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RICCI ITERATIONS ON KÄHLER CLASSES

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Abstract In this paper we consider the dynamical system involved by the Ricci operator on the space of Kähler metrics of a Fano manifold. Nadel has defined an iteration scheme given by the Ricci operator and asked whether it has some non-trivial periodic points. First, we prove that no such periodic points can exist. We define the inverse of the Ricci operator and consider the dynamical behaviour of its iterates for a Fano Kähler–Einstein manifold. Then we define a finite-dimensional procedure to give an approximation of Kähler–Einstein metrics using this iterative procedure and apply it on $\mathbb{CP}^2$ blown up in three points.

Keywords: Kähler–Einstein metrics; dynamics of the Ricci operator; Kähler–Ricci flow; numerical approximations; Bergman kernel; Fano manifold

AMS 2000 *Mathematics subject classification:* Primary 32Q20; 53C44; 14J45; 34K28; 11Y16

Introduction

Let M be a Fano manifold of complex dimension n. For any Kähler metric g we denote

$$\omega := \omega_g = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}}(z)\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$$

its corresponding Kähler form, a closed positive $(1,1)$-form on M. For a Kähler form ω, we consider the space of strictly ω-plurisubharmonic potentials

$$Ka_{[\omega]} = \{\varphi \in C^\infty(M) : \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\},$$

and the space Ka_{c_1} of Kähler forms cohomologous to $c_1(M)$. For any Kähler metric ω, we let

$$\mathrm{Ric}(\omega) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det(g_{i\bar{j}})$$

denote the Ricci form of ω. It is well defined globally and lies in the $c_1(M) > 0$ class. Following [**17**], if $\mathrm{Ric}(\omega)$ is a Kähler form, we let $\mathrm{Ric}^{(2)}(\omega)$ denote its Ricci form, and in a similar way, we define higher powers of the Ricci operator as long as the positivity is preserved. The motivation for this construction comes from the simple fact that, when they exist, Kähler–Einstein metrics are by definition fixed points for this iteration process.

A long time ago, Nadel asked whether these are all periodic points and proved the absence of periodic points of order 2 and 3. Furthermore, he raised the question whether the existence of Kähler–Einstein metrics could be related to this iteration procedure. This question is also very natural. Actually, as we will explain later, one can define an inverse of the Ricci operator using the celebrated Calabi–Yau theorem and see its iterations as a kind of naive *discretization* of the (normalized) Kähler–Ricci flow

$$\frac{\partial w_t}{\partial t} = -\operatorname{Ric}(\omega_t) + \omega_t. \tag{0.1}$$

We now explain the organization of this paper. We will show that some natural energy functionals are decreasing along these iterations. This will give us a simple proof of the non-existence of (non-trivial) periodic points and thus answer Nadel's first question. Then we study the question of the existence of periodic points of infinite order. The behaviour of our dynamical system is closely tied with the existence of Kähler–Einstein metrics on the Fano manifold. Note that a part of the results presented in §§ 2 and 3 have been published recently in [**20**] and we refer to this reference for more advanced progress on that topic. Then, we generalize Nadel's iteration scheme and define a family of natural operators $\operatorname{Ric}_\epsilon$, $\epsilon \geqslant 0$, and see that their behaviour is particularly simple when the manifold is Kähler–Einstein and $\epsilon < 1$. Furthermore, we relate the iteration procedure to the notion of canonically balanced metric studied by Donaldson in [**10**]. This gives us an approximation procedure in a finite-dimensional setup of the Kähler–Einstein metric (when it does exist *a priori* on the considered Fano manifold) which is the main motivation of our work. We study the efficiency of this procedure in details. Finally, we apply our techniques to the case of the Del Pezzo surface given by $\mathbb{P}^2$ blown up in three points, and give a numerical approximation of the Kähler–Einstein metric living on it. The main part of this article appeared in a preprint in early 2007, during the stay of the author at Imperial College.

1. Positive and negative Ricci iterations

Firstly, we look for a natural way to discretize the Kähler–Ricci flow (0.1). This leads us to consider for a given Kähler form ω_0 the sequence

$$\frac{\omega_{k+1} - \omega_j}{(k+1) - k} = -\operatorname{Ric}(\omega_{k+1}) + \omega_{k+1}$$

for any integer $k \geqslant 1$. Of course, this can be also written as

$$\operatorname{Ric}(\omega_{k+1}) = \omega_k. \tag{1.1}$$

Our first observation lies in the fact that this last equation can always be solved. Actually, from the Calabi–Yau theorem [**26**], for all $k \geqslant 0$, there exists a unique smooth solution to this non-linear partial differential equation, which means that there exists a Kähler form ω_{k+1} in Ka_{c_1} whose Ricci form equals ω_k. This can be rephrased by saying that there exists an inverse of the Ricci operator, that we shall denote

$$\operatorname{Ric}^{(-1)} : Ka_{c_1} \to Ka_{c_1}$$

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and hence it is natural to consider the dynamical system induced by the higher powers $\mathrm{Ric}^{(-k)} := \mathrm{Ric}^{(-1)} \circ \cdots \circ \mathrm{Ric}^{(-1)}$ of this operator. We expect the behaviour of the iterates of $\mathrm{Ric}^{(-1)}$ to be related to the corresponding Kähler–Ricci flow (0.1).

Let us now give some notation. For a positive integer k we will denote $Ka_{c_1}^{(k)}$ the maximal domain of definition of $\mathrm{Ric}^{(k)}$. Thus, we obtain naturally a filtration of $Ka_{c_1}^{(0)} = Ka_{c_1}$. We also let $Ka_{c_1}^{(k)}$ denote the set of all metrics in $Ka_{c_1}^{(k-1)}$ whose image under $\mathrm{Ric}^{(k-1)}$ is non-negative. We let $Ka_{c_1}^{(\infty)}$ denote the set of all $L^\infty(M)$ limits $\lim_{k\to\infty} \mathrm{Ric}^{(-k)}\,\alpha$, for $\alpha \in Ka_{c_1}^{(0)}$ when those limits do exist. Finally, let J be the fixed complex structure on M and G be any connected compact subgroup of the group $\mathrm{Aut}(M, J)$ of holomorphic diffeomorphisms of (M, J). We denote by $Ka_{\Omega}^{(G)}$ the space of G-invariant Kähler forms in Ka_Ω. Such forms exists as can be seen by averaging over orbits of G with respect to the Haar measure of G. We notice that $\mathrm{Ric}^{(-1)}$ maps $Ka_{c_1}^{(G)}$ into itself.

2. Energy functionals on the space of Kähler potentials

In §§ 2.1 and 2.2, we present some properties of some well-known energy functionals. Let $\Omega \in H^2(M, \mathbb{R})$ denote a Kähler class. We call a function $A : Ka_\Omega \times Ka_\Omega \to \mathbb{R}_+$ an *energy functional* if it is zero only on the diagonal. By an *exact energy functional* we will mean one which satisfies in addition the cocycle condition (see [**16**])

$$A(\omega_1, \omega_2) + A(\omega_2, \omega_3) = A(\omega_1, \omega_3).$$

Throughout the paper, V will denote the volume of the manifold with respect to $[\omega]$, i.e. $V = \int_M \omega^n/n!$ and we define $\omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi$ for simplicity. The energy functionals I, J, introduced by Aubin in [**1**], are defined for each pair $(\omega, \omega_\varphi) \in Ka_\Omega \times Ka_\Omega$ by

$$I(\omega, \omega_\varphi) = \frac{1}{V}\int_M \sqrt{-1}\partial\varphi \wedge \bar\partial\varphi \wedge \sum_{i=0}^{n-1} \omega^i \wedge \omega_\varphi^{n-1-i} = \frac{1}{V}\int_M \varphi(\omega^n - \omega_\varphi^n), \tag{2.1}$$

$$J(\omega, \omega_\varphi) = \frac{1}{V(n+1)}\int_M \sqrt{-1}\partial\varphi \wedge \bar\partial\varphi \wedge \sum_{i=0}^{n-1} (n-i)\omega^i \wedge \omega_\varphi^{n-1-i}. \tag{2.2}$$

We note some of their basic properties for which we refer the reader to [**1**, **21**, **24**]. Note that I, J and $I - J$ are all non-negative and equivalent. One may also define them via a variational formula. Connect each pair $(\omega, \omega_{\varphi_1} := \omega + \sqrt{-1}\partial\bar\partial\varphi_1)$ with a piecewise smooth path $\{\omega_{\varphi_t}\}$. Then, for example, for $I - J$, we have for any such path

$$(I - J)(\omega, \omega_{\varphi_1}) = -\frac{1}{V}\int_{[0,1]\times M} \varphi_t \Delta_t \dot\varphi_t \omega_{\varphi_t}^n \wedge \mathrm{d}t. \tag{2.3}$$

2.1. The F_1 functional

Let us define

$$F^0(\omega_0, \varphi) = -(I - J)(\omega_0, \omega_\varphi) - \frac{1}{V}\int_M \varphi\omega_\varphi^n. \tag{2.4}$$

For a Kähler manifold of positive Chern class, we define the following functional on $Ka_{c_1} \times Ka_{c_1}$

$$F_1(\omega, \omega_\varphi) = F^0(\omega, \varphi) - \log\left(\frac{1}{V}\int_M \mathrm{e}^{f_\omega - \varphi}\omega^n\right).$$

Here f_ω is the Ricci deviation defined up to a constant by

$$\sqrt{-1}\partial\bar{\partial} f_\omega = \mathrm{Ric}(\omega) - \omega.$$

The critical points of the functionals F_1 are Kähler–Einstein metrics. Moreover, the second variation of F_1 at a critical point in the direction of the plane spanned by $\psi_1, \psi_2 \in T_\varphi Ka_{[\omega]}$ is given by the expression

$$\frac{1}{V}\int_M (\tfrac{1}{2} g_\varphi(\nabla\psi_1, \nabla\psi_2) - \psi_1\psi_2)\omega_\varphi^n. \tag{2.5}$$

This is non-negative and vanishes precisely when ψ_1 and ψ_2 are proportional and eigenfunctions of eigenvalue -1 of $\Delta_{\bar{\partial}}$ (see [**24**, p. 64]). Thus this infinitesimal variation corresponds to holomorphic automorphisms and to moving within the set of Kähler–Einstein forms.

2.2. K-energy and E_k functionals

The Chen–Tian energy functionals E_k, $k = 0, \dots, n$, are defined in a similar manner by

$$\begin{aligned} E_k(\omega, \omega_{\varphi_1}) = \frac{(k+1)}{V}\int_{[0,1]\times M} \Delta_{\varphi_t}\dot{\varphi}_t \, \mathrm{Ric}(\omega_{\varphi_t})^k \wedge \omega_{\varphi_t}^{n-k} \wedge \mathrm{d}t \\ - \frac{(n-k)}{V}\int_{[0,1]\times M} \dot{\varphi}_t(\mathrm{Ric}(\omega_{\varphi_t})^{k+1} - \mu_k\omega_{\varphi_t}^{k+1})\omega_{\varphi_t}^{n-1-k}\,\mathrm{d}t, \end{aligned} \tag{2.6}$$

where one has defined the topological term

$$\mu_k = \frac{c_1(M)^{k+1} \cup [\omega]^{n-k-1}([M])}{[\omega]^n([M])}.$$

This gives rise to well-defined *exact* energy functionals independent of the choice of path [**6**]. The *K-energy*, E_0, was introduced by Mabuchi [**16**]. The following formula is taken from [**23**, § 7.2].

Proposition 2.1. *Let f be a function satisfying* $\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} f$. *One has*

$$E_0(\omega, \omega_\varphi) = \frac{1}{V}\int_M \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n - (I - J)(\omega, \omega_\varphi) + \frac{1}{V}\int_M f(\omega^n - \omega_\varphi^n).$$

Remark that the E_k functionals vanish on pairs joined by a one parameter subgroup of automorphisms through the identity [**6**, Corollary 5.5].

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Definition 2.2. We say that an exact functional $\mathcal{F}$ is bounded from below if $\mathcal{F}(\omega, \omega_\varphi) \geqslant C$ for every $\omega_\varphi \in Ka_\Omega$. We say it is proper on $Ka_\Omega^{(G)}$ (in the sense of Tian) if there exists a function $\rho : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{s\to\infty} \rho(s) = \infty$ such that

$$\mathcal{F}(\omega, \omega_\varphi) \geqslant \rho((I - J)(\omega, \omega_\varphi)),$$

for every $\omega_\varphi \in Ka_\Omega^{(G)}$.

This is well defined, in other words depends only on the class $[\omega]$ since the failure of $I - J$ to satisfy the cocycle condition is under control with respect to the two base metrics,

$$(I - J)(\omega, \omega_{\varphi_2}) - (I - J)(\omega_{\varphi_1}, \omega_{\varphi_2}) = (I - J)(\omega, \omega_{\varphi_1}) - \frac{1}{V}\int_M \varphi_1(\omega_{\varphi_2}^n - \omega_{\varphi_1}^n).$$

2.3. A lower bound for the energy functionals

In the next three sections, we study the dynamics of the Ric^{-1} operator by analysing the behaviour of the functionals that we have just introduced. Firstly, we recall a result of non-negativity of Bando and Mabuchi.

Theorem 2.3 (Bando and Mabuchi [4, Theorem A], Bando [3, Theorem 1] and Song and Weinkove [22, Theorem 1.2]). *Let M be a Fano manifold and assume that M carries a Kähler–Einstein metric ω_{KE}. Then, for $k = 0, 1$,*

$$E_k(\omega_{\mathrm{KE}}, \omega) \geqslant 0$$

for all $\omega \in Ka_{c_1}$ with equality if and only if ω is Kähler–Einstein. In that case there exists a holomorphic automorphism homotopic to the identity χ such that $\chi^\star\omega_{\mathrm{KE}} = \omega$.

Proof. We give a sketch of the proof for $k = 0$ with an emphasis on the features that will be useful in later sections. Consider the deformation $\{\omega_{\varphi_t}\} \subseteq Ka_{c_1}$ constructed from two paths, solutions of the following Monge–Ampère equations

$$\begin{aligned} \omega_{\varphi_t}^n &= \mathrm{e}^{tf + c_t}\omega^n \qquad (t \in [0, 1]) \\ &= \mathrm{e}^{f-(t-1)\varphi_t}\omega^n \quad (t \in [1, 2]), \end{aligned} \tag{2.7}$$

where $\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}f$ with the normalizations

$$\int_M \mathrm{e}^{tf+c_t}\omega^n = \int_M \mathrm{e}^{f-(t-1)\varphi_t}\omega^n = V.$$

Note that the first path is the one used in Yau's continuity method proof [**26**]. It connects any point ω in Ka_{c_1} to $\mathrm{Ric}^{(-1)}(\omega)$ in $Ka_{c_1}^{(2)}$. The second path, introduced by Aubin in [**1**], is used to connect any point in $Ka_{c_1}^{(2)}$ to a Kähler–Einstein metric.

The existence of the first path is equivalent to the Calabi–Yau theorem. The second path may not exist in the presence of non-trivial holomorphic vector fields but Bando and Mabuchi show that arbitrarily close to ω in the C^∞-topology, there exist metrics for which such a path exists. Since the K-energy is continuous this will be sufficient for the argument (cf. [**4**] and [**22**, § 3]).

Now, for $t \in [0,1]$ one has

$$\operatorname{Ric}(\omega_{\varphi_t}) = (1-t)\operatorname{Ric}(\omega) + t\omega \tag{2.8}$$

and

$$\Delta_t \dot{\varphi}_t = f + \dot{c}_t, \tag{2.9}$$

hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_0(\omega_{\mathrm{KE}},\omega_{\varphi_t}) &= -\frac{1}{V}\int_M \dot{\varphi}_t(\operatorname{Ric}(\omega_{\varphi_t}) - \omega_{\varphi_t}) \wedge n\omega_{\varphi_t}^{n-1}\\ &= -\frac{1}{V}\int_M \dot{\varphi}_t((1-t)\sqrt{-1}\partial\bar{\partial}f - \sqrt{-1}\partial\bar{\partial}\omega_{\varphi_t}) \wedge n\omega_{\varphi_t}^{n-1}\\ &= -(1-t)\frac{1}{V}\int_M \dot{\varphi}_t(\Delta_{\omega_{\varphi_t}}\dot{\varphi}_t)^2\omega_{\varphi_t}^n - \frac{\mathrm{d}}{\mathrm{d}t}(I-J)(\omega,\omega_{\varphi_t})\end{aligned} \tag{2.10}$$

with $t \in [0,1]$, from which we conclude

$$E_0(\omega_{\mathrm{KE}},\omega_{\varphi_1}) \leqslant E_0(\omega_{\mathrm{KE}},\omega). \tag{2.11}$$

Next, for $t \in [1,2]$,

$$\operatorname{Ric}(\omega_{\varphi_t}) = (2-t)\omega + (t-1)\omega_{\varphi_t}$$

and

$$\Delta_t \dot{\varphi}_t = -\varphi_t + t\dot{\varphi}_t,$$

hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_0(\omega_{\mathrm{KE}},\omega_{\varphi_t}) &= -\frac{1}{V}\int_M \dot{\varphi}_t(-(2-t)\sqrt{-1}\partial\bar{\partial}\omega_{\varphi_t}) \wedge n\omega_{\varphi_t}^{n-1}\\ &= -(2-t)\frac{\mathrm{d}}{\mathrm{d}t}(I-J)(\omega_{\mathrm{KE}},\omega_{\varphi_t}) \leqslant 0\\ &= -(2-t)\frac{1}{V}\int_M((\Delta_t\dot{\varphi}_t)^2 + t|\partial\dot{\varphi}_t|_t^2)\omega_{\varphi_t}^n \leqslant 0,\end{aligned} \tag{2.12}$$

where $t \in [1,2]$. The theorem now follows for E_0.

Song and Weinkove extended this argument to E_1 using two detailed computations. The first shows that while E_k may not necessarily be monotone (when the path exists), one still has $E_k(\omega_{\mathrm{KE}},\omega_{\varphi_1}) \geqslant E_k(\omega_{\mathrm{KE}},\omega_{\varphi_2}) = 0$. In other words, we have the following theorem.

Theorem 2.4 (Song and Weinkove [22, Theorem 1.1]). *Let (M,ω_{KE}) be a Fano Kähler–Einstein manifold. Then for any $\omega \in \overline{Ka_{c_1}^{(2)}}$ and for each $k = 0,\dots,n$ one has*

$$E_k(\omega_{\mathrm{KE}},\omega) \geqslant 0,$$

with equality if and only if ω is Kähler–Einstein and $\chi^\star\omega_{\mathrm{KE}} = \omega$ with χ a holomorphic automorphism homotopic to the identity.

The second calculation shows that when $k = 1$, one has

$$E_1(\omega_{\varphi_1}, \omega) \geqslant E_1(\omega_0, \omega) = 0.$$

Explicitly, their computation shows that

$$\begin{aligned} E_k(\omega_{\varphi_1}, \omega) = {} & \frac{1}{V}\int_M \sqrt{-1}\partial\varphi_1 \wedge \bar\partial\varphi_1 \wedge \sum_{i=0}^{n-1} a_i \omega^i \wedge \omega_{\varphi_1}^{n-1-i} \\ & + (k+1)\frac{1}{V}\int_{M\times[0,1]} (1-t)(\Delta_{\omega_{\varphi_t}}\dot\varphi_t)^2 \omega_{\varphi_t}^n \wedge \mathrm{d}t \\ & - \frac{1}{V}\int_M \sum_{i=1}^{k} \binom{i+1}{k+1} f(\sqrt{-1}\partial\bar\partial f)^i \wedge \omega^{n-i}, \end{aligned} \tag{2.13}$$

with

$$a_i = \begin{cases} \dfrac{(n-k)(i+1)}{n+1} & \text{if } 0 \leqslant i \leqslant k-1, \\ \dfrac{(k+1)(n-i)}{n+1} & \text{if } k \leqslant i \leqslant n. \end{cases}$$

Since the last term is positive on Ka_{c_1} for $k = 1$ they conclude their proof. □

2.4. A system of Monge–Ampère equations

Now, let $\omega = \omega_0 \in Ka_{c_1}$ denote an initial Kähler metric for our iterations. We present the iterative procedure defined by (1.1) in terms of Monge–Ampère equations. Let φ_1 be a Kähler potential with

$$\mathrm{Ric}(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi_1) = \omega_0.$$

By denoting $f := f_{\omega_0}$ the Ricci deviation of ω_0, this equation becomes

$$\omega_{\varphi_1}^n = \mathrm{e}^f \omega^n$$

together with the volume normalization

$$\frac{1}{V}\int_M \mathrm{e}^f \omega_0^n = 1.$$

This determines φ_1 only up to a constant, which will be fixed in the next step. Put $\omega_1 = \omega_{\varphi_1}$. In the second step we solve

$$\mathrm{Ric}(\omega_1 + \sqrt{-1}\partial\bar\partial\varphi_2) = \omega_1$$

and $\omega_1 - \mathrm{Ric}(\omega_1) = \omega_1 - \omega_0 = \sqrt{-1}\partial\bar\partial\varphi_1$. The Monge–Ampère equation is now

$$\omega_{\varphi_1+\varphi_2}^n = \mathrm{e}^{-\varphi_1}\omega_{\varphi_1}^n = \mathrm{e}^{f-\varphi_1}\omega_0^n,$$

with φ_1 determined uniquely by

$$\frac{1}{V}\int_M \mathrm{e}^{f-\varphi_1}\omega_0^n = 1.$$

Iterating this procedure we have $\mathrm{Ric}^{(-k)}(\omega_0) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\sum_{j=1}^{k}\varphi_j$ for each $k \in \mathbb{N}$ where φ_k is the solution of the Monge–Ampère equation

$$\left(\omega_0 + \sqrt{-1}\partial\bar{\partial}\sum_{j=1}^{k}\varphi_j\right)^n = \mathrm{e}^{f-\sum_{j=1}^{k-1}\varphi_j}\omega_0^n, \tag{2.14}$$

and each of the φ_j is uniquely determined by

$$\frac{1}{V}\int_M \mathrm{e}^{f-\sum_{j=1}^{k-1}\varphi_j}\omega_0^n = 1. \tag{2.15}$$

The complex Monge–Ampère system (2.14) is equivalent to (1.1). From now on we set

$$\Phi_k = \sum_{j=1}^{k}\varphi_j$$

and

$$\omega_k = \omega_{\Phi_k} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi_k.$$

2.5. Monotonicity of the energy functionals

The following proposition describes the monotonicity of the K-energy and F_1 along the iteration. Note that the equivalent for the Kähler–Ricci flow is well known.

Proposition 2.5. *With notation as above,*

$$\begin{aligned} &E_0(\omega_0, \omega_k) = -(I-J)(\omega_0, \omega_k) - \frac{1}{V}\int_M \Phi_{k-1}\omega_k^n + \frac{1}{V}\int_M h\omega_0^n \leqslant 0,\\ &F_1(\omega_0, \omega_k) = F^0(\omega_0, \Phi_k) \leqslant 0,\\ &E_1(\omega_0, \omega_k) \leqslant 0, \end{aligned}$$

with equality if and only if ω is Kähler–Einstein.

Furthermore, all the E_j with $j = 0, \dots, n$ decrease along the iteration starting from the second iteration.

Proof. To prove the first inequality we note that

$$\begin{aligned} E_0(\omega_{k-1}, \omega_k) &= \frac{-1}{V}\int_M \varphi_{k-1}\omega_k^n - (I-J)(\omega_{k-1}, \omega_k) - \frac{1}{V}\int_M \varphi_{k-1}(\omega_{k-1}^n - \omega_k^n)\\ &= -(I-J)(\omega_{k-1}, \omega_k) - \frac{1}{V}\int_M \varphi_{k-1}\omega_{k-1}^n. \end{aligned} \tag{2.16}$$

The first term is non-positive with equality if and only if $\omega_k = \omega_{k-1} = \mathrm{Ric}(\omega_k)$, while the second term is non-positive since

$$1 = \frac{1}{V}\int_M \omega_k^n = \frac{1}{V}\int_M \mathrm{e}^{-\varphi_{k-1}}\omega_{k-1}^n \geqslant \frac{1}{V}\int_M (1-\varphi_{k-1})\omega_{k-1}^n. \tag{2.17}$$

Since by the cocyclicity property

$$E_0(\omega,\omega_k)=\sum_{j=1}^{k}E_0(\omega_{j-1},\omega_j),$$

the conclusion follows.

The second inequality follows similarly, since

$$F_1(\omega_{k-1},\omega_k)=-(I-J)(\omega_{k-1},\omega_k)-\frac{1}{V}\int_M\varphi_k\omega_k^n,$$

and now one can use (2.17).

The third inequality follows from

$$E_1(\omega_{k-1},\omega_k)=2F_1(\omega_{k-1},\omega_k)+\frac{1}{V}\int_M\varphi_k(\omega_k^n+\omega_k^{n-1}\wedge\omega_{k-1})$$

from the formula relating the functionals F_1 and E_j. Since both summands are non-negative (note that $\omega_{k-1}-\omega_k=-\sqrt{-1}\partial\bar{\partial}\varphi_k$), we are done. Finally, the decrease of the functionals is proved using (2.11) and (2.13) and the previous steps of the proof. □

3. The dynamics of the Ricci operator

We are now ready to answer the question raised by Nadel [**17**]. Note that some of the results presented in that section were discussed in detail by the author and Rubinstein. These results have been obtained independently but with a similar approach. We refer to [**20**] for further refinements in that direction.

Theorem 3.1. *Let (M,ω) be a Kähler manifold with positive first Chern class. Assume that $\mathrm{Ric}^{(k)}(\omega)=\omega$ for some $k\in\mathbb{N}$. Then ω is Kähler–Einstein.*

Proof. Note that the nonexistence of fixed points of negative order implies that of positive order, and vice versa. Therefore, assume that for some $\omega\in Ka_{c_1}$ and some $l\in\mathbb{N}$ one has $\mathrm{Ric}^{(-l)}(\omega)=\omega$. By the cocycle condition we therefore have

$$0=E_0(\omega,\mathrm{Ric}^{(-l)}\,\omega)=\sum_{i=0}^{l-1}E_0(\mathrm{Ric}^{(-i)}\,\omega,\mathrm{Ric}^{(-i-1)}\,\omega).\tag{3.1}$$

On the other hand, from the first part of (2.7),

$$E_0(\omega,\mathrm{Ric}^{(-1)}\,\omega)=-\frac{1}{V}\int_{M\times[0,1]}(1-t)(\Delta_{\omega_{\varphi_t}}\dot{\varphi}_t)^2\omega_{\varphi_t}^n\wedge\mathrm{d}t-(I-J)(\omega,\mathrm{Ric}^{(-1)}\,\omega).$$

Thus $E_0(\omega,\mathrm{Ric}^{(-1)}\,\omega)\leqslant 0$, with equality if and only if $\mathrm{Ric}^{(-1)}\,\omega=\omega$. Therefore, each of the terms in (3.1) must vanish identically and we conclude that (M,ω) is Kähler–Einstein. □

Proposition 3.2. *Let M be as above and assume that $\omega \in Ka_{c_1}^{(k)}$ for all $k \in \mathbb{N}$ and that ω is not Kähler–Einstein. Then $\lim_{k\to\infty} \mathrm{Ric}^{(k)}(\omega)$ does not exist in $Ka_{c_1}^{(0)}$.*

Proof. If $\omega_\infty = \lim_{k\to\infty} \mathrm{Ric}^{(k)}(\omega)$ exists and is smooth it satisfies $\mathrm{Ric}(\omega_\infty) = \omega_\infty$. But $E_0(\omega, \omega_\infty) > 0$ contradicting (2.3). □

Let G_0 denote the Green function for $\Delta = \Delta_{\bar\partial}$ with respect to (M, ω_0) with $\int_M G_0(x,y)\omega_0^n(y) = 0$ and $A(\omega_0) = -\inf_M G_0$ such that

$$f(x) - \frac{1}{V}\int_M f\omega_0^n = -\frac{1}{V}\int_M G_0(x,y)\Delta f(y)\omega_0^n(y), \quad \forall f \in C^\infty(M).$$

Then, one has the following estimate due to Bando and Mabuchi.

Theorem 3.3 (Bando and Mabuchi [4]). *For ω a Kähler form, one has*

$$A(\omega) \leqslant \tfrac{1}{2} c_n \operatorname{diam}(M, \omega)^2.$$

If $\mathrm{Ric}(\omega) \geqslant \epsilon\omega$ *for some* $\epsilon > 0$ *then* $\operatorname{diam}(M,\omega)^2 \leqslant (\pi^2(2n-1))/\epsilon$ *by Myers's theorem.*

As an immediate corollary we have the following lemma.

Lemma 3.4. *Let M be a Fano manifold. Assume that the K-energy is proper. Let $(\omega_l)_{l\in\mathbb{N}}$ be a sequence of Kähler forms on which the K-energy is bounded from above and such that there exists $l_0(\omega_0) \in \mathbb{N}$ and $\epsilon > 0$ with*

$$\mathrm{Ric}(\omega_l) \geqslant \epsilon\omega_l, \quad \forall l \geqslant l_0.$$

Then there exists a constant C_1 depending only on (M, ω_0) such that

$$\|\Phi_l\|_{C^0} \leqslant C_1, \quad \forall l \in \mathbb{N},$$

where

$$\omega_l = \omega_0 + \sqrt{-1}\partial\bar\partial\Phi_l \quad \text{and} \quad \frac{1}{V}\int_M \mathrm{e}^{f-\Phi_l}\frac{\omega_0^n}{n!} = 1.$$

Proof. Let G_l be the Green function for $\Delta_l = \Delta_{\bar\partial,\omega_l}$ (i.e. the Laplacian with respect to (M, ω_l)) satisfying $\int_M G_l(x,y)\omega_l^n(y) = 0$. Set $A_l = -\inf_{M\times M} G_l$.

Since $-n < \Delta_1\Phi_l$ and $n > \Delta_l\Phi_l$ the Green formula gives

$$\Phi_l(x) - \frac{1}{V}\int_M \Phi_l\omega_0^n = -\frac{1}{V}\int_M G_1(x,y)\Delta\Phi_l(y)\omega_0^n(y) \leqslant nA_1, \tag{3.2}$$

$$\Phi_l(x) - \frac{1}{V}\int_M \Phi_l\omega_l^n = -\frac{1}{V}\int_M G_l(x,y)\Delta\Phi_l(y)\omega_l^n(y) \geqslant -nA_l. \tag{3.3}$$

Hence

$$\operatorname{osc}_M \Phi_l = \sup_M \Phi_l - \inf_M \Phi_l \leqslant n(A_1 + A_l) + I(\omega, \omega_l). \tag{3.4}$$

Since E_0 is proper on $Ka_{c_1}^{(G)}$ in the sense of Tian, if $E_0(\omega_0, \cdot)$ is uniformly bounded from above on a subset of $Ka_{c_1}^{(G)}$ so is the functional $I(\omega, \cdot)$. We conclude that $I(\omega_0, \omega_l)$ is uniformly bounded independently of l. Finally, the proposition follows from Theorem 3.3 which provides a uniform bound for A_l. □

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As a consequence of the properness of the K-energy for Fano Einstein manifolds [**19**,**24**] we obtain the following corollary.

Corollary 3.5. *Let M be a Fano Einstein manifold with no non-trivial holomorphic vector field. Consider the sequence of Kähler metrics $(\omega_k)_{k\in\mathbb{N}}$ defined by the iterations (1.1) and assume that for k sufficiently large there exists a constant $\epsilon > 0$ with*

$$\mathrm{Ric}(\omega_k) \geqslant \epsilon\omega_k.$$

Then ω_k converges to the Kähler–Einstein metric when k tends to infinity in C^∞ topology.

Finally, inspired by [**18**], we derive the following proposition.

Proposition 3.6. *Let M be a Fano manifold with G a maximal compact subgroup of $\mathrm{Aut}(M, J)$. Consider the sequence of G-invariant Kähler metrics ω_k defined by the system (1.1). Assume that there exists a constant $1 > \kappa > 0$ such that for k sufficiently large,*

$$(2-\kappa)\omega_0^n \geqslant \omega_k^n \geqslant \kappa\omega_0^n,$$

where ω_0 is G-invariant Kähler metric. Then M is Kähler–Einstein and ω_k converges to a G-invariant Kähler–Einstein metric when k tends to infinity.

Proof. Thanks to the proof of Lemma 3.4 and Theorem 3.3, we are reduced to prove an upper bound for $I(\omega, \omega_k)$. But if we denote $\Phi_k^+(x) = \sup\{0, \Phi_k(x)\}$ and $\Phi_k^-(x) = \inf\{0, \Phi_k(x)\}$, we obtain

$$\begin{aligned} I(\omega,\omega_k) &\leqslant \frac{1}{V}\int_{\{\omega_k^n \geqslant \omega_0^n\}} (-\Phi_k^-)(\omega_k^n - \omega_0^n) + \frac{1}{V}\int_{\{\omega_0^n \geqslant \omega_k^n\}} (\Phi_k^+)(\omega_0^n - \omega_k^n) \\ &\leqslant (1-\kappa)\frac{1}{V}\int_M (\Phi_k^+ - \Phi_k^-)\omega_0^n \\ &\leqslant (1-\kappa)\operatorname{osc}_M \Phi_k. \end{aligned}$$

Together with

$$\operatorname{osc}_M \Phi_k \leqslant n(A_1 + A_k) + I(\omega, \omega_k),$$

this gives us to the C^0 bound for Φ_k. Now this is a standard procedure to derive the convergence in C^∞ topology [**2**, **12**, **21**, **24**]. □

The Aubin operators

For a Kähler manifold M, we consider the family of Monge–Ampère equations (2.7). We introduce the *Aubin operators* Ric_ϵ by setting

$$\mathrm{Ric}_\epsilon(\omega) = \omega_{\varphi_{1+\epsilon}}$$

for each $\epsilon \in [0, 1]$ such that $\varphi_{1+\epsilon}$, a solution of (2.7), exists. Note that $\mathrm{Ric}_0(\omega) = \mathrm{Ric}^{(-1)}(\omega)$ and $\mathrm{Ric}_1(\omega) = \omega_{\mathrm{KE}}$. Formally, one can think that

$$\mathrm{Ric}_\epsilon = \left(\frac{1}{1-\epsilon}(\mathrm{Ric} - \epsilon\,\mathrm{Id})\right)^{-1}$$

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and that we have defined the following sequence of Monge–Ampère equations

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi_k)^n = \mathrm{e}^{(\epsilon-1)\Phi_{k-1}-\epsilon\Phi_k}(\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi_{k-1})^n. \tag{3.5}$$

Let $G \subseteq \mathrm{Aut}(M,J)$ be a maximal connected compact subgroup as before. Then from uniqueness of solutions of the family of the Monge–Ampère equations (2.7) we conclude that Ric_ϵ maps $Ka_{c_1}^{(G)}$ into itself.

We recall the definition of α-invariant introduced by Tian:

$$\alpha_M = \sup\biggl\{\alpha \geqslant 0 : \sup_{\varphi\in Ka_{c_1}} \int_M \mathrm{e}^{-\alpha\varphi}\omega^n < \infty\biggr\}. \tag{3.6}$$

In [**23**] it is proved that α_M is a positive holomorphic invariant of Fano manifolds. Regarding the existence of the Aubin operators we state the following proposition.

Proposition 3.7.

(i) *Assume that M is Fano. Then the operators Ric_ϵ exist for all $\epsilon \in [0, \min\{1, ((n+1)/n)\alpha_M\})$ [**23**].*

(ii) *Assume in addition that the K-energy is bounded from below. Then the operators Ric_ϵ exist for any $\epsilon \in [0,1)$ [**4**, Theorem 5.7].*

We now recover by a conceptually simpler method a theorem of Tian [**24**].

Corollary 3.8. *Let M be a Fano manifold. Let G be a maximal compact subgroup of $\mathrm{Aut}(M,J)$ and assume that the K-energy is proper on $Ka_{c_1}^{(G)}$ and let $\epsilon \in (0,1)$. Then there exist G-invariant Kähler–Einstein metrics. All such metrics are the limit points of the iterates of Ric_ϵ on $Ka_{c_1}^{(G)}$ in the C^∞-topology.*

Proof. Since the K-energy is proper on $Ka_{c_1}^{(G)}$ and in particular bounded from below, Ric_ϵ are defined for each $\epsilon \in (0,1)$. Let us denote $\omega_k = \mathrm{Ric}_\epsilon^{(k)}(\omega)$ where $\omega_k \in Ka_{c_1}^{(G)}$. By the same reasoning as in the proof of Lemma 3.4 we obtain

$$\mathrm{osc}_M\, \Phi_k \leqslant n(A(\omega_0) + A(\omega_k)) + I(\omega, \omega_k). \tag{3.7}$$

Since by (2.11) and (2.12) the K-energy decreases along iterates we still have a uniform bound on the functional I along the orbits.

Now, on another hand, we have a uniform bound (depending on ϵ) on $A(\omega_k)$. Actually, from the equality

$$\mathrm{Ric}(\omega_k) = (1-\epsilon)\omega_{k-1} + \epsilon\omega_k,$$

one can apply Lemma 3.4. This gives a C^0 bound on the potentials Φ_k.

We shall now prove that one has a C^2 estimate for Φ_k. Since $\|\Phi_k\|_{C^2} \leqslant \max(n+\Delta\Phi_k, n)$, where Δ is the Laplacian with respect to ω_0, we just need to give a bound on $\Delta\Phi_k$. We follow the techniques of [**5**, § 4] and [**2**, Chapter 7]. First of all, we have the obvious bound $\Delta\Phi_k > -n$ since ω_k is Kähler. We shall prove that we have an upper bound for $\Delta\Phi_k$. In local coordinates $\zeta \in \mathbb{C}^n$, we can define

$$u = \Phi_k + \theta,$$

where θ stands for a local potential for the Kähler form ω_0. Then one can define the continuous function on M

$$\eta = \max_{\zeta \neq 0} \frac{u_{\zeta\bar{\zeta}}}{\theta_{\zeta\bar{\zeta}}}.$$

We will apply the maximum principle to the function $\alpha(p) = \log \eta(p) - \kappa \Phi_k(p)$ where κ is a constant that we shall fix later. Note that without loss of generality, we may assume that $u_{i\bar{j}}$ is diagonal and $u_{1\bar{1}} \geqslant \cdots \geqslant u_{n\bar{n}}$ at a point $p_{\max}$ where a maximum is attained. Finally, we fix ζ such that $\eta = u_{\zeta\bar{\zeta}}/\theta_{\zeta\bar{\zeta}}$ at $p_{\max}$. We will need the following technical lemma [**5**, Lemma 4.2].

Lemma 3.9. *Let u be a C^4 plurisubharmonic function with $F = \det(u_{i\bar{j}})$. Then for any direction ζ,*

$$u^{i\bar{j}}(\log u_{\zeta\bar{\zeta}})_{i\bar{j}} \geqslant \frac{(\log F)_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}}.$$

Thanks to the maximum principle applied at $p_{\max}$, one obtains from the lemma

$$\begin{aligned} 0 &\geqslant \frac{(\log(\mathrm{e}^{-(1-\epsilon)\Phi_{k-1}-\epsilon\Phi_k}\det(\theta_{p\bar{q}})))_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}} + \kappa u^{i\bar{j}}\theta_{i\bar{j}} - n\kappa \\ &\geqslant -\frac{(1-\epsilon)(\Phi_{k-1})_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}} - \frac{\epsilon(\Phi_k)_{\zeta\bar{\zeta}}}{u_{\zeta\bar{\zeta}}} - \frac{C_0}{u_{\zeta\bar{\zeta}}} + \left(-C_1 + \frac{\kappa}{C_2}\right)\sum_i \frac{1}{u_{i\bar{i}}} - n\kappa \\ &\geqslant -\frac{(1-\epsilon)(\Phi_{k-1})_{\zeta\bar{\zeta}}}{u_{1\bar{1}}} - \frac{C_0}{u_{1\bar{1}}} + \max(1, C_0)\sum_i \frac{1}{u_{i\bar{i}}} - n\kappa - C_3(\epsilon), \end{aligned}$$

where C_0, C_1, C_2 depend only on ω_0 and M, while κ has been chosen depending on those three constants. Here we have used the fact that Φ_k is C^0 bounded. On the other hand, by the arithmetico-geometric inequality,

$$\sum_{2\leqslant i\leqslant n} \frac{1}{u_{i\bar{i}}} \geqslant \frac{n-1}{(u_{2\bar{2}}\cdots u_{n\bar{n}})^{1/(n-1)}} \geqslant \frac{(n-1)u_{1\bar{1}}^{1/(n-1)}}{\mathrm{e}^{-(1-\epsilon)\Phi_{k-1}-\epsilon\Phi_k}},$$

which induces, using the C^0 bound again, that

$$u_{1\bar{1}}^{n/(n-1)} - C_4 u_{1\bar{1}} - C_5 \leqslant (1-\epsilon)(\Phi_{k-1})_{\zeta\bar{\zeta}},$$

where C_3, C_4, C_5 are positive constants depending only on M, ω_0 and ϵ. Finally, this last inequality shows by taking the trace that

$$(\Delta\Phi_k)^{n/(n-1)} \leqslant C_4'\Delta\Phi_k + C_5' + C_6'\Delta\Phi_{k-1}.$$

But this means in particular that $\sup_M \Phi_k$ cannot tend to $+\infty$ when k varies. Thus, the C^2 estimate for Φ_k holds. In order to get a $C^{2,\beta}(M)$-estimate, one can invoke now directly [**5**, Theorem 5.1]. We may therefore extract a converging subsequence $\mathrm{Ric}_\epsilon^{(k_j)}(\omega)$

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in the $C^{2,\beta}(M,\omega)$-topology whose limit ω_∞ lies in $C^2(M)$. Moreover, since the K-energy is bounded from below we have

$$\lim_{j\to\infty} E_0(\mathrm{Ric}_\epsilon^{(k_j)}(\omega), \mathrm{Ric}_\epsilon^{(k_{j+1})}(\omega)) = \lim_{j\to\infty} \sum_{l=k_j}^{k_{j+1}-1} E_0(\mathrm{Ric}_\epsilon^{(l)}(\omega), \mathrm{Ric}_\epsilon^{(l+1)}(\omega)) = 0.$$

As each of the summands is non-positive one has $E_0(\omega_\infty, \mathrm{Ric}_\epsilon(\omega_\infty)) = 0$ and it follows that $\omega_\infty \in Ka_{c_1}^{(G)}$ is smooth and Kähler–Einstein by (2.11) and (2.12). Since this is true for each converging subsequence we conclude that, in fact, the sequence of iterates itself converges. □

In fact we also get the following theorem.

Theorem 3.10. *Let M be a Fano manifold with no non-trivial holomorphic vector field. Assume that M carries a Kähler–Einstein metric ω_{KE}. Then for any $\epsilon \in (0,1)$ and $\omega \in Ka_{c_1}^{(G)}$ one has $\lim_{k\to\infty} \mathrm{Ric}_\epsilon^{(k)}(\omega) = \omega_{\mathrm{KE}}$ in the C^∞-topology.*

4. Applications and numerical results

4.1. The finite-dimensional picture

In [**10**], Donaldson introduced the notion of a ν-balanced metric for a fixed volume form ν and proved its existence [**10**, Proposition 4] under some very general conditions [**10**, p. 10]. This gives a sequence of canonical Kähler metrics on the manifold that lie in the same Kähler class. As we shall see now, these metrics have the properties to solve the Calabi problem, i.e. to converge towards the Kähler metric that has volume form ν in a given Kähler class.

Let us fix a volume form ν on a smooth projective manifold X with a polarization L, i.e. L is an ample line bundle. Choose $r \in \mathbb{N}$ sufficiently large such that X is embedded by the holomorphic sections of $L^r := L^{\otimes^r}$ in the projective space $\mathbb{P}H^0(X, L^r)^\star$.

Notation. For a smooth hermitian metric $h \in \mathrm{Met}(L)$ on the line bundle L, we denote $c_1(h) \in 2\pi[L]$ its curvature. Furthermore, we set $V = [c_1(L)]^n(X)$ the volume of L, and note

$$N_r = h^0(X, L^r),$$

which is finite since X is compact. In all the following $\mathrm{Met}(\Xi)$ means the space of (smooth) hermitian metrics on the vector space or bundle Ξ.

Definition 4.1. A ν-balanced metric of order r is a fixed point of the map T_ν : $\mathrm{Met}(H^0(X, L^r)) \to \mathrm{Met}(H^0(X, L^r))$,

$$T(\mathsf{H})_{i,j} = \frac{N_r}{V} \int_X \frac{\langle S_i, S_j \rangle}{\sum_{i=1}^{N_r} |S_i|^2} \, \mathrm{d}\nu,$$

where H is a hermitian metric of $H^0(X, L^r)$ and $(S_i)_{i=1,\dots,N_r}$ is an orthonormal basis of $H^0(X, L^r)$ with respect to H. A ν-balanced metric is unique up to action of $\mathrm{SU}(N_r)$.

Donaldson's proof shows that the dynamical system induced by the compositions of the T_ν map has a fixed attractive point in $\mathrm{Met}(H^0(X, L^r))$. Let us consider the Fubini–Study map $\mathrm{FS} : \mathrm{Met}(H^0(X, L^r)) \to \mathrm{Met}(L^r)$ [**10**, p. 4, § 2.2],

$$\mathrm{FS} : \mathsf{H} \mapsto \frac{N_r}{V} \frac{1}{\sum_{i=1}^{N_r} |S_i|^2}$$

with $(S_i)_{i=1,\dots,N_r} \in H^0(X, L^r)$ orthonormal basis with respect to H. Then a ν-balanced metric on $\mathrm{Met}(H^0(X, L^r))$ induces an algebraic metric on $\mathrm{Met}(L^r)$ that we still call ν-balanced.

Theorem 4.2. *Let X be a smooth projective manifold and L be a polarization on X. For r large enough, let us denote $H_r \in \mathrm{Met}(H^0(X, L^r))$ the sequence of ν-balanced metrics of order r. Then the sequence*

$$\frac{1}{2\pi} c_1(\mathrm{FS}(H_r)^{1/r})$$

converges to a Kähler form $\omega_\infty \in [c_1(L)]$ in C^∞ topology that satisfies

$$\omega_\infty^n = \nu.$$

Proof. To prove this theorem, we use the powerful Calabi–Yau theorem. Hence we know the existence of a Kähler form ω in $[c_1(L)]$ such that $\omega^n = \nu$. We use Wang's theorem [**25**] with the trivial bundle and L. There is a Hermite–Einstein metric on these bundles and the metrics H_r are 'balanced' with respect to ω in the sense studied by Wang. This is due to the obvious fact that the considered bundles are Gieseker stable. Thus, one obtains directly the convergence of the sequence of metrics $\mathrm{FS}(H_r)^{1/r} \in \mathrm{Met}(L)$ to the metric h_L with $(1/2\pi)c_1(h_L) = \omega$. □

We now assume that M is a Fano manifold and consider the polarization $L = -K_M > 0$ given by the anticanonical line bundle. Let us consider a smooth hermitian metric h_0 on L with $(1/2\pi)c_1(h_0) = \omega_0$ and let us call f_{ω_0} the Ricci deviation of ω_0. Now for any integer $k \geqslant 1$, we call ν_k the volume form induced by the Kähler metric $\mathrm{Ric}^{(-k)}(\omega_0)$,

$$\nu_k = (\mathrm{Ric}^{(-k)}(\omega_0))^n.$$

We define for each k, the ν_k-balanced metric of order r in the following way.

First, we consider $\widetilde{\mathrm{Hilb}}_{\omega_0}(\langle \cdot\,, \cdot \rangle)$ the L^2-metric on the space $H^0(M, L^r)$ induced from the metric $\langle \cdot\,, \cdot \rangle \in \mathrm{Met}(L^r)$,

$$\widetilde{\mathrm{Hilb}}_{\omega_0}(\langle \cdot\,, \cdot \rangle)(S_i, S_j) = \langle S_i, S_j \rangle_{\widetilde{\mathrm{Hilb}}_{\omega_0}(\langle \cdot\,, \cdot \rangle)} = \int_M \langle S_i, S_j \rangle \mathrm{e}^{f_{\omega_0}} \omega_0^n.$$

Now, for a given hermitian metric H_0 on $H^0(M, L^r)$, we define the metric $\mathrm{FS}(H_0)$ on L^r by

$$\sum_{i=1}^{N_r} |S_i|^2_{\mathrm{FS}(H_0)} = \frac{N_r}{V},$$

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where the $(S_i)_{i=1,\dots,N_r}$ form an H_0-orthonormal basis of $H^0(M, L^r)$. Then, from [**10**, Proposition 4], we know that the dynamical system $\mathrm{FS} \circ \widetilde{\mathrm{Hilb}}_{\omega_0}$ has an attractive fixed point $h_{\omega_0,r}$. We call

$$H_{0,r} = \widetilde{\mathrm{Hilb}}_{\omega_0}(h_{\omega_0,r}) \in \mathrm{Met}(H^0(M, L^r)).$$

Moreover, we obtain a Kähler form

$$\omega_{1,r} = \frac{1}{2\pi} c_1(h^{1/r}_{\omega_0,r}) \in [c_1(L)].$$

For the second step, i.e. in order to find the balanced metric $h_{\mathrm{Ric}^{-1}(\omega_0),r}$, we introduce the operator $\widetilde{\mathrm{Hilb}}_{\omega_{1,r}}$,

$$\widetilde{\mathrm{Hilb}}_{\omega_{1,r}}(\langle \cdot\,,\cdot \rangle)(S_i, S_j) = \int_M \langle S_i, S_j \rangle \mathrm{e}^{-\varphi_{0,r}} \omega^n_{1,r},$$

where $\varphi_{0,r}$ is the potential of the metric $h^{1/r}_{\omega_0,r} \in \mathrm{Met}(L)$. Iterating this procedure leads us to define at each step k a dynamical system

$$\mathrm{FS} \circ \widetilde{\mathrm{Hilb}}_{\omega_{k,r}} : \mathrm{Met}(L^r) \to \mathrm{Met}(L^r)$$

which has an attractive fixed point $h_{\omega_{k,r}} \in \mathrm{Met}(L^r)$ and we write $h_{\omega_{k,r}} = \mathrm{e}^{-r\varphi_{k,r}} h_{\omega_{0,r}}$. Hence, we have

$$\widetilde{\mathrm{Hilb}}_{\omega_{k,r}}(\langle \cdot\,,\cdot \rangle)(S_i, S_j) = \int_M \langle S_i, S_j \rangle \mathrm{e}^{-\varphi_{k-1,r}} \omega^n_{k,r} \tag{4.1}$$

and

$$H_{k,r} = \widetilde{\mathrm{Hilb}}_{\omega_k}(h_{\omega_k,r}) \in \mathrm{Met}(H^0(M, L^r)).$$

Corollary 4.3. *Under the above assumptions, and for r sufficiently large, the sequence*

$$\frac{1}{2\pi} c_1(h^{1/r}_{\omega_{k,r}})$$

converges when $r \to +\infty$ to the solution of the Monge–Ampère equation (2.14) in C^∞ topology with exponential speed of convergence.

Conjecture 4.4. *Under the above assumptions, the sequence $(1/2\pi)c_1(h^{1/r}_{\omega_{k,r}})$ converges when $k \to +\infty$ to a Kähler metric $\omega_r \in Ka_{c_1}$ in C^∞ topology with exponential speed of convergence. If M has a Kähler–Ricci soliton, then ω_r converges to a Kähler–Ricci soliton in the sense of Cheeger–Gromov.*

Let us describe now how our discussion can be useful for numerical approximations of Kähler–Einstein metrics on Fano manifolds. One has to notice at this stage that we can write (4.1) as

$$\widetilde{\mathrm{Hilb}}_{\omega_{k,r}}(\langle \cdot\,,\cdot \rangle)(S_i, S_j) = \int_M \langle S_i, S_j \rangle \biggl(\frac{V}{N_r} \sum_i |\widetilde{S_{i,k-1}}|^2 \biggr)^{-1/r}, \tag{4.2}$$

where the $(\widetilde{S_{i,k-1}}) \in H^0(M, L^r)$ form an orthonormal basis with respect to the metric $H_{k-1,r}$, i.e. the ν_{k-1}-balanced metric of order r. Note that the right-hand side of (4.2) makes sense since the term $(\sum_i |\widetilde{S_{i,k-1}}|^2)^{-1}$ can be considered as a section of $K_M^r \otimes \overline{K_M}^r$.

In order to obtain the ν_k-balanced metric of order r, i.e. $H_{k,r}$, one needs to iterate the operator $\widetilde{\mathrm{Hilb}}_{\omega_{k,r}} \circ FS$. This gives a sequence $(H_{k,r,p})_{p\in\mathbb{N}} \in \mathrm{Met}(H^0(M, L^r))$ and we choose the first term to be $H_{k,r,0} = H_{k-1,r}$. Hence, with our notation, $H_{k,r} = H_{k,r,\infty}$.

We now remark that if one expects the algorithm to be convergent, and thus $H_{k-1,r}$ to be close to $H_{k,r}$ for large k and r, then it is natural to assume that $H_{k-1,r,1}$ is close to $H_{k-1,r,\infty} = H_{k,r,0}$, i.e. just one step is sufficient to get the ν_k-balanced metric. This is precisely what is done implicitly in [**10**, § 2.2.2]. Thus, this justifies at least formally the definition of Ricci* balanced metrics.

Definition 4.5. Let M be a Fano manifold. Fix $r \in \mathbb{N}^*$ sufficiently large. We define the operator $\tilde{T} : \mathrm{Met}(H^0(M, -K_M^r)) \to \mathrm{Met}(H^0(M, -K_M^r))$ by

$$\tilde{T}(\mathsf{H})_{ij} = \left(\frac{N_r}{V}\right)^{1+1/r} \int_M \frac{\langle S_i, S_j\rangle}{(\sum_{i=1}^{N_r} |S_i|^2)^{1+1/r}} \tag{4.3}$$

for $(S_i) \in H^0(M, -K_M^r)$ a H-orthonormal basis. A Ricci balanced metric is a fixed point of the operator $\tilde{T}$.

When Ricci balanced metrics exist *a priori*, we expect their behaviour to be understood for large r via the iterations of the Ric^{-1} operator, and thus to be related to the Kähler–Ricci flow. In that direction, we obtain the following theorem.

Theorem 4.6. *Assume that M is a Fano manifold with no non-trivial holomorphic vector field. If M has a Kähler–Einstein metric, then for r sufficiently large, there exists a sequence of Ricci balanced metrics $H_{\mathrm{Ric},r} \in \mathrm{Met}(H^0(M, -K_M^r))$ unique up to action of $\mathrm{SU}(N_r)$. Each Ricci balanced metric is an attractive fixed point of the map $\tilde{T}$. Furthermore, the sequence*

$$\frac{1}{2\pi} c_1(\mathrm{FS}(H_{\mathrm{Ric},r})^{1/r})$$

converges when $r \to +\infty$ to the Kähler–Einstein metric ω_{KE} in C^∞ topology and the speed of convergence is $O(1/r)$.

Proof. The proof is similar to the proof of the main result of [**8**,**9**]. Actually, a Ricci balanced metric $h_{\mathrm{Ric},r} \in \mathrm{Met}(-K_M^r)$ satisfies that its associated Bergman kernel is constant on the manifold, i.e. for all $p \in M$,

$$\sum_{i=1}^{N_r} |S_i|^2_{h_{\mathrm{Ric},r}}(p) = \frac{N_r}{V},$$

where the sections $(S_i)_{i=1,\dots,N_r} \in H^0(M, -K_M^r)$ are orthonormal with respect to the inner product

$$\langle a, b\rangle = \int_M (h_{\mathrm{Ric},r})^{(r+1)/r} \otimes a \otimes \bar{b},$$

* In [**10**], these metrics are called ‘canonically’ balanced but we consider our denomination more enlightening on the role of the Ricci operator.

and $N_r = \dim H^0(M, -K_M^r)$. Note that one can generalize the results of Lu [**14**] on the asymptotic of the Bergman kernel in that case. The details will appear in a forthcoming paper where the relationship with GIT stability will be studied. □

In our implementations for finding numerical approximations of Kähler–Einstein metrics, we use this notion and consider the iterations of the $\tilde{T}$ operator. Our tests on toric manifolds have shown that the sequence of metrics defined by (4.2) and (4.3) have similar behaviours (i.e. iterating many times $\widetilde{\mathrm{Hilb}}_{\omega_{k,r}} \circ \mathrm{FS}$ or just once) and converge to the Kähler–Einstein metric when it exists *a priori*. These procedures have the advantage to skip the computation of the determinant of the Fubini–Study metric as required by the original notion of balanced metric studied in [**8**, **15**, **27**]. Finally, we remark that even in the case of $(\mathbb{CP}^1, \mathcal{O}(k))$, the sequence of balanced and Ricci balanced metrics converge at different speeds towards the Fubini–Study metric [**10**]. We shall give now an argument to explain this fact.

Proposition 4.7. *Assume that we are under the conditions of the above theorem. Then a sequence of iterates of the map $\tilde{T}$ converges with exponential speed towards the Ricci balanced metric $H_{\infty,r}$. If λ_1 is the smallest eigenvalue of the Laplacian Δ_{KE} of the Kähler–Einstein metric, then the ratio of convergence is controlled by $\mathrm{e}^{-\lambda_1/(4\pi r)}$ for r large enough.*

Proof. Actually, if one considers $H_{\mathrm{Ric},r}$ a Ricci-balanced metric of order r, and $\tilde{H} = H_{\mathrm{Ric},r} + \epsilon$ another metric, then

$$\tilde{T}(\tilde{H})_{i\bar{j}} = \left(\frac{N_r}{V}\right)^{1+1/r} \int_M \frac{\langle S_i, S_j\rangle}{((N_r/V) - \sum_{\alpha,\beta} \epsilon_{\alpha\beta}\langle S_\alpha, S_\beta\rangle)^{1+1/r}},$$

and, up to the first order, one has

$$\tilde{T}(\tilde{H})_{i\bar{j}} = (H_{\mathrm{Ric},r})_{i\bar{j}} + \left(1 + \frac{1}{r}\right)\frac{V}{N_r}\int_M \sum_{\alpha,\beta} \langle S_i, S_j\rangle \epsilon_{\alpha\beta}\langle S_\alpha, S_\beta\rangle \,\mathrm{FS}(H_{\mathrm{Ric},r})^{1/r} + O(\epsilon^2).$$

From the previous theorem, $\mathrm{FS}(H_{\infty,r})^{1/r}$ converges to the volume form ω_{KE}^n of the Kähler–Einstein metric, and the speed of convergence is in $O(1/r)$. The operator

$$\tilde{Q} : \epsilon \mapsto \tilde{Q}(\epsilon)_{i\bar{j}} = \left(1 + \frac{1}{r}\right)\frac{V}{N_r}\int_M \sum_{\alpha,\beta} \epsilon_{\alpha\beta}\langle S_\alpha, S_\beta\rangle\langle S_i, S_j\rangle \,\mathrm{FS}(H_{\mathrm{Ric},r})^{1/r}$$

represents the linearization of our algorithm close to the Ricci balanced point. Now, from [**10**, **13**], when r tends to infinity, $\tilde{Q}$ is a quantification of the operator

$$\left(1 + O\left(\frac{1}{r}\right)\right)\exp\left(-\frac{\Delta_{\mathrm{KE}}}{4\pi r}\right).$$

Thus, the speed of convergence for finding the Ricci balanced metric is controlled by

$$\left(1 + O\left(\frac{1}{r}\right)\right)\exp\left(-\frac{\lambda_1}{4\pi r}\right),$$

where $\lambda_1 > 0$ is the smallest eigenvalue of the Laplacian Δ_{KE}. □

We shall now compute the complexity of our algorithm for finding Ricci balanced metrics. Suppose we are looking for an approximation of a Kähler–Einstein metric with a small error ε for the C^∞ topology. Hence, in view of Theorem 4.6, we choose $r \sim 1/((1-\lambda)\varepsilon)$ (where $0 < \lambda < 1$ is a constant that we shall fix later) and we compute a Ricci balanced metric of order r. With $n = \dim M$, one has by Riemann–Roch $N_r \sim r^n$. It is necessary to fix at least N_p points on our manifold M with $N_p > \frac{1}{2}N_r(N_r+1)$ since we are looking for an hermitian metric in $\mathrm{Met}(H^0(L^k))$ and thus need to solve at least $\frac{1}{2}N_r(N_r+1)$ equations for that. We believe that $N_p \sim r^{2n}$ is a reasonable choice. We do not take into account the complexity of finding the points on the manifold since it needs to be done once for all. At each iteration of the $\tilde{T}$ map, we inverse a hermitian matrix of size $N_r \times N_r$ to obtain a basis of orthonormal sections, which asks *a priori* $N_r^2 \log(N_r) \sim nr^{2n}\log(r)$ operations. Now, we also need to compute the Bergman function $\sum_i |S_i|^2$ for all the N_p points, and this can be considered as evaluating N_r polynomials of degree r in n variables. Finally, each iteration of the $\tilde{T}$ map has complexity

$$\mathcal{C}(\tilde{T}) \sim nr^{2n}\log(r) + r^{2n}\left(r^n \binom{n}{r}\right) = \Theta(r^{4n}).$$

Now, we have exponential speed of convergence towards the Ricci balanced metric thanks to the previous proposition. Thus, one needs approximatively $k(\lambda,\varepsilon)$ iterations of the $\tilde{T}$ map if we want to approximate the Ricci balanced metric with error $\lambda\varepsilon$ where

$$k = \Theta\left(r \log\left(\frac{1}{\lambda\varepsilon}\right)\right).$$

Finally, the whole process has complexity

$$\mathcal{C}_{\mathrm{Ricci}} \sim k\mathcal{C}(\tilde{T}) = \Theta\left(\frac{\log(1/\lambda\varepsilon)}{((1-\lambda)\varepsilon)^{4n+1}}\right).$$

Since we have freedom on the choice of λ we obtain that there exist positive constants c_1, c_2 such that asymptotically when $\varepsilon \to 0$,

$$\frac{c_1}{\varepsilon^{4n+1}} \leqslant \mathcal{C}_{\mathrm{Ricci}} \leqslant \frac{c_2}{\varepsilon^{4n+2}}.$$

For the sake of clarity, we now compare the efficiency of the algorithm for finding Ricci balanced metrics and the algorithm for finding balanced metrics in the sense of [**8**,**15**,**27**].

Definition 4.8. Let X be a projective manifold and L a polarization on X. For $r \in \mathbb{N}^*$ sufficiently large, we consider the operator $T : \mathrm{Met}(H^0(X,L^r)) \to \mathrm{Met}(H^0(X,L^r))$,

$$T(\mathsf{H})_{ij} = \frac{N_r}{V}\int_X \frac{\langle S_i, S_j\rangle}{\sum_{i=1}^{N_r}|S_i|^2} c_1(\mathrm{FS}(\mathsf{H})^{1/r})^n \tag{4.4}$$

for $(S_i) \in H^0(X,L^r)$ a H-orthonormal basis. A balanced metric is a fixed point of the operator T.

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Assume now that M is a Fano manifold with no non-trivial holomorphic vector field and carries a Kähler–Einstein metric. Fixing $L=-K_M$ in the previous definition, the main theorem of [**8**] shows the existence and the convergence of a sequence of balanced metrics towards the Kähler–Einstein metric. This convergence is proved to be with speed $O(1/r)$. We have an analogue of Proposition 4.7.

Proposition 4.9. *Let X be a smooth projective manifold, L a polarization on X such that $\mathrm{Aut}(X,L)$ is discrete and there exists a constant scalar curvature Kähler metric ω_∞ in $[c_1(L)]$. Then the iterates of the map T converge with an exponential speed of convergence. If λ_1 is the smallest eigenvalue of the Laplacian Δ_∞ of the metric ω_∞, then the ratio of convergence is controlled by*

$$(1+\tfrac{1}{2}n\pi)\mathrm{e}^{-\lambda_1/(4\pi r)}$$

for r large enough.

Proof. We do essentially the same computation as before. Close to the balanced point, we need to compute to the linearization of the T map. If one considers H_r a balanced metric of order r, ω_r the curvature of $\mathrm{FS}(H_r)^{1/r}$ and $H'=H_r+\epsilon$ another metric, then

$$T(H')_{i\bar{j}}=\int_X\frac{\langle S_i,S_j\rangle}{1-(V/N_r)\sum_{\alpha,\beta}\epsilon_{\alpha\beta}\langle S_\alpha,S_\beta\rangle}\times\biggl(\omega_r+\frac{\sqrt{-1}}{r}\partial\bar{\partial}\log\biggl(1-\frac{V}{N_r}\sum_{\alpha,\beta}\epsilon_{\alpha\beta}\langle S_\alpha,S_\beta\rangle\biggr)\biggr)^n,$$

and, up to the first order, one has

$$T(H')_{i\bar{j}}=(H_r)_{i\bar{j}}+\frac{V}{N_r}\int_X\sum_{\alpha,\beta}\langle S_i,S_j\rangle\epsilon_{\alpha\beta}\langle S_\alpha,S_\beta\rangle\omega_r^n \tag{4.5}$$

$$+\frac{V}{N_r}\int_X\langle S_i,S_j\rangle\frac{1}{2r}\Delta_r\biggl(\sum_{\alpha,\beta}\epsilon_{\alpha\beta}\langle S_\alpha,S_\beta\rangle\biggr)\omega_r^n+O(\epsilon^2). \tag{4.6}$$

Here Δ_r means the Laplacian with respect to the Kähler metric ω_r. Since $\|\omega_r-\omega_\infty\|_{C^\infty}=O(1/r)$, [**13**] shows that the second term of the right-hand side of (4.5) is a quantification of the operator

$$\biggl(1+O\biggl(\frac{1}{r}\biggr)\biggr)\exp\biggl(-\frac{\Delta_\infty}{4\pi r}\biggr).$$

We now briefly explain what the asymptotic behaviour of the operator is:

$$Q_\Delta:\epsilon\mapsto Q_\Delta(\epsilon)_{i\bar{j}}=\frac{V}{rN_r}\int_X\langle S_i,S_j\rangle\Delta_r\biggl(\sum_{\alpha,\beta}\epsilon_{\alpha\beta}\langle S_\alpha,S_\beta\rangle\biggr)\omega_r^n.$$

In order to do that, we apply the localization techniques of [**13**].

Let $P_r(z,z')$ be the smooth kernel of the orthogonal projection from $C^\infty(X,L^r)$ to $H^0(X,L^r)$ with respect to the L^2 metric induced by $\mathrm{FS}(H_r)$ and the volume form ω_r^n.

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We are interested in the behaviour along the diagonal of the integral operator associated to $P_r(z,z')\Delta_r P_r(z,z')$, i.e.

$$\boldsymbol{Q}_\Delta(f)(z) \mapsto \frac{1}{r^{n+1}} \int_X P_r(z,z')\Delta_r P_r(z,z') f(z')\omega_r^n,$$

for $f \in C^\infty(X)$ and $\Delta_r = \Delta_{r,z}$. On the other hand, we also introduce the Bergman kernel on $\mathbb{C}^n$ by

$$P(Z,Z') = \exp\biggl(-\tfrac{1}{2}\pi\sum_{i=1}^{n}(|z_i|^2 + |z_i'|^2 - 2z_i\bar{z}_i')\biggr) = \mathrm{e}^{-(\pi/2)(|Z|^2+|Z'|^2-2Z*Z')},$$

where $Z = (z_1,\dots,z_n)$ and $Z' = (z_1',\dots,z_n')$ and $Z * Z' = \sum_{i=1}^n z_i\bar{z}_i'$. The localization techniques of [**7**] show the convergence (at first order) when $r \to +\infty$,

$$\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{1}{r^n} P_r(Z,Z') \to \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} P(\sqrt{r}Z, \sqrt{r}Z'),$$

where $|Z|$, $|Z'|$ are small enough. In particular, under the same conditions, we have the convergence when $r \to +\infty$

$$\frac{1}{r^{2n}} P_r(Z,Z')\Delta_Z P_r(Z,Z') \to P(\sqrt{r}Z,\sqrt{r}Z')\Delta_Z P(\sqrt{r}Z,\sqrt{r}Z'), \tag{4.7}$$

and we are led to identify the right-hand side of (4.7). We denote $(Z_r, Z_r') = (\sqrt{r}Z, \sqrt{r}Z')$. In normal coordinates at $Z = 0$ and with respect to

$$\Delta = -\sum_i \frac{\partial^2}{\partial Z_j^2}$$

we obtain

$$\begin{aligned}
&(P(Z_r,Z_r')\Delta P(Z_r,Z_r'))_{|Z=0} \\
&\qquad = \pi r \mathrm{e}^{-\pi r(|Z|^2+|Z'|^2-2Z*Z')}\biggl(n - \pi r\biggl(\biggl|\sum_i z_i\biggr|^2 - 2\sum_{i,j} z_i\bar{z}_j'\biggr)\biggr)_{|Z=0} \\
&\qquad = \pi r n \mathrm{e}^{-\pi r|Z'|^2}.
\end{aligned}$$

From [**13**], one knows that $r^n \int_{|Z'|<\epsilon} \mathrm{e}^{-r\pi|Z'|^2} f(Z')\omega^n$ is a quantification of the operator $\exp(-\Delta_\omega f/4\pi r)$ when $r \to +\infty$. Thus, we get now that at first order

$$\biggl|\boldsymbol{Q}_\Delta(f) - n\pi\exp\biggl(-\frac{\Delta_\infty f}{4\pi r}\biggr)\biggr| \leqslant \frac{C}{r}\|f\|_2. \tag{4.8}$$

Note that we could also have derived this expression by using the Lichnerowicz formula. Finally, with (4.5), (4.6) and (4.8), we are done. □

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Finally, we estimate the complexity $\mathcal{C}_{\mathrm{bal}}$ of the algorithm for finding a balanced metric. The computation is very similar to the previous case of Ricci balanced metric. The main difference is the computation of the term giving the volume form

$$\left(\frac{1}{r}\sqrt{-1}\partial\bar{\partial}\log\sum_{i=1}^{N_r}|S_i|^2\right)^n.$$

This requires the evaluation over N_p points of N_r sections of degree r, of degree $r-2$ (derivatives with respect to $\partial\bar{\partial}$) and of degree $r-1$ in n variables (derivatives with respect to ∂ and $\bar{\partial}$). Thus, the complexity of one iteration of the T map is

$$\begin{aligned}\mathcal{C}(T) &\sim nr^{2n}\log(r) + r^{2n}\left(r^n\left(\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}\right)\right) \\ &\sim 4\mathcal{C}(\tilde{T}),\end{aligned} \tag{4.9}$$

and of course

$$\mathcal{C}_{\mathrm{Ricci}} \leqslant \tfrac{1}{4}\mathcal{C}_{\mathrm{bal}}.$$

4.2. The case of the projective plane blown up in three points

Let us consider the toric Fano manifold M_0 given by blowing up $\mathbb{P}^2$ in three (non-aligned) points. From a result of Song, its α-invariant is 1 and thus M_0 possesses a Kähler–Einstein metric (this is also a consequence of Tian's work of classification of Einstein Del Pezzo surfaces). Let us mention that the Kähler–Einstein metric on this manifold has been very recently studied in [**11**] by simulating the Ricci flow with partial differential equation techniques.

We implement our algorithms (i.e. in order to find balanced and Ricci balanced metrics) using the special symmetries on M_0. The computations of the points on the manifolds are relatively quick since we are essentially reduced to a two-dimensional real manifold and we can use the fact that the polytope has a dihedral symmetry group D_6 generated by the action of $\mathbb{Z}_2$ (reflections) and $\mathbb{Z}_6$ (rotations). The fan of this toric variety is given by the six rays spanned by

$$v_0 = (1,0), \quad v_1 = (1,1), \quad v_2 = (0,1), \quad v_3 = (-1,0), \quad v_4 = (-1,-1), \quad v_5 = (0,-1),$$

and as it is well known that the polytope is actually the hexagon. There are different ways to choose the points on M_0 but we decided to just generate the points on one of the six affine charts associated to the cone formed by pairs of rays (v_i, v_{i+1}) (i.e. by defining a certain cut-off function).

Our program is written in C++ (compiler gcc 3.4.6) and can be launched essentially with four different algorithms. For each algorithm we print the scalar curvature of the computed metric at each point of the polytope associated to M_0. This gives pictures with different colours depending on how much the scalar curvature is close to 1 (see Figure 1). The computation of the scalar curvature is possible with exact precision (up to the machine precision) since our metrics are all algebraic. It has the disadvantage

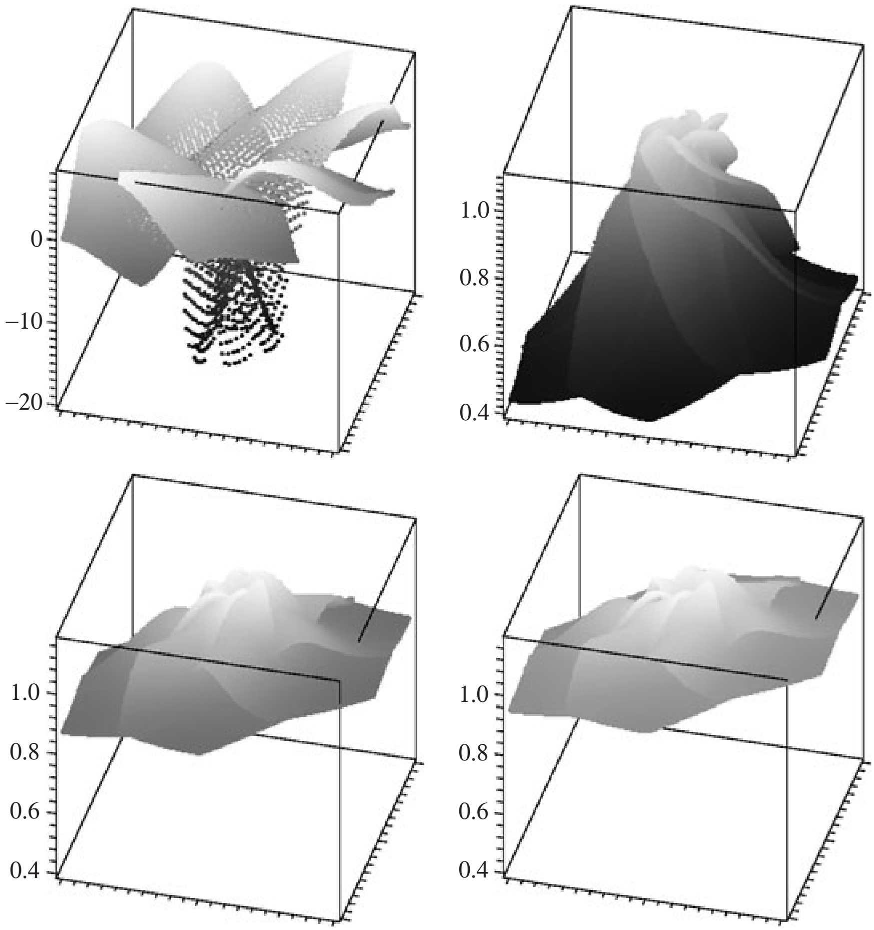

Figure 1. Value of the scalar curvature for the metric obtained after 1, 10, 20 and 40 iterations of the $\tilde{T}$ map over M_0.

to take time since it involves derivatives of order 4, but on the other hand is necessary to check the accuracy of our program and we will see later how this can be useful to improve our algorithms. Some animated pictures generated by the program and the program itself can be downloaded from the author's website: www.latp.univ-mrs.fr/~jkeller/julien-keller-progs.html. (Some other programs for other Fano Einstein surfaces will be made available at this address.) Despite this loss of time, all the four algorithms (for the given parameters below) can be run in one minute or less on a decent desktop computer. We now describe the results for each algorithm.

For the computation of the balanced metric as defined in [**8**–**10**], we choose the parameter $r = 8$ (see § 4.1) and compute approximatively 10^4 points. After 50 iterations, the average scalar curvature on the manifold is 0.95 and the maximum error is 16%.

For the computation of the Ricci balanced metric as defined in [**10**] or our discretization of the Ricci flow (4.2), we fix again $r = 8$ and compute approximatively 5×10^4 points.

After 35 iterations, the average scalar curvature on the manifold is now 0.99 and the maximum error is less than 4%.

Finally, we try to improve the our first two algorithms by using the metric of order r to compute the metric of order $r+1$. This is based on a very simple argument that we discuss briefly. If one knows the balanced metric $h_r \in \mathrm{Met}(L^r)$ (or Ricci balanced metric) of order r, then one has in particular the relation

$$\sum_i |S_i|^2_{h_r}(p) = \frac{N_r}{V}$$

for all the points p on the manifold. Here the sections $(S_i)_{i=1,\dots,N_r}$ form an orthonormal basis of $H^0(M, L^r)$ with respect to the L^2-metric corresponding to the choice of our algorithm. But now, by using the asymptotic of the Bergman function (see [**14**]) one can also write on this surface

$$\sum_i |S_i|^2_{h_r}(p) = r^2 + \tfrac{1}{2} r\,\mathrm{scal}(c_1(h_r))(p) + \Gamma(p) + O(1/r),$$

where $\Gamma(p)$ is a certain function which in fact is an algebraic expression of the curvature of $c_1(h_r)$ and its derivatives. Hence, once we have computed h_r, it is clear that we can deduce the value of Γ at each point of the manifold. Now at order $r+1$, we look for a metric $\tilde{h}_{r+1}$ such that

$$\begin{aligned}\sum_{i=1}^{N_{r+1}} |S_i|^2_{\tilde{h}_{r+1}}(p) &= \frac{N_{r+1}}{V} + \Gamma(p)\\ &= \frac{1}{V}(N_{r+1} + \tfrac{1}{2} r(1 - \mathrm{scal}(c_1(h_r))(p)))\end{aligned}$$

with respect to the corresponding L^2-metric. Roughly speaking, it corresponds to force the algorithm to get a metric with constant scalar curvature up to an error of size

$$O\biggl(\frac{1}{(r+1)^2}\biggr), \tag{4.10}$$

instead of only $O(1/(r+1))$ with the previous algorithms. We apply the same trick for both balanced and Ricci balanced metrics. We call these new sequence of metrics one-step recursively balanced ('1-s.bal.' for short) or one-step recursively Ricci balanced ('1-s.R.bal.'). The advantage of this method is that it is particularly simple (at least for dimension 2) to program it, since we have already coded the computation of the scalar curvature.

Note that the computation of the scalar curvature in the process gives a complexity similar to $\mathcal{C}(T)$, see (4.9). Hence, thanks to (4.10), the complexity of the computation of one-step recursively Ricci balanced is

$$\mathcal{C}(\text{1-s.R.bal.}) \simeq \sqrt{\mathcal{C}(\text{Ricci})} + 4\sqrt{\mathcal{C}(\text{Ricci})} \simeq 5\sqrt{\mathcal{C}(\text{Ricci})}.$$

This is particularly efficient and let us hope that it is possible to compute Kähler–Einstein metric on 3-folds and 4-folds with reasonable precision and very few symmetries.

Table 1. *Results for the different algorithms for the third Del Pezzo surface.*

Method ($r = 4$)	Balanced	1-s.bal.	Ricci balanced	1-s.R.bal.
Average scalar curvature	0.786	0.888	0.949	0.984
Maximum scalar curvature	1.079	1.041	1.012	1.041
Minimum scalar curvature	0.618	0.717	0.820	0.867
Time (s)	9.1	16.1	8.0	15.5

Table 1 gives an overview of the results for $r = 4$, a choice of 10^4 points on the manifold M_0 and 15 iterations for each algorithm.

As an example, we obtain Figure 1 for the scalar curvature (printed in Z coordinate) over M_0 after various iterations of the Ricci balanced algorithm. Note that here $r = 12$ and the first metric is chosen randomly. One can see that for 40 iterations we get a metric with scalar curvature almost equal to 1 everywhere over the hexagonal polytope. Using this metric, we can find numerical approximations (see www.latp.univ-mrs.fr/~jkeller/julien-keller-progs.html) of geodesics for the Kähler–Einstein metric on M_0. It seems to give a numerical evidence that the geodesic equations on this manifold form an integrable system.

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References

1. T. Aubin, Réduction du cas positif de l'équation de Monge–Ampère sur les variétés kählériennes compactes à la démonstration d'une inégalité, *J. Funct. Analysis* **57** (1984), 143–153.
2. T. Aubin, *Some nonlinear problems in Riemannian geometry* (Springer, 1998).
3. S. Bando, The K-energy map, almost Kähler–Einstein metrics and an inequality of the Miyaoka–Yau type, *Tohoku Math. J.* **39** (1987), 231–235.
4. S. Bando and T. Mabuchi, Uniqueness of Kähler–Einstein metrics modulo connected group actions, in *Algebraic Geometry, Sendai, 1985*, pp. 11–40, Advanced Studies in Pure Mathematics, Volume 10 (North-Holland, Amsterdam, 1987).
5. Z. Blocki, The complex Monge–Ampère equation on compact Kähler manifolds, notes available at http://gamma.im.uj.edu.pl/~blocki/publ/ln/index.html (2007).
6. X. X. Chen and G. Tian, Ricci flow on Kähler–Einstein surfaces, *Invent. Math.* **147** (2002), 487–544.
7. X. Dai, K. Liu and X. Ma, On the asymptotic expansion of Bergman kernel, *J. Diff. Geom.* **72**(1) (2006), 1–41.

8. S. K. Donaldson, Scalar curvature and projective embeddings, I, *J. Diff. Geom.* **59** (2001), 479–522.
9. S. K. Donaldson, Scalar curvature and projective embeddings, II, *Q. J. Math.* **56** (2005), 345–356.
10. S. K. Donaldson, Some numerical results in complex differential geometry, preprint (available at arxiv.org/abs/math/0512625, 2005).
11. C. Doran, M. Headrick, C. P. Herzog, J. Kantor and T. Wiseman, Numerical Kähler–Einstein metric on the third del Pezzo, *Commun. Math. Phys.* **282**(2) (2008), 357–393.
12. S. Kolodziej, *The complex Monge–Ampère equation and pluripotential theory*, Memoirs of the American Mathematical Society, Volume 178 (American Mathematical Society, Providence, RI, 2005).
13. K. Liu and X. Ma, A remark on 'Some numerical results in complex differential geometry' [arxiv.org/abs/math/0512625] by S. K. Donaldson, *Math. Res. Lett.* **14**(2) (2007), 165–171.
14. Z. Lu, On the lower order terms of the asymptotic expansion of Tian–Yau–Zelditch, *Am. J. Math.* **122** (2000), 235–273.
15. H. Luo, Geometric criterion for Gieseker–Mumford stability of polarized manifolds, *J. Diff. Geom.* **49** (1998), 577–599.
16. T. Mabuchi, K-energy maps integrating Futaki invariants, *Tohoku Math. J.* **38** (1986), 575–593.
17. A. Nadel, On the absence of periodic points for the Ricci curvature operator acting on the space of Kähler metrics, in *Modern Methods in Complex Analysis: Princeton Conference in Honor of Gunning and Kohn*, Annals of Mathematics Studies, Volume 137 (Princeton University Press, 1995).
18. N. Pali, Characterization of Einstein–Fa no manifolds via the Kähler–Ricci flow, preprint arXiv:math/0607581v2 (2007).
19. D. H. Phong, J. Song, J. Sturm and B. Weinkove, The Moser–Trudinger inequality on Kähler–Einstein manifolds, *Am. J. Math.* **130**(4) (2008), 1067–1081.
20. Y. Rubinstein, Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kahler metrics, *Adv. Math.* **218** (2008), 1526–1565.
21. Y.-T. Siu, *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler–Einstein metrics* (Birkhäuser, 1987).
22. J. Song and B. Weinkove, Energy functionals and canonical Kähler metrics, *Duke Math. J.* **137** (2007), 159–184.
23. G. Tian, On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, *Invent. Math.* **89** (1987), 225–246.
24. G. Tian, *Canonical metrics in Kähler geometry*, Lectures in Mathematics, ETH Zürich (Birkhäuser, 2000).
25. X. Wang, Canonical metrics on stable vector bundles, *Commun. Analysis Geom.* **13**(2) (2005), 253–285.
26. S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I, *Commun. Pure Appl. Math.* **31** (1978), 339–411.
27. S. Zhang, Heights and reductions of semi-stable varieties, *Compositio Math.* **104** (1996), 77–105.