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SINGULARITY OF ORIENTED GRAPHS FROM SEVERAL CLASSES

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Abstract

A digraph is called oriented if there is at most one arc between two distinct vertices. An oriented graph D is nonsingular if its adjacency matrix $A(D)$ is nonsingular. We characterise all nonsingular oriented graphs from three classes: graphs in which cycles are vertex disjoint, graphs in which all cycles share exactly one common vertex and graphs formed by cycles sharing a common path. As a straightforward corollary, the singularity of oriented bicyclic graphs is determined.

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1. Introduction

A digraph $D = (V(D), E(D))$ consists of a nonempty finite set $V(D)$ of elements called vertices and a finite set $E(D)$ of ordered pairs of distinct vertices called arcs. Its adjacency matrix $A(D)$ is the $(0, 1)$-matrix with $a_{ij} = 1$ if and only if v_iv_j is an arc in D. The rank $r(D)$ of D is the rank of its adjacency matrix $A(D)$ and D is called singular (respectively, nonsingular) if $A(D)$ is singular (respectively, nonsingular). A digraph D can be obtained from an undirected simple graph G by replacing each edge uv of G by an arc uv (or vu) or a pair of arcs uv and vu; we call G the underlying graph of D. Note that if $v_iv_j \in E(D)$ whenever $v_jv_i \in E(D)$, then D becomes an undirected graph and $A(D)$ is symmetric. The rank of $A(D)$ is equal to the number of nonzero eigenvalues and $A(D)$ is nonsingular if and only if $A(D)$ has no zero eigenvalue. We deal with digraphs where loops are not permitted and there is at most one arc between two distinct vertices; these are usually called oriented graphs.

Singularity of undirected graphs is important in chemistry. For a conjugated hydrocarbon molecule, its chemical stability depends on the singularity of the corresponding molecule graph [11]. In 1957, Collatz and Sinogowitz [2] posed the

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problem of characterising all singular (or nonsingular) graphs. The problem seems to be very hard and only a few particular results are known. For undirected graphs, it is easy to see that each complete graph is nonsingular [3]. Cvetković and Gutman [4, 5] showed that if B is bipartite and has no cycles with length 0 (mod 4), then $r(B) = 2m(B)$, where $m(B)$ is the matching number of B. This means that B is nonsingular if and only if B has a perfect matching. Guo *et al.* [7] determined all nonsingular unicyclic graphs. Li *et al.* [12] characterised the singularity of line graphs of unicyclic graphs with depth one. Gutman and Sciriha [9] presented a beautiful result showing that the line graph $L(T)$ of a tree T is either nonsingular or it has exactly one zero eigenvalue. For bipartite graphs [6], bicyclic graphs [10] and tricyclic graphs [1], the rank sets of each type have been determined, but characterisation of nonsingular graphs remains open. For further results on the singularity of undirected graphs, see [8].

There have been few investigations of the singularity of oriented graphs. Zhang *et al.* [14] recently characterised the oriented graphs with rank no larger than two. We aim to characterise the singularity of several types of oriented graphs and treat three classes: the graphs in which cycles are vertex disjoint, the graphs in which all cycles share exactly one common vertex and the graphs formed by cycles sharing a common path. As a corollary, we determine all nonsingular oriented bicyclic graphs.

2. Preliminaries

Some notations and definitions are needed throughout this paper. An oriented graph D is connected if its underlying graph is connected. Two vertices are adjacent if they are connected by an arc. We indicate an arc from u to v by writing uv. For a vertex v of D, we denote by $N_+(v)$ (respectively, $N_-(v)$) the set of vertices u such that vu (respectively, uv) is an arc of D. The out-degree (respectively, in-degree) of v is $d_+(v) = |N_+(v)|$ (respectively, $d_-(v) = |N_-(v)|$). The degree of v in D is $d(v) = d_+(v) + d_-(v)$. If $d(v) = 1$, we call v a pendant vertex. We call v a sink (respectively, source) vertex if $d_+(v) = 0$ (respectively, $d_-(v) = 0$) and a sink-source vertex if $d_+(v)d_-(v) = 0$.

Let $D = (V(D), E(D))$ be an oriented graph. An oriented graph $H = (V(H), E(H))$ is called a subgraph of D if $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$. We denote by $H = D - uv$ the subgraph of D formed by deleting the arc uv from D. When $uv \in E(H)$ if and only if $uv \in E(D)$ for each ordered pair of vertices u and v in H, then H is called an induced subgraph of D. If H is an induced subgraph of D, then $D - H$ is the induced subgraph of D with $V(D - H) = V(D) \setminus V(H)$. If $V(H) = \{x\}$, we write $D - H$ as $D - x$ for brevity.

Since we focus on oriented graphs, we call an oriented cycle (respectively, oriented path) a cycle (respectively, path) for brevity. A cycle is called directed if it contains no sink-source vertex. A path is called directed if it contains exactly two sink-source vertices. We write P_n (respectively, C_n) to denote a directed path (respectively, directed cycle) of order n. For an oriented connected graph D, set $c(D) = |E(D)| - |V(D)| + 1$.

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Now D is an oriented tree if and only if $c(D) = 0$. If $c(D) = 1$, D is said to be unicyclic. If $c(D) = 2$, D is said to be bicyclic.

Lemma 2.1 [14]. *Let D be an oriented graph. If $D = D_1 \cup D_2 \cup \cdots \cup D_k$ is the union of its connected components, then $r(D) = \sum_{i=1}^{k} r(D_i)$.*

Lemma 2.2. *Let D be an oriented graph. If D has a sink-source vertex, then D is singular. In particular, if D has a pendant vertex, then D is singular.*

Proof. Let v be a sink-source vertex of D and let $A(D)$ be the adjacency matrix of D. Then v corresponds to a zero row or zero column in $A(D)$. Thus $A(D)$ and D are singular. □

Let D be an oriented graph with a vertex v. Let $D(v)$ be the oriented graph obtained from D by replacing v by two new vertices v' and v'' and by replacing any arc e containing v by $v'w$ if $e = vw \in E(D)$ or by wv'' if $e = wv \in E(D)$. We say that $D(v)$ is a stretching of D at the vertex v.

Lemma 2.3 [13]. *Let D be an oriented graph with a vertex v. Then $r(D(v)) = r(D)$.*

Lemma 2.4 [14]. *Let D be an oriented graph and let x, y be two distinct vertices of D.*

(i) *If $N_+(x) \subseteq N_+(y)$, then $r(D_1) = r(D)$, where D_1 is the subgraph of D obtained by deleting all yz for $z \in N_+(x)$.*
(ii) *If $N_-(x) \subseteq N_-(y)$, then $r(D_2) = r(D)$, where D_2 is the subgraph of D obtained by deleting all wy for $w \in N_-(x)$.*
(iii) *If $N_+(x) \subseteq N_+(y)$ and $N_-(x) \subseteq N_-(y)$, then $r(D_3) = r(D)$, where D_3 is the subgraph of D obtained by deleting all yz for $z \in N_+(x)$ and all wy for $w \in N_-(x)$.*

Now we characterise the singularity of oriented paths, cycles, trees or unicyclic graphs. These results are also used in the proofs of our main theorems.

Lemma 2.5. *Let P be a directed path. Then $r(P) = |V(P)| - 1$.*

Proof. We proceed by induction on $|V(P)|$. If $P = P_2$, then $r(P) = 1 = |V(P)| - 1$. Assume that $|V(P)| \geq 3$, and denote $H = P - x$, where x is a pendant vertex of P. Let y be a neighbour of x in P. By stretching P at y we have the oriented graph $P(y)$. Now $r(P) = r(P(y))$ by Lemma 2.3. Note that $P(y) = H \cup P_2$. By Lemma 2.1 and the induction hypothesis,

$$r(P) = r(P(y)) = r(H \cup P_2) = r(H) + r(P_2) = |V(H)| - 1 + |V(P_2)| - 1 = |V(P)| - 1.$$

The result, therefore, follows by induction. □

Lemma 2.6. *Let C be an oriented cycle. Then C is nonsingular if and only if C is directed.*

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PROOF. For the sufficiency, suppose that $C = C_n$ and v is a vertex of C_n. We get a directed path P_{n+1} by stretching C_n at v. By Lemmas 2.3 and 2.5,

$$r(C) = r(P_{n+1}) = |V(P_{n+1})| - 1 = n + 1 - 1 = n = |V(C)|,$$

and thus C is nonsingular.

For the necessity, suppose, by contradiction, that C is not directed and x is a sink-source vertex. Now C is singular by Lemma 2.2, contrary to the hypothesis. □

The next two lemmas are straightforward corollaries of Lemma 2.2 and Lemmas 2.2 and 2.6, respectively.

LEMMA 2.7. *Each oriented tree is singular.*

LEMMA 2.8. *Let D be an oriented unicyclic graph. Then D is nonsingular if and only if D is a directed cycle.*

3. Oriented graphs in which cycles are vertex disjoint

In this section, we characterise nonsingular oriented graphs when cycles are vertex disjoint. For an oriented graph D belonging to this class, we call two cycles adjacent if there is a path connecting them using only vertices outside cycles. A cycle C is called pendant in D if C contains exactly one vertex of degree three, while all other vertices in C are of degree two.

THEOREM 3.1. *Let D be an oriented connected graph in which cycles are disjoint. Then D is nonsingular if and only if D satisfies all the following conditions:*

(i) *D has no pendant vertices;*
(ii) *each cycle in D is directed;*
(iii) *two cycles are adjacent in D if and only if they are connected by an arc.*

PROOF. If D is acyclic or unicyclic, the conclusion follows from Lemmas 2.7 and 2.8. Now we assume that D has at least two cycles.

For the sufficiency, we proceed by induction on the number of cycles in D. Let C be a pendant cycle in D which satisfies all three conditions and let $v \in V(C)$ be a vertex with degree three. Let x and y be two neighbours of v in C and let u be the neighbour of v not in C. Note that C is a directed cycle. Assume that $uv, xv, vy \in E(D)$. Since $N_+(x) = \{v\} \subseteq N_+(u)$, it follows that $r(D) = r(D - uv) = r(D - C) + r(C)$ by Lemma 2.4. Since C and $D - C$ both satisfy all three conditions, $D - C$ and C are nonsingular by the induction hypothesis. Now

$$r(D) = r(D - C) + r(C) = |V(D - C)| + |V(C)| = |V(D)|$$

and so D is nonsingular.

For the necessity, item (i) follows immediately from Lemma 2.2. For (iii), we first prove the following claim.

Claim. For an arbitrary vertex $v \in V(D)$, if $d(v) \geq 3$, then v is in some cycle of D.

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To prove our claim, suppose, by contradiction, that v is not in any cycle and $d(v) \geq 3$. Note that v is not a sink-source vertex by Lemma 2.2. The stretching $D(v)$ of D is a disjoint union of two graphs, say, D_1 and D_2, where D_1 contains v' and D_2 contains v''. Since v' is a sink-source vertex of D_1 and v'' is a sink-source vertex of D_2, both D_1 and D_2 are singular. Now

$$r(D) = r(D(v)) = r(D_1) + r(D_2) \leq |V(D_1)| - 1 + |V(D_2)| - 1 = |V(D)| - 1.$$

Thus D is singular which gives a contradiction. This proves our claim.

From the claim, any pair of adjacent cycles in D are connected by a path. Assume that P is a path connecting two adjacent cycles such that $|V(P)| \geq 3$ and let u be a vertex in P with degree two. Note that u cannot be a sink-source vertex. The stretching $D(u)$ of D is a disjoint union of two graphs, say, D_3 and D_4, where D_3 contains a pendant vertex u' and D_4 contains a pendant vertex u''. Since u' is a sink-source vertex of D_3 and u'' is a sink-source vertex of D_4, both D_3 and D_4 are singular by Lemma 2.2. Again

$$r(D) = r(D(v)) = r(D_3) + r(D_4) \leq |V(D_3)| - 1 + |V(D_4)| - 1 = |V(D)| - 1.$$

Thus D is singular, which gives a contradiction and (iii) follows.

With (i) and (iii), we prove (ii) by using a minimum counterexample method. Assume, by contradiction, that (ii) fails, so that counterexamples (while still admitting (i) and (iii)) exist. Let D be such a nonsingular oriented graph containing some cycle which is not directed and such that the order $|V(D)|$ is minimal among all counterexamples that admit (i) and (iii). By (i), D has no pendant vertex. Let C be a pendant cycle of D. If C is not directed, then at least two vertices are sink-source vertices in C. Thus at least one of them is a sink-source vertex of D. But then D is singular, which is a contradiction. Thus C is directed and nonsingular. By (iii), C is connected with $D - C$ by an arc. As in the proof for sufficiency, $r(D) = r(D - C) + r(C)$. Now

$$r(D - C) = r(D) - r(C) = |V(D)| - |V(C)| = |V(D - C)|.$$

Thus $D - C$ is nonsingular, admits (i) and (iii) and contains some cycle that is not directed. This contradicts the assumption that D is a minimum counterexample and proves (ii). □

4. Oriented graphs in which all cycles share exactly one common vertex

In this section, we show that when all cycles share exactly one common vertex in an oriented graph, it must be singular.

THEOREM 4.1. *Let D be an oriented connected graph in which all ($\geq$2) cycles share exactly one common vertex. Then D is singular.*

PROOF. Assume, by contradiction, that D is nonsingular. Let $k \geq 2$ and $C^1, C^2, \ldots, C^k$ be all the cycles of D sharing a vertex x in D, where $|V(C^i)| \geq 3$ for $i = 1, 2, \ldots, k$. Then D has no pendant vertex and each cycle is directed by Lemma 2.2. Now there

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exist $2k$ neighbours of x, say, $u_i, v_i \in V(C^i)$ for $i = 1, 2, \ldots, k$, such that $N_+(u_i) = \{x\}$ and $N_-(v_i) = \{x\}$. Since $N_+(u_k) = \{x\} = N_+(u_i)$ and $N_-(v_k) = \{x\} = N_-(u_i)$ for $i = 1, 2, \ldots, k-1$, it follows that $r(D) = r(D - \sum_{i=1}^{k-1} u_i x - \sum_{i=1}^{k-1} xv_i)$ by Lemma 2.4. Note that $D - \sum_{i=1}^{k-1} u_i x - \sum_{i=1}^{k-1} xv_i = P^1 \cup \cdots \cup P^{k-1} \cup C^k$, where $P^i = C^i - x$ is a directed path for $i = 1, 2, \ldots, k-1$ satisfying $|V(P^i)| = |V(C^i)| - 1$. Then

$$\begin{aligned} r(D) &= r\Big(D - \sum_{i=1}^{k-1} u_i x - \sum_{i=1}^{k-1} xv_i\Big) = r(P^1 \cup \cdots \cup P^{k-1} \cup C^k) = \sum_{i=1}^{k-1} r(P^i) + r(C^k) \\ &= \sum_{i=1}^{k-1} (|V(C^i)| - 2) + |V(C^k)| = |V(D)| - k + 1 \le |V(D)| - 1, \end{aligned}$$

where the fourth equality follows from Lemmas 2.5 and 2.6. But then D is singular, which gives a contradiction. □

5. Oriented graphs formed by cycles sharing a path

In this section, we characterise nonsingular oriented graphs formed by cycles sharing a path. When the path is an edge, such graphs are usually called book graphs. For a cycle C in an oriented graph D, if $u, v \in V(C)$ and $uv \in E(D)\backslash E(C)$, then we say that uv is a chord of C.

THEOREM 5.1. *Let D be an oriented connected graph formed by (≥ 2) cycles sharing a path. Then D is nonsingular if and only if D is obtained from a directed cycle by adding a chord.*

PROOF. For sufficiency, let D be obtained from a directed cycle C (with length at least four) by adding a chord uv. Let x and y be two neighbours of v in C. Note that C is a directed cycle. Assume that $xv, vy \in E(D)$. Since $N_+(x) = \{v\} \subseteq N_+(u)$, it follows that $r(D) = r(D - uv)$ by Lemma 2.4. Since $D - uv$ is a directed cycle, it is nonsingular by Lemma 2.6. Now

$$r(D) = r(D - uv) = |V(D - uv)| = |V(D)|$$

and D is nonsingular.

For the necessity, assume that D is obtained from k (≥ 2) paths $P^1, P^2, \ldots, P^k$ by identifying all initial vertices as v, all terminal vertices as u and then connecting u and v with a path P^{k+1}. By Lemma 2.2, D has no pendant vertex, each path is directed and u, v are not sink-source vertices. We first prove a claim.

Claim. $k = 2$.

To prove our claim, assume, by contradiction, that $k \geq 3$. Then there exist two neighbours of v, say, $x_1 \in V(P^1), x_2 \in V(P^2)$, such that $N_+(x_1) = N_+(x_2) = \{v\}$ or $N_-(x_1) = N_-(x_2) = \{v\}$. Without loss of generality, assume that $N_+(x_1) = N_+(x_2) = \{v\}$. Now $r(D) = r(D - x_1 v)$ by Lemma 2.4. Note that x_1 is a sink vertex of the graph $D - x_1 v$. By Lemma 2.2, $D - x_1 v$ is singular. Now

$$r(D) = r(D - x_1 v) \le |V(D - x_1 v)| - 1 = |V(D)| - 1$$

and D is singular, which leads to a contradiction. This proves our claim.

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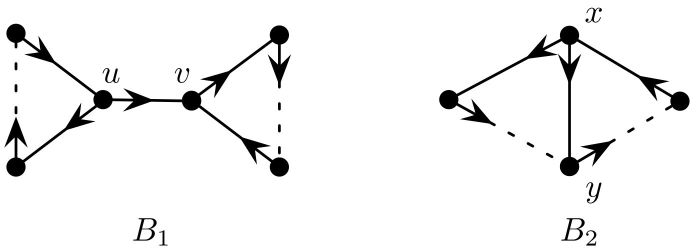

Figure 1. Two nonsingular oriented bicyclic graphs: $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

With the above claim, we assume that D is obtained from a cycle C (with length at least four) by connecting two of its vertices u and v with a path P. By Lemma 2.2, P is directed. The proof of the claim also shows that C must be directed. Now we aim to prove that the directed path P must be an arc uv or vu. Assume that P is directed from u to v via w, where $wv \in E(D)$. Note that C is directed. Suppose $x \in V(C)$ and $x \in N_-(v)$. Now $N_+(x) = N_+(w) = \{v\}$ and $r(D) = r(D - wv)$. By Lemma 2.2, $r(D - wv) \leq |V(D - wv)| - 1 = |V(D)| - 1$ since w is a sink of $D - wv$. But now $r(D) \leq |V(D)| - 1$ and D is singular. This leads to a contradiction and proves the necessity. □

6. Remarks

As we have mentioned, singularity for undirected trees and unicyclic graphs has been determined, but for undirected bicyclic graphs the problem has not been solved. In Section 2, we characterise the singularity of oriented trees and oriented unicyclic graphs. Now we determine the singularity of oriented bicyclic graphs. Note that if an oriented connected graph is bicyclic, it has two cycles sharing no common vertex, or sharing exactly one common vertex, or sharing a common path. The proof of the final theorem follows immediately from Theorems 3.1, 4.1 and 5.1.

Theorem 6.1. *Let D be an oriented bicyclic graph. Then D is nonsingular if and only if $D \in \mathcal{B}_1$ or $D \in \mathcal{B}_2$, where $\mathcal{B}_1$ is the set of oriented graphs obtained from two directed cycles by connecting an arc and $\mathcal{B}_2$ is the set of oriented graphs obtained from a directed cycle by adding a chord (see Figure 1).*

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