campana and Peternell [13] and Demailly, Peternell and Schneider [17] generalised the Howard–Smyth–Wu structure theorem to nef tangent bundles as an algebraic analogue of semipositive bisectional curvature, and further classified surfaces and 3-folds with nef tangent bundle. (See [13] and [33] for the Campana–Peternell conjecture).

It is of interest to consider pseudo-effective tangent bundles as a natural generalisation of these structure results. The theory of singular Hermitian metrics on vector bundles, which has been rapidly developed, is a crucial tool to understanding pseudo-effective vector bundles. Hence we first develop the theory of singular Hermitian metrics on vector bundles (more generally, torsion-free sheaves). As one of the main applications, we obtain the following structure theorem for projective manifolds with pseudo-effective tangent bundle (see also see Theorem 3.12 for compact Kähler manifolds):

Theorem 1.1. *Let X be a projective manifold with pseudo-effective tangent bundle. Then X admits a (surjective) morphism $\phi : X \to Y$ with connected fibre to a smooth manifold Y with the following properties:*

(1) *The morphism $\phi : X \to Y$ is smooth (that is, all the fibres are smooth).*
(2) *The image Y admits a finite étale cover $A \to Y$ by an abelian variety A.*
(3) *A general fibre F of ϕ is rationally connected.*
(4) *A very general fibre F of ϕ also has the pseudo-effective tangent bundle.*

Moreover, if we further assume that T_X admits a positively curved singular Hermitian metric, then we have the following:

(5) *The standard exact sequence of tangent bundles*

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \phi^* T_Y \longrightarrow 0$$

splits.

(6) *The morphism $\phi : X \to Y$ is locally trivial (that is, all the fibres are smooth and isomorphic).*

See Definition 2.1 and Proposition 2.2 for pseudo-effective vector bundles, and see subsection 2.1 for positively curved singular Hermitian metrics.

A similar structure theorem holds when $-K_X$ is nef [11], or more generally, when there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is nef [10]. Theorem 1.1 can be seen as an analogue of [9], [11] or [10] by considering the pseudo-effective tangent bundle T_X instead of $-K_X$. By applying [18, Theorem 1.4] to

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the situation of Theorem 1.1, we can also see that the non-nef locus of $-K_X$ is dominant over Y if it is not empty.

The proof of Theorem 1.1 is based on the strategy in [29, 30] and the theory of singular Hermitian metrics on vector bundles developed in this paper. In particular, Theorems 1.2, 1.3 and 1.4 play an important role in the proof. Theorem 1.2, which can be seen as a generalisation of [10], gives a characterisation of numerically flat vector bundles in terms of pseudo-effectivity (see Definition 2.3 for numerically flat vector bundles, and [39] for a generalisation to Kähler manifolds). The proof depends on the theory of admissible Hermitian–Einstein metrics in [7]. Theorems 1.3 and 1.4 were proved in [25] under the additional assumption of the minimal extension property. Our contribution is to remove this assumption, which enables us to use the notion of singular Hermitian metrics flexibly.

Theorem 1.2. *Let X be a projective manifold and let $\mathcal{E}$ be a reflexive coherent sheaf on X. If $\mathcal{E}$ is pseudo-effective and the first Chen class $c_1(\mathcal{E})$ is zero, then $\mathcal{E}$ is locally free on X and numerically flat.*

Theorem 1.3. *Let E be a vector bundle with positively curved (singular) Hermitian metric g on a (not necessarily compact) complex manifold X. Let*

$$0 \to S \to E \to Q \to 0$$

be an exact sequence by vector bundles S and Q on X. Then this exact sequence splits if the induced quotient metric on Q is Hermitian flat on X (which is satisfied when X is compact and the first Chern class $c_1(Q)$ is zero, by Lemma 3.5).

Theorem 1.4. *Let X be a compact complex manifold and let*

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0$$

be an exact sequence of reflexive coherent sheaves $\mathcal{S}$, $\mathcal{E}$ and $\mathcal{Q}$ on X. If $\mathcal{E}$ admits a positively curved (singular) Hermitian metric and the first Chen class $c_1(\mathcal{Q}) = 0$, then we have the following:

(1) *$\mathcal{Q}$ is locally free and Hermitian flat.*
(2) *$\mathcal{E} \to \mathcal{Q}$ is a surjective bundle morphism on $X_{\mathcal{E}}$.*
(3) *The exact sequence splits on X.*

Here $X_{\mathcal{E}}$ is the maximal Zariski open set where $\mathcal{E}$ is locally free.

It is natural to attempt to classify all the surfaces X with pseudo-effective tangent bundle, as an application of Theorem 1.1. In the case where the tangent bundle is nef, a surface X has no curve with negative self-intersection, and thus X is always minimal. However, a surface X with pseudo-effective tangent bundle is not necessarily minimal, which is one of the difficulties in classifying them. In this paper, we classify all the minimal surfaces (see subsection 4.1 for more detail):

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Theorem 1.5. (1) *If a (not necessarily minimal) ruled surface $X \to C$ has the pseudo-effective tangent bundle T_X, then the base C is the projective line $\mathbb{P}^1$ or an elliptic curve.*

(2) *Further, in the case where C is an elliptic curve, the surface X is a minimal ruled surface (that is, the ruling $X \to C$ is a smooth morphism).*

(3) *Conversely, any minimal ruled surfaces $X \to C$ over an elliptic curve and over the projective line $C = \mathbb{P}^1$ have the pseudo-effective tangent bundle T_X.*

Moreover, we study the remaining problem (that is, the classification of blowups of Hirzebruch surfaces) in detail. As a result, we determine whether the blowups of Hirzebruch surfaces at general points have pseudo-effective tangent bundle, except for the blowup at four general points. These studies provide nontrivial examples of pseudo-effective or singular positively curved vector bundles.

2. Preliminaries

2.1. Singular Hermitian metrics

In this subsection, we recall the notion of singular Hermitian metrics on vector bundles (or more generally, on torsion-free sheaves).

Let E be a (holomorphic) vector bundle on a complex manifold X. Following [25], we first recall the definition of a singular Hermitian metric on E. A function $|\bullet| : V \to [0,\infty]$ on a finite-dimensional vector space V is called a *singular Hermitian inner product* if it satisfies the definition of seminorms and the parallelogram law (see [25, Definition 16.1] for details). Note that $|\bullet|$ determines the usual Hermitian inner product on V when $0 < |v| < \infty$ holds for any $0 \neq v \in V$. A *singular Hermitian metric* on E is a family $g := \{|\bullet|_x\}_{x\in X}$ of singular Hermitian inner products $|\bullet|_x$ on the vector space E_x satisfying the following conditions:

- $|\bullet|_x$ determines the usual Hermitian inner product outside a set of measure zero.
- The function

$$|u|_g : U \to [0,\infty] \text{ defined by } |u|_g(x) := |u(x)|_x$$

is measurable for any open set $U \subset X$ and any section $u \in H^0(U,E)$.

From the definition, it directly follows that $0 < \det g < \infty$ holds almost everywhere, and thus $\det g$ determines the singular Hermitian metric on $\det E$. Further, g induces the dual singular Hermitian metric on the dual vector bundle $E^\vee$ by $g^\vee = {}^t g^{-1}$. The metric g on E is said to be *positively curved* if $\log|u|_{g^\vee}$ is a psh function (that is, $\sqrt{-1}\partial\overline{\partial}\log|u|_{g^\vee} \geq 0$ in the sense of currents) for any local section u of $E^\vee$.

In this paper, for a torsion-free sheaf $\mathcal{E}$, we denote by $X_\mathcal{E}$ the maximal Zariski open set where $\mathcal{E}$ is locally free and denote by $\mathcal{E}^\vee$ the dual reflexive sheaf $\mathrm{Hom}(\mathcal{E},\mathcal{O}_X)$. These notions can be defined also for a torsion-free sheaf $\mathcal{E}$ by considering them on the locally free sheaf $\mathcal{E}|_{X_\mathcal{E}}$ [25, 35].

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2.2. Positivity of torsion-free sheaves

In this subsection, we recall the notions of positivity of vector bundles and torsion-free coherent sheaves. We first confirm the definition of a pseudo-effective coherent sheaf:

Definition 2.1. A torsion-free coherent sheaf $\mathcal{E}$ on a compact complex manifold X is said to be *pseudo-effective* if for any integer $m > 0$, there exists a singular Hermitian metric h_m on $\mathrm{Sym}^m \mathcal{E}$ such that $\sqrt{-1}\partial\overline{\partial}\log|u|^2_{h_m^\vee} \geq -\omega$ on $X_{\mathcal{E}}$ for any local section u of $(\mathrm{Sym}^m \mathcal{E})^\vee$. Here ω is a fixed Hermitian form on X and $\mathrm{Sym}^m \mathcal{E}$ is the mth symmetric power of $\mathcal{E}$.

The notion of pseudo-effectivity is often used with a different meaning. For example, in other papers, a vector bundle E on a projective manifold X may be called pseudo-effective when $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a pseudo-effective line bundle. Here $\mathcal{O}_{\mathbb{P}(E)}(1)$ is the hyperplane bundle of the projective space bundle $\mathbb{P}(E) \to X$ (that is, the set of hyperplanes in E). However, our definition of pseudo-effective vector bundles is stronger: it additionally requires that the image of the non-nef locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$ be properly contained in X.

The following characterisations of pseudo-effective vector bundles may help readers understand our definition:

Proposition 2.2 ([5, Proposition 3.1, Proposition 5.3], [27, Theorem 1.2], [35, subsection 2.3])**.** *Let E be a vector bundle on a projective manifold X. Then the following are equivalent:*

- *E is pseudo-effective in the sense of Definition 2.1.*
- *There exists an ample line bundle A such that $\mathrm{Sym}^m E \otimes A$ is generically globally generated for any integer $m > 0$ – that is, $\mathrm{Sym}^m E \otimes A$ is generated by global sections at a general point.*
- *The non-nef locus B of the hyperplane bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not dominant over X – that is, the image $f(B)$ under $f : \mathbb{P}(E) \to X$ is properly contained in X.*

Now we summarise the notions of positivity of vector bundles and torsion-free coherent sheaves:

Definition 2.3 ([3, Definition 7.1], [16, Definition 6.4], [17, Definition 1.17], [34, Definition 3.20])**.** Let E (resp., $\mathcal{E}$) be a vector bundle (resp., a torsion-free coherent sheaf) on a projective manifold X.

(1) E is *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$.

(2) E is *numerically flat* if E is nef and $c_1(E) = 0$, or equivalently, if both E and $E^\vee$ are nef.

(3) $\mathcal{E}$ is *almost nef* if there exists a countable family of proper subvarieties Z_i of X such that $\mathcal{E}|_C$ is nef for any curve $C \not\subset \bigcup_i Z_i$.

(4) $\mathcal{E}$ is *weakly positive at* $x \in X$ if, for any $a \in \mathbb{N}_{>0}$ and for any ample line bundle A on X, there exists $b \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^{ab}(\mathcal{E})^{\vee\vee} \otimes A^b$ is globally generated at x, where $\mathrm{Sym}^{ab}(\mathcal{E})^{\vee\vee}$ is the double dual of the abth symmetric power of $\mathcal{E}$.

(5) $\mathcal{E}$ is *pseudo-effective* if $\mathcal{E}$ is weakly positive at some $x \in X$.

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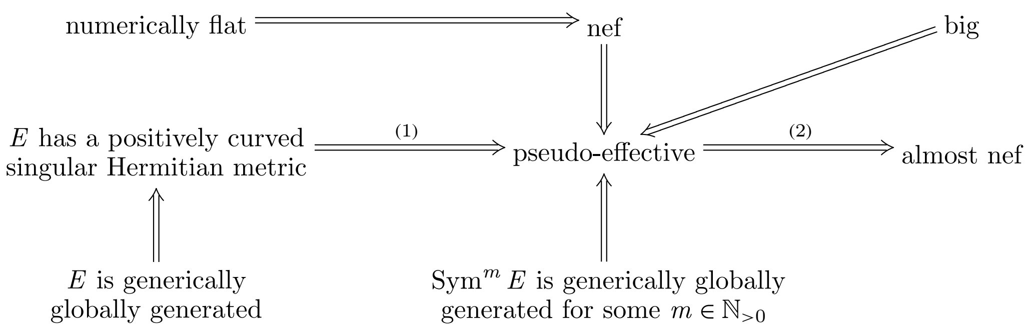
(6) $\mathcal{E}$ is *big* if there exist $a \in \mathbb{N}_{>0}$ and an ample line bundle A on X such that $\mathrm{Sym}^a(\mathcal{E})^{\vee\vee} \otimes A^{-1}$ is pseudo-effective.

(7) $\mathcal{E}$ is *generically globally generated* if the stalk $\mathcal{E}_x$ at a general point x in X is generated by global sections.

Note that the definitions of nef, big and or pseudo-effective vector bundles coincide with the usual one in the case where E is a line bundle. Relationships among them can be summarised by the following diagram:



The converse implications of (1) and (2) hold when E is a line bundle (see [3, Theorem 0.2] and [14, Section 6]), but the converse of (1) is not always true in the higher-rank case [24, Example 5.4], and the converse of (2) is unknown.

3. Proof of the main results

This section is devoted to the proof of the main results.

3.1. Numerically flat vector bundles

In this subsection, we give a proof for Theorem 1.2 after we prove Lemmas 3.1 and 3.3. Lemma 3.1, which easily follows from the result of [17, Proposition 1.16], is quite useful and often used in this paper.

Lemma 3.1. *Let X be a projective manifold and let $\mathcal{E}$ be an almost-nef torsion-free coherent sheaf on X.*

(1) *Any nonzero section $\tau \in H^0(X, \mathcal{E}^\vee)$ is nonvanishing on $X_{\mathcal{E}}$.*

(2) *Let $\mathcal{S}$ be a reflexive coherent sheaf such that $\det \mathcal{S}$ is pseudo-effective, and let $0 \to \mathcal{S} \to \mathcal{E}^\vee$ be an injective sheaf morphism. Then $\mathcal{S}$ is locally free on $X_{\mathcal{E}}$ and the morphism is an injective bundle morphism on $X_{\mathcal{E}}$.*

Proof. In [17], the same conclusion was proved for nef vector bundles. We denote by Z a countable union of proper subvarieties of X satisfying the definition of almost-nef sheaves. We may assume that $X \setminus X_{\mathcal{E}} \subset Z$ by adding the subvariety $X \setminus X_{\mathcal{E}}$ into Z.

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(1) Let $\tau \in H^0(X, \mathcal{E}^\vee)$ be a nonzero section. For any $p \in X_{\mathcal{E}}$, by taking a complete intersection of ample hypersurfaces we construct a curve C passing through p such that $C \not\subset Z$. We may assume that $C \subset X_{\mathcal{E}}$ by $\operatorname{codim}(X \setminus X_{\mathcal{E}}) \geq 2$. Then $\mathcal{E}|_C$ is a nef vector bundle thanks to $C \subset X_{\mathcal{E}}$, and thus it follows from [17, Proposition 1.16] that the nonzero section $\tau|_C$ is nonvanishing. In particular, τ is nonvanishing at p.

(2) Following the argument in [17], we obtain the nonzero section

$$\tau \in H^0(X, \Lambda^p \mathcal{E}^\vee \otimes \det \mathcal{S}^\vee)$$

from the induced morphism $\det \mathcal{S} \to \Lambda^p \mathcal{E}^\vee$. Here $p := \operatorname{rank} \mathcal{S}$. We remark that $\Lambda^p \mathcal{E} \otimes \det \mathcal{S}$ is also almost nef, by the assumption for $\mathcal{S}$. Hence, by applying the first conclusion and [17, Lemma 1.20] to τ we can obtain the desired conclusion. □

Lemma 3.2. *Let X be a compact complex manifold and let $\mathcal{E}$ be a pseudo-effective torsion-free coherent sheaf on X. Then the same conclusion as in Lemma 3.1 holds.*

Proof. We will prove only conclusion (1). For the metric h_m on $\operatorname{Sym}^m \mathcal{E}$ satisfying the property in Definition 2.1, we consider the function f_m on X defined by

$$f_m := \frac{1}{m} \log |\tau^m|_{h_m^\vee}.$$

By the construction of h_m, we have

$$\sqrt{-1}\partial\overline{\partial} f_m \geq -\frac{1}{m}\omega,$$

and thus its weak limit (after we take a subsequence) should be zero. On the other hand, when we assume that τ has a zero point at some point $p \in X_{\mathcal{E}}$, it can be shown that the Lelong number of f_m is greater than or equal to 1. This is a contradiction of the fact that the weak limit is zero. Indeed, the section τ^m can be locally written as $\tau^m = \sum_I \tau_I e_I$. Here $\{e_i\}_{i=1}^r$ is a local frame of $\mathcal{E}$, I is a multi-index of degree m and $e_I := \prod_{i \in I} e_i$. It follows that the holomorphic function τ_I has multiplicity $\geq m$ at p from $\tau = 0$ at $p \in X_{\mathcal{E}}$. It can be seen that $|\langle e_I, e_J \rangle_{h_m^\vee}|$ is bounded, since $\log |u|_{h_m^\vee}$ is almost psh for any local section u (see, for example, [35, Lemma 2.2.4]). Hence we can easily check that

$$|\tau^m|_{h_m^\vee} \leq C \sum_I |\tau_I|.$$

This implies that the Lelong number of f_m is greater than or equal to 1. □

Lemma 3.3. *Let X be a projective manifold and E be a vector bundle on X. Let X_0 be a Zariski open set in X with $\operatorname{codim}(X \setminus X_0) \geq 2 + i$. Then the morphism induced by the restriction*

$$H^j(X, E) \to H^j(X_0, E)$$

is an isomorphism for any $j \leq i$.

Proof. The proof is given by the standard argument in terms of ample hypersurfaces and induction on dimension. □

Theorem 3.4 is a slight generalisation of [10, Proposition 2.7], and it has already been proved in [11, Corollary 2.12] for the case where X is a surface; but we do not know whether it can be reduced to [11, Corollary 2.12] by the argument of restriction to a general surface. Our proof heavily depends on the theory of admissible Hermitian–Einstein metrics developed in [7].

Theorem 3.4 (=Theorem 1.2; cf. [10])**.** *Let X be a projective manifold and let $\mathcal{E}$ be a reflexive coherent sheaf. If $\mathcal{E}$ is pseudo-effective and the first Chen class $c_1(\mathcal{E})$ is zero, then $\mathcal{E}$ is locally free and numerically flat.*

Proof. We will use induction on the rank r of $\mathcal{E}$. Reflexive coherent sheaves of rank 1 are always line bundles [21], and thus the conclusion is obvious in the case of $r = 1$. It is not so difficult to check the numerical flatness of $\mathcal{E}$ if $\mathcal{E}$ is shown to be locally free (see the argument to follow, or the proof in [17, Theorem 1.18]). We will focus on proving local freeness.

We fix an ample line bundle A on X. In the case of $r > 1$, we take a coherent subsheaf $\mathcal{S}$ with minimal rank among coherent subsheaves of $\mathcal{E}$ satisfying $\int_X c_1(\mathcal{S}) \cdot c_1(A)^{n-1} \geq 0$. We may assume that $\mathcal{S}$ is reflexive by taking the double dual if necessary. Now we consider the following exact sequence of sheaves:

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} := \mathcal{E}/\mathcal{S} \to 0. \tag{3.1}$$

The quotient sheaf $\mathcal{Q} := \mathcal{E}/\mathcal{S}$ is pseudo-effective. In particular, the first Chern class $c_1(\mathcal{Q})$ is also pseudo-effective. On the other hand, we have

$$0 = c_1(\mathcal{E}) = c_1(\mathcal{S}) + c_1(\mathcal{Q}).$$

Then it follows that $c_1(\mathcal{S}) = c_1(\mathcal{Q}) = 0$, since $c_1(\mathcal{Q})$ is pseudo-effective, and we have

$$\int_X c_1(\mathcal{Q}) \cdot c_1(A)^{n-1} = -\int_X c_1(\mathcal{S}) \cdot c_1(A)^{n-1} \leq 0.$$

By applying Lemma 3.1 to $\mathcal{Q}^\vee \to \mathcal{E}^\vee$, we can see that $\mathcal{Q}$ (and thus $\mathcal{S}$) is a vector bundle on $X_\mathcal{E}$ and the morphism is a bundle morphism on $X_\mathcal{E}$.

We first consider the case where the rank of $\mathcal{S}$ is equal to $r = \operatorname{rank} \mathcal{E}$. In this case, we obtain $\mathcal{S} = \mathcal{E}$. Indeed, it follows that $\mathcal{S} \cong \mathcal{E}$ on $X_\mathcal{E}$, since the bundle morphism $\mathcal{S} \to \mathcal{E}$ on $X_\mathcal{E}$ is an isomorphism. Then we can easily check $\mathcal{S} = \mathcal{E}$ by reflexivity and $\operatorname{codim}(X \setminus X_\mathcal{E}) \geq 3$. Further, we can prove that

$$\int_X c_2(\mathcal{E}) \cdot c_1(A)^{n-2} = 0.$$

Indeed, for a surface $S := H_1 \cap H_2 \cap \cdots \cap H_{n-2}$ in X constructed by general members H_i of a complete linear system A, it follows that $\mathcal{E}|_S$ is a pseudo-effective vector bundle from $\operatorname{codim}(X \setminus X_\mathcal{E}) \geq 3$. Hence $\mathcal{E}|_S$ is numerically flat on S, and thus $c_2(\mathcal{E}|_S) = 0$ (see [11, Corollary 2.12] or [17]). We can easily check that

$$\int_X c_2(\mathcal{E}) \cdot c_1(A)^{n-2} = \int_S c_2(\mathcal{E}|_S) = 0.$$

By the assumption of $c_1(\mathcal{E}) = 0$ and the result of [7, Corollary 3], we can conclude that $\mathcal{E}$ is a Hermitian flat vector bundle on X from the stability of the reflexive sheaf $\mathcal{S} = \mathcal{E}$. Therefore $\mathcal{E}$ is locally free and numerically flat.

It remains to consider the case of $\operatorname{rank} \mathcal{S} < \operatorname{rank} \mathcal{E}$. We consider the surjective bundle morphism

$$\Lambda^{m+1}\mathcal{E} \otimes \det \mathcal{Q}^\vee \to \mathcal{S}$$

on $X_\mathcal{E}$. By $\operatorname{codim}(X \setminus X_\mathcal{E}) \geq 3$ and $c_1(\mathcal{Q}) = 0$, the reflexive sheaf $\mathcal{S}$ is pseudo-effective. Therefore we can conclude that $\mathcal{S}$ is a numerically flat vector bundle on X by the induction hypothesis.

On the other hand, the reflexive hull $\mathcal{Q}^{\vee\vee}$ is a vector bundle on X by the induction hypothesis. The extension class obtained from the exact sequence (3.1) on $X_\mathcal{E}$ can be extended to the extension class (defined on X) of $\mathcal{S}$ and $\mathcal{Q}^{\vee\vee}$ by Lemma 3.3. The extended class determines the vector bundle whose restriction to $X_\mathcal{E}$ corresponds to $\mathcal{E}$. This implies that $\mathcal{E}$ is a vector bundle by the reflexivity of $\mathcal{E}$. □

3.2. Splitting theorem for positively curved vector bundles

In this subsection, we prove Theorems 1.3 and Theorem 1.4.

Lemma 3.5. *Let $\mathcal{Q}$ be a reflexive coherent sheaf on a compact complex manifold X. If $\mathcal{Q}$ admits a positively curved singular Hermitian metric $g_\mathcal{Q}$ and $c_1(\mathcal{Q}) = 0$, then we have the following:*

(1) *$(\mathcal{Q}, g_\mathcal{Q})$ is Hermitian flat on $X_\mathcal{Q}$.*

(2) *$\mathcal{Q}$ is a locally free sheaf on X and $g_\mathcal{Q}$ extends to a Hermitian flat metric on X.*

Proof. (1) The proof follows from an argument in [12] and the following lemma:

Lemma 3.6 ([36, Theorem 1.6])**.** *Let E be a holomorphic vector bundle and h_E be a positively curved singular Hermitian metric on E. If the induced metric $\det h_E$ on the determinant bundle $\det E$ is nonsingular (that is, smooth metric), then the curvature current $\sqrt{-1}\Theta_{h_E}$ of h_E is well defined as an* $\operatorname{End}(E)$*-valued* $(1,1)$*-form with measure coefficients.*

In our situation, the singular Hermitian metric $\det g_\mathcal{Q}$ on the determinant bundle $\det \mathcal{Q}$ is positively curved. By $c_1(\mathcal{Q}) = 0$, the curvature $\sqrt{-1}\Theta_{\det g_\mathcal{Q}}$ of $\det g_\mathcal{Q}$ is identically zero on $X_\mathcal{Q}$. In particular, it can be seen that $\det g_\mathcal{Q}$ is nonsingular. Then, by Lemma 3.5, the curvature current $\sqrt{-1}\Theta_{g_\mathcal{Q}}$ of $g_\mathcal{Q}$ is well defined on $X_\mathcal{Q}$.

The curvature $\sqrt{-1}\Theta_{g_{\mathcal{Q}}}$ is locally written as

$$\sqrt{-1}\Theta_{g_{\mathcal{Q}}} = \sum_{j,k,\alpha,\beta} \mu_{j\overline{k}\alpha\overline{\beta}} dz^j \wedge d\overline{z}^k e_\alpha \otimes e_\beta^\vee,$$

where $(z_1, \ldots, z_n)$ denotes a local coordinate and $\{e_1, \ldots, e_r\}$ denotes a local frame of $\mathcal{Q}$. Then by $\sqrt{-1}\Theta_{\det g_{\mathcal{Q}}} = \sqrt{-1}\,\mathrm{tr}\,\Theta_{g_{\mathcal{Q}}} = 0$, we obtain

$$\sum_{j,k}\sum_{\alpha} \mu_{j\overline{k}\alpha\overline{\alpha}} dz^j \wedge d\overline{z}^k = 0.$$

Since $g_{\mathcal{Q}}$ is positively curved, we obtain

$$\sum_{j,k} \mu_{j\overline{k}\alpha\overline{\alpha}} dz^j \wedge d\overline{z}^k \geq 0$$

for every α. Then we have that $\mu_{j\overline{k}\alpha\overline{\alpha}} = 0$ for every j, k, α. For every α and β, we have

$$\mathrm{Re}(\xi^\alpha \overline{\xi}^\beta \sum_{j,k} \mu_{j\overline{k}\alpha\overline{\beta}} v^j \overline{v}^k) \geq 0.$$

Hence we can conclude that $\mu_{j\overline{k}\alpha\overline{\beta}} = 0$ for every j, k, α, β, and thus $\sqrt{-1}\Theta_{g_{\mathcal{Q}}} = 0$.

(2) The vector bundle $\mathcal{Q}|_{X_{\mathcal{Q}}}$ is a local system on $X_{\mathcal{Q}}$, since $\mathcal{Q}$ is Hermitian flat on $X_{\mathcal{Q}}$. On the other hand, we have $\pi_1\big(U \setminus (X \setminus X_{\mathcal{Q}})\big) \cong \pi_1(U)$ for any open set U in X by $\mathrm{codim}(X \setminus X_{\mathcal{Q}}) \geq 3$. Hence $\mathcal{Q}|_{X_{\mathcal{Q}}}$ can be extended to the local system on X, which coincides with $\mathcal{Q}$ by reflexivity. □

We prepare the following lemma for the proof of Theorem 1.3:

Lemma 3.7. *Let (E, h) be a Hermitian flat vector bundle on a complex manifold X. Then for any point $x \in X$ and a basis $\{e_{1,x}, \ldots, e_{r,x}\}$ on the fibre E_x, there exists a local holomorphic frame $\{e_1, \ldots, e_r\}$ near x such that $e_j(x) = e_{j,x}$ and $\langle e_i, e_j\rangle_h$ is constant.*

Proof. Let D be the Chern connection associated to (E, h). Then, by flatness, we can take a local frame $\{e_j\}$ around x such that $De_j \equiv 0$. We may assume that $e_j(x) = e_{j,x}$. Since D is compatible with h, we have $d\langle e_i, e_j\rangle_h = \{De_i, e_j\}_h + \{e_i, De_j\}_h = 0$, and thus $\langle e_i, e_j\rangle_h$ is constant. Moreover, taking the $(0,1)$-part of $De_j \equiv 0$, we obtain $\overline{\partial} e_j \equiv 0$, which says that e_j is holomorphic. □

Theorem 3.8 (=Theorem 1.3 Hermitian metric g)**.** *Let E be a vector bundle with positively curved (singular) Hermitian metric g on a (not necessarily compact) complex manifold X. Let*

$$0 \to S \to E \to Q \to 0$$

be an exact sequence of vector bundles on X. Then the exact sequence splits if the induced quotient metric on Q is Hermitian flat on X (which is satisfied when X is compact and the first Chern class $c_1(Q)$ is zero, by Lemma 3.5).

Remark 3.9. When g is smooth, this theorem is obvious by a classical result. Indeed, the curvature of (Q, gQ) is more positive in the sense of Griffiths than that

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of (E, g) by by the inequality of Gauss–Codazzi type. It follows that the equality of the positivity of the curvatures actually holds from the assumptions, which leads to the splitting. Nevertheless, this argument does not work when g is singular, because the curvature forms or currents cannot be defined for singular Hermitian metrics.

Proof of Theorem 3.8. This proof is a generalisation of [24, Theorem 5.1]. We will work on dual bundles. By taking the dual, we have the exact sequence

$$0 \to Q^\vee \to E^\vee \to S^\vee \to 0. \tag{3.2}$$

Then we have a negatively curved singular Hermitian metric $h^\vee$ whose restriction to $Q^\vee$ is flat, by (the dual of) Lemma 3.5(1). Therefore, by Lemma 3.7, we can take a holomorphic orthonormal frame $(\kappa_1^\alpha, \ldots, \kappa_q^\alpha)$ of $Q^\vee$ on a small open set U^α. Let ϵ_j^α be the image of κ_j^α in $E^\vee$. Take $\epsilon_{q+1}^\alpha, \ldots, \epsilon_{q+s}^\alpha$ such that $(\epsilon_1^\alpha, \ldots, \epsilon_{q+s}^\alpha)$ is a local frame of $E^\vee$. Let σ_j^α be the image of ϵ_j^α in $S^\vee$. We remark that $(\sigma_{q+1}^\alpha, \ldots, \sigma_{q+s}^\alpha)$ is a local frame of $S^\vee$. We will write the transition functions of $Q^\vee$ and $S^\vee$ as follows:

$$\begin{aligned}
\kappa_1^\alpha &= \Phi_{1,1}^{Q^\vee,\alpha\beta}\kappa_1^\beta + \cdots + \Phi_{1,q}^{Q^\vee,\alpha\beta}\kappa_q^\beta,\\
&\vdots\\
\kappa_q^\alpha &= \Phi_{q,1}^{Q^\vee,\alpha\beta}\kappa_1^\beta + \cdots + \Phi_{q,q}^{Q^\vee,\alpha\beta}\kappa_q^\beta,\\
\\
\sigma_{q+1}^\alpha &= \Phi_{q+1,q+1}^{S^\vee,\alpha\beta}\sigma_{q+1}^\beta + \cdots + \Phi_{q+1,q+s}^{S^\vee,\alpha\beta}\sigma_{q+s}^\beta,\\
&\vdots\\
\sigma_{q+s}^\alpha &= \Phi_{q+s,q+1}^{S^\vee,\alpha\beta}\sigma_{q+1}^\beta + \cdots + \Phi_{q+s,q+s}^{S^\vee,\alpha\beta}\sigma_{q+s}^\beta.
\end{aligned}$$

The transition functions for $E^\vee$ can be written as

$$\begin{aligned}
\epsilon_1^\alpha &= \Phi_{11}^{Q^\vee,\alpha\beta}\epsilon_1^\beta + \cdots + \Phi_{1q}^{Q^\vee,\alpha\beta}\epsilon_q^\beta,\\
&\vdots\\
\epsilon_q^\alpha &= \Phi_{q1}^{Q^\vee,\alpha\beta}\epsilon_1^\beta + \cdots + \Phi_{qq}^{Q^\vee,\alpha\beta}\epsilon_q^\beta,\\
\epsilon_{q+1}^\alpha &= \Phi_{q+1,1}^{E^\vee,\alpha\beta}\epsilon_1^\beta + \cdots + \Phi_{q+1,q}^{E^\vee,\alpha\beta}\epsilon_q^\beta + \Phi_{q+1,q+1}^{S^\vee,\alpha\beta}\epsilon_{q+1}^\beta + \cdots + \Phi_{q+1,q+s}^{S^\vee,\alpha\beta}\epsilon_r^\beta,\\
&\vdots\\
\epsilon_{q+s}^\alpha &= \Phi_{q+s,1}^{E^\vee,\alpha\beta}\epsilon_1^\beta + \cdots + \Phi_{q+s,q}^{E^\vee,\alpha\beta}\epsilon_q^\beta + \Phi_{q+s,q+1}^{S^\vee,\alpha\beta}\epsilon_{q+1}^\beta + \cdots + \Phi_{q+s,q+s}^{S^\vee,\alpha\beta}\epsilon_{q+s}^\beta.
\end{aligned}$$

For short, we will write the coefficient matrix as

$$\Phi^{E^\vee,\alpha\beta} = \begin{pmatrix} \Phi^{Q^\vee,\alpha\beta} & 0 \\ \Psi^{\alpha\beta} & \Phi^{S^\vee,\alpha\beta} \end{pmatrix}.$$

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Next, let h^α be the matrix

$$h^\alpha := \begin{pmatrix} \langle \epsilon_1^\alpha, \epsilon_1^\alpha \rangle_h & \langle \epsilon_1^\alpha, \epsilon_2^\alpha \rangle_h & \cdots & \langle \epsilon_1^\alpha, \epsilon_{q+s}^\alpha \rangle_h \\ \vdots & \ddots & & \vdots \\ \langle \epsilon_{q+s}^\alpha, \epsilon_1^\alpha \rangle_h & \cdots & & \langle \epsilon_{q+s}^\alpha, \epsilon_{q+s}^\alpha \rangle_h \end{pmatrix}.$$

Note that the upper-left $q \times q$ matrix is constant by the choice of $\epsilon_1^\alpha, \ldots, \epsilon_q^\alpha$. Since h is negatively curved, the coefficients of the lower-left $s \times q$ matrix are holomorphic (say ϕ^α), by [24, Proposition 5.2]. Then we can write

$$h^\alpha = \begin{pmatrix} C^\alpha & \overline{\phi^\alpha} \\ \phi^\alpha & * \end{pmatrix},$$

where C^α is a $q \times q$ matrix whose coefficients are constant on U^α. By the equality

$$h^\alpha = \Phi^{E^\vee, \alpha\beta} h^\beta (\overline{{}^t\Phi^{E^\vee, \alpha\beta}}),$$

we have

$$C^\alpha = \Phi^{Q^\vee, \alpha\beta} C^\beta (\overline{{}^t\Phi^{Q^\vee, \alpha\beta}}),$$
$$\phi^\alpha = \Psi^{\alpha\beta} C^\beta (\overline{{}^t\Phi^{Q^\vee, \alpha\beta}}) + \Phi^{S^\vee, \alpha\beta} \phi^\beta (\overline{{}^t\Phi^{Q^\vee, \alpha\beta}}).$$

From these equalities, it follows that

$$\phi^\alpha (C^\alpha)^{-1} = \Psi^{\alpha\beta} (\Phi^{Q^\vee, \alpha\beta})^{-1} + \Phi^{S^\vee, \alpha\beta} \phi^\beta (C^\beta)^{-1} (\Phi^{Q^\vee, \alpha\beta})^{-1}.$$

On the other hand, the extension class of the given exact sequence can be calculated as the cohomology class of the Čech 1-cocycle:

$$\left\{ \sum_{\lambda=q+1}^{q+s} \sum_{\mu=1}^{q} \Psi_{\lambda,\mu}^{\alpha\beta} \kappa_\mu^\beta \otimes (\sigma_\lambda^\alpha)^\vee \in H^0(U_{\alpha\beta}, \mathcal{O}(Q^\vee \otimes S)) \right\}_{\alpha\beta}$$
$$= \left\{ \sum_{\lambda=q+1}^{q+s} \sum_{\mu=1}^{q} \sum_{\nu=1}^{q} \Psi_{\lambda,\mu}^{\alpha\beta} ((\Phi^{Q^\vee, \alpha\beta})^{-1})_{\mu\nu} \kappa_\nu^\alpha \otimes (\sigma_\lambda^\alpha)^\vee \right\}_{\alpha\beta}.$$

It is the differential of the Čech 0-cochain

$$\left\{ \sum_{\nu=1}^{q} \sum_{\lambda=q+1}^{q+s} (\phi^\alpha (C^\alpha)^{-1})_{\lambda\nu} \kappa_\nu^\alpha \otimes (\sigma_\lambda^\alpha)^\vee \in H^0(U_\alpha, \mathcal{O}(Q^\vee \otimes S)) \right\}_\alpha,$$

and thus the extension class is zero. Therefore sequence (3.2) splits. □

Theorem 3.10 (=Theorem 1.4). *Let X be a compact complex manifold and let*

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0$$

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be an exact sequence of reflexive coherent sheaves $\mathcal{S}$, $\mathcal{E}$ and $\mathcal{Q}$ on X. If $\mathcal{E}$ admits a positively curved (singular) Hermitian metric and the first Chen class $c_1(\mathcal{Q}) = 0$, then we have the following:

(1) *$\mathcal{Q}$ is locally free and Hermitian flat.*

(2) *$\mathcal{E} \to \mathcal{Q}$ is a surjective bundle morphism on $X_{\mathcal{E}}$.*

(3) *The exact sequence splits on X.*

Proof. Conclusion (1) follows from Lemma 3.5, and conclusion (2) from Lemma 3.1. Also, from Theorem 3.8 it follows that there exists a bundle morphism $j : \mathcal{Q} \to \mathcal{E}$ on $X_{\mathcal{E}}$ such that $\mathcal{E} = \mathcal{S} \oplus j(\mathcal{Q})$ on $X_{\mathcal{E}}$. By taking the push-forward i_* by the natural inclusion $i : X_{\mathcal{E}} \to X$ and the double dual, we obtain

$$(i_*\mathcal{E})^{\vee\vee} = (i_*\mathcal{S})^{\vee\vee} \oplus (i_*j(\mathcal{Q}))^{\vee\vee} \text{ on } X.$$

By $\operatorname{codim}(X \setminus X_{\mathcal{E}}) \geq 3$ and reflexivity, we have $\mathcal{E} \cong (i_*\mathcal{E})^{\vee\vee}$, $\mathcal{S} \cong (i_*\mathcal{S})^{\vee\vee}$ and $\mathcal{Q} \cong (i_*j(\mathcal{Q}))^{\vee\vee}$. This finishes the proof. □

3.3. Pseudo-effective tangent bundles

This subsection is devoted to the proof of Theorem 1.1.

Theorem 3.11 (=Theorem 1.1)**.** *Let X be a projective manifold with pseudo-effective tangent bundle. Then X admits a morphism $\phi : X \to Y$ with connected fibre to a smooth manifold Y with the following properties:*

(1) *The morphism $\phi : X \to Y$ is smooth (that is, all the fibres are smooth).*

(2) *The image Y admits a finite étale cover $A \to Y$ by an abelian variety A.*

(3) *A general fibre F of ϕ is rationally connected.*

(4) *A very general fibre F of ϕ also has pseudo-effective tangent bundle.*

Moreover, if we further assume that T_X admits a positively curved singular Hermitian metric, then we have the following:

(5) *The following exact sequence splits:*

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \phi^* T_Y \longrightarrow 0.$$

(6) *The morphism $\phi : X \to Y$ is locally trivial (that is, all the fibres are smooth and isomorphic).*

Proof. For a projective manifold X with the pseudo-effective tangent bundle T_X, we consider an MRC fibration $\phi : X \dashrightarrow Y$ to a projective manifold Y, and take a resolution $\pi : \bar{X} \to X$ of the indeterminacy locus of ϕ (see [28] and [8] for MRC fibrations). Here we

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have the following commutative diagram:

$$\begin{array}{ccc} \bar{X} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\bar{\phi}} & \\ X & \overset{\phi}{\dashrightarrow} & Y. \end{array}$$

(1), (3) To prove (1) and (3) using [23, Corollary 2.11], we will construct a foliation on X (that is, an integrable subbundle of T_X) whose general leaf is rationally connected. We will show that the relative tangent bundle $T_{X/Y} \subset T_X$ (which is defined only on a Zariski open set of X) can be extended to a subbundle of T_X on X. If it can be shown, it is not so difficult to check that this subbundle is integrable and its general leaf is rationally connected (that is, all the assumptions in [23, Corollary 2.11] are satisfied).

Now we have the exact sequence of coherent sheaves

$$0 \longrightarrow \bar{\phi}^*\Omega_Y \longrightarrow \Omega_{\bar{X}} \longrightarrow \Omega_{\bar{X}/Y} := \Omega_{\bar{X}}/\bar{\phi}^*\Omega_Y \longrightarrow 0.$$

Then we obtain the injective sheaf morphism $0 \to \pi_*\bar{\phi}^*\Omega_Y \to \Omega_X$ by taking the push-forward. Here we use the formula $\pi_*\Omega_{\bar{X}} = \Omega_X$. By taking the dual, we obtain the exact sequence on X

$$0 \longrightarrow \mathcal{S} := \operatorname{Ker} r \longrightarrow T_X \xrightarrow{\ r\ } \mathcal{Q} := (\pi_*\bar{\phi}^*\Omega_Y)^\vee. \tag{3.3}$$

We remark that this sequence corresponds to the standard exact sequence of tangent bundles on a Zariski open set where ϕ is a smooth morphism.

The morphism r is generically surjective, and thus the reflexive sheaf $\mathcal{Q}$ is also pseudo-effective. In particular, the first Chern class $c_1(\mathcal{Q})$ is also pseudo-effective. On the other hand, it follows from [3, 19] that the image Y of MRC fibrations has the pseudo-effective canonical bundle K_Y. Further, $\mathcal{Q}$ coincides with the usual pullback of T_Y on X_0. Here X_0 is the maximal Zariski open set where ϕ is a morphism. Hence, by $\operatorname{codim}(X \setminus X_0) \geq 2$, it can be shown that

$$-c_1(\mathcal{Q}) = c_1(\pi_*\bar{\phi}^*\Omega_Y) = c_1(\pi_*\bar{\phi}^*K_Y)$$

is pseudo-effective.

By this argument, we can see that $\mathcal{Q}$ is a pseudo-effective reflexive sheaf with $c_1(\mathcal{Q}) = 0$, and thus we can conclude by Theorem 1.2 that $\mathcal{Q}$ is a numerically flat vector bundle on X. On the other hand, we obtain the injective sheaf morphism $0 \to \mathcal{Q}^\vee \to \Omega_X$ on X from sequence (3.3), since $\mathcal{Q}^\vee$ is torsion-free and r is a generically surjective morphism defined on X (not only on X_0). Then by applying Lemma 3.1 to $0 \to \mathcal{Q}^\vee \to \Omega_X$, we can see that sequence (3.3) is a bundle morphism on X. In particular, ϕ is smooth on X_0 (since sequence (3.3) is not a bundle morphism on the nonsmooth locus of ϕ). The subbundle $\mathcal{S}$ defined by the kernel corresponds to the relative tangent bundle $T_{X/Y}$ defined on X_0. Hence $\mathcal{S}$ determines the foliation on X, since $T_{X/Y}$ is integrable on X_0 (see, for example, [30, subsection 2.2]). Further, its general leaf is rationally connected. Indeed, there exists a Zariski open set Y_1 in Y such that $\phi : X_1 := \phi^{-1}(Y_1) \to Y_1$ is a proper morphism, since $\phi : X \dashrightarrow Y$ is an almost-holomorphic map (that is, general fibres are compact). A general leaf of $\mathcal{S}$ corresponds to a general fibre of ϕ by $\mathcal{S} = T_{X/Y}$ on X_1, and thus it is rationally connected. Therefore, we can choose an MRC fibration to be holomorphic and smooth by [23, Corollary 2.11]. We use the same notation $\phi : X \to Y$ for the smooth MRC fibration.

(2) By (1), we have the standard exact sequence

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \phi^* T_Y \longrightarrow 0,$$

and we have already checked that $\phi^* T_Y$ is pseudo-effective and $c_1(\phi^* T_Y) = 0$. The pullback $\phi^* T_Y$ is numerically flat by Theorem 1.2, and thus T_Y is also numerically flat. The Beauville–Bogomolov decomposition [4] asserts that there exists a finite étale cover $Y' \to Y$ such that Y' is the product of hyper-Kähler manifolds, Calabi–Yau manifolds and abelian varieties. Let Z be a component of Y' of hyper-Kähler manifolds or Calabi–Yau manifolds. We remark that T_Z is also numerically flat. In general, numerically flat vector bundles are local systems (see, for example, [17]). Hence T_Z should be a trivial vector bundle on Z, since Z is simply connected and T_Z is also numerically flat. This is a contradiction of the definition of hyper-Kähler manifolds or Calabi–Yau manifolds. Hence the image Y admits a finite étale cover $A \to Y$ by an abelian variety A.

(4) By considering the restriction of the standard exact sequence of the tangent bundle to a fibre F of $\phi : X \to Y$, we obtain

$$0 \longrightarrow T_{X/Y}|_F = T_F \longrightarrow T_X|_F \longrightarrow \phi^* T_Y|_F = N_{F/X} = \mathcal{O}_F^{\oplus m} \longrightarrow 0.$$

When we consider the projective space bundle $f : \mathbb{P}(T_X) \to X$ and the non-nef locus $B \subset \mathbb{P}(T_X)$ of $\mathcal{O}_{\mathbb{P}(T_X)}(1)$, it can be seen that $f(B)$ is properly contained in X by the pseudo-effectivity of T_X (see Proposition 2.2 and its references). By the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(T_X|_F) & \hookrightarrow & \mathbb{P}(T_X) \\ \downarrow f & & \downarrow f \\ F & \hookrightarrow & X, \end{array}$$

we have $\mathcal{O}_{\mathbb{P}(T_X|_F)}(1) = \mathcal{O}_{\mathbb{P}(T_X)}(1)|_{f^{-1}(F)}$. Hence we obtain

$$\mathbb{B}_-\big(\mathcal{O}_{\mathbb{P}(T_X|_F)}(1)\big) = \mathbb{B}_-\big(\mathcal{O}_{\mathbb{P}(T_X)}(1)|_{f^{-1}(F)}\big) \subset f^{-1}(F) \cap B.$$

Here we use the fact that $\mathbb{B}_-(L|_Z) \subset \mathbb{B}_-(L) \cap Z$ holds for any line bundle L and any subvariety Z. This implies that the image of the non-nef locus of $\mathcal{O}_{\mathbb{P}(T_X|_F)}(1)$ is contained in $f(B \cap F)$. The image $f(B \cap F)$ is properly contained in F for a very general fibre F. Note that B (and $f(B)$) is a countable union of proper subvarieties. Hence $T_X|_F$ is pseudo-effective, by another application of Proposition 2.2.

(5), (6) we finally show that the MRC fibration $\phi : X \to Y$ is locally trivial if we further assume that X admits a positively curved singular Hermitian metric. Under the assumption of such a metric, the exact sequence of the tangent bundle splits (that is, $T_X \cong T_{X/Y} \oplus \phi^* T_Y$) by Theorem 1.4. Then by Ehresmann's theorem (see also [23, Lemma 3.19]), we can see that $\phi : X \to Y$ is locally trivial. □

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Theorem 3.12. *Let X be a compact Kähler manifold with pseudo-effective tangent bundle, and let $\phi : X \to Y := \mathrm{Alb}(X)$ be its Albanese map. Then the Albanese map ϕ is a surjective smooth morphism and satisfies all the conclusions in Theorem 3.11 except* (3) *and* (6) *by replacing an abelian variety in* (2) *with a compact complex torus.*

Proof. In the proof of Theorem 3.11, the assumption of projectivity was used only for (3) and (6). The other arguments work even if we replace MRC fibrations with the Albanese map. Hence it is sufficient to prove that the Albanese map ϕ is a surjective smooth morphism. This is easy to check. Indeed, for a basis $\{\eta_k\}_{k=1}^q$ of $H^0(X,\Omega_X)$, it follows that any nontrivial linear combination of them is nonvanishing by Lemma 3.2. This implies that ϕ is a surjective smooth morphism (see, for example, [13]). □

It was proved in [17] that X is a Fano manifold if it is a rationally connected manifold with nef tangent bundle. As an analogue of this result, we suggest the following problem. We remark that the geometry of a general fibre F in Theorem 1.1 can be determined if this problem can be affirmatively solved.

Problem 3.13. If a projective manifold X is rationally connected and has pseudo-effective tangent bundle, is the anticanonical bundle $-K_X$ big?

4. Surfaces with pseudo-effective tangent bundle

Toward the classification of surfaces with pseudo-effective tangent bundle, we study minimal ruled surfaces in subsection 4.1 and their blowups in subsection 4.2, which provide interesting examples of positively curved vector bundles.

4.1. On minimal ruled surfaces

In this subsection, we consider a ruled surface $\phi : X \to C$ over a smooth curve C. If T_X is pseudo-effective, the base C should be either the projective line or an elliptic curve, by Theorem 1.1. Conversely, it follows that any minimal ruled surface $\phi : X \to \mathbb{P}^1$ over $\mathbb{P}^1$ (that is, a Hirzebruch surface) has the pseudo-effective tangent bundle from the following proposition. However, such surfaces do not have nef tangent bundle except, for the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$, since they have a curve with negative self-intersection.

Proposition 4.1. *If X is a projective toric manifold, then T_X is generically globally generated. In particular, any Hirzebruch surface has pseudo-effective tangent bundle.*

Proof. For a toric manifold X, we have an inclusion $(\mathbb{C}^*)^n \subset X$ as a Zariski open dense subset and an action $(\mathbb{C}^*)^n \curvearrowright X$. Consider a family of actions $(e^{i\theta}, 1, \cdots, 1)$. Differentiating it by θ at $\theta = 0$, we obtain a holomorphic vector field on X. Similarly, we can construct n vector fields which generate $T_X|_{(\mathbb{C}^*)^n}$, and thus T_X is generically globally generated. □

Now we consider a ruled surface $\phi : X \to C$ over an elliptic curve C. Thanks to Theorem 1.1, we can see that the ruling $\phi : X \to C$ should be a smooth morphism when X has pseudo-effective tangent bundle. The minimal ruled surface X over C can be classified by [1, 2, 37]: X is isomorphic to S_n, A_0, A_{-1} or a surface in $\mathcal{S}_0$. Here a surface X in $\mathcal{S}_0$ means that the projective space bundle $\mathbb{P}(\mathcal{O}_C \oplus L)$ for some $L \in \mathrm{Pic}^0(C)$ and A_0 (resp., A_{-1}) is the projective space bundle associated with a vector bundle of rank 2 that is the nonsplit extension of $\mathcal{O}_C$ by $\mathcal{O}_C$ (resp., $\mathcal{O}_C(p)$), where p is a point in C. It can be seen that A_0, A_{-1} and surfaces in $\mathcal{S}_0$ have nef tangent bundle by [13], and thus the remaining problem is the case of $X = S_n$. The ruled surface S_n is the projective space bundle associated with the vector bundle $\mathcal{O}_C \oplus \mathcal{O}_C(np)$. Note that the tangent bundle of $S_0 = \mathbb{P}^1 \times C$ is nef. By this observation, it is enough for our purpose to investigate $X = S_n$ in the case were $n \geq 1$. By the following proposition, we can see that S_n has pseudo-effective tangent bundle (which is not nef), and further that it admits no positively curved singular Hermitian metric.

Proposition 4.2. *Let $\phi : X \to C$ be a minimal ruled surface over an elliptic curve C. Then we have the following:*

(1) *The tangent bundle of S_n is pseudo-effective, but it does not admit positively curved singular Hermitian metrics when $n \geq 1$.*

(2) *The tangent bundle of S_0, A_0, A_{-1} and a surface in $\mathcal{S}_0$ is nef.*

Proof. All the ruled surfaces with nef tangent bundle are classified in [13], which implies that (2) holds and the tangent bundle of S_n is not nef for $n \geq 1$.

From now on, let X be the projective space bundle S_n associated with the vector bundle $E_n := \mathcal{O}_C \oplus \mathcal{O}_C(np)$. We first check the latter statement in (1). If $X = S_n$ admits a positively curved singular Hermitian metric, the exact sequence

$$0 \to T_{X/C} \to T_X \to \phi^* T_C \to 0$$

splits by Theorem 1.4, and thus we have

$$h^0(X, T_X) = h^0(X, T_{X/C}) + h^0(X, \phi^* T_C). \tag{4.1}$$

On the other hand, we have $h^0(X, T_X) = n+1$ from [37, Theorem 3]. Also we can easily check that

$$\phi_*(T_{X/C}) = \phi_*(-K_X) = \mathrm{Sym}^2(E_n) \otimes \det E_n^{\vee}.$$

This implies that

$$h^0(X, T_{X/C}) = h^0(C, \mathcal{O}_C(-np) \oplus \mathcal{O}_C \oplus \mathcal{O}_C(np)) = n+1.$$

This is a contradiction of equation (4.1).

We will prove that T_X is pseudo-effective. For this purpose, it is sufficient to prove that $\mathrm{Sym}^m(T_X) \otimes \phi^*\mathcal{O}(2p)$ is generically globally generated for any $m \geq 0$. Our strategy is to observe a gluing condition of $X = S_n$ carefully to construct holomorphic sections that generate $\mathrm{Sym}^m(T_X) \otimes \phi^*\mathcal{O}(2p)$ at general points.

Let v be a local coordinate centred at p and let $V \subset C$ be a sufficiently small open neighbourhood of p. Further, let U be the open set $U := C \setminus \{p\}$ and u be the standard coordinate of the universal cover $\mathbb{C} \to C$. The ruled surface X can be constructed by

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gluing $(u,\zeta) \in U \times \mathbb{P}^1$ and $(v,\eta) \in V \times \mathbb{P}^1$ with the following identification:

$$\zeta = v^n \eta \quad \text{and} \quad [u] = p + v, \tag{4.2}$$

where ζ and η are the inhomogeneous coordinates of $\mathbb{P}^1$.

Let θ be a meromorphic section of $\mathrm{Sym}^m(T_X)$ with pole along the fibre $\phi^{-1}(p)$ of p. Our strategy is as follows: we first look for a sufficient condition for the pole of θ being of order at most 2. Then we concretely construct θ satisfying this condition, which can be regarded as a holomorphic section of $\mathrm{Sym}^m(T_X) \otimes \phi^*\mathcal{O}(2p)$, and we show that such sections generate $\mathrm{Sym}^m(T_X) \otimes \phi^*\mathcal{O}(2p)$ on a Zariski open set.

Now θ is a meromorphic section of $\mathrm{Sym}^m(T_X)$ whose pole appears only along the fibre $\phi^{-1}(p)$. Hence by expanding θ on $U \times \mathbb{P}^1$, we have

$$\theta = \sum_{p=0}^{m} a_p(u,\zeta) \Big(\frac{\partial}{\partial \zeta}\Big)^{m-p} \Big(\frac{\partial}{\partial u}\Big)^{p} \text{ on } U \times \mathbb{P}^1. \tag{4.3}$$

Here a_p is a meromorphic function on X. The gluing condition (4.2) yields

$$\frac{\partial}{\partial \zeta} = \frac{1}{v^n}\frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial u} = -n\frac{\eta}{v}\frac{\partial}{\partial \eta} + \frac{\partial}{\partial v}. \tag{4.4}$$

Then we can obtain the following expansion of θ on $V \times \mathbb{P}^1$ by an involved but straightforward computation:

$$\theta = \sum_{\ell=0}^{m} \Big\{ \sum_{p=\ell}^{m} d_{p,\ell}\, a_p(v,\eta) \frac{\eta^{p-\ell}}{v^{n(m-p)+p-\ell}} \Big\} \Big(\frac{\partial}{\partial \eta}\Big)^{m-\ell} \Big(\frac{\partial}{\partial v}\Big)^{\ell} \text{ on } V \times \mathbb{P}^1. \tag{4.5}$$

Here, $d_{p,\ell}$ is the nonzero constant defined by $d_{p,\ell} := (-n)^{p-\ell}\binom{p}{p-\ell}$. The ruling $X \to C$ is locally trivial, and sections of $\mathrm{Sym}^p(T_F)$ on a fibre F are polynomials of degree (at most) $2p$. This implies that the meromorphic function $a_{m-k}(u,\zeta)$ is a polynomial of degree $2k$ with respect to ζ, and thus we can write a_{m-k} as

$$a_{m-k}(v,\eta) = \sum_{q=0}^{2k} a_{m-k}^{(q)}(v)\, \zeta^q = \sum_{q=0}^{2k} a_{m-k}^{(q)}(v)\, v^{nq}\eta^q, \quad \text{for any } 0 \le k \le m, \tag{4.6}$$

for some meromorphic function $a_{m-k}^{(q)}(v)$ on C with pole only at p. Here we use gluing condition (4.2) again.

We will find a sufficient condition for $a_{m-k}^{(q)}(v)$ for guaranteeing that the coefficients in equation (4.5) have a pole of order at most 2. We remark that the section θ satisfying this condition determines the holomorphic section of $\mathrm{Sym}^m(T_X) \otimes \phi^*\mathcal{O}(2p)$. By substituting equation (4.6) for equation (4.5) and rearranging it concerning the powers of η, we can obtain a sufficient and necessary condition, but this method needs such complicated computation that we want to avoid writing it down. Here, to improve our prospect, we focus only on a sufficient condition by considering the restricted situation where $a_{m-k}^{(q)} = 0$ for $q \neq k$. In this situation, it is not so difficult to show that θ determines the holomorphic section of $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ if $a^{(q)}_{m-q}$ satisfies the condition that

$$\sum_{p=0}^{q} d_{m-p,\,m-q}\,a^{(p)}_{m-p}(v)\,\frac{1}{v^{q-p}} \text{ has a pole of order} \leq 2 \text{ at } p \text{ for any } 0\leq q\leq m. \tag{4.7}$$

See the following table for $q=0,1,2$.

$q=q$	coeff. of $(\partial/\partial\eta)^q(\partial/\partial v)^{m-q}$	$\sum_{p=0}^{q} d_{m-p,\,m-q}\,a^{(p)}_{m-p}/v^{q-p}$
$q=0$	coeff. of $(\partial/\partial\eta)^0(\partial/\partial v)^{m}$	$d_{m,\,m}\,a^{(0)}_{m}$
$q=1$	coeff. of $(\partial/\partial\eta)^1(\partial/\partial v)^{m-1}$	$d_{m,\,m-1}\,a^{(0)}_{m}/v+d_{m-1,\,m-1}\,a^{(1)}_{m-1}$
$q=2$	coeff. of $(\partial/\partial\eta)^2(\partial/\partial v)^{m-2}$	$d_{m,\,m-2}\,a^{(0)}_{m}/v^2+d_{m-1,\,m-2}\,a^{(1)}_{m-1}/v+d_{m-2,\,m-2}\,a^{(2)}_{m-2}$

To construct meromorphic functions $a^{(p)}_{m-p}$ on C satisfying condition (4.7), for every $n\geq 2$, we take meromorphic functions P_n on the elliptic curve C such that P_n has a pole only at p and its Laurent expansion at p can be written as

$$P_n(v)=\frac{1}{v^n}+\sum_{k\geq n+1}\frac{a_k}{v^k}.$$

Note that we can easily find these functions by using Weierstrass elliptic functions and their differentials.

We first put $a^{(0)}_{m}:=P_2/d_{m,m}$. Then the second row of the table satisfies condition (4.7) (that is, it has a pole of order at most 2) if we define $a^{(1)}_{m-1}$ by $a^{(1)}_{m-1}:=-d_{m,\,m-1}/d_{m-1,\,m-1}P_3$. In the same way, the third row also satisfies condition (4.7) if we define $a^{(2)}_{m-2}$ by an appropriate linear combination of P_3 and P_4. By repeating this process, we can construct meromorphic functions $a^{(p)}_{m-p}$ on C satisfying condition 4.7 by a linear combination of $\{P_3\}_{k=3}^{p+2}$. We denote by θ_0 the holomorphic section of $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ obtained from this construction. The section θ_0 generates the vector $(\partial/\partial\eta)^0(\partial/\partial v)^m$ on a Zariski open set, since $a^{(0)}_{m}=P_2/d_{m,m}$ is nonzero.

Now we put $a^{(0)}_{m}:=0$ and $a^{(1)}_{m-1}:=P_2/d_{m-1,\,m-1}$, so that the first and second rows in the table have a pole of order at most 2. Then, by the same argument as before, we can construct meromorphic functions $a^{(p)}_{m-p}$ satisfying condition (4.7) by defining them by an appropriate linear combination of $\{P_k\}_{k=3}^{p+2}$ (for example, $a^{(2)}_{m-2}:=-d_{m-1,\,m-2}/d_{m-2,\,m-2}P_3$). We denote by θ_1 the obtained holomorphic section of $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$. By this construction, the function $a^{(0)}_{m}$ is zero and $a^{(1)}_{m-1}$ is nonzero. Hence it follows that the sections θ_0 and θ_1 generate the vectors $(\partial/\partial\eta)^0(\partial/\partial v)^m$ and $(\partial/\partial\eta)^1(\partial/\partial v)^{m-1}$ on a Zariski open set.

By repeating this process, we can construct holomorphic sections $\{\theta_p\}_{p=0}^{m}$ of $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ generating $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ on a Zariski open set. □

In the rest of this subsection, we suggest the following problem, which seems to be important not only for the proof of Proposition 4.2 without local coordinates but also for the study of a gap between being almost nef and pseudo-effective:

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Problem 4.3. We consider an exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0.$$

When S and Q are pseudo-effective, is E pseudo-effective?

Remark 4.4. When S and Q are nef, the extension E is also nef [17, Proposition 1.15]. Hence we can easily show that E is almost nef if S and Q are almost nef. In particular, it can be shown that $\mathcal{O}_E(1)$ is pseudo-effective by [3], but we do not know whether or not E itself is pseudo-effective. The difficulty is in showing that the image of the non-nef locus $\mathcal{O}_E(1)$ to X is properly contained in X. If Problem 4.3 can be affirmatively solved, the pseudo-effectivity of the tangent bundle of $X = S_n$ is easily obtained, by applying it to the standard exact sequence of the tangent bundle. In fact, we tried some methods from [17, 20, 37] to solve Problem 4.3, but did not succeed. This problem seems to be subtle, since we do not know whether there is a gap between being almost nef and pseudo-effective.

4.2. On rational surfaces

By the results in subsection 4.1, it is enough for the classification of surfaces to determine when the blowup of a Hirzebruch surface has pseudo-effective tangent bundle. However, it seems to be too hard a problem to classify all the blowups completely, since X delicately depends on the position and number of blowup points. In this subsection, we study only blowups along *general points*. A complete classification cannot be achieved even in this case, but we obtain an interesting relation between the positivity of a tangent bundle and the geometry of Hirzebruch surfaces. The following proposition gives the requirement for the blowup having pseudo-effective tangent bundle:

Proposition 4.5. *Let $\phi : \mathbb{F}_n \to \mathbb{P}^1$ be a Hirzebruch surface and let $\pi : X \to \mathbb{F}_n$ be the blowup along the set Σ of general points on $\mathbb{F}_n$. Then we have the following:*

(1) *If the tangent bundle T_X of X is generically globally generated, then $\sharp\Sigma \leq 2$.*

(2) *If the tangent bundle T_X of X is pseudo-effective, then $\sharp\Sigma \leq 4$.*

Remark 4.6. The interesting point here is that the conclusion $\sharp\Sigma \leq 2$ in (1) is optimal, and further, that generic global generation and pseudo-effectivity behave differently for $\sharp\Sigma$. Indeed, it follows that the tangent bundle T_X in the case where $\sharp\Sigma \leq 3$ is pseudo-effective but not generically globally generated, from Proposition 4.8.

Proof of Proposition 4.5. (1) Fix a holomorphic vector field ξ on X. We will define a holomorphic vector field θ_ξ on $\mathbb{P}^1$ as follows. Let t be a local holomorphic coordinate on $U \subset \mathbb{P}^1$. By pulling back dt, we obtain a holomorphic 1-form $\pi^*\phi^* dt$ on $U' := (\pi \circ \phi)^{-1}(U)$. Then $\langle \xi, \pi^*\phi^* dt\rangle$ is a holomorphic function on U'. Thus it is constant along each fibre and defines a holomorphic function on U. Now we define the holomorphic vector field θ_ξ on $\mathbb{P}^1$ to be

$$\theta_\xi := \langle \theta_\xi, dt\rangle \frac{\partial}{\partial t} \quad \text{and} \quad \langle \theta_\xi, dt\rangle := \langle \xi, \pi^*\phi^* dt\rangle.$$

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Since we assumed that T_X is generically globally generated, we can choose ξ with $\theta_\xi \not\equiv 0$ on $\mathbb{P}^1$.

We claim that θ_ξ has zeros on the set $\phi(\Sigma) \subset \mathbb{P}^1$. To prove the claim, we take a local coordinate (t,s) on $\mathbb{F}_n$ centred at a point in Σ such that t is the pullback of a local coordinate on $\mathbb{P}^1$. If we put $v := t/s$, then (v,s) is a coordinate on X. Then we have

$$\langle \xi, \pi^*\phi^* dt\rangle = \langle \xi, d(vs)\rangle = \langle \xi, sdv + vds\rangle.$$

The last term vanishes at $(v,s) = (0,0)$, and thus $\langle \theta_\xi, dt\rangle = 0$ at $t = 0$. This shows the claim.

In the case of $\sharp\Sigma \geq 3$, the vector field θ_ξ has at least three zeros on $\mathbb{P}^1$. This contradicts the fact that $\deg T_{\mathbb{P}^1} = 2$, and thus we have $\sharp\Sigma \leq 2$.

(2) Since T_X is pseudo-effective, we can choose an ample line bundle A and a sequence of positively curved singular Hermitian metrics h_m on $(\mathrm{Sym}^m T_X) \otimes A$. Fix a smooth Hermitian metric h_A on A with positive curvature. Then $h_m \otimes h_A^{-1}$ is a (possibly not positively curved) singular Hermitian metric on $\mathrm{Sym}^m T_X$. Define a singular Hermitian metric g_m on $\pi^*\phi^* T_{\mathbb{P}^1}$ by the mth root of the quotient metric of $h_m \otimes h_A^{-1}$ induced by the morphism $\mathrm{Sym}^m T_X \to (\pi^*\phi^* T_{\mathbb{P}^1})^{\otimes m}$. Since $(h_m \otimes h_A^{-1}) \otimes h_A$ is positively curved, the metric $g_m^m \otimes h_A$ is also positively curved. The curvature current $\sqrt{-1}\Theta_{g_m}$ of g_m satisfies

$$\sqrt{-1}\Theta_{g_m} \geq -\frac{1}{m}\omega_A.$$

Then by taking a subsequence (if necessary), we can assume that $\sqrt{-1}\Theta_{g_m}$ weakly converges to a positive current $T \in c_1(\pi^*\phi^* T_{\mathbb{P}^1})$. By a similar argument to (1), we obtain a d-closed positive $(1,1)$-current S in $c_1(T_{\mathbb{P}^1})$ such that $T = \phi^*\pi^* S$. Hence we have

$$\sqrt{-1}\Theta_{g_m} \to \pi^*\phi^* S = T \in c_1(\phi^*\pi^* T_{\mathbb{P}^1}).$$

We take a point $p \in \Sigma$ and put $p_0 := \phi(p)$. We claim that the bound of the Lelong number

$$\nu(S, p_0) \geq \frac{1}{2}. \tag{4.8}$$

We fix a local coordinate t near $p_0 \in \mathbb{P}^1$. Let (t,s) be a coordinate on $\mathbb{F}_n$ centred at p. As before, since $v = t/s$, (v,s) is a coordinate on X. Let $p' \in X$ be a point defined by $(v,s) = (0,0)$. Let C be a (local) holomorphic curve on X defined by $\{v = s\}$. We will denote $\overline{C} := \pi(C)$. The defining equation of $\overline{C}$ is $\{t/s = s\} = \{t = s^2\}$. Then we have

$$\nu(S, p_0) = \frac{1}{2}\nu(\phi^* S|_{\overline{C}}, p). \tag{4.9}$$

Indeed, the function $\phi^*\gamma$ is a local potential of $\phi^* S$ for a local potential γ of S. Note that s is a local coordinate on $\overline{C}$ while $t = s^2$ is a local coordinate on $\mathbb{P}^1$. By the formula of Lelong numbers

$$\nu(S, p_0) = \liminf_{t\to 0} \frac{\gamma(t)}{\log|t|},$$

we can obtain

$$\nu(\phi^* S|_{\overline{C}}, p) = \liminf_{s\to 0} \frac{\phi^*\gamma(s^2, s)}{\log|s|} = \liminf_{s\to 0} \frac{\gamma(s^2)}{\log|s|} = 2\nu(S, p_0).$$

This proves equation (4.9). Since the Lelong number will increase after we take the restriction, we have

$$\nu(\phi^* S|_{\overline{C}}, p) = \nu(T|_C, p') \geq \nu(T, p').$$

Lelong numbers will also increase after taking a weak limit of currents, and thus we obtain

$$\nu(T, p') \geq \limsup_{m\to+\infty} \nu(\sqrt{-1}\Theta_{g_m}, p').$$

The local weight of g_m is written as

$$\frac{1}{2m} \log|(\pi^*\phi^*(dt))^m|^2_{h_m^{-1}\otimes h_A}.$$

Since $t = vs$ on X, we can calculate as follows:

$$|(\pi^*\phi^*(dt))^m|^2_{h_m^{-1}\otimes h_A} = |(vds + sdv)^m|^2_{h_m^{-1}\otimes h_A}. \tag{4.10}$$

Since h_m^{-1} is negatively curved and h_A is smooth, it follows that $|\cdot|^2_{h_m^{-1}\otimes h_A} \leq C_0|\cdot|^2_{h_{\mathrm{sm}}}$ for a smooth Hermitian metric h_{sm} and some constant $C_0 > 0$ (both depending on m). Then the right-hand side of equation (4.10) is bounded as

$$\begin{aligned} &|(\pi^*\phi^*(dt))^m|^2_{h_m^{-1}\otimes h_A} \\ \leq &C_0|(vds + sdv)^m|^2_{h_{\mathrm{sm}}} \\ \leq &C_0|(v, s)|^{2m}. \end{aligned}$$

Thus, the Lelong number of $\sqrt{-1}\Theta_{g_m}$ is bounded as

$$\nu(\sqrt{-1}\Theta_{g_m}, p') \geq \frac{1}{2m} \liminf_{(v,s)\to 0} \frac{C_0|(v, s)|^{2m}}{\log|(v, s)|} = 1.$$

This proves expression (4.8). By $\deg T_{\mathbb{P}^1} = 2$, the current S has at most four points at which its Lelong number greater than or equal to $1/2$. Therefore, $\sharp\Sigma \leq 4$. □

We finally prove Proposition 4.8, by applying the following lemma. The lemma is useful when we compare a vector field on a given manifold with its blowup.

Lemma 4.7. *Let $\pi : Y \to \mathbb{C}^2$ be the blowup at $(\alpha, \beta) \in \mathbb{C}^2$ with the exceptional divisor E, and let (x, y) be the standard coordinate of $\mathbb{C}^2$. We consider a holomorphic section θ of* $\mathrm{Sym}^m T_{\mathbb{C}^2}$ *and its expansion*

$$\theta = \sum_{k=0}^{m} f_k(x, y)\Big(\frac{\partial}{\partial x}\Big)^k \Big(\frac{\partial}{\partial y}\Big)^{m-k}.$$

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Then the pullback $(\pi|_{Y\setminus E})^*\theta$ by the isomorphism $\pi|_{Y\setminus E}$ on $Y\setminus E$ can be extended to the holomorphic section of $\mathrm{Sym}^m T_Y$ *if and only if*

$$\sum_{k=0}^{m} f_k(s+\alpha, st+\beta)\Big(\frac{\partial}{\partial s}-\frac{t}{s}\frac{\partial}{\partial t}\Big)^k\Big(\frac{1}{s}\frac{\partial}{\partial t}\Big)^{m-k}$$

is holomorphic with respect to $(s,t)\in\mathbb{C}^2$.

Proof. We first remark that any holomorphic section ξ of $\mathrm{Sym}^m T_Y$ determines the section θ_ξ of $\mathrm{Sym}^m T_{\mathbb{C}^2}$. Indeed, a given section ξ induces the section θ_ξ of $\mathrm{Sym}^m T_{\mathbb{C}^2}$ on $\mathbb{C}^2\setminus\{(\alpha,\beta)\}$ via the isomorphism $\pi|_{Y\setminus E}$, which can be extended on $\mathbb{C}^2$ since the blowup centre has codimension 2.

We describe Y and E by the coordinates

$$Y=\{(x,y,[z:w])\in\mathbb{C}^2\times\mathbb{P}^1\,|\,(x-\alpha)w=(y-\beta)z\},$$
$$E=\{(\alpha,\beta,[z:w])\,|\,[z:w]\in\mathbb{P}^1\},$$

and put the Zariski open set $Y':=Y\cap\{w\neq 0\}$. The following map r gives a coordinate of Y', and $\pi|_{Y'}$ can be written as follows:

$$\begin{array}{ccccccc} r:\mathbb{C}^2 & \to & Y' & \qquad & \pi|_{Y'}:Y' & \to & \mathbb{C}^2\\ (s,t) & \mapsto & (s+\alpha, st+\beta,[1:t]), & & (x,y,[z:w]) & \mapsto & (x,y). \end{array}$$

If $(\pi\circ r)^*\theta$ is holomorphic on $\mathbb{C}^2$, then $(\pi|_{Y\setminus E})^*\theta$ can be extended to the holomorphic section of $\mathrm{Sym}^m T_Y$. Indeed, in this case the section $(\pi|_{Y\setminus E})^*\theta$ can be extended to the holomorphic section of $\mathrm{Sym}^m T_{Y'}$. Hence it can also be extended on Y, since the codimension of $E\cap\{w=0\}$ is 2.

By calculation, we obtain

$$(\pi\circ r)^*\theta=\sum_{k=0}^{m} f_k(s+\alpha, st+\beta)\Big(\frac{\partial}{\partial s}-\frac{t}{s}\frac{\partial}{\partial t}\Big)^k\Big(\frac{1}{s}\frac{\partial}{\partial t}\Big)^{m-k}.$$

Hence $(\pi\circ r)^*\theta$ is holomorphic on $\mathbb{C}^2$ if and only if the right-hand side is holomorphic in $(s,t)\in\mathbb{C}^2$, which completes the proof. □

Proposition 4.8. *The blowup of the Hirzebruch surface $\mathbb{F}_n$ along one, two or three general points has pseudo-effective tangent bundle. More precisely, the following are true:*

(1) *The blowup of the Hirzebruch surface $\mathbb{F}_n$ along one or two general points has generically globally generated tangent bundle.*

(2) *The blowup of the Hirzebruch surface $\mathbb{F}_n$ along three general points has generically globally generated second symmetric power of the tangent bundle.*

We first show Proposition 4.8 in the simplest case, $n=0$.

Proof. (1) In general, for a birational morphism $f:Y\to Z$ between projective manifolds, we have the natural inclusion $f_*T_Y\subset T_Z$. Since the natural inclusion is of course generically an isomorphism, the tangent bundle T_Z is generically globally generated if

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the tangent bundle T_Y is. Therefore, it is sufficient for the proof to treat only the blowup $\pi : X \to \mathbb{F}_0$ along two general points p_1, p_2.

We take a Zariski open set $\mathbb{C}\times\mathbb{C} = W_0 \subset \mathbb{F}_0$ with the local coordinate (x, y). We may assume that $p_1 = (0,0)$ and $p_2 = (1,1)$ by using the action of the automorphism group of $\mathbb{F}_0$. We define the set of holomorphic vector fields on W_0:

$$\mathcal{T} := \left\{ \sum_{k=0}^{2} a_k x^k \frac{\partial}{\partial x} + \sum_{l=0}^{2} b_l y^l \frac{\partial}{\partial y} \,\middle|\, a_k, b_l \in \mathbb{C} \right\}.$$

We remark that any $\theta \in \mathcal{T}$ can be extended to a global holomorphic section of $T_{\mathbb{F}_0}$. From Lemma 4.7, for a holomorphic vector field $\theta := a(x)\partial/\partial x + b(y)\partial/\partial y \in \mathcal{T}$, the section θ can be lifted to the holomorphic section of $T_{\mathbb{F}_0}$ if and only if

$$\frac{1}{s}\big(-a(s+\alpha)t + b(st+\beta)\big) \text{ is holomorphic with respect to } (s,t)$$

for $(\alpha, \beta) = (0,0)$ and $(\alpha, \beta) = (1,1)$. We choose θ_1 and θ_2 in $\mathcal{T}$ as follows:

$$\theta_1 = (x^2 - x)\frac{\partial}{\partial x} \text{ and } \theta_2 = (y^2 - y)\frac{\partial}{\partial y}.$$

Then we can easily see that $\pi^*\theta_1$ and $\pi^*\theta_2$ can be extended to the global holomorphic sections of T_X. For a point $q = (x, y) \in W_0$ such that $x \neq 0,1$ and $y \neq 0,1$, the vectors $\theta_1(q)$ and $\theta_2(q)$ at q give a basis of $T_{W_0,q}$. Therefore T_X is generically globally generated.

(2) Let $\pi : X \to \mathbb{F}_0$ be a blowup of $\mathbb{F}_0$ along three general points p_1, p_2, p_3. Our goal in this proof is to show that $\mathrm{Sym}^2(T_X)$ is generically globally generated. Since p_1, p_2, p_3 are in general position, we may assume $p_1, p_2, p_3 \in W_0$, $p_1 = (0,0)$, $p_2 = (1,1)$ and $p_3 = (-1,-1)$ by the action of the automorphism group of $\mathbb{F}_0$.

We define $\mathcal{T}$ by

$$\mathcal{T} := \left\{ \sum_{k=0}^{4} a_k x^k \left(\frac{\partial}{\partial x}\right)^2 + \sum_{0\le k,\, l\le 2} b_{kl} x^k y^l \frac{\partial}{\partial x}\frac{\partial}{\partial y} + \sum_{k=0}^{4} c_k y^k \left(\frac{\partial}{\partial y}\right)^2 \,\middle|\, a_k, b_{kl}, c_k \in \mathbb{C} \right\}.$$

It is easy to show that any $\theta \in \mathcal{T}$ can be extended to a holomorphic global section of $\mathrm{Sym}^2 T_{\mathbb{F}_0}$.

By Lemma 4.7, for a holomorphic section

$$\theta = a(x)\left(\frac{\partial}{\partial x}\right)^2 + b(x,y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + c(y)\left(\frac{\partial}{\partial y}\right)^2 \in \mathcal{T},$$

the section θ can be lifted to the section of $\mathrm{Sym}^2 T_{\mathbb{F}_0}$ if and only if the following are holomorphic with respect to $(s,t) \in \mathbb{C}\times\mathbb{C}$:

$$\frac{1}{s}\big(-2a(s+\alpha, st+\beta)t + b(s+\alpha, st+\beta)\big),$$
$$\frac{1}{s^2}\big(a(s+\alpha, st+\beta)t^2 - b(s+\alpha, st+\beta)t + c(s+\alpha, st+\beta)\big),$$

for $(\alpha, \beta) = (0,0), (1,1), (-1,-1)$.

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Here we set

$$\theta_1 = y^2(x^2-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y^2(y^2-1)\left(\frac{\partial}{\partial y}\right)^2,$$
$$\theta_2 = x^2(x^2-1)\left(\frac{\partial}{\partial x}\right)^2 + x^2(y^2-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y},$$
$$\theta_3 = (x-y)^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}.$$

Then $\pi^*\theta_1$, $\pi^*\theta_2$ and $\pi^*\theta_3$ can be extended to global holomorphic sections of $\mathrm{Sym}^2 T_X$. For a general point $q \in W_0$, it is easy to see that $\theta_1(q)$, $\theta_2(q)$ and $\theta_3(q)$ give a basis of $\mathrm{Sym}^2 T_{W_0,q}$. Therefore $\mathrm{Sym}^2 T_X$ is generically globally generated. □

As a preliminary of the proof for $\mathbb{F}_n$, we prove the following claim. We regard the Hirzebruch surface $\mathbb{F}_n$ for $n \geq 1$ as the hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$:

$$\mathbb{F}_n = \{([X_1 : X_2], [Y_0 : Y_1 : Y_2]) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid Y_1 X_2^n = Y_2 X_1^n\}.$$

We set $U = \{Y_1 \neq 0 \text{ or } Y_2 \neq 0\}$. We first observe the automorphism group of $\mathbb{F}_n$ so that three general points move to specific points, which makes our computation not so hard.

Claim 4.9. *Three general points $p_1, p_2, p_3 \in U$ move to $([1:0],[1:1:0]), ([1:1],[1:1:1]), ([1:-1],[1:1:(-1)^n])$ by the action of the automorphism group of $\mathbb{F}_n$.*

Proof. Let S, T be variables and P_n be a vector subspace of homogeneous polynomials of degree n in $\mathbb{C}[S,T]$. The linear group $\mathrm{GL}(2,\mathbb{C})$ acts on P_n as follows: for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$ and any $\sum_{k=0}^n a_k S^k T^{n-k} \in P_n$, we define the action by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \left(\sum_{k=0}^n a_k S^k T^{n-k}\right) := \sum_{k=0}^n a_k (aS+bT)^k (cS+dT)^{n-k}.$$

This induces the semidirect product $G_n := P_n \rtimes \mathrm{GL}(2,\mathbb{C})$.

For any $g = \left(\sum_{k=0}^n a_k S^k T^{n-k}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in G_n$, we define the action of $\mathbb{F}_n$ as follows: for any $q = ([X_1 : X_2], [Y_0 : Y_1 : Y_2]) \in \mathbb{F}_n$, we define $g(q)$ by

$$([aX_1 + bX_2 : cX_1 + dX_2], [Y_0 X_1^n + Y_1 \sum_{k=0}^n a_k X_1^k X_2^{n-k} : Y_1(aX_1+bX_2)^n : Y_1(cX_1+dX_2)^n])$$

if $X_1 \neq 0$ and by

$$([aX_1 + bX_2 : cX_1 + dX_2], [Y_0 X_2^n + Y_2 \sum_{k=0}^n a_k X_1^k X_2^{n-k} : Y_2(aX_1+bX_2)^n : Y_2(cX_1+dX_2)^n])$$

if $X_2 \neq 0$ (see [6, Section 6.1], [15, Theorem 4.10]).

Note that the ruling $\phi : \mathbb{F}_n \to \mathbb{P}^1$ coincides with the first projection. We may assume that p_1, p_2, $p_3 \in U$, and that their images in $\mathbb{P}^1$ are different from each other. By the

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action of $g=\left(0,\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)$, we obtain

$$\phi(g(p_1))=[1:0],\quad \phi(g(p_2))=[1:1],\quad \phi(g(p_3))=[1:-1]$$

if we properly choose g. Therefore we may assume

$$p_1=([1:0],[x_1:y_1:0]), p_2=([1:1],[x_2:y_2:y_2]), p_3=([1:-1],[x_3:y_3:(-1)^n y_3]).$$

It follows that $y_k\neq 0$ for $k=1,2,3$, since we have $g\cdot U\subset U$ for any $g\in G_n$.

In the case of $n=1$, we set

$$a=\frac{x_1}{y_1}-\frac{x_2}{2y_2}-\frac{x_3}{2y_3},\quad a_0=-\frac{x_2}{2y_2}-\frac{x_3}{2y_3},\quad a_1=\frac{x_1}{y_1}-\frac{x_2}{y_2}.$$

Then p_1, p_2 and p_3, respectively, move to $([1:0],[1:1:0])$, $([1:1],[1:1:1])$ and $([1:-1],[1:1:-1])$ by the action of $\left(a_0 S+a_1 T,\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right)\in G_1$, since we may assume $x_1/y_1-x_2/2y_2-x_3/2y_3\neq 0$, since p_1,p_2,p_3 are general points.

In the case of $n\geq 2$, we set $m=2\lfloor n/2\rfloor$,

$$a_0=\frac{x_1-y_1}{y_1},\quad a_1=-\frac{x_2-y_2}{2y_2}+\frac{x_3+y_3}{2y_3},\quad a_m=-\frac{x_1-y_1}{y_1}-\frac{x_2-y_2}{2y_2}-\frac{x_3+y_3}{2y_3}$$

and $a_k=0$ for $k\neq 0,1,m$. Then p_1, p_2 and p_3, respectively move to $([1:0],[1:1:0])$, $([1:1],[1:1:1])$ and $([1:-1],[1:1:(-1)^n])$ by the action of $\left(\sum_{k=0}^n a_k S^k T^{n-k},\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\right)\in G_n$. □

Proof of Proposition 4.8 for $\mathbb{F}_n$. (1) We define the Zariski open sets $W_k\cong\mathbb{C}\times\mathbb{C}$ in $\mathbb{F}_n$ for $k=1,2,3$ as follows:

$$\begin{array}{rcl} i_1: W_1 & \to & \mathbb{F}_n \\ (x,y) & \mapsto & ([1:x],[1:y:x^n y]), \end{array}$$

$$\begin{array}{rcl} i_2: W_2 & \to & \mathbb{F}_n \\ (u,v) & \mapsto & ([1:u],[v:1:u^n]), \end{array}$$

$$\begin{array}{rcl} i_3: W_3 & \to & \mathbb{F}_n \\ (\zeta,\eta) & \mapsto & ([\zeta:\eta],[1:\zeta^n\eta:\eta]). \end{array}$$

We take $\theta=a(x,y)\partial/\partial x+b(x,y)\partial/\partial y\in H^0(W_1,T_{W_1})$. The section θ extends to a holomorphic global section of $T_{\mathbb{F}_n}$ if and only if θ is holomorphic on W_2 and W_3, since the codimension of $\mathbb{F}_n\setminus\cup_{k=1,2,3}W_k$ is 2. A straightforward computation yields

$$\theta=a(u,1/v)\frac{\partial}{\partial u}-v^2 b(u,1/v)\frac{\partial}{\partial v}\quad \text{on } W_1\cap W_2,$$
$$\theta=-\zeta^2 a(1/\zeta,\zeta^n\eta)\frac{\partial}{\partial\zeta}+\left(n\zeta\eta a(1/\zeta,\zeta^n\eta)+\frac{b(1/\zeta,\zeta^n\eta)}{\zeta^n}\right)\frac{\partial}{\partial\eta}\quad \text{on } W_1\cap W_3.$$

Hence the section θ can be extended to a global holomorphic section of $T_{\mathbb{F}_n}$ if and only if we define $a(x,y)$ and $b(x,y)$ to be

$$a(x,y) = a_0 + a_1 x + a_2 x^2 \quad \text{and} \quad b(x,y) = (b_0 - n a_2 x) y + b_1(x) y^2$$

for some $a_0, a_1, a_2, b_0 \in \mathbb{C}$ and some $b_1(x) \in \mathbb{C}[x]$ with $\deg(b_1) \leq n$. We define

$$\mathcal{T} := \left\{ (a_0 + a_1 x + a_2 x^2) \frac{\partial}{\partial x} + (b_0 y - n a_2 x y + b_1 y^2 + b_2 x y^2) \frac{\partial}{\partial y} \,\middle|\, a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{C} \right\}.$$

Then it can be seen that any $\theta \in \mathcal{T}$ extends to a holomorphic global section of $T_{\mathbb{F}_n}$.

Let $\pi : X \to \mathbb{F}_n$ be the blowup of $\mathbb{F}_n$ along two general points p_1, p_2. By Claim 4.9, we may assume $p_1, p_2 \in W_1$, $p_1 = (0,1)$ and $p_2 = (1,1)$. We choose θ_1 and θ_2 in $\mathcal{T}$ as follows:

$$\theta_1 = y(y-1) \frac{\partial}{\partial y} \quad \text{and} \quad \theta_2 = x(x-1) \frac{\partial}{\partial x} + n x y (y-1) \frac{\partial}{\partial y}.$$

By Lemma 4.7, the sections θ_1 and θ_2 can be lifted to holomorphic global sections of T_X. For any point $q = (x,y) \in W_1$ such that $x \neq 0,1$ and $y \neq 0,1$, $(\theta_1)_q$ and $(\theta_2)_q$ give a basis of $T_{W_1,q}$. Therefore T_X is generically globally generated.

(2) Let $\pi : X \to \mathbb{F}_n$ be a blowup of $\mathbb{F}_n$ along three general points p_1, p_2, p_3. We show that $\mathrm{Sym}^2(T_X)$ is generically globally generated. By Claim 4.9, we may assume $p_1, p_2, p_3 \in W_1$, $p_1 = (0,1)$, $p_2 = (1,1)$ and $p_3 = (-1,-1)$.

We take

$$\theta = a(x,y) \left(\frac{\partial}{\partial x} \right)^2 + b(x,y) \frac{\partial}{\partial x} \frac{\partial}{\partial y} + c(x,y) \left(\frac{\partial}{\partial y} \right)^2 \in H^0(W_1, \mathrm{Sym}^2 T_{W_1}).$$

First we investigate the condition when θ extends to a global holomorphic section of $\mathrm{Sym}^2 T_{\mathbb{F}_n}$. We have

$$\theta = a(u,1/v) \left(\frac{\partial}{\partial u} \right)^2 - v^2 b(u,1/v) \frac{\partial}{\partial u} \frac{\partial}{\partial v} + v^4 c(u,1/v) \left(\frac{\partial}{\partial v} \right)^2 \text{ on } W_1 \cap W_2 \text{ and,}$$

$$\begin{aligned} \theta = {} & \zeta^4 a(1/\zeta, \zeta^n \eta) \left(\frac{\partial}{\partial \zeta} \right)^2 + \left(-2n\zeta^3 \eta a(1/\zeta, \zeta^n \eta) - \frac{1}{\zeta^{n-2}} b(\zeta, \zeta^n \eta) \right) \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \eta} \\ & + \left(n^2 \zeta^2 \eta^2 a(1/\zeta, \zeta^n \eta) + \frac{n\eta}{\zeta^{n-1}} b(1/\zeta, \zeta^n \eta) + \frac{1}{\zeta^{2n}} c(1/\zeta, \zeta^n \eta) \right) \left(\frac{\partial}{\partial \eta} \right)^2 \text{ on } W_1 \cap W_3. \end{aligned}$$

In the case of $n = 1$, the section θ extends to a global holomorphic section of $\mathrm{Sym}^2 T_{\mathbb{F}_n}$ if we have

- $a(x,y) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$,
- $b(x,y) = (b_0 + b_1 x + b_2 x^2 - 2a_4 x^3) y + (b_3 + b_4 x + b_5 x^2 + b_6 x^3) y^2$,
- $c(x,y) = (c_0 - (a_3 + b_2) x + a_4 x^2) y^2 + (c_1 + c_2 x - b_6 x^2) y^3 + (c_3 + c_4 x + c_5 x^2 + c_6 x^3 + c_7 x^4) y^4$,

where all coefficients are constant. Here we set

- $$\theta_1 = x(x^2-1) \left(\frac{\partial}{\partial x} \right)^2 + y(-3x^2 + y(x^3 + x^2 + x - 1) + 1) \frac{\partial}{\partial x} \frac{\partial}{\partial y} + y^2 (2x + y^2 (x^2+1) - y(x+1)^2) \left(\frac{\partial}{\partial y} \right)^2,$$

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- $\theta_2 = x^2(-x^2+1)\left(\frac{\partial}{\partial x}\right)^2 + 2x^2y(x-y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + x^2y^2(y^2-1)\left(\frac{\partial}{\partial y}\right)^2$,
- $\theta_3 = x(-x^2+1)\left(\frac{\partial}{\partial x}\right)^2 + y(3x^2+y(-x^2-2x+1)-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y}$
  $+ y^2(-2x+y^2(2x-1)+1)\left(\frac{\partial}{\partial y}\right)^2$.

Then using Lemma 4.7 again, the sections $\pi^*\theta_1$, $\pi^*\theta_2$ and $\pi^*\theta_3$ extend to holomorphic global sections of $\mathrm{Sym}^2\, T_X$. For a general point $q \in W_1$, $\theta_1(q)$, $\theta_2(q)$ and $\theta_3(q)$ give a basis of $\mathrm{Sym}^2\, T_{W_1,q}$. Therefore $\mathrm{Sym}^2\, T_X$ is generically globally generated.

In the case where $n \geq 2$, the section θ extends to a holomorphic global section of $\mathrm{Sym}^2\, T_{\mathbb{F}_n}$ if

- $a(x,y) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$,
- $b(x,y) = (b_0 + b_1x + b_2x^2 - 2na_4x^3)y + (b_3 + b_4x + b_5x^2 + b_6x^3)y^2$,
- $c(x,y) = (c_0 - (n^2a_3 + nb_2)x + n^2a_4x^2)y^2 + (c_1 + c_2x + c_3x^2)y^3 + (c_4 + c_5x + c_6x^2 + c_7x^3 + c_8x^4)y^4$,

where all coefficients are constant. We set

- $\theta_1 = xy^2(x^2-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y^3(-3x^2+y(-x^4+2x^3+2x^2-1)+1)\left(\frac{\partial}{\partial y}\right)^2$,
- $\theta_2 = xy^2(x^2-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y^2(xy^2(x+2)-y(x+1)^2+1)\left(\frac{\partial}{\partial y}\right)^2$,
- $\theta_3 = x(x^3-2x^2-x+2)\left(\frac{\partial}{\partial x}\right)^2$

$$+ y(-2nx^3+6x^2+2x(n-1)-2+y(nx(n-6)+x^3(-n^2+6n-4)+2))\frac{\partial}{\partial x}\frac{\partial}{\partial y}$$

$$+ y^2(nx(nx+2n-6)+2n+1+y(-n^2(x+1)^2+y(n^2+6nx-2n-1)))\left(\frac{\partial}{\partial y}\right)^2.$$

Then $\pi^*\theta_1$, $\pi^*\theta_2$ and $\pi^*\theta_3$ extend to holomorphic global sections of $\mathrm{Sym}^2\, T_X$. For a general point $q \in W_1$, $\theta_1(q)$, $\theta_2(q)$ and $\theta_3(q)$ give a basis of $\mathrm{Sym}^2\, T_{W_1,q}$. Therefore $\mathrm{Sym}^2\, T_X$ is generically globally generated. □

All surfaces with pseudo-effective tangent bundle can be classified except for the blowup of Hirzebruch surfaces at special points if the following problem is solved:

Problem 4.10. Does the blowup of Hirzebruch surfaces at four general points have pseudo-effective tangent bundle?

Remark 4.11. Höring, Liu and Shao found some examples of pseudo-effective tangent bundles in [22], by a method based on varieties of minimal rational tangents, which is rather different from our method. Note that their definition of a pseudo-effective tangent bundle is weaker than ours.

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