Probability in the Engineering and Informational Sciences, **25**, 2011, 187–204.
doi:10.1017/S0269964810000343

ON STOCHASTIC AND AGING PROPERTIES OF GENERALIZED ORDER STATISTICS

Mahdi Tavangar and Majid Asadi
Department of Statistics
University of Isfahan
Isfahan, 81744, Iran
E-mails: m.tavangar@stat.ui.ac.ir; m.asadi@stat.ui.ac.ir

The generalized order statistics (GOS) model is a unified model that contains the well-known ordered random data such as order statistics and record values. In the present article, we investigate some stochastic ordering results and aging properties of the conditional GOS. The results of the article subsume some of the existing results, which recently are obtained in the literature, on conditional GOS. In particular, our results hold for the model of progressively type II right censored order statistics without any restriction on the censoring scheme.

1. INTRODUCTION

In the recent years, several results have appeared in the literature that deal with the stochastic comparisons and aging properties of the residual lifetime or the inactivity time of order statistics, record values, and other models of ordered random variables. Asadi and Bayramoglu [4,5], Li and Zhao [22,23], Asadi [3], Khaledi and Shaked [20], and Tavangar and Asadi [30] are among the researchers who investigated properties of conditional order statistics. The stochastic properties of the conditional record values are studied by several authors such as Khaledi and Shojaei [21], Raqab and Asadi [26,27], Asadi and Raqab [6], and Tavangar and Asadi [29]. Franco, Ruiz, and Ruiz [14], Belzunce, Mercader, and Ruiz [9], Hu and Zhuang [17], Hu, Jin, and Khaledi [16], Xie and Hu [32], Zhao and Balakrishnan [34], and Hashemi, Tavangar, and Asadi [15] have generalized some ordering results on order statistics and record values to arrive at the generalized order statistics (GOS) under some restrictions on the parameters of the model.

 0269-9648/11 $25.00

This article is an attempt to generalize some of the existing results in the literature on GOS to compare the residual lifetime or the inactivity time of GOS under less restrictions on the parameters of the GOS model. In the process of doing so, we also provide some results on the aging properties of conditional GOS. As we avoid restrictions that the previous studies put on the parameters of the GOS model, it will be shown that our findings yield new results for various useful models of ordered random variables, such as the sequential order statistics and Pfeifer record model. Some interesting comparisons on the residual lifetime and the inactivity time of progressively type II right censored order statistics, with an arbitrary censoring scheme are also established.

The concept of GOS has been introduced by Kamps [18]. He showed that order statistics, record values, progressively type II right censored order statistics, and some other ordered random variables can be considered as special cases of the GOS. For more details on properties of these models, readers can refer to Kamps [18], Arnold, Balakrishnan, and Nagaraja [1,2], David and Nagaraja [13], and Balakrishnan and Aggarwala [7]. Let $F(x)$ be an absolutely continuous distribution function with density function $f(x)$. Also let $\bar{F}(x) = 1 - F(x)$ be the survival function. The random variables

$$X(1, n, \tilde{m}, k), X(2, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$$

are called the GOS based on F if the joint density function is given by

$$\begin{aligned} &f_{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, \ldots, x_n) \\ &\quad = k \left[\prod_{i=1}^{n-1} \gamma_j\right] \left[\prod_{i=1}^{n-1} \{\bar{F}(x_i)\}^{m_i} f(x_i)\right] \left\{\bar{F}(x_n)\right\}^{k-1} f(x_n), \\ &\quad F^{-1}(0) < x_1 \le x_2 \le \cdots \le x_n < F^{-1}(1-), \end{aligned}$$

where $n \in \mathbb{N}, k > 0$ and $m_1, m_2, \ldots, m_{n-1} \in \mathbb{R}$ such that $\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j \ge 1$ for all $r \in \{1, 2, \ldots, n-1\}$ and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1})$ if $n \ge 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary if $n = 1$). When F is uniform on (0,1), the GOS are denoted by $U(1, n, \tilde{m}, k), U(2, n, \tilde{m}, k), \ldots, U(n, n, \tilde{m}, k)$. It is easy to show that

$$(U(1, n, \tilde{m}, k), \ldots, U(n, n, \tilde{m}, k)) \stackrel{\mathrm{d}}{=} (F(X(1, n, \tilde{m}, k)), \ldots, (F(X(n, n, \tilde{m}, k))), \quad \textbf{(1)}$$

where $\stackrel{\mathrm{d}}{=}$ stands for the equality in distribution. The joint density function of the first r ($r < n$) GOS based on F is then given by

$$\begin{aligned} &f_{X(1,n,\tilde{m},k),\ldots,X(r,n,\tilde{m},k)}(x_1, \ldots, x_r) \\ &\quad = c_{r-1} \left[\prod_{i=1}^{r-1} \{\bar{F}(x_i)\}^{m_i} f(x_i)\right] \left\{\bar{F}(x_r)\right\}^{\gamma_r - 1} f(x_r), \\ &\quad F^{-1}(0) < x_1 \le x_2 \le \cdots \le x_r < F^{-1}(1-), \end{aligned} \quad \textbf{(2)}$$

where $c_{r-1} = \prod_{j=1}^{r} \gamma_j$.

https://doi.org/10.1017/S0269964810000343 Published online by Cambridge University Press

Recently, Tavangar and Asadi [31] have presented an expression for the joint density function of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$. For $u \in (0,1)$, we define $\xi_1 \equiv 1$ and

$$\xi_r(u) = \int_{\mathcal{A}_{r-1}} \prod_{i=1}^{r-1} (1-u_i)^{m_i}\, du_1\, du_2 \cdots du_{r-1}, \qquad r = 2,3,\ldots,n,$$

where $\mathcal{A}_{r-1} = \{(u_1,u_2,\cdots,u_{r-1}) : 0 \le u_1 \le \cdots \le u_{r-1} \le u\}$. Then, using (2), we can obtain the marginal density function of the rth GOS as

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1} f(x)\{1-F(x)\}^{\gamma_r - 1} \xi_r(F(x)), \qquad -\infty < x < \infty. \tag{3}$$

Now, the joint density function of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$, without any restriction on $\tilde{m}$ can be written as

$$\begin{aligned} f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(u,v) &\\ = c_{s-1}\{1-F(u)\}^{\gamma_r-\gamma_s-1} \xi_r(F(u))\{1-F(v)\}^{\gamma_{s-1}} &\\ \times \Psi_{s-r-1}\left(\frac{1-F(v)}{1-F(u)}\right) f(u) f(v), \qquad u < v, & \end{aligned} \tag{4}$$

where

$$\Psi_1(v) = g_{m_{r+1}}(1-v)$$

and for $\ell = 2,3,\ldots,$

$$\Psi_l(v) = \int_{\mathcal{B}} g_{m_{r+1}}(1-u_1) \left[\prod_{i=1}^{l-1} u_i^{m_{r+i+1}}\right] du_1 \cdots du_{l-2}\, du_{l-1}, \tag{5}$$

where $\mathcal{B} = \{(u_1,u_2,\ldots,u_{l-1}) : v \le u_{l-1} \le \cdots \le u_2 \le u_1 < 1\}$, and for $x \in [0,1)$,

$$g_m(x) = \begin{cases} \left[1-(1-x)^{m+1}\right]/(m+1) & \text{if } m \ne -1 \\ -\log(1-x), & \text{if } m = -1. \end{cases}$$

Before giving the main results, we first recall some aging concepts and stochastic orders that are pertinent to the developments of the article.

Definition 1.1:

(a) *A lifetime distribution with survival function $\bar{F}$ is said to belong to the class of increasing failure rate (IFR) (decreasing failure rate (DFR)) distributions if $\bar{F}(x+t)/\bar{F}(t)$ is decreasing (increasing) in t for each $x > 0$. Distributions with a Lebesgue density belong to the IFR (DFR) class if and only if their hazard rates $\lambda(t) = f(t)/\bar{F}(t)$ are increasing (decreasing).*

https://doi.org/10.1017/S0269964810000343 Published online by Cambridge University Press

(b) *A lifetime distribution with survival function $\bar{F}$ is said to be new better than used (NBU) (new worse than used (NWU)) if for every $t, x \geq 0$, we have $\bar{F}(t+x) \leq (\geq) \bar{F}(t)\bar{F}(x)$.*

For more details on these concepts, we refer the reader to Barlow and Proschan [8].

DEFINITION 1.2: *Let X and Y be two random variables with density functions f and g, distribution functions F and G, and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. As the ratios in the following statements are well defined, X is said to be less than Y in*

(a) *likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing in x;*
(b) *hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;*
(c) *reversed hazard rate order (denoted by $X \leq_{\mathrm{rh}} Y$) if $G(x)/F(x)$ is increasing in x;*
(d) *stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\bar{G}(x) \geq \bar{F}(x)$.*

For a comprehensive discussion on these stochastic orders, we refer the reader to Shaked and Shanthikumar [28] and Müller and Stoyan [25]. It is important to mention that when the supports of X and Y have a common left end point, we have the following chain of implication among the above stochastic orders:

$$X \leq_{\mathrm{lr}} Y \Rightarrow X \leq_{\mathrm{hr}} Y (X \leq_{\mathrm{rh}} Y) \Rightarrow X \leq_{\mathrm{st}} Y.$$

DEFINITION 1.3: *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors with density functions f and g, respectively. Then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the multivariate likelihood ratio order (denoted by $\boldsymbol{X} \leq_{\mathrm{lr}} \boldsymbol{Y}$) if ($\wedge$ and $\vee$ denote respectively the minimum and the maximum operations)*

$$\begin{aligned} &f(x_1, x_2, \ldots, x_n) g(y_1, y_2, \ldots, y_n) \\ &\quad \leq f(x_1 \wedge y_1, x_2 \wedge y_2, \cdots, x_n \wedge y_n) g(x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n), \end{aligned}$$

for all $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ (see Shaked and Shanthikumar [28] and references therein).

DEFINITION 1.4: *A nonnegative function $h(x, y)$ is said to be totally positive of order 2 (TP_2) (reverse regular of order 2) (RR_2)) if for $x_1 < x_2$ and $y_1 < y_2$,*

$$h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1) \geq 0 \ (\leq 0).$$

For more details and applications of these concepts, we refer to Karlin [19].

We have organized the article as follows. Section 2 is devoted to stochastic comparisons of residual lifetime and inactivity time of GOS when the sampling is based on one population and two populations. In Section 3 we investigate the aging properties of GOS. It is shown, for example, that when the underlying distribution has an increasing failure rate, then, under some mild conditions, the residual lifetime of the GOS are stochastically decreasing in time. We will show that the results of the article subsume some of the results that have recently been published in the literature.

Throughout the article, "increasing" stands for "nondecreasing" and "decreasing" stands for "nonincreasing." Additionally, for any random variable W, f_W and $\bar{F}_W$ denote the probability density function and the survival function of W, respectively.

2. STOCHASTIC COMPARISONS

In the following, we first prove a theorem in which in part (a) we given an extended version of part (a) of Theorem 3.1 in Hu and Zhuang [17] and in parts (b) and (c) we prove a different version of parts (b) and (c) of Theorem 3.1 of [17]. For our derivation, which we use in the next section, we consider a parameter vector $\tilde{\mu}$ in the GOS that is different from that used by Hu and Zhuang [17].

THEOREM 2.1: *Let $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \dots, X(n,n,\tilde{m},k)$ be the GOS based on a continuous distribution function $F(x)$ and assume that $\tilde{\mu} = (m_2, \dots, m_{n-1}), n \in \mathbb{N}$. Then*

(a) $X(r-1,n,\tilde{m},k) \le_* X(r,n,\tilde{m},k), r = 2,3,\dots,n$;
(b) $X(r,n,\tilde{m},k) \le_* X(r,n-1,\tilde{\mu},k), r = 1,2,\dots,n-1$, if $m_i \ge -1$ for each $i = 1,2,\dots,r$,
(c) $X(r-1,n-1,\tilde{\mu},k) \le_* X(r,n,\tilde{m},k), r = 2,3,\dots,m$,

where the order "$\le_*$" is "$\le_{\rm lr}$" if F is absolutely continuous and is "$\le_{\rm hr}$" or "$\le_{\rm rh}$" if F is continuous.

PROOF: Let $\tilde{\mu} = (\mu_1, \dots, \mu_{n-2})$. For fixed $n \in \mathbb{N}$, define

$$\gamma_{j,n-1}^{\tilde{\mu}} = k + \sum_{i=j}^{n-2}(\mu_i + 1), \qquad 1 \le j \le n-2,$$

$$\gamma_{j,n}^{\tilde{m}} = k + \sum_{i=j}^{n-1}(m_i + 1), \qquad 1 \le j \le n-1,$$

and $\gamma_{n-1,n-1}^{\tilde{\mu}} = \gamma_{n,n}^{\tilde{m}} = k$. Let $\{B_{i,n-1}^{\tilde{\mu}},\ i = 1,2,\dots,n-1\}$ and $\{B_{i,n}^{\tilde{m}},\ i = 1,2,\dots,n\}$ be two independent sequences of random variables such that $B_{i,n-1}^{\tilde{\mu}}$ and $B_{i,n}^{\tilde{m}}$ have exponential distributions with parameters $\gamma_{i,n-1}^{\tilde{\mu}}$ and $\gamma_{i,n}^{\tilde{m}}$, respectively. It is easy to show that $\gamma_{j,n-1}^{\tilde{\mu}} = \gamma_{j+1,n}^{\tilde{m}}$, $j = 1,2,\dots,n-1$, and $\gamma_{j,n}^{\tilde{m}} = \gamma_{j,n-1}^{\tilde{\mu}} + (m_j + 1)$, $j = 1,2,\dots,n-1$.

Hence, $B^{\tilde{\mu}}_{j,n-1} \overset{\mathrm{d}}{=} B^{\tilde{m}}_{j+1,n}$, $j = 1, 2, \ldots, n-1$, and $B^{\tilde{m}}_{j,n} \leq_{\mathrm{lr}} B^{\tilde{\mu}}_{j,n-1}$, $j = 1, 2, \ldots, n-1$, if $m_j \geq -1$. The first result implies that

$$\sum_{j=1}^{r-1} B^{\tilde{\mu}}_{j,n-1} \overset{d}{=} \sum_{j=2}^{r} B^{\tilde{m}}_{j,n}. \tag{6}$$

Since the exponential distribution has a logconcave density, by Theorem 1.C.9 of Shaked and Shanthikumar [28], we have

$$\sum_{j=1}^{r-1} B^{\tilde{m}}_{j,n} \leq_{\mathrm{lr}} \sum_{j=1}^{r} B^{\tilde{m}}_{j,n}, \qquad r = 2, 3, \ldots, n-1, \tag{7}$$

$$\sum_{j=1}^{r} B^{\tilde{m}}_{j,n} \leq_{\mathrm{lr}} \sum_{j=1}^{r} B^{\tilde{\mu}}_{j,n-1}, \qquad r = 1, 2, \ldots, n-1, \quad \text{if } m_j \geq -1$$
$$\text{for } j = 1, 2, \ldots, r, \tag{8}$$

and

$$\sum_{j=2}^{r} B^{\tilde{m}}_{j,n} \leq_{\mathrm{lr}} \sum_{j=1}^{r} B^{\tilde{m}}_{j,n}.$$

It follows from (6) and the last inequality that

$$\sum_{j=1}^{r-1} B^{\tilde{\mu}}_{j,n-1} \leq_{\mathrm{lr}} \sum_{j=1}^{r} B^{\tilde{m}}_{j,n}. \tag{9}$$

From the fact that the likelihood ratio order implies both the hazard rate order and the reversed hazard rate order, it is obvious that one can replace the order "$\leq_{\mathrm{lr}}$" by the orders "$\leq_{\mathrm{hr}}$" and "$\leq_{\mathrm{rh}}$" in relations (7), (8) & (9). Now, the required result follows from Cramer and Kamps [12] (see also Cramer and Kamps [11]) and the facts that the likelihood ratio order, the hazard rate order, and the reversed hazard rate order are closed under increasing transformations (see Theorems 1.C.8, 1.B.2, and 1.B.43 in Shaked and Shanthikumar [28]). ■

It should be mentioned here that part (a) of Theorem 2.1 is proved in Hu and Zhuang [17] under the assumption that $m_j \geq -1$ for each j and parts (b) and (c) are proved by Hu and Zhuang [17] under the condition that $\tilde{\mu} = (m_1, \ldots, m_{n-2})$. We will use this theorem in Section 3 to study some aging and monotonicity properties of the conditional GOS.

Assume that X and Y are two continuous random variables with the distribution functions F and G, the density functions f and g, and the hazard rates λ_1 and λ_2, respectively. Further, let $X(r, n, \tilde{m}, k)$ and $Y(r, n, \tilde{m}', k')$ denote the GOS based on F and G, with parameters k and m_i, and k' and m'_i, $i = 1, 2, \ldots, n-1$, respectively. The following theorem extends a result of Zhao and Balakrishnan [34] under more general assumptions on the parameters of the model of GOS.

THEOREM 2.2: *Let $k \geq k'$ and $m_i \geq m_i'$ for all $i = 1, 2, \ldots, n-1$. Suppose that (a) $m_i \geq 0$ or $m_i' \geq 0$ for all $i = 1, 2, \ldots, n-1$ and $X \leq_{\rm lr} Y$ or (b) $m_i \geq -1$ or $m_i' \geq -1$ for all $i = 1, 2, \ldots, n-1$, $X \leq_{\rm hr} Y$, and $\lambda_2(x)/\lambda_1(x)$ is increasing in x. Then for $1 \leq r \leq s \leq n$ and any $t \in \mathbb{R}$,*

(i) $[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t] \leq_{\rm lr} [Y(s,n,\tilde{m}',k') - t | Y(r,n,\tilde{m}',k') > t]$;

(ii) $[t - X(r,n,\tilde{m},k) | X(s,n,\tilde{m},k) \leq t] \geq_{\rm lr} [t - Y(r,n,\tilde{m}',k') | Y(s,n,\tilde{m}',k') \leq t]$.

PROOF: We only prove part (i). The proof of part (ii) can be established similarly. First note that by Theorem 3.10 in Belzunce et al. [9], we have

$$(X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)) \\ \leq_{\rm lr} (Y(1,n,\tilde{m}',k'), Y(2,n,\tilde{m}',k'), \ldots, Y(n,n,\tilde{m}',k')).$$

Since multivariate likelihood ratio order is closed under marginalization (see Theorem 6.E.4 in Shaked and Shanthikumar [28]), we get, for $1 \leq r \leq n$,

$$X(r,n,\tilde{m},k) \leq_{\rm lr} Y(r,n,\tilde{m}',k')$$

and for $1 \leq r < s \leq n$,

$$(X(r,n,\tilde{m},k), X(s,n,\tilde{m},k)), \leq_{\rm lr} (Y(r,n,\tilde{m}',k'), Y(s,n,\tilde{m}',k'))$$

The first result implies that

$$[X(r,n,\tilde{m},k) | X(r,n,\tilde{m},k) > t] \leq_{\rm lr} [Y(r,n,\tilde{m}',k') | Y(r,n,\tilde{m}',k') > t]$$

(see Theorem 1.C.6 in Shaked and Shanthikumar [28]), whereas the second one implies that

$$[X(s,n,\tilde{m},k) | X(r,n,\tilde{m},k) > t] \leq_{\rm lr} [Y(s,n,\tilde{m}',k') | Y(r,n,\tilde{m}',k') > t]$$

(see Theorem 6.E.1 in Shaked and Shanthikumar [28] from which the required result follows immediately. ■

REMARK 2.3: The condition that $\lambda_2(x)/\lambda_1(x)$ be increasing in Theorem 2.2 arises in reliability theory naturally. Consider a series system with two independent components. If X and Y denote the lifetimes of the components, then clearly the lifetime of the system is $T = \min(X, Y)$. Cha and Mi [10] have considered the probability function $p(t) = P(Y = T | T = t)$—the probability that component Y causes the system failure given that the system fails at time t. They showed that $\lambda_2(x)/\lambda_1(x)$ is increasing if and only if $p(t)$ is increasing function of t. If we define $R(t) = P(Y = T | T > t)$, then $R(t)$ shows the probability that the component with lifetime Y causes the system failure given that the system has survived up to time t. Zardasht and Asadi [33] showed that if $\lambda_2(x)/\lambda_1(x)$ is increasing, then $R(t)$ is increasing in t.

https://doi.org/10.1017/S0269964810000343 Published online by Cambridge University Press

The next corollary follows from Theorem 2.2.

COROLLARY 2.4: *Let* $k \geq k'$ *and* $m_i \geq m_i'$ *for all* $i = 1, 2, \ldots, n-1$. *Suppose that* $m_i \geq -1$ *or* $m_i' \geq -1$ *for all* $i = 1, 2, \ldots, n-1$. *Then for* $1 \leq r \leq s \leq n$ *and any* $t \in \mathbb{R}$,

(a) $[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t] \leq_{\text{lr}} [X(s,n,\tilde{m}',k') - t | X(r,n,\tilde{m}',k') > t]$;

(b) $[t - X(r,n,\tilde{m},k) | X(s,n,\tilde{m},k) \leq t] \geq_{\text{lr}} [t - X(r,n,\tilde{m}',k') | X(s,n,\tilde{m}',k') \leq t]$.

Our assumptions on the parameters of GOS model enable us to get a result for progressively censored data, as a corollary to Theorem 2.2, which we have not seen in the literature so far. Progressive censoring is an important mechanism of data collection in reliability. Let $X_1, X_2, \ldots, X_n$ denote the failure times of n independent and identically distributed (i.i.d.) items that are placed on a life test. Suppose that $R_1, R_2, \ldots, R_m$ are some fixed nonnegative integers such that $\sum_{\nu=1}^{m} R_\nu = n - m$. It is planned that only m failures will be observed and the remaining $n - m$ lifetimes will be censored progressively according to the censoring scheme $\tilde{R} = (R_1, R_2, \ldots, R_m)$. More specifically, at the time of the ith failure, $R_i, i = 1, 2, \ldots, m$, surviving items will be randomly withdrawn from the experiment. The resulting ordered observed failure times, denoted by $X_{1:m:n}^{\tilde{R}}, X_{2:m:n}^{\tilde{R}}, \ldots, X_{m:m:n}^{\tilde{R}}$, are called the progressively type II right censored order statistics of size m from a sample of size n with progressive censoring scheme $\tilde{R}$. These ordered random variables form a special case of GOS if we set $n = m, m_i = R_i, i = 1, 2, \ldots, m-1$, and $k = R_m + 1$ (in the model of GOS). We refer the reader to Balakrishnan and Aggarwala [7] and references therein for a comprehensive discussion and inferential procedures based on progressive censoring.

Consider an experiment with arbitrary censoring scheme $\tilde{R}$. For reliability engineers and survival analysts, it would be of interest to get information on the stochastic properties of the residual life of the items that are alive at time t. For this reason, one can consider the conditional random variable

$$[X_{s:m:n}^{\tilde{R}} - t | X_{r:m:n}^{\tilde{R}} \geq t], \qquad 1 \leq r \leq s \leq m \leq n,$$

which is the residual life to the next sth failure time given that at most $(r-1)$ failures occurred at or before time t. From Theorem 2.2 we obtain the following corollary on the residual lifetimes and the inactivity times of progressively type II right censored order statistics.

COROLLARY 2.5: *Let* $X_{1:m:n}^{\tilde{R}}, X_{2:m:n}^{\tilde{R}}, \ldots, X_{m:m:n}^{\tilde{R}}$ *be m progressively type II right censored order statistics arising from n i.i.d. nonnegative random variables distributed as X, and similarly let* $Y_{1:m:N}^{\tilde{S}}, Y_{2:m:N}^{\tilde{S}}, \ldots, Y_{m:m:N}^{\tilde{S}}$ *be m progressively type II right censored order statistics arising from N i.i.d. nonnegative random variables distributed as Y. Let* $R_i \geq S_i$ *for all* $i = 1, 2, \ldots, m-1$ *and* $X \leq_{\text{lr}} Y$. *Then, for* $1 \leq r \leq s \leq m \leq \min\{n, N\}$ *and* $t \in \mathbb{R}_+$,

(a) $[X_{s:m:n}^{\tilde{R}} - t | X_{r:m:n}^{\tilde{R}} > t] \leq_{\text{lr}} [Y_{s:m:N}^{\tilde{S}} - t | Y_{r:m:N}^{\tilde{S}} > t]$;

(b) $[t - X^{\tilde{R}}_{r:m:n} | X^{\tilde{R}}_{s:m:n} \le t] \ge_{\rm lr} [t - Y^{\tilde{S}}_{r:m:N} | Y^{\tilde{S}}_{s:m:N} \le t]$.

The next result reveals that, without any restrictions on the parameter vector $\tilde{m}$, $[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t]$ is decreasing in k in the sense of likelihood ratio order and $[t - X(r,n,\tilde{m},k) | X(s,n,\tilde{m},k) \le t]$ is increasing in k in the sense of likelihood ratio order. Before proving the theorem, we need the following lemma.

LEMMA 2.6: *Let X be a random variable with absolutely continuous survival fucntion $\bar{F}(x)$. Then $\Psi_\ell(\bar{F}(x)/\bar{F}(u))$ is TP_2 in $(u,x) \in \mathbb{R}^2$, $\ell = 1, 2, \ldots, n-1$ where Ψ_ℓ is defined in (5).*

PROOF: We prove the result by an inductive argument. In the case for which $\ell = 1$, after some algebra, one can show that

$$\Psi_1\left(\frac{\bar{F}(x)}{\bar{F}(u)}\right) = g_{m_{r+1}}\left(1 - \frac{\bar{F}(x)}{\bar{F}(u)}\right)$$

is TP_2 in $(u,x) \in \mathbb{R}^2$. Now, assume that $\Psi_{\ell-1}(\bar{F}(x)/\bar{F}(u))$ is TP_2 in $(u,x) \in \mathbb{R}^2$. After some manipulations, we can obtain the following recurrence relation:

$$\Psi_\ell\left(\frac{\bar{F}(x)}{\bar{F}(u)}\right) = \int_{\mathbb{R}} I_{\{u \le w \le x\}} \left(\frac{\bar{F}(w)}{\bar{F}(u)}\right)^{m_{r+\ell}} \Psi_{\ell-1}\left(\frac{\bar{F}(w)}{\bar{F}(u)}\right) \frac{f(w)}{\bar{F}(u)}\, dw,$$

where $f(x)$ is the density function corresponding to $\bar{F}(x)$ and I denotes the indicator function. It is easy to verify that $I_{\{u \le w \le x\}}$ is TP_2 in each pairs of its arguments when the third argument is fixed. By applying Theorem 5.1 of Karlin [19, p. 123], it follows that $\Psi_\ell(\bar{F}(x)/\bar{F}(u))$ is TP_2 in $(u,x) \in \mathbb{R}^2$ and hence we obtain the desired result. ■

THEOREM 2.7: *For $1 \le r \le s \le n, k \ge k' \ge 1, \tilde{m} \in \mathbb{R}^{n-1}$, and any $t \in \mathbb{R}$,*

(a) $[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t] \le_{\rm lr} [X(s,n,\tilde{m},k') - t | X(r,n,\tilde{m},k') > t]$;

(b) $[t - X(r,n,\tilde{m},k) | X(s,n,\tilde{m},k) \le t] \ge_{\rm lr} [t - X(r,n,\tilde{m},k') | X(s,n,\tilde{m},k') \le t]$.

PROOF: We first prove part (a). Clearly, we only need to prove that

$$[X(s,n,\tilde{m},k) | X(r,n,\tilde{m},k) > t] \le_{\rm lr} [X(s,n,\tilde{m},k') | X(r,n,\tilde{m},k') > t]$$

for $1 \le r \le s \le n, k \ge k'$, and any $t \in \mathbb{R}$. To this end, it is enough to show that the conditional density function of $[X(s,n,\tilde{m},k) | X(r,n,\tilde{m},k) > t]$ is RR_2 in $(k,x) \in \mathbb{R}_+ \times \mathbb{R}$. In the case for which $r = s$, from (3) the density function of $[X(r,n,\tilde{m},k) | X(r,n,\tilde{m},k) > t]$ can be written as

$$f_{[X(r,n,\tilde{m},k)|X(r,n,\tilde{m},k)>t]}(x) = \frac{c_{r-1}}{P\{X(r,n,\tilde{m},k) > t\}} \{\bar{F}(x)\}^{\gamma_r - 1} \xi_r(F(x)) f(x), \qquad x \ge t.$$

From the fact that $\{\bar{F}(x)\}^{\gamma_r-1}$ is RR_2 in $(k,x) \in \mathbb{R}_+ \times \mathbb{R}$, it follows that $f_{[X(r,n,\tilde{m},k)|X(r,n,\tilde{m},k)>t]}(x)$ is also RR_2 in $(k,x) \in \mathbb{R}_+ \times [t, \infty)$. For the case $r < s$, from

(4) the density function $f_{[X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)>t]}(x)$ can be expressed as

$$
\begin{aligned}
&f_{[X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)>t]}(x)\\
&\quad = \frac{c_{s-1}}{P\{X(r,n,\tilde{m},k) > t\}} f(x)\{\bar{F}(x)\}^{\gamma_s-1}\\
&\quad\quad \times \int_{\mathbb{R}} I_{\{t\le u\le x\}}\xi_r(F(u))f(u)\{\bar{F}(u)\}^{\gamma_r-\gamma_s-1}\Psi_{s-r-1}\left(\frac{\bar{F}(x)}{\bar{F}(u)}\right)du
\end{aligned}
$$

for $x > t$. Note that the integral on the right-hand side does not depend on k and that $\{\bar{F}(x)\}^{\gamma_s-1}$ is RR_2 in $(k,x) \in \mathbb{R}_+ \times \mathbb{R}$. We conclude that $f_{[X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)>t]}(x)$ is RR_2 in $(k,x) \in \mathbb{R}_+ \times [t,\infty)$ and this proves part (a).

To prove part (b), note that, using (4), the conditional density of $[X(r,n,\tilde{m},k)|X(s,n,\tilde{m},k) \le t]$ can be expressed as

$$
\begin{aligned}
&f_{[X(r,n,\tilde{m},k)|X(s,n,\tilde{m},k)\le t]}(x)\\
&\quad = \frac{c_{s-1}}{P\{X(s,n,\tilde{m},k) \le t\}} f(x)\{\bar{F}(x)\}^{\gamma_r-\gamma_s-1}\xi_r(F(x))\\
&\quad\quad \times \int_{\mathbb{R}} I_{\{x\le u\le t\}}f(u)\{\bar{F}(u)\}^{\gamma_s-1}\Psi_{s-r-1}\left(\frac{\bar{F}(u)}{\bar{F}(x)}\right)du. \qquad \textbf{(10)}
\end{aligned}
$$

We only need to show that $f_{[X(r,n,\tilde{m},k)|X(s,n,\tilde{m},k)\le t]}$ is RR_2 in $(k,x) \in \mathbb{R}_+ \times (-\infty,t]$ for $1 \le r \le s \le n$ and any $t \in \mathbb{R}$. The result for the case $r = s$ is easy to verify. For the case $r < s$, consider the conditional density function in (10). Upon noting that $\gamma_r - \gamma_s$ does not depend on k, we just need to show that

$$
V(k,x) = \int_{\mathbb{R}} I_{\{x\le u\le t\}}f(u)\{\bar{F}(u)\}^{\gamma_s-1}\Psi_{s-r-1}\left(\frac{\bar{F}(u)}{\bar{F}(x)}\right)du
$$

is RR_2 in $(k,x) \in \mathbb{R}_+ \times (-\infty,t]$. By Lemma 2.6, $\Psi_{s-r-1}(\bar{F}(u)/\bar{F}(x))$ is TP_2 in $(u,x) \in \mathbb{R}^2$. It is not difficult to show that $I_{\{x\le u\le t\}}$ is TP_2 in $(x,u) \in \mathbb{R}_+ \times [x,t]$ and $\{\bar{F}(u)\}^{\gamma_s-1}$ is RR_2 in $(k,u) \in \mathbb{R}_+ \times [x,t]$. This implies that $V(k,x)$ is RR_2 in $(k,x) \in \mathbb{R}_+ \times (-\infty,t)$ (see Karlin [19] or Lemma 2.3 in Zhao and Balakrishna [34]). The proof is complete. ∎

3. AGING PROPERTIES

In this section we obtain some aging properties of conditional GOS

$$
[X(s,n,\tilde{m},k) - t|X(r,n,\tilde{m},k) > t], \qquad 1 \le r \le s \le n. \qquad \textbf{(11)}
$$

For $t \in \mathbb{R}_+$, let $F_t(x) = 1 - \bar{F}(t+x)/\bar{F}(t), x > 0$, denote the distribution function of the residual life random variable $X_t = [X - t|X > t]$. First, we prove the following lemma.

LEMMA 3.1: *For* $1 \le r \le n$ *and* $t \in \mathbb{R}$,

$$[X(r,n,\tilde{m},k) - t | X(1,n,\tilde{m},k) > t] \stackrel{\mathrm{d}}{=} X_t(r,n,\tilde{m},k),$$

where $X_t(r,n,\tilde{m},k)$ *is the rth GOS of sample size n based on the distribution* $F_t(x)$.

PROOF: From the joint density function in (2) it follows that

$$\begin{aligned} &P\{X(r,n,\tilde{m},k) - t \le x | X(1,n,\tilde{m},k) > t\} \\ &\quad = \frac{P\{F(t) < U(1,n,\tilde{m},k) \le U(r,n,\tilde{m},k) \le F(t+x)\}}{P\{U(1,n,\tilde{m},k) > F(t)\}} \\ &\quad = \frac{c_{r-1}}{\{\bar{F}(t)\}^{\gamma_1}} \int_{\mathcal{B}} \left\{ \prod_{i=1}^{r-1} (1-u_i)^{m_i} \right\} (1-u_r)^{\gamma_r - 1}\, du_1\, du_2 \cdots du_r, \end{aligned}$$

where $\mathcal{B} = \{(u_1, \cdots, u_r) : F(t) < u_1 \le u_2 \le \cdots \le u_r \le F(t+x)\}$. Upon making the transformation $u_i = F(t) + \bar{F}(t) v_i, i = 1, 2, \cdots, r$, which maps the domain $\mathcal{B}$ into $\mathcal{C} = \{(v_1, \cdots, v_r) : 0 < v_1 \le v_2 \le \cdots \le v_r \le F_t(x)\}$, we obtain

$$\begin{aligned} &P\{X(r,n,\tilde{m},k) - t \le x | X(1,n,\tilde{m},k) > t\} \\ &\quad = c_{r-1} \int_{\mathcal{C}} \left\{ \prod_{i=1}^{r-1} (1-v_i)^{m_i} \right\} (1-v_r)^{\gamma_r - 1}\, dv_1\, dv_2 \cdots dv_r \\ &\quad = P\{U(r,n,\tilde{m},k) \le F_t(x)\} \\ &\quad = P\{X_t(r,n,\tilde{m},k) \le x\} \end{aligned}$$

This completes the proof. ■

In the following theorem, we give a mixture representation for the survival function of conditional random variable in (11). The result of the theorem is a useful tool in establishing the main results of this section.

THEOREM 3.2: *The survival function of* $[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t], 1 \le r \le s \le n$, *denoted by* $\bar{H}^{\tilde{m}}_{r,s,n,t}(x)$, *can be represented as*

$$\bar{H}^{\tilde{m}}_{r,s,n,t}(x) = \frac{\sum_{j=0}^{r-1} \gamma_{j+1}^{-1} f_{X(j+1,n,\tilde{m},k)}(t) \bar{F}_{X_t(s-j,n-j,\tilde{\mu}_j,k)}(x)}{\sum_{j=0}^{r-1} \gamma_{j+1}^{-1} f_{X(j+1,n,\tilde{m},k)}(t)}, \qquad \textbf{(12)}$$

where $\tilde{\mu}_j = (\mu_1, \cdots, \mu_{n-j-1}) = (m_{j+1}, \cdots, m_{n-1})$.

https://doi.org/10.1017/S0269964810000343 Published online by Cambridge University Press

PROOF: From (1) and (2) we obtain

$$
\begin{aligned}
P\{X(r,n,\tilde{m},k) > t\} &= \sum_{j=0}^{r-1} P\{X(j,n,\tilde{m},k) \le t < X(j+1,n,\tilde{m},k)\} \\
&= \sum_{j=0}^{r-1} P\{U(j,n,\tilde{m},k) \le F(t) < U(j+1,n,\tilde{m},k)\} \\
&= \sum_{j=0}^{r-1} c_j \int_{\mathcal{B}_{j+1}} \left\{\prod_{i=1}^{j}(1-u_i)^{m_i}\right\} (1-u_{j+1})^{\gamma_{j+1}-1}\, du_1\, du_2 \cdots du_{j+1} \\
&= \sum_{j=0}^{r-1} \frac{c_j}{\gamma_{j+1}} \{\bar{F}(t)\}^{\gamma_{j+1}} \xi_{j+1}(F(t)), \qquad \textbf{(13)}
\end{aligned}
$$

where $U(0,n,\tilde{m},k) = 0, X(0,n,\tilde{m},k) = -\infty$ and $\mathcal{B}_{j+1} = \{(u_1,u_2,\cdots,u_{j+1}) : 0 < u_1 \le \cdots \le u_j \le F(t) < u_{j+1}\}$. In a similar way, we obtain

$$
\begin{aligned}
&P\{t < X(r,n,\tilde{m},k) < X(s,n,\tilde{m},k) \le t+x\} \\
&\quad = P\{F(t) < U(r,n,\tilde{m},k) < U(s,n,\tilde{m},k) \le F(t+x)\} \\
&\quad = \sum_{j=0}^{r-1} P\{U(j,n,\tilde{m},k) \le F(t) < U(j+1,n,\tilde{m},k) < U(s,n,\tilde{m},k) \le F(t+x)\} \\
&\quad = \sum_{j=0}^{r-1} c_{s-1}\xi_{j+1}(F(t)) \int_{\mathcal{C}_{j+1}} \left\{\prod_{i=j+1}^{s-1}(1-u_i)^{m_i}\right\} (1-u_s)^{\gamma_s-1}\, du_{j+1} \cdots du_s, \qquad \textbf{(14)}
\end{aligned}
$$

where $\mathcal{C}_{j+1} = \{(u_{j+1},\cdots,u_s) : F(t) < u_{j+1} \le \cdots \le u_s \le F(t+x)\}$. Note that the integral on the right-hand side of (14) can be written as

$$
\begin{aligned}
&\int_{\mathcal{C}_{j+1}} \left\{\prod_{i=1}^{s-j-1}(1-v_i)^{m_{i+j}}\right\} (1-v_{s-j})^{\gamma_s-1}\, dv_1\, dv_2 \cdots dv_{s-j} \\
&\quad = \int_{\mathcal{C}_{j+1}} \left\{\prod_{i=1}^{s-j-1}(1-v_i)^{\mu_i}\right\} (1-v_{s-j})^{\gamma^*_{s-j,n-j}-1}\, dv_1\, dv_2 \cdots dv_{s-j} \\
&\quad = \frac{1}{c^*_{s-j-1}} P\{t < X(1,n-j,\tilde{\mu}_j,k) < X(s-j,n-j,\tilde{\mu}_j,k) \le t+x\},
\end{aligned}
$$

where $\gamma^*_{\ell,n-j} = k + \sum_{i=\ell}^{n-j-1}(\mu_i+1)$ and $c^*_{s-j-1} = \prod_{i=1}^{s-j} \gamma^*_{i,n-j}$. One can easily show that $\gamma^*_{\ell,n-j} = \gamma_{\ell+j,n}$ and $c^*_{s-j-1} = c_{s-1}/c_{j-1}$. On the other hand,

$$
P\{X(1,n-j,\tilde{\mu}_j,k) > t\} = \{\bar{F}(t)\}^{\gamma^*_{1,n-j}} = \{\bar{F}(t)\}^{\gamma_{j+1}}.
$$

Taking into account these observations, the integral on the right-hand side of (14) can be rephrased as

$$\frac{c_{j-1}}{c_{s-1}} P\{X(s-j, n-j, \tilde{\mu}_j, k) - t \leq x | X(1, n-j, \tilde{\mu}_j, k) > t\}\{\bar{F}(t)\}^{\gamma_{j+1}}$$
$$= \frac{c_j}{c_{s-1}\gamma_{j+1}} \{\bar{F}(t)\}^{\gamma_{j+1}}\{1 - \bar{H}^{\tilde{\mu}_j}_{1,s-j,n-j,t}(x)\}. \quad \textbf{(15)}$$

Substituting (15) in (14) and using (13), we get

$$P\{X(s, n, \tilde{m}, k) - t \leq x | X(r, n, \tilde{m}, k) > t\}$$
$$= \frac{\sum_{j=0}^{r-1} c_j/\gamma_{j+1}\{\bar{F}(t)\}^{\gamma_{j+1}} \xi_{j+1}(F(t))\{1 - \bar{H}^{\tilde{\mu}_j}_{1,s-j,n-j,t}(x)\}}{\sum_{j=0}^{r-1} c_j/\gamma_{j+1}\{\bar{F}(t)\}^{\gamma_{j+1}} \xi_{j+1}(F(t))},$$

and hence

$$\bar{H}^{\tilde{m}}_{r,s,n,t}(x) = \frac{\sum_{j=0}^{r-1} c_j/\gamma_{j+1}\{\bar{F}(t)\}^{\gamma_{j+1}} \xi_{j+1}(F(t))\bar{H}^{\tilde{\mu}_j}_{1,s-j,n-j,t}(x)}{\sum_{j=0}^{r-1} c_j/\gamma_{j+1}\{\bar{F}(t)\}^{\gamma_{j+1}} \xi_{j+1}(F(t))}. \quad \textbf{(16)}$$

On applying Lemma 3.1, it is seen that $\bar{H}^{\tilde{m}}_{r,s,n,t}(x)$ can be rewritten as (12). ■

REMARK 3.3: The representation (12) shows that the survival function of $[X(s, n, \tilde{m}, k) - t | X(r, n, \tilde{m}, k) > t]$ is a mixture of the survival functions of the residual random variables corresponding to GOS $X_t(s - r + 1, n - r + 1, \tilde{\mu}_{r-1}, k)$, $X_t(s - r + 2, n - r + 2, \tilde{\mu}_{r-2}, k), \cdots, X_t(s, n, \tilde{m}, k)$. This representation generalizes the result of Lemma 2.2 of Hu et al. [16] to the case of arbitrary parameter vector $\tilde{m}$. Some specialized versions of this representation for order statistics and record values have also been already obtained by Asadi and Bayramoglu [4], Li and Zhao [22], and Raqab and Asadi [27].

We will now focus on aging properties of $[X(s, n, \tilde{m}, k) - t | X(r, n, \tilde{m}, k) > t]$ with respect to t. The following lemma will be useful in proving the next result.

LEMMA 3.4: *For any integer r such that* $1 \leq r \leq n$, $[X(r, n, \tilde{m}, k) - t | X(1, n, \tilde{m}, k) > t]$ *is stochastically decreasing (increasing) in* $t \in \mathbb{R}_+$ *if and only if X is IFR (DFR).*

PROOF: By definition, X is IFR (DFR) if and only if $F_t(x)$ is increasing (decreasing) in $t \in \mathbb{R}_+$. Using Lemma 3.1, and (1), the survival function $\bar{H}^{\tilde{m}}_{1,r,n,t}(x)$ can be expressed as

$$\bar{H}^{\tilde{m}}_{1,r,n,t}(x) = P\{U(r, n, \tilde{m}, k) > F_t(x)\}.$$

Thus, X is IFR (DFR) if and only if $\bar{H}^{\tilde{m}}_{1,r,n,t}(x)$ is decreasing (increasing) in $t \in \mathbb{R}_+$. This completes the result. ■

THEOREM 3.5: *For any two integers r and s such that* $1 \leq r \leq s \leq n$ *and arbitrary* $\tilde{m} \in \mathbb{R}^{n-1}$, *the following hold:*

(a) If X is IFR, then $[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t]$ *is stochastically decreasing in* $t \in \mathbb{R}_+$.
(b) If $[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t]$ *is stochastically increasing in* $t \in \mathbb{R}_+$, *then X is DFR.*

PROOF: First note that, by representation (12), the survival function $\bar{H}^{\tilde{m}}_{r,s,n,t}(x)$ can be rewritten as

$$\bar{H}^{\tilde{m}}_{r,s,n,t}(x) = E_t\{\psi(U,t)\},$$

where $\{\psi(j,t)\} = \bar{H}^{\tilde{\mu}_j}_{1,s-j,n-j,t}(x)$ and the distribution function of the random variable U belongs to the family $\mathcal{P} = \{H(\cdot|t), t \in \mathbb{R}_+\}$ with densities

$$h(j|t) = \frac{\gamma_{j+1}^{-1} f_{X(j+1,n,\tilde{m},k)}(t)}{\sum_{j=0}^{r-1} \gamma_{j+1}^{-1} f_{X(j+1,n,\tilde{m},k)}(t)}, \quad j = 0, 1, \cdots, r-1.$$

(a) Let X be IFR. Then, by Lemma 3.4 $\psi(j,t)$ is decreasing in $t \in \mathbb{R}_+$ for all j. From part (c) of Theorem 2.1, we have

$$X_t(s-j-1, n-j-1, \tilde{\mu}_{j+1}, k) \leq_{\text{st}} X_t(s-j, n-j, \tilde{\mu}_j, k).$$

This, in turn, implies that $\psi(j+1,t) \leq \psi(j,t)$ and hence $\psi(j,t)$ is decreasing in j for all $t \in \mathbb{R}_+$. It follows from part (a) of Theorem 2.1 that $h(j+1|t)/h(j|t)$ is increasing in $t \in \mathbb{R}_+$. Thus, $h(j|t)$ is TP_2 in $(j,t) \in \{0,1,\cdots,r-1\} \times \mathbb{R}_+$, which, in turn, implies that $\bar{H}(j|t_1) \leq \bar{H}(j|t_2), j = 0, 1, \ldots, r-1$, whenever $t_1 < t_2$. Taking into account these observations, from Lemma 2.2 in Misra and van der Meulen [24] we can conclude that $E_t\{\psi(U,t)\}$ is a decreasing function of t, from which the result of part (a) of the theorem follows.

(b) If $E_t\{\psi(U,t)\}$ is increasing in $t \in \mathbb{R}_+$, then we must have $\psi(j,t)$ is increasing in $t \in \mathbb{R}_+$ for some $j \in \{0,1,\ldots,r-1\}$. Hence, there exists some $j \in \{0,1,\ldots,r-1\}$ such that $\bar{H}^{\tilde{\mu}_j}_{1,s-j,n-j,t}(x)$ is increasing in $t \in \mathbb{R}_+$, which, by Lemma 3.4 implies that X is DFR. The proof of the theorem is then complete. ■

THEOREM 3.6: *For any two integers r and s such that* $1 \leq r \leq s \leq n$ *and arbitrary* $\tilde{m} \in \mathbb{R}^{n-1}$ *we have the following results:*

(a) If X is NBU, then

$$[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t] \leq_{\text{st}} X(s,n,\tilde{m},k)$$

https://doi.org/10.1017/S0269964810000343 Published online by Cambridge University Press

for $t \in \mathbb{R}_+$*, and if* X *is NWU, then*

$$[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t] \geq_{\text{st}} X(s-r+1, n-r+1, \tilde{\mu}_{r-1}, k)$$

for $t \in \mathbb{R}_+$.

(b) If

$$[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t] \geq_{\text{st}} X(s,n,\tilde{m},k) \tag{17}$$

for $t \in \mathbb{R}_+$*, then* X *is NWU, and if*

$$[X(s,n,\tilde{m},k) - t | X(r,n,\tilde{m},k) > t] \leq_{\text{st}} X(s-r+1, n-r+1, \tilde{\mu}_{r-1}, k) \tag{18}$$

for $t \in \mathbb{R}_+$*, then* X *is NBU.*

PROOF:

(a) For $1 \leq r \leq n$, let $\bar{H}^{\tilde{m}}_{r,n}(x)$ denote the survival function of $X(r,n,\tilde{m},k)$. If X is NBU (NWU), then $F_t(x) \geq (\leq)\ F(x)$ and hence

$$\begin{aligned}\bar{H}^{\tilde{m}}_{1,s,n,t}(x) &= P\{U(s,n,\tilde{m},k) > F_t(x)\} \\ &\leq (\geq) P\{U(s,n,\tilde{m},k) > F(x)\} = \bar{H}^{\tilde{m}}_{s,n}(x).\end{aligned}$$

According to Theorem 2.1 (c) we have

$$\bar{H}^{\tilde{\mu}_{r-1}}_{1,s-r+1,n-r_{+1,t}}(x) \leq \bar{H}^{\tilde{\mu}_j}_{1,s-j,n-j,t}(x) \leq \bar{H}^{\tilde{m}}_{1,s,n,t}(x)$$

for $0 \leq j \leq r-1$. If we multiply the terms of the above inequalities by $c_j/\gamma_{j+1}\{\bar{F}(t)\}^{\gamma_{j+1}}\xi_{j+1}(F(t))$ and add the terms from 0 to $r-1$, then using (16) we get the following inequalities:

$$\bar{H}^{\tilde{\mu}_{r-1}}_{1,s-r+1,n-r+1,t}(x) \leq \bar{H}^{\tilde{m}}_{r,s,n,t}(x) \leq \bar{H}^{\tilde{m}}_{1,s,n,t}(x).$$

By combining these results, we obtain $\bar{H}^{\tilde{m}}_{r,s,n,t}(x) \leq \bar{H}^{\tilde{m}}_{s,n}(x)$ when X is NBU and $\bar{H}^{\tilde{m}}_{r,s,n,t}(x) \geq \bar{H}^{\tilde{\mu}_{r-1}}_{s-r+1,n-r+1}(x)$ when X is NWU. These prove part (a) of the theorem.

(b) Let (17) hold for $t \in \mathbb{R}_+$. Then

$$\bar{H}^{\tilde{m}}_{s,n}(x) \leq \bar{H}^{\tilde{m}}_{r,s,n,t}(x) \leq \bar{H}^{\tilde{m}}_{1,s,n,t}(x),$$

which implies that

$$P\{U(s,n,\tilde{m},k) > F(x)\} \leq P\{U(s,n,\tilde{m},k) > F_t(x)\}.$$

This means that $F_t(x) \leq F(x)$ for all t, $x \in \mathbb{R}_+$; that is, X is NWU. If (18) holds for $t \in \mathbb{R}_+$, then

$$\bar{H}^{\tilde{\mu}_{r-1}}_{1,s-r+1,n-r+1,t}(x) \leq \bar{H}^{\tilde{m}}_{r,s,n,t}(x) \leq \bar{H}^{\tilde{\mu}_{r-1}}_{s-r+1,n-r+1}(x)$$

and, hence,

$$P\{U(s-r+1,n-r+1,\tilde{\mu}_{r-1},k) > F_t(x)\}$$
$$\leq P\{U(s-r+1,n-r+1,\tilde{\mu}_{r-1},k) > F(x)\}.$$

Thus, $F(x) \leq F_t(x)$ for all $t, x \in \mathbb{R}_+$ and X is NBU. Hence, the result follows.

■

Now, we can apply Theorems 3.5 and 3.6 to obtain the following corollary regarding the residual life of progressively type II right censored order statistics defined above.

COROLLARY 3.7: *For any two integers r and s such that* $1 \leq r \leq s \leq m \leq n$ *and arbitrary censoring scheme* $\tilde{R}$*, we have the following results:*

(a) If F is IFR, then $[X^{\tilde{R}}_{s:m:n} - t | X^{\tilde{R}}_{r:m:n} \geq t]$ *is stochastically decreasing in* $t \in \mathbb{R}_+$.

(b) If F is NBU, then

$$[X^{\tilde{R}}_{s:m:n} - t | X^{\tilde{R}}_{r:m:n} \geq t] \leq_{\mathrm{st}} X^{\tilde{R}}_{s:m:n},$$

and if F is NWU, then

$$[X^{\tilde{R}}_{s:m:n} - t | X^{\tilde{R}}_{r:m:n} \geq t] \geq_{\mathrm{st}} X^{(R_r, R_{r+1}, \cdots, R_m)}_{s-r+1:m-r+1:\gamma_r}.$$

REMARK 3.8: Using the results of this article one can also establish some stochastic orderings and aging monotonicity results for other models of ordered random variables such as Pfeifer records and sequential order statistics based on general distributions $F_1, F_2, \cdots, F_n$, which are special cases of the GOS model. We refer the reader to Kamps [18] for the definition of these models.

Acknowledgements

The authors would like to thank the Editor and an anonymous referee, for their valuable and constructive comments, which improved the presentation of the paper. This work is supported in part by the "Ordered and Spacial Data Center of Excellence of Ferdowsi University of Mashhad".

References

1. Arnold, B.C., Balakrishnan, N., & Nagaraja, H.N. (1992). *A first course in order statistics*. New York: Wiley.
2. Arnold, B.C., Balakrishnan, N., & Nagaraja, H.N. (1998). *Records*. New York: Wiley.
3. Asadi, M. (2006). On the mean past lifetime of the components of a parallel system. *Journal of Statistical Planning and Inference* 136: 1197–1206.

4. Asadi, M. & Bayramoglu, I. (2005). A note on the mean residual life function of a parallel system. *Communications in statistics: Theory and Methods* 34: 475–484.
5. Asadi, M. & Bayramoglu, I. (2006). On the mean residual life function of the k-out-of-n systems at the system level. *IEEE Transactions on Reliability* 55: 314–318.
6. Asadi, M. & Raqab, M.Z. (2010). The mean residual of record values at the level of previous records. *Metrika* 72: 251–264.
7. Balakrishnan, N. & Aggarwala, R. (2000). *Progressive censoring: Theory, methods and applications*. Boston: Birkhauser.
8. Barlow, R.E. & Proschan, F. (1975). *Statistical theory of reliability and life testing*. New York: Holt, Rinehart, Winston.
9. Belzunce, F., Mercader, J.A., & Ruiz, J.M. (2005). Stochastic comparisons of generalized order statistics. *Probability in the Engineering and Informational Sciences* 19: 99–120.
10. Cha, J.H. & Mi, J. (2007). Some probability functions in reliability and their applications. *Naval Research Logistics* 54: 128–135.
11. Cramer, E. & Kamps, U. (2001). Estimation with sequential order statistics from exponential distributions. *Annals of the Institute of Statistical Mathematics* 53: 307–324.
12. Cramer, E. & Kamps, U. (2003). Marginal distributions of sequential and generalized order statistics. *Metrika* 58: 293–310.
13. David, H.A. & Nagaraja, H.N. (2003). *Order statistics*. 3rd ed. Hoboken, NJ: Wiley.
14. Franco, M., Ruiz, J.M., & Ruiz, M.C. (2001). Stochastic orderings between spacings of generalized order statistics. *Probability in the Engineering and Informational Sciences* 16: 471–484.
15. Hashemi, M., Tavangar, M., & Asadi, M. (2010). Some properties of the residual lifetime of progressively type II right censored order statistics. *Statistics and Probability Letters* 80: 845–859.
16. Hu, T., Jin, W., & Khaledi, B.-E. (2007). Ordering conditinoal ditributions of generalized order statistics. *Probability in the Engineering and Informational Sciences* 21: 401–417.
17. Hu, T. & Zhuang, W. (2005). A note on comparisons of generalized order statistics. *Statistics and Probability Letters* 72: 163–170.
18. Kamps, U. (1995). *A concept of generalized order statistics*. Stuttgart: Teubner.
19. Karlin, S. (1968). *Total positivity*. Palo Alto, CA: Stanford University Press.
20. Khaledi, B.-E. & Shaked, M. (2007). Ordering conditional lifetimes of coherent systems. *Journal of Statistical Planning and Inference* 137: 1173–1184.
21. Khaledi, B.-E. & Shojaei, R. (2007). On stochastic ordering between residual record values. *Statistics and Probability Letters* 77: 1467–1472.
22. Li, X. & Zhao, P. (2006). Some aging properties of the residual life of k-out-of-n systems. *IEEE Transaction on Reliability* 55(3): 535–541.
23. Li, X. & Zhao, P. (2008). Stochastic comparison on general inactivity time and general residual life of k-out-of-n systems. *Communications in Statistics: Simulation and Computation* 37: 1005–1019.
24. Misra, N. & van der Meulen, E.C. (2003). On stochastic properties of m-spacings. *Journal of Statistical Planning and Inference* 115: 683–697.
25. Müller, A. & Stoyan, D. (2002). *Comparison methods for stochastic models and risks*. New York: Wiley.
26. Raqab, M.Z. & Asadi, M. (2008). On the mean residual life of records. *Journal of Statistical Planning and Inference* 138: 3660–3666.
27. Raqab, M.Z. & Asadi, M. (2010). Some results on the mean residual waiting time of records. *Statistics* 44(5): 493–504.
28. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders*. New York: Springer.
29. Tavangar, M. & Asadi, M. (2011). Some results on conditional expectations of lower record values. *Statistics*. doi:10.1080/02331880903348481.
30. Tavangar, M. & Asadi, M. (2010). A study on the mean past lifetime of the components of $(n-k+1)$-out-of-n system at the system level. *Metrika* 72: 59–73.
31. Tavangar, M. & Asadi, M. (2011). Some unified characterization results on the generalized Pareto distributions based on generalized order statistics. *Metrika*. Technical report.

32. Xie, H. & Hu, T. (2008). Conditional ordering of generalized order statistics revisited. *Probability in the Engineering and Informational Sciences* 22: 334–346.
33. Zardasht, V. & Asadi, M. (2010). Evaluation of $P(X_t > Y_t)$ when both X_t and Y_t are residual lifetimes of two systems. *Statistica Neerlandica* 64(4): 460–481.
34. Zhao, P. & Balakrishnan, N. (2009). Stochastic comparisons and properties of conditional generalized order statistics. *Journal of Statistical Planning and Inference* 139: 2920–2932.